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A study of Milne-type inequalities for several convex function classes with applications

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Abstract. Fractional integral operators have indeed been the subject of significant research in various mathematical and scientific disciplines over the past few decades. The main aim of this article is to establish a new identity employing the Atangana Baleanu fractional integral operator for the case of differentiable functions. Moreover, we present several fractional Milne-type inequalities for bounded function by fractional integrals. Furthermore, we obtain fractional Milne-type inequalities for the case of Lipschitzian functions. Lastly, we explore applications related to special means, and quadrature formulas.

1. The first section

Convexity is a fundamental concept in mathematics that plays a crucial role in various areas of pure and applied mathematics. Convexity plays a central concept in the development of inequality theory. The concept of a function is one of the fundamental ideas in mathematics and plays a central role in many areas. In additional many researchers have focused on developing new classes of functions and classifying them into different types. These are just a few examples of the classes of functions, convex function, bounded functions, and Lipschitz function that researchers have considered to obtain new error bounds. The convex function, which has applications in statistics, inequality theory, convex programming, and numerical analysis, is one of the types of functions that were defined as the outcome of this significant research. The definition of this fascinating class of functions as follows.

Definition 1.1. *The mapping of $\lambda : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if:*

$$\lambda\left(\xi\delta_1 + (1-\xi)\delta_2\right) \leq \xi\lambda\left(\delta_1\right) + (1-\xi)\lambda\left(\delta_2\right),$$

for all $\delta_1, \delta_2 \in I$ and $\xi \in [0,1]$.

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Fractional calculus has recently been widely recognized as a potent mathematical tool for modelling and analysing many physical and real-world events. The notion of fractional derivatives and integrals is not a fresh revelation, but the methodical advancement and popular acknowledgment of fractional calculus have gained impetus in recent decades. The concept of fractional derivatives and integrals has been used in several mathematical fields, including complex analysis, numerical analysis, function analysis, and Stochastic. Numerous scholars have dedicated their efforts to examining integral inequalities using various fractional operators. The investigation of integral inequalities using fractional operators has been a dynamic and productive domain of study, resulting in notable progress in several branches of mathematics and beyond. Researchers have recently discovered and established a novel limit in integral inequalities, which has created fresh opportunities for further study. The examination of error limits for functions that are differentiable, twice differentiable, or n-times differentiable is a basic component of mathematical analysis. Several trapezoid-type inequalities were derived for other kinds of functions, such as differentiable convex functions [1], bounded functions [2, 10], Lipschitzian functions [3], functions with bounded variation [4], and twice differentiable convex functions [5]. The papers [6, 7] primarily focused on the fractional variations of trapezoid-type inequalities. Various writers have established midpoint-type inequalities for fractional integrals [8, 9], differentiable convex functions [12], and doubly differentiable convex functions [11]. A number of articles have been presented recently that focus on establishing significant inequalities [13]-[23]. The literature includes the following inequality, also known as Simpson's type.

$$\left|\frac{1}{3}\left[\frac{\lambda(\delta_1)+\lambda(\delta_2)}{2}+2\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\lambda(x)\,dx\right]\right|\leq\frac{(\delta_2-\delta_1)^4}{2880}\left\|\lambda^{(4)}\right\|_\infty .$$

where $\lambda:[\delta_1,\delta_2]\to\mathbb{R}$ is a four time differentiable function on (δ_1,δ_2) and $\left\|\lambda^{(4)}\right\|_\infty=\sup_{x\in(\delta_1,\delta_2)}\left|\lambda^{(4)}(x)\right|<\infty$.

Milne's formula, which is of open type, is parallel to Simpson's formula, which is of closed type, in terms of Newton-Cotes formulas, because they hold under the same conditions.

Definition 1.2. *Suppose that $\lambda:[\delta_1,\delta_2]\to\mathbb{R}$ is a four-times continuously differentiable mapping on (δ_1,δ_2), and let $\left\|\lambda^{(4)}\right\|_\infty=\sup_{x\in(\delta_1,\delta_2)}\left|\lambda^{(4)}(x)\,dx\right|<\infty$. Then, one has the inequality [24].*

$$\left|\left[\frac{2}{3}\lambda(\delta_1)-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)+\frac{2}{3}\lambda(\delta_2)\right]-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\lambda(x)\,dx\right|\leq\frac{7(\delta_2-\delta_1)^4}{23040}\left\|\lambda^{(4)}\right\|_\infty .$$

Definition 1.3. *[25] Let $H^1(\delta_1,\delta_2)$ be the Sobolev space of order one defined as*

$$H^1(\delta_1,\delta_2)=\left\{g\in L^2(\delta_1,\delta_2):g'\in L^2(\delta_1,\delta_2)\right\}$$

where

$$L^2(\delta_1,\delta_2)=\left\{g(z):\left(\int_{\delta_1}^{\delta_2}g^2(z)\,dz\right)^{\frac{1}{2}}<\infty\right\}.$$

Let $\lambda\in H^1(\delta_1,\delta_2)$, $\delta_1<\delta_2$, $\alpha\in[0,1]$, then the notion of left derivative in the sense of Caputo-Fabrizio is defined as:

$$\left({}_{\delta_1}^{CFD}D^{\alpha}\lambda\right)(x)=\frac{\beta(\alpha)}{1-\alpha}\int_{\delta_1}^{x}\lambda'(\xi)\,e^{\frac{-\alpha(x-\xi)^{\alpha}}{1-\alpha}}\,d\xi,$$

$x>\alpha$ and the associated integral operator is

$$\left({}_{\delta_1}^{CF}I^{\alpha}\lambda\right)(x)=\frac{1-\alpha}{\beta(\alpha)}\lambda(x)+\frac{\alpha}{\beta(\alpha)}\int_{\delta_1}^{x}\lambda(\xi)\,d\xi$$

where $\beta(\alpha) > 0$ is the normalization function satisfying $\beta(0) = \beta(1) = 1$. For $\alpha = 0, \alpha = 1$, the left derivative is defined as follows, respectively

$$\begin{aligned}\left({}_{\delta_1}^{CFD}D^0\lambda\right)(x) &= \lambda'(x)\\ \left({}_{\delta_1}^{CFD}D^1\lambda\right)(x) &= \lambda(x) - \lambda(\delta_1).\end{aligned}$$

For the right derivative operator

$$\left({}_{\delta_2}^{CFD}D^\alpha\lambda\right)(x) = \frac{\beta(\alpha)}{1-\alpha}\int_x^{\delta_2}\lambda'(\xi)\,e^{\frac{-\alpha(\xi-x)^\alpha}{1-\alpha}}d\xi,$$

$x < \delta_2$ and the associated integral operator is

$$\left({}^{CF}I_{\delta_2}^\alpha\lambda\right)(x) = \frac{1-\alpha}{\beta(\alpha)}\lambda(x) + \frac{\alpha}{\beta(\alpha)}\int_x^{\delta_2}\lambda(\xi)\,d\xi,$$

where $\beta(\alpha) > 0$ is a normalization function that satisfies $\beta(0) = \beta(1) = 1$.

