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Confluent Appell polynomials of class $A^{(2)}$ and generalization to Szász operators

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Abstract. In this paper, we define confluent Appell polynomials of class $A^{(2)}$, construct a generalization of Szász operators using these polyomials and derive some approximation properties of this generalization on the semi infinite interval in a weighted function space. Finally, some graphical results are given to show the approximation process of constructed operators to a given function f.

1. Introduction

Szász operators [15] are an extension of Bernstein operators to infinite intervals. These operators have a significant impact in the field of approximation theory. Recently, there has been a significant amount of research on the study of generalizations of Szász operators, particularly those defined using polynomials and generating functions. These generalizations offer a variety of novel sequences of operators for approximation theory. Jakimovski and Leviatan [10] proposed an extension of Szász operators using Appell polynomials. Ismail [7] introduced an additional form of Szász operators and also established Jakimovski and Leviatan operators using Sheffer polynomials. On the other hand, Kazmin [11] has defined that a sequence of polynomials $\{P_n(z)\}$, $P_n^{(n)}(z) \equiv c_n$, where $c_n \neq 0$ are constants, $n = 0, 1, 2, ...,$ is called a system of generalized Appell polynomials (or a system of polynomials of class $A^{(2)}$) if any one of the following equivalent conditions holds:

1. $P_n''(z) = P_{n-2}(z)$;
2. There exist two formal power series

$$A(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \text{ and } B(t) = \sum_{k=0}^{\infty} \frac{b_k}{k!} t^k,$$

which formally satisfy the identity

$$A(t) e^{zt} + B(t) e^{-zt} = \sum_{n=0}^{\infty} P_n(z) t^n.$$

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The requirement that the n-th Appell polynomial $P_n(z)$ have degree n is equivalent to requiring that $a_0^2 - b_0^2 \neq 0$. Various properties of polynomials in the class $A^{(2)}$ were studied by Ozhegov [13]. Then, Varma and Sucu [16] have introduced a generalization of Szász operators with the help of the Appell polynomials of class $A^{(2)}$ defined by Kazmin [11];

$$T_n(f;x) = \frac{1}{e^{nx}A(1) + e^{-nx}B(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right)$$

where $A(1) > 0$, $B(1) \geq 0$, $p_k(x) > 0$ and $x \in [0, \infty)$. Özarslan and Çekim [14] have introduced the confluent Appell polynomials $\left\{P_n^{(a,b)}(x)\right\}_{n=0}^{\infty}$,

$$A(t)\, {}_1F_1(a;b;xt) = \sum_{n=0}^{\infty} P_n^{(a,b)}(x) \frac{t^n}{n!},$$

where $A(t)$ is an analytic function in the disc $|t| < R$, $R > 1$,

$$A(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}, \quad a_0 \neq 0$$

and

$${}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}.$$

Subsequently, Özarslan and Çekim [14] have developed approximation operators utilizing confluent Appell polynomials, facilitating the approximation of a function defined on the semi-infinite interval within a weighted function space. One can find more generalizations of Szász operators using similar methods in the literature [5],[8],[9],[12] and [17].

Now, we introduce confluent Appell polynomials of class $A^{(2)}$ and utilize them to develop a generalized form of Szász operators. This is accomplished by leveraging the properties of confluent Appell polynomials of class $A^{(2)}$.

2. The Confluent Appell Polynomials of Class $A^{(2)}$

In this chapter, we introduce univariate confluent Appell polynomials of class $A^{(2)}$. We give them the generating function and properties we have obtained for them.