Definition 1.4. *[26] The fractional integral operator with non-local kernel of function $\lambda \in H^1(\delta_1, \delta_2)$ is defined as:*

$${}_{\delta_1}^{AB}I_x^\alpha(\lambda(x)) = \frac{1-\alpha}{\beta(\alpha)}\lambda(x) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\int_{\delta_1}^{x}\lambda(\xi)(x-\xi)^{\alpha-1}d\xi,$$

and right hand sides of AB-fractional operator

$${}_{\delta_2}^{AB}I_x^\alpha(\lambda(x)) = \frac{1-\alpha}{\beta(\alpha)}\lambda(x) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\int_x^{\delta_2}\lambda(\xi)(\xi-x)^{\alpha-1}d\xi,$$

where $\delta_2 > \delta_1$, $\alpha \in [0, 1]$ and $\beta(\alpha) > 0$ is a normalization function.

Theorem 1.5. *[28] Let $\lambda : [\delta_1, \delta_2] \to \mathbb{R}$ be a differentiable function on I^o, $\delta_1, \delta_2 \in I^o$ with $\delta_1 < \delta_2$, where $\lambda' \in L[\delta_1, \delta_2]$. If $|\lambda'|$ is convex function on $[\delta_1, \delta_2]$, then we get the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\lambda(\delta_1) - \lambda\left(\frac{\delta_1+\delta_2}{2}\right) + 2\lambda(\delta_2)\right]\right.\\ &\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta_2-\delta_1)^\alpha}\left[J_{\delta_1^+}^\alpha\lambda\left(\frac{\delta_1+\delta_2}{2}\right) + J_{\delta_2^-}^\alpha\lambda\left(\frac{\delta_1+\delta_2}{2}\right)\right]\right|\\ \leq\ &\frac{\delta_2-\delta_1}{12}\left(\frac{\alpha+4}{\alpha+1}\right)(|\lambda'(\delta_1)| + |\lambda'(\delta_2)|).\end{aligned}$$

Proposition 1.6. *If we take $\alpha = 1$ in Theorem 1.5, we have*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\lambda(\delta_1) - \lambda\left(\frac{\delta_1+\delta_2}{2}\right) + 2\lambda(\delta_1)\right] - \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\lambda(u)\,du\right|\\ \leq\ &\frac{5(\delta_2-\delta_1)}{24}(|\lambda'(\delta_1)| + |\lambda'(\delta_2)|).\end{aligned}$$

Theorem 1.7. *[28] Let $\lambda : [\delta_1, \delta_2] \to \mathbb{R}$ be a differentiable function on I^o, $\delta_1, \delta_2 \in I^o$ with $\delta_1 < \delta_2$, where $\lambda' \in L[\delta_1, \delta_2]$. If $|\lambda'|$ is an L-Lipschitz function on $[\delta_1, \delta_2]$, then we get the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\lambda\left(\delta_1\right)-\lambda\left(\frac{\delta_1+\delta_2}{2}\right)+2\lambda\left(\delta_2\right)\right]\right.\\
&\left.-\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(\delta_2-\delta_1\right)^{\alpha}}\left[J^{\alpha}_{\delta_1^+}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)+J^{\alpha}_{\delta_2^-}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)\right]\right|\\
\leq\ &\frac{\left(\delta_2-\delta_1\right)^2}{24}\left(\frac{\alpha+8}{\alpha+2}\right)L.
\end{aligned}$$

Proposition 1.8. *If we take $\alpha = 1$ in Theorem 1.7, we have*

$$\left|\frac{1}{3}\left[2\lambda\left(\delta_1\right)-\lambda\left(\frac{\delta_1+\delta_2}{2}\right)+2\lambda\left(\delta_2\right)\right]-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\lambda\left(u\right)du\right|\leq\frac{\left(\delta_2-\delta_1\right)^2}{8}L.$$

Motivated by ongoing studies, we are established a new identity for Atangana Baleanu integral operator. Then utilizing this identity we are presented a novel fractional version of the Milne type inequalities for functions whose absolute value of differentiable are convex. Employing well knows inequalities like Power-mean, Hölder, and bounded functions are obtained to our auxiliary result. Finally, we are discussed some applications to special means, and quadrature formula.

2. An Identity for Differentiable Function

In this section we deal with identity, which is necessary to attain our main result.

Lemma 2.1. *Let $\lambda : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o, $\delta_1, \delta_2 \in I^o$ with $\delta_1 < \delta_2$, and $\lambda' \in L^1[\delta_1, \delta_2]$, then the following fractional equality holds:*

$$\begin{aligned}
&\frac{2}{3}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{2^{\alpha-1}\left(1-\alpha\right)}{\beta\left(\alpha\right)\left(\delta_2-\delta_1\right)^{\alpha}}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)\\
&-\frac{2^{\alpha-1}}{\left(\delta_2-\delta_1\right)^{\alpha}}\left(\left({}^{AB}_{\delta_1}I^{\alpha}_{\frac{\delta_1+\delta_2}{2}}\lambda\left(\delta_1\right)\right)+\left({}^{AB}_{\frac{\delta_1+\delta_2}{2}}I^{\alpha}_{\delta_2}\lambda\left(\delta_2\right)\right)\right)\\
=\ &\frac{\delta_2-\delta_1}{4}\left[\int_0^1\left(\xi^{\alpha}-\frac{4}{3}\right)\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)\right.\\
&\left.-\left(\xi^{\alpha}-\frac{4}{3}\right)\lambda'\left(\left(\frac{\xi}{2}\right)\delta_1+\lambda'\left(\frac{2-\xi}{2}\right)\delta_2\right)d\xi\right].
\end{aligned}$$

Proof. Let

$$\begin{aligned}
&\int_0^1\left(\xi^{\alpha}-\frac{4}{3}\right)\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)d\xi\\
&-\int_0^1\left(\xi^{\alpha}-\frac{4}{3}\right)\lambda'\left(\left(\frac{\xi}{2}\right)\delta_1+\lambda'\left(\frac{2-\xi}{2}\right)\delta_2\right)d\xi\\
=\ &I_1-I_2.
\end{aligned}$$

Integration by parts

$$\begin{aligned}
I_1 =\ &\int_0^1\left(\xi^{\alpha}-\frac{4}{3}\right)\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)d\xi\\
=\ &\frac{2}{\delta_2-\delta_1}\left(\xi^{\alpha}-\frac{4}{3}\right)\lambda\left(\left(\frac{2-\xi}{2}\right)\delta_1+\frac{\xi}{2}\delta_2\right)\Big|_0^1-\frac{2\alpha}{\delta_2-\delta_1}\int_0^1\xi^{\alpha-1}\lambda\left(\left(\frac{2-\xi}{2}\right)\delta_1+\frac{\xi}{2}\delta_2\right)d\xi\\
=\ &\frac{-2}{3\left(\delta_2-\delta_1\right)}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)+\frac{8}{3\left(\delta_2-\delta_1\right)}\lambda\left(\delta_1\right)-\frac{2\alpha}{\delta_2-\delta_1}\int_0^1\xi^{\alpha-1}\lambda\left(\left(\frac{2-\xi}{2}\right)\delta_1+\frac{\xi}{2}\delta_2\right)d\xi
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-2}{3(\delta_2-\delta_1)}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)+\frac{8}{3(\delta_2-\delta_1)}\lambda(\delta_1)-\left(\frac{2}{\delta_2-\delta_1}\right)^{\alpha+1}\alpha\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}(u-\delta_1)^{\alpha-1}\lambda(u)\,du\\
&= \frac{-2}{3(\delta_2-\delta_1)}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)+\frac{8}{3(\delta_2-\delta_1)}\lambda(\delta_1)-\left(\frac{2}{\delta_2-\delta_1}\right)^{\alpha+1}\\
&\quad\times\left[\frac{\alpha\beta(\alpha)\Gamma\alpha}{\beta(\alpha)\Gamma\alpha}\int_{\delta_1}^{\frac{\delta_1+\delta_2}{2}}(u-\delta_1)^{\alpha-1}+\frac{1-\alpha}{\beta(\alpha)}\lambda(\delta_1)-\frac{1-\alpha}{\beta(\alpha)}\lambda(\delta_1)\right]\\
&= \frac{-2}{3(\delta_2-\delta_1)}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)+\frac{8}{3(\delta_2-\delta_1)}\lambda(\delta_1)-\left(\frac{2}{\delta_2-\delta_1}\right)^{\alpha+1}\frac{1-\alpha}{\beta(\alpha)}\lambda(\delta_1)\\
&\quad-\left(\frac{2}{\delta_2-\delta_1}\right)^{\alpha+1}\left({}^{AB}_{\delta_1}I^{\alpha}_{\frac{\delta_1+\delta_2}{2}}\lambda(\delta_1)\right). \qquad (1)
\end{aligned}
$$

Similarly I_2, we get

$$
\begin{aligned}
I_2 &= \int_0^1\left(\xi^{\alpha}-\frac{4}{3}\right)\lambda'\left(\left(\frac{\xi}{2}\right)\delta_1+\lambda'\left(\frac{2-\xi}{2}\right)\delta_2\right)d\xi\\
&= \frac{2}{\delta_2-\delta_1}\left(\xi^{\alpha}-\frac{4}{3}\right)\lambda\left(\frac{\xi}{2}\delta_1+\left(\frac{2-\xi}{2}\right)\delta_2\right)\Bigg|_0^1+\frac{2\alpha}{\delta_2-\delta_1}\int_0^1\xi^{\alpha-1}\lambda\left(\frac{\xi}{2}\delta_1+\left(\frac{2-\xi}{2}\right)\delta_2\right)d\xi\\
&= \frac{2}{3(\delta_2-\delta_1)}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{8}{3(\delta_2-\delta_1)}\lambda(\delta_2)+\frac{2\alpha}{\delta_2-\delta_1}\int_0^1\xi^{\alpha-1}\lambda\left(\frac{\xi}{2}\delta_1+\left(\frac{2-\xi}{2}\right)\delta_2\right)d\xi\\
&= \frac{2}{3(\delta_2-\delta_1)}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{8}{3(\delta_2-\delta_1)}\lambda(\delta_2)+\left(\frac{2}{\delta_2-\delta_1}\right)^{\alpha+1}\alpha\int_{\frac{\delta_1+\delta_2}{2}}^{\delta_2}(\delta_2-u)^{\alpha-1}\lambda(u)\,du\\
&= \frac{2}{3(\delta_2-\delta_1)}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{8}{3(\delta_2-\delta_1)}\lambda(\delta_2)+\left(\frac{2}{\delta_2-\delta_1}\right)^{\alpha+1}\\
&\quad\times\left[\frac{\alpha\beta(\alpha)\Gamma\alpha}{\beta(\alpha)\Gamma\alpha}\int_{\frac{\delta_1+\delta_2}{2}}^{\delta_2}(\delta_2-u)^{\alpha-1}\lambda(u)\,du+\frac{1-\alpha}{\beta(\alpha)}\lambda(\delta_2)-\frac{1-\alpha}{\beta(\alpha)}\lambda(\delta_2)\right]\\
&= \frac{2}{3(\delta_2-\delta_1)}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{8}{3(\delta_2-\delta_1)}\lambda(\delta_2)+\left(\frac{2}{\delta_2-\delta_1}\right)^{\alpha+1}\frac{1-\alpha}{\beta(\alpha)}\lambda(\delta_2)\\
&\quad+\left(\frac{2}{\delta_2-\delta_1}\right)^{\alpha+1}\left({}^{AB}_{\frac{\delta_1+\delta_2}{2}}I^{\alpha}_{\delta_2}\lambda(\delta_2)\right). \qquad (2)
\end{aligned}
$$