Definition 2.1. *A polynomial system $\left\{P_n^{(a,b)}(x)\right\}_{n=0}^{\infty}$ is called confluent Appell of class $A^{(2)}$ if there exists a generating function of the form*

$$A(t)\, {}_1F_1(a;b;xt) + B(t)\, {}_1F_1(a;b;-xt) = \sum_{n=0}^{\infty} P_n^{(a,b)}(x) \frac{t^n}{n!}, \tag{1}$$

where $A(t)$ and $B(t)$ are an analytic functions in the disc $|t| < R$, $R > 1$, $a_0^2 - b_0^2 \neq 0$

$$A(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}, B(t) = \sum_{k=0}^{\infty} b_k \frac{t^k}{k!}$$

and

$$ {}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \tag{2} $$

is the confluent hypergeometric function. This function is convergent for all finite z and the Pochhammer symbol is defined by

$$ (a)_n = \begin{cases} a(a+1)\dots(a+n-1) & ;\ n \geq 1 \\ 1 & ;\ n=0 \end{cases} $$

Theorem 2.2. *Let $\left\{P_n^{(a,b)}(x)\right\}_{n=0}^{\infty}$ be a confluent polynomial system where $b \notin \{0,-1,-2,\dots\}$. The following assertions are equivalent.*

1. $\left\{P_n^{(a,b)}(x)\right\}_{n=0}^{\infty}$ *is a set of confluent Appell of class $A^{(2)}$ polynomial system.*
2. *There exists a sequence $\{c_k\}_{k\geq 0}$ independent of n with $c_0 \neq 0$ such that*

$$ P_n^{(a,b)}(x) = \sum_{k=0}^{n} c_{n-k} \binom{n}{k} \frac{(a)_k}{(b)_k} x^k \tag{3} $$

where $c_{n-k} = a_{n-k} + (-1)^k b_{n-k}$

Proof. (1) $\Leftrightarrow$ (2) : Let $\left\{P_n^{(a,b)}(x)\right\}_{n=0}^{\infty}$ be a sequence of confluent Appell polynomials of class $A^{(2)}$. If we use series expansions and Cauchy product in generating functions, we obtain the equality

$$ \begin{aligned} \sum_{n=0}^{\infty} P_n^{(a,b)}(x) \frac{t^n}{n!} &= A(t)\,{}_1F_1(a;b;xt) + B(t)\,{}_1F_1(a;b;-xt) \\ &= \left(\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{(xt)^k}{k!}\right) \\ &+ \left(\sum_{n=0}^{\infty} b_n \frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{(-xt)^k}{k!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(a_{n-k} + (-1)^k b_{n-k}\right) \binom{n}{k} \frac{(a)_k}{(b)_k} x^k\right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_{n-k} \binom{n}{k} \frac{(a)_k}{(b)_k} x^k\right) \frac{t^n}{n!} \end{aligned} $$

and from this equality is obtained.
□

Theorem 2.3. *Let $\left\{P_n^{(a,b)}(x)\right\}_{n=0}^{\infty}$ be a confluent polynomial system where $b \notin \{0,-1,-2,\dots\}$. If $P_n^{(a,b)}(x)$ holds properties of given in (3) then it satisfies the following equality*

$$ \left(P_n^{(a,b)}(x)\right)'' = \frac{a(a+1)}{b(b+1)} n(n-1) P_{n-2}^{(a+2,b+2)}(x),\ n \geq 2. \tag{4} $$

and $P_n^{(a,b)}(0)$ is independent of a and b.

Proof. If we take twice derivate of both sides of equation (3) with respect to x, we get

$$\begin{aligned}\left(P_n^{(a,b)}(x)\right)'' &= \sum_{k=2}^{n} c_{n,k}(a,b)\frac{(a)_k}{(b)_k}k(k-1)x^{k-2}\\ &= \sum_{k=0}^{n-2} c_{n,k+2}(a,b)\frac{(a)_{k+2}}{(b)_{k+2}}(k+1)(k+2)x^k\\ &= \frac{a(a+1)}{b(b+1)}n(n-1)\sum_{k=0}^{n-2} c_{n-k-2}\binom{n-2}{k}\frac{(a+2)_k}{(b+2)_k}x^k\\ &= \frac{a(a+1)}{b(b+1)}n(n-1)P_{n-2}^{(a+2,b+2)}(x)\end{aligned}$$

where

$$c_{n,k+2} = \frac{n}{k+2}\frac{n-1}{k+1}\cdots\frac{n-k-1}{1}c_{n-k-2,0}(a,b) = \binom{n-2}{k}c_{n-k-2,0}(a,b) = \binom{n-2}{k}c_{n-k-2}.$$