By equalities (1) and (2), we get

$$
\begin{aligned}
&\frac{(\delta_2-\delta_1)}{4}(I_1-I_2)\\
&= \frac{2}{3}(\lambda(\delta_1)+\lambda(\delta_2))-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{2^{\alpha-1}(1-\alpha)}{\beta(\alpha)(\delta_2-\delta_1)^{\alpha}}(\lambda(\delta_1)+\lambda(\delta_2))\\
&\quad-\frac{2^{\alpha-1}}{(\delta_2-\delta_1)^{\alpha}}\left(\left({}^{AB}_{\delta_1}I^{\alpha}_{\frac{\delta_1+\delta_2}{2}}\lambda(\delta_1)\right)+\left({}^{AB}_{\frac{\delta_1+\delta_2}{2}}I^{\alpha}_{\delta_2}\lambda(\delta_2)\right)\right).
\end{aligned}
$$

The proof of Lemma 2.1 is completed. □

Theorem 2.2. *Suppose that assumptions of Lemma 2.1 hold. If $|\lambda'|$ is a convex function on $[\delta_1, \delta_2]$, then the following fractional inequality holds:*

$$
\begin{aligned}
&\left|\frac{2}{3}(\lambda(\delta_1)+\lambda(\delta_2))-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{2^{\alpha-1}(1-\alpha)}{\beta(\alpha)(\delta_2-\delta_1)^{\alpha}}(\lambda(\delta_1)+\lambda(\delta_2))\right.\\
&\quad\left.-\frac{2^{\alpha-1}}{(\delta_2-\delta_1)^{\alpha}}\left(\left({}^{AB}_{\delta_1}I^{\alpha}_{\frac{\delta_1+\delta_2}{2}}\lambda(\delta_1)\right)+\left({}^{AB}_{\frac{\delta_1+\delta_2}{2}}I^{\alpha}_{\delta_2}\lambda(\delta_2)\right)\right)\right|\\
&\leq \frac{\delta_2-\delta_1}{12}\left(\frac{4\alpha+1}{\alpha+1}\right)[|\lambda'(\delta_1)|+|\lambda'(\delta_2)|].
\end{aligned}
$$

Proof. By taking modulus in Lemma 2.1, with the help of the convexity $|\lambda'|$, we get

$$\begin{aligned}
&\left|\frac{2}{3}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{2^{\alpha-1}\left(1-\alpha\right)}{\beta\left(\alpha\right)\left(\delta_2-\delta_1\right)^{\alpha}}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)\right.\\
&\left.-\frac{2^{\alpha-1}}{\left(\delta_2-\delta_1\right)^{\alpha}}\left(\left({}_{\delta_1}^{AB}I_{\frac{\delta_1+\delta_2}{2}}^{\alpha}\lambda\left(\delta_1\right)\right)+\left({}_{\frac{\delta_1+\delta_2}{2}}^{AB}I_{\delta_2}^{\alpha}\lambda\left(\delta_2\right)\right)\right)\right|\\
\leq\;&\frac{\delta_2-\delta_1}{4}\left[\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|\left|\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)\right|d\xi\right.\\
&\left.+\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|\left|\lambda'\left(\left(\frac{\xi}{2}\right)\delta_1+\lambda'\left(\frac{2-\xi}{2}\right)\delta_2\right)\right|d\xi\right]\\
\leq\;&\frac{\delta_2-\delta_1}{4}\left[\left(\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|\left(\frac{2-\xi}{2}\left|\lambda'\left(\delta_1\right)\right|+\frac{\xi}{2}\left|\lambda'\left(\delta_2\right)\right|\right)d\xi\right)\right.\\
&\left.+\left(\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|\left(\frac{\xi}{2}\left|\lambda'\left(\delta_1\right)\right|+\frac{2-\xi}{2}\left|\lambda'\left(\delta_2\right)\right|\right)d\xi\right)\right]\\
\leq\;&\frac{\delta_2-\delta_1}{4}\left[\left(\frac{2\alpha^2+5\alpha+1}{2\left(\alpha+1\right)\left(\alpha+2\right)}\right)\left|\lambda'\left(\delta_1\right)\right|+\left(\frac{2\alpha+1}{6\left(\alpha+2\right)}\right)\left|\lambda'\left(\delta_2\right)\right|\right.\\
&\left.+\left(\frac{2\alpha+1}{6\left(\alpha+2\right)}\right)\left|\lambda'\left(\delta_1\right)\right|+\left(\frac{2\alpha^2+5\alpha+1}{2\left(\alpha+1\right)\left(\alpha+2\right)}\right)\left|\lambda'\left(\delta_2\right)\right|\right]\\
\leq\;&\frac{\delta_2-\delta_1}{12}\left(\frac{4\alpha+1}{\alpha+1}\right)\left[\left|\lambda'\left(\delta_1\right)\right|+\left|\lambda'\left(\delta_2\right)\right|\right].
\end{aligned}$$

The proof of Theorem 2.2 is completed. □

Remark 2.3. *Assume that $\alpha = 1$ in Theorem 2.2, we have*

$$\begin{aligned}
&\left|\frac{1}{3}\left(2\lambda\left(\delta_1\right)-\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-2\lambda\left(\delta_2\right)\right)-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\lambda\left(u\right)du\right|\\
\leq\;&\frac{5\left(\delta_2-\delta_1\right)}{24}\left(\left|\lambda'\left(\delta_1\right)\right|+\left|\lambda'\left(\delta_2\right)\right|\right),
\end{aligned}$$

which is proved by Alomari in [27].

Theorem 2.4. *Suppose that assumptions of Lemma 2.1 hold. If $|\lambda'|^q$ is a convex function on $[\delta_1, \delta_2]$ and $q > 1$, then the following fractional inequality holds:*

$$\begin{aligned}
&\left|\frac{2}{3}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{2^{\alpha-1}\left(1-\alpha\right)}{\beta\left(\alpha\right)\left(\delta_2-\delta_1\right)^{\alpha}}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)\right.\\
&\left.-\frac{2^{\alpha-1}}{\left(\delta_2-\delta_1\right)^{\alpha}}\left(\left({}_{\delta_1}^{AB}I_{\frac{\delta_1+\delta_2}{2}}^{\alpha}\lambda\left(\delta_1\right)\right)+\left({}_{\frac{\delta_1+\delta_2}{2}}^{AB}I_{\delta_2}^{\alpha}\lambda\left(\delta_2\right)\right)\right)\right|\\
\leq\;&\frac{\delta_2-\delta_1}{4}\left(\left(\frac{4}{3}\right)^{p}-\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\lambda'\left(\delta_1\right)\right|^q+\left|\lambda'\left(\delta_2\right)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\lambda'\left(\delta_1\right)\right|^q+3\left|\lambda'\left(\delta_2\right)\right|^q}{4}\right)^{\frac{1}{q}}\right]\\
\leq\;&\frac{\delta_2-\delta_1}{4}\left(\left(\frac{16}{3}\right)^{p}-\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left|\lambda'\left(\delta_1\right)\right|+\left|\lambda'\left(\delta_2\right)\right|\right),
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By taking modulus in Lemma 2.1, we have

$$\begin{aligned}
&\left|\frac{2}{3}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{2^{\alpha-1}\left(1-\alpha\right)}{\beta\left(\alpha\right)\left(\delta_2-\delta_1\right)^{\alpha}}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)\right.\\
&\left.-\frac{2^{\alpha-1}}{\left(\delta_2-\delta_1\right)^{\alpha}}\left(\left({}^{AB}_{\delta_1}I^{\alpha}_{\frac{\delta_1+\delta_2}{2}}\lambda\left(\delta_1\right)\right)+\left({}^{AB}_{\frac{\delta_1+\delta_2}{2}}I^{\alpha}_{\delta_2}\lambda\left(\delta_2\right)\right)\right)\right|\\
\leq\;&\frac{\delta_2-\delta_1}{4}\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|\left[\left|\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)\right|d\xi\right. \qquad (3)\\
&\left.+\left|\lambda'\left(\left(\frac{\xi}{2}\right)\delta_1+\lambda'\left(\frac{2-\xi}{2}\right)\delta_2\right)\right|d\xi\right].
\end{aligned}$$

By Hölder inequality in (3) and using the convexity of $|\lambda'|^q$, we get

$$\begin{aligned}
&\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|\left|\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)\right|d\xi\\
\leq\;&\left(\int_0^1\left(\frac{4}{3}-\xi^{\alpha}\right)^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)\right|^q d\xi\right)^{\frac{1}{q}}\\
\leq\;&\left(\int_0^1\left(\frac{4}{3}-\xi^{\alpha}\right)^p d\xi\right)^{\frac{1}{p}}\left[\int_0^1\left(\frac{2-\xi}{2}\left|\lambda'\left(\delta_1\right)\right|^q+\frac{\xi}{2}\left|\lambda'\left(\delta_2\right)\right|^q\right)d\xi\right]^{\frac{1}{q}}\\
\leq\;&\left(\left(\frac{4}{3}\right)^p-\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{3\left|\lambda'\left(\delta_1\right)\right|^q+\left|\lambda'\left(\delta_2\right)\right|^q}{4}\right)^{\frac{1}{q}}. \qquad (4)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|\left|\lambda'\left(\left(\frac{\xi}{2}\right)\delta_1+\lambda'\left(\frac{2-\xi}{2}\right)\delta_2\right)\right|d\xi\\
\leq\;&\left(\left(\frac{4}{3}\right)^p-\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left|\lambda'\left(\delta_1\right)\right|^q+3\left|\lambda'\left(\delta_2\right)\right|^q}{4}\right)^{\frac{1}{q}}. \qquad (5)
\end{aligned}$$

Here we have used the fact that

$$|A-B|<A^p-B^p,$$

for $A>B$ and $p\geq 1$. By putting (4), (5) in (3), then we obtain the first inequality.

For the proof of the second inequality, let $a_1=3\left|\lambda'\left(\delta_1\right)\right|^q$, $b_1=\left|\lambda'\left(\delta_2\right)\right|^q$, $a_2=\left|\lambda'\left(\delta_1\right)\right|^q$, and $b_2=3\left|\lambda'\left(\delta_2\right)\right|^q$ utilizing the fact that

$$\sum_{k=1}^{n}\left(a_k+b_k\right)^s\leq\sum_{k=1}^{n}a_k^s+\sum_{k=1}^{n}b_k^s,\ 0\leq s<1,$$

and $1+3^{\frac{1}{q}}$ the required result can be established straightforwardly.

The proof of Theorem 2.4 is completed. □

Remark 2.5. *Assume that $\alpha=1$ in Theorem 2.4, we have*

$$\begin{aligned}
&\left|\frac{1}{3}\left(2\lambda\left(\delta_1\right)-\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-2\lambda\left(\delta_2\right)\right)-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\lambda\left(u\right)du\right|\\
\leq\;&\frac{\delta_2-\delta_1}{4}\left(\left(\frac{16}{3}\right)^p-\frac{4}{p+1}\right)^{\frac{1}{p}}\left(\left|\lambda'\left(\delta_1\right)\right|+\left|\lambda'\left(\delta_2\right)\right|\right),
\end{aligned}$$

which is proved by Budak and Kösem [29, Corollary 1].

Theorem 2.6. *Suppose that assumptions of Lemma 2.1 hold. If $|\lambda'|^q$ is a convex function on $[\delta_1, \delta_2]$ and $q \geq 1$, then the following fractional inequality holds:*

$$\begin{aligned}
&\left|\frac{2}{3}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{2^{\alpha-1}(1-\alpha)}{\beta(\alpha)\left(\delta_2-\delta_1\right)^{\alpha}}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)\right.\\
&\left.-\frac{2^{\alpha-1}}{\left(\delta_2-\delta_1\right)^{\alpha}}\left(\left({}_{\delta_1}^{AB}I^{\alpha}_{\frac{\delta_1+\delta_2}{2}}\lambda\left(\delta_1\right)\right)+\left({}^{AB}_{\frac{\delta_1+\delta_2}{2}}I^{\alpha}_{\delta_2}\lambda\left(\delta_2\right)\right)\right)\right|\\
\leq\ &\frac{\delta_2-\delta_1}{4}\left(\frac{4\alpha+1}{3(1+\alpha)}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{2\alpha^2+5\alpha+1}{2(\alpha+1)(\alpha+2)}\right)\left|\lambda'\left(\delta_1\right)\right|^q+\left(\frac{2\alpha+1}{6(\alpha+2)}\right)\left|\lambda'\left(\delta_2\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left(\frac{2\alpha+1}{6(\alpha+2)}\right)\left|\lambda'\left(\delta_1\right)\right|^q+\left(\frac{2\alpha^2+5\alpha+1}{2(\alpha+1)(\alpha+2)}\right)\left|\lambda'\left(\delta_2\right)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. By using of Lemma 2.1, the power-mean inequality and using the convexity of $|\lambda'|^q$, we have

$$\begin{aligned}
&\left|\frac{2}{3}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{2^{\alpha-1}(1-\alpha)}{\beta(\alpha)\left(\delta_2-\delta_1\right)^{\alpha}}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)\right.\\
&\left.-\frac{2^{\alpha-1}}{\left(\delta_2-\delta_1\right)^{\alpha}}\left(\left({}_{\delta_1}^{AB}I^{\alpha}_{\frac{\delta_1+\delta_2}{2}}\lambda\left(\delta_1\right)\right)+\left({}^{AB}_{\frac{\delta_1+\delta_2}{2}}I^{\alpha}_{\delta_2}\lambda\left(\delta_2\right)\right)\right)\right|\\
\leq\ &\frac{\delta_2-\delta_1}{4}\left[\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|\left|\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)\right|d\xi\right.\\
&\left.+\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|\left|\lambda'\left(\left(\frac{\xi}{2}\right)\delta_1+\lambda'\left(\frac{2-\xi}{2}\right)\delta_2\right)\right|d\xi\right]\\
\leq\ &\frac{\delta_2-\delta_1}{4}\left[\left(\int_0^1\left(\frac{4}{3}-\xi^{\alpha}\right)d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|\left|\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)\right|^q d\xi\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\left(\frac{4}{3}-\xi^{\alpha}\right)d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|\left|\lambda'\left(\left(\frac{\xi}{2}\right)\delta_1+\lambda'\left(\frac{2-\xi}{2}\right)\delta_2\right)\right|^q d\xi\right)^{\frac{1}{q}}\right]\\
\leq\ &\frac{\delta_2-\delta_1}{4}\left[\left(\int_0^1\left(\frac{4}{3}-\xi^{\alpha}\right)d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|\left(\left(\frac{2-\xi}{2}\right)\left|\lambda'\left(\delta_1\right)\right|^q+\frac{\xi}{2}\left|\lambda'\left(\delta_2\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\left(\frac{4}{3}-\xi^{\alpha}\right)d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|\left(\frac{\xi}{2}\left|\lambda'\left(\delta_1\right)\right|^q+\left(\frac{2-\xi}{2}\right)\left|\lambda'\left(\delta_2\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right]\\
\leq\ &\frac{\delta_2-\delta_1}{4}\left(\frac{4\alpha+1}{3(1+\alpha)}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{2\alpha^2+5\alpha+1}{2(\alpha+1)(\alpha+2)}\right)\left|\lambda'\left(\delta_1\right)\right|^q+\left(\frac{2\alpha+1}{6(\alpha+2)}\right)\left|\lambda'\left(\delta_2\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left(\frac{2\alpha+1}{6(\alpha+2)}\right)\left|\lambda'\left(\delta_1\right)\right|^q+\left(\frac{2\alpha^2+5\alpha+1}{2(\alpha+1)(\alpha+2)}\right)\left|\lambda'\left(\delta_2\right)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

The proof of Theorem 2.6 is completed. □

Remark 2.7. *Assume that $\alpha = 1$ in Theorem 2.6, we have*

$$\left|\frac{1}{3}\left(2\lambda\left(\delta_1\right)-\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-2\lambda\left(\delta_2\right)\right)-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\lambda(u)\,du\right|$$

$$\leq \frac{(\delta_2-\delta_1)}{4}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left[\left(\frac{4\left|\lambda'(\delta_1)\right|^q+\left|\lambda'(\delta_2)\right|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|\lambda'(\delta_1)\right|^q+4\left|\lambda'(\delta_2)\right|^q}{6}\right)^{\frac{1}{q}}\right],$$

which is obtained by Budak and Kösem [29, Remark 2].

Theorem 2.8. *Suppose that assumptions of Lemma 2.1 hold. If their exist constant $-\infty < m < M < +\infty$ such that $m \leq |\lambda'(x)| \leq M$ for all $x \in [\delta_1, \delta_2]$, then the following fractional inequality holds:*

$$\begin{aligned}&\left|\frac{2}{3}(\lambda(\delta_1)+\lambda(\delta_2))-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{2^{\alpha-1}(1-\alpha)}{\beta(\alpha)(\delta_2-\delta_1)^\alpha}(\lambda(\delta_1)+\lambda(\delta_2))\right.\\&\left.-\frac{2^{\alpha-1}}{(\delta_2-\delta_1)^\alpha}\left(\left({}^{AB}_{\delta_1}I^{\alpha}_{\frac{\delta_1+\delta_2}{2}}\lambda(\delta_1)\right)+\left({}^{AB}_{\frac{\delta_1+\delta_2}{2}}I^{\alpha}_{\delta_2}\lambda(\delta_2)\right)\right)\right|\\&\leq \frac{(\delta_2-\delta_1)(4\alpha+1)}{12(\alpha+1)}(M-m).\end{aligned}$$

Proof. By using the Lemma 2.1, we have

$$\begin{aligned}&\frac{2}{3}(\lambda(\delta_1)+\lambda(\delta_2))-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{2^{\alpha-1}(1-\alpha)}{\beta(\alpha)(\delta_2-\delta_1)^\alpha}(\lambda(\delta_1)+\lambda(\delta_2))\\&-\frac{2^{\alpha-1}}{(\delta_2-\delta_1)^\alpha}\left(\left({}^{AB}_{\delta_1}I^{\alpha}_{\frac{\delta_1+\delta_2}{2}}\lambda(\delta_1)\right)+\left({}^{AB}_{\frac{\delta_1+\delta_2}{2}}I^{\alpha}_{\delta_2}\lambda(\delta_2)\right)\right)\\&=\frac{\delta_2-\delta_1}{4}\left[\int_0^1\left(\xi^\alpha-\frac{4}{3}\right)\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)d\xi\right.\\&\left.-\int_0^1\left(\xi^\alpha-\frac{4}{3}\right)\lambda'\left(\left(\frac{\xi}{2}\right)\delta_1+\lambda'\left(\frac{2-\xi}{2}\right)\delta_2\right)d\xi\right]\\&=\frac{\delta_2-\delta_1}{4}\int_0^1\left(\xi^\alpha-\frac{4}{3}\right)\left[\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)-\frac{m+M}{2}\right]d\xi\\&+\frac{\delta_2-\delta_1}{4}\int_0^1\left(\xi^\alpha-\frac{4}{3}\right)\left[\frac{m+M}{2}-\lambda'\left(\left(\frac{\xi}{2}\right)\delta_1+\lambda'\left(\frac{2-\xi}{2}\right)\delta_2\right)\right]d\xi. \qquad (6)\end{aligned}$$

Employing the absolute value on both sides of equality (6), we have

$$\begin{aligned}&\left|\frac{2}{3}(\lambda(\delta_1)+\lambda(\delta_2))-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{2^{\alpha-1}(1-\alpha)}{\beta(\alpha)(\delta_2-\delta_1)^\alpha}(\lambda(\delta_1)+\lambda(\delta_2))\right.\\&\left.-\frac{2^{\alpha-1}}{(\delta_2-\delta_1)^\alpha}\left(\left({}^{AB}_{\delta_1}I^{\alpha}_{\frac{\delta_1+\delta_2}{2}}\lambda(\delta_1)\right)+\left({}^{AB}_{\frac{\delta_1+\delta_2}{2}}I^{\alpha}_{\delta_2}\lambda(\delta_2)\right)\right)\right|\\&\leq\frac{\delta_2-\delta_1}{4}\left[\int_0^1\left|\xi^\alpha-\frac{4}{3}\right|\left|\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)-\frac{m+M}{2}\right|d\xi\right.\\&\left.+\int_0^1\left|\xi^\alpha-\frac{4}{3}\right|\left|\frac{m+M}{2}-\lambda'\left(\left(\frac{\xi}{2}\right)\delta_1+\lambda'\left(\frac{2-\xi}{2}\right)\delta_2\right)\right|d\xi\right]. \qquad (7)\end{aligned}$$

Since $m \leq |\lambda'(x)| \leq M$ for all $x \in [\delta_1, \delta_2]$, we get

$$\left|\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)-\frac{m+M}{2}\right|\leq\frac{M-m}{2}, \qquad (8)$$

and

$$\left|\frac{m+M}{2}-\lambda'\left(\left(\frac{\xi}{2}\right)\delta_1+\lambda'\left(\frac{2-\xi}{2}\right)\delta_2\right)\right|\leq\frac{M-m}{2}. \qquad (9)$$

Utilizing these equalities (8), (9) in (7), we have

$$\begin{aligned}
&\left|\frac{2}{3}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{2^{\alpha-1}(1-\alpha)}{\beta(\alpha)\left(\delta_2-\delta_1\right)^{\alpha}}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)\right.\\
&\left.-\frac{2^{\alpha-1}}{\left(\delta_2-\delta_1\right)^{\alpha}}\left(\left({}^{AB}_{\delta_1}I^{\alpha}_{\frac{\delta_1+\delta_2}{2}}\lambda\left(\delta_1\right)\right)+\left({}^{AB}_{\frac{\delta_1+\delta_2}{2}}I^{\alpha}_{\delta_2}\lambda\left(\delta_2\right)\right)\right)\right|\\
\leq\ & \frac{\left(\delta_2-\delta_1\right)(M-m)}{4}\left[\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|d\xi\right]\\
\leq\ & \frac{\left(\delta_2-\delta_1\right)(4\alpha+1)}{12(\alpha+1)}(M-m).
\end{aligned}$$

This completes the proof. □

Remark 2.9. *Assume that $\alpha=1$ in Theorem 2.8, we have*

$$\left|\frac{1}{3}\left(2\lambda\left(\delta_1\right)-\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-2\lambda\left(\delta_2\right)\right)-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\lambda(u)\,du\right|\leq\frac{5\left(\delta_2-\delta_1\right)}{24}(M-m),$$

which is obtained by Budak and Kösem [29, Corollary 2].

Theorem 2.10. *Suppose that assumptions of Lemma 2.1 hold. If $\lambda'$is L-Lipschitzian function on $[\delta_1,\delta_2]$, then the following fractional inequality holds:*

$$\begin{aligned}
&\left|\frac{2}{3}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{2^{\alpha-1}(1-\alpha)}{\beta(\alpha)\left(\delta_2-\delta_1\right)^{\alpha}}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)\right.\\
&\left.-\frac{2^{\alpha-1}}{\left(\delta_2-\delta_1\right)^{\alpha}}\left(\left({}^{AB}_{\delta_1}I^{\alpha}_{\frac{\delta_1+\delta_2}{2}}\lambda\left(\delta_1\right)\right)+\left({}^{AB}_{\frac{\delta_1+\delta_2}{2}}I^{\alpha}_{\delta_2}\lambda\left(\delta_2\right)\right)\right)\right|\\
\leq\ & \frac{\left(\delta_2-\delta_1\right)^2L}{4}\left[\frac{2}{3}-\frac{1}{(\alpha+1)(\alpha+2)}\right].
\end{aligned}$$

Proof. By using the Lemma 2.1, λ' is L-Lipschitzian function, we get

$$\begin{aligned}
&\left|\frac{2}{3}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)-\frac{1}{3}\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-\frac{2^{\alpha-1}(1-\alpha)}{\beta(\alpha)\left(\delta_2-\delta_1\right)^{\alpha}}\left(\lambda\left(\delta_1\right)+\lambda\left(\delta_2\right)\right)\right.\\
&\left.-\frac{2^{\alpha-1}}{\left(\delta_2-\delta_1\right)^{\alpha}}\left(\left({}^{AB}_{\delta_1}I^{\alpha}_{\frac{\delta_1+\delta_2}{2}}\lambda\left(\delta_1\right)\right)+\left({}^{AB}_{\frac{\delta_1+\delta_2}{2}}I^{\alpha}_{\delta_2}\lambda\left(\delta_2\right)\right)\right)\right|\\
=\ & \frac{\delta_2-\delta_1}{4}\int_0^1\left|\xi^{\alpha}-\frac{4}{3}\right|\left[\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)\right.\\
&\left.-\lambda'\left(\left(\frac{2-\xi}{2}\right)\delta_1+\lambda'\left(\frac{\xi}{2}\right)\delta_2\right)\right]d\xi\\
\leq\ & \frac{\left(\delta_2-\delta_1\right)L}{4}\int_0^1\left(\frac{4}{3}-\xi^{\alpha}\right)\left(\delta_2-\delta_1\right)(1-\xi)\,d\xi\\
\leq\ & \frac{\left(\delta_2-\delta_1\right)^2}{4}L\left[\int_0^1\left(\frac{4}{3}-\xi^{\alpha}\right)(1-\xi)\,d\xi\right]\\
\leq\ & \frac{\left(\delta_2-\delta_1\right)^2L}{4}\left[\frac{2}{3}-\frac{1}{(\alpha+1)(\alpha+2)}\right].
\end{aligned}$$

This completes the proof. □

Remark 2.11. *Assume that* $\alpha = 1$ *in Theorem 2.10, we have*

$$\left|\frac{1}{3}\left(2\lambda\left(\delta_1\right)-\lambda\left(\frac{\delta_1+\delta_2}{2}\right)-2\lambda\left(\delta_2\right)\right)-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\lambda\left(u\right)du\right|\leq\frac{\left(\delta_2-\delta_1\right)^2}{8}L,$$

which is obtained by Budak and Kösem [29, Corollary 4].

3. Application to special means

We shall consider the following special means:
(a) The Arithmetic Mean:

$$A=A\left(\delta_1,\delta_2\right):=\frac{\delta_1+\delta_2}{2},\ \delta_1,\delta_2\geq 0;$$

(b) The Logarithmic Mean:

$$L=\ L(\delta_1,\delta_2):=\frac{\delta_2-\delta_1}{\ln\delta_2-\ln\delta_1}\ \delta_1,\delta_2>0,\delta_1\neq\delta_2;$$

(c) The Generalized logarithmic Mean:

$$L_r^r=\ L_r^r\left(\delta_1,\delta_2\right):=\left[\frac{\delta_2^{r+1}-\delta_1^{r+1}}{\left(r+1\right)\left(\delta_2-\delta_1\right)}\right]^{1/r}\ r\in\mathbb{R}\backslash\{-1,0\},\ \delta_1,\delta_2>0$$

Proposition 3.1. *Let* $\delta_1,\delta_2\in\mathbb{R}$ *with* $0<\delta_1<\delta_2$, *then we have*

$$\left|4A\left(\delta_1^2,\delta_2^2\right)-A^2\left(\delta_1,\delta_2\right)-3L_2^2\left(\delta_1,\delta_2\right)\right|\leq\frac{5\left(\delta_2-\delta_1\right)}{24}\left[\delta_1^2+\delta_2^2\right].$$

Proof. The assertion follows from Theorem 2.2, applying the $\lambda\left(x\right)=\frac{1}{2}x^2$ and $\alpha=1$. □

4. Application to quadrature formula

Considering Z is the partition of the points $\delta_1=x_0<x_1<..........<x_n=\delta_2$ of the interval $[\delta_1,\delta_2]$ and let

$$\int_{\delta_1}^{\delta_2}\lambda\left(x\right)dx=\mu\left(\lambda,Z\right)+R\left(\lambda,Z\right),$$

where

$$\mu\left(\lambda,Z\right)=\sum_{i=0}^{n-1}\left(\frac{x_{i+1}-x_i}{3}\right)\left(2\lambda\left(x_i\right)-\lambda\left(\frac{x_i+x_{i+1}}{2}\right)+2\lambda\left(x_{i+1}\right)\right),$$

and $R\left(\lambda,Z\right)$ constitute the considering approximation error.

Proposition 4.1. *Let* $\lambda:[\delta_1,\delta_2]\to\mathbb{R}$ *is a differentiable function on* (δ_1,δ_2) *with* $0\leq\delta_1<\delta_2$ *and* $\lambda'\in L\left[\delta_1,\delta_2\right]$. *If* $|\lambda'|$ *is convex function, we have*

$$\begin{aligned}\left|R\left(\lambda,Z\right)\right| \leq\ & \sum_{i=0}^{n-1}\frac{\left(x_{i+1}-x_i\right)}{4}\left(\frac{4}{3}-\frac{1}{1+\alpha}\right)^{1-\frac{1}{q}}\left[\left(\frac{2\alpha^2+5\alpha+1}{2\left(\alpha+1\right)\left(\alpha+2\right)}\right)\left|\lambda'\left(x_i\right)\right|^q+\left(\frac{2\alpha+1}{6\left(\alpha+2\right)}\right)\left|\lambda'\left(x_{i+1}\right)\right|^q\right.\\ & \left.+\left(\frac{2\alpha+1}{6\left(\alpha+2\right)}\right)\left|\lambda'\left(x_i\right)\right|^q+\left(\frac{2\alpha^2+5\alpha+1}{2\left(\alpha+1\right)\left(\alpha+2\right)}\right)\left|\lambda'\left(x_{i+1}\right)\right|^q\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. The assertion follows from Theorem 2.6 on the subintervals $[x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ of the partition Z and $\alpha = 1$, we have

$$\begin{aligned}&\left|\frac{1}{3}\left(2\lambda\left(x_i\right)-\lambda'\left(\frac{x_i+x_{i+1}}{2}\right)+2\lambda\left(x_{i+1}\right)\right)-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\lambda\left(u\right)du\right|\\ \leq\ &\frac{\left(x_{i+1}-x_i\right)}{4}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left[\left(\frac{4\left|\lambda'\left(x_i\right)\right|^q+\left|\lambda'\left(x_{i+1}\right)\right|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|\lambda'\left(x_i\right)\right|^q+4\left|\lambda'\left(x_{i+1}\right)\right|^q}{6}\right)^{\frac{1}{q}}\right].\end{aligned} \tag{10}$$

By multiplying both sides of the inequality (10) by $(x_{i+1-}x_i)$ summing the resulting inequalities for $i = 0, 1, ..., n-1$, and then applying the triangular inequality, the desired result is obtained. □

5. Conclusion

Fractional calculus is a fascinating subject with many applications in the modelling of natural problems. In this paper, we have established a new identity and given the fractional version of Milne-type inequality using the Atangana Baleanu fractional operator. Then considering this identity, are obtained new bounds and estimates using well known inequalities such as power-mean, Hölder and bounded functions. Furthermore, we have discussed some applications to special means, and quadrature formula. It appears that our results generalization the inequality obtained by [27, 28]. In the future, scholars may explore inequality with modified Caputo-Fabrizio fractional operators and modified A-B fractional operators and other convexity.

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