□

3. Construction of Operators ζ_n

Let $P_n^{(a,b)}(x)$ be confluent Appell polynomials of class $A^{(2)}$. We define a new generalization of Szász operators by

$$\zeta_n(f;x) = \frac{1}{A(1)\,{}_1F_1(a;b;nx) + B(1)\,{}_1F_1(a;b;-nx)}\sum_{k=0}^{\infty}\frac{p_k^{(a,b)}(nx)}{k!}f\left(\frac{k}{n}\right) \tag{5}$$

where $f \in C[0,\infty)$, $x \geq 0$, $n \in \mathbb{N}$, $b > a > 0$.
With the help of following assumptions

(i) $A(t)$ and $B(t)$ are analytics functions given in (1),
(ii) $A(1) > 0$ and $B(1) \geq 0$,
(iii) $p_k^{(a,b)}(x) > 0$ for all $k = 0, 1, ...$ such that $0 \leq k \leq n$, $c_{n-k} = a_{n-k} + (-1)^k b_{n-k} > 0$.

It is clear that these operators defined in (5) are linear positive operators.
We note that in the special case $A(t) = 1$ and $B(t) = 0$, ζ_n operators will be reduced to confluent Szász operators in [14]. In the special case $A(t) = 1$, $B(t) = 0$ and $a = b$, we discover the well-known Szász operators.

Lemma 3.1. *For the function given in (1), we have the following equalities*

1. $\sum_{k=0}^{\infty}\frac{p_k^{(a,b)}(nx)}{k!} = A(1)\,{}_1F_1(a;b;nx) + B(1)\,{}_1F_1(a;b;-nx)$,
2. $\sum_{k=0}^{\infty}\frac{p_{k+1}^{(a,b)}(nx)}{k!} = A'(1)\,{}_1F_1(a;b;nx) + B'(1)\,{}_1F_1(a;b;-nx)$
$+nx\frac{a}{b}\left[A(1)\,{}_1F_1(a+1;b+1;nx) - B(1)\,{}_1F_1(a+1;b+1;-nx)\right]$,
3. $\sum_{k=0}^{\infty}\frac{p_{k+2}^{(a,b)}(nx)}{k!} = A''(1)\,{}_1F_1(a;b;nx) + B''(1)\,{}_1F_1(a;b;-nx)$
$+2nx\frac{a}{b}\left[A(1)\,{}_1F_1(a+1;b+1;nx) - B(1)\,{}_1F_1(a+1;b+1;-nx)\right]$
$+n^2x^2\frac{a(a+1)}{b(b+1)}\left[A(1)\,{}_1F_1(a+2;b+2;nx) + B(1)\,{}_1F_1(a+2;b+2;-nx)\right]$.

Lemma 3.2. *Let $\zeta_n(f;x)$ be the operator introduced in (5). By using Lemma 3.1, we get*

1. $\zeta_n(1;x) = 1$,
2. $\zeta_n(t;x) = x\frac{a}{b}\frac{A(1)_1F_1(a+1;b+1;nx)-B(1)_1F_1(a+1;b+1;-nx)}{A(1)_1F_1(a;b;nx)+B(1)_1F_1(a;b;-nx)} + \frac{1}{n}\frac{A'(1)_1F_1(a;b;nx)+B'(1)_1F_1(a;b;-nx)}{A(1)_1F_1(a;b;nx)+B(1)_1F_1(a;b;-nx)}$,
3. $\zeta_n(t^2;x) = x^2\frac{a(a+1)}{b(b+1)}\frac{A(1)_1F_1(a+2;b+2;nx)+B(1)_1F_1(a+2;b+2;-nx)}{A(1)_1F_1(a;b;nx)+B(1)_1F_1(a;b;-nx)} + \frac{x}{n}\frac{a}{b}\frac{(2A'(1)+A(1))_1F_1(a+1;b+1;nx)-(2B'(1)+B(1))_1F_1(a+1;b+1;-nx)}{A(1)_1F_1(a;b;nx)+B(1)_1F_1(a;b;-nx)}$
$+\frac{1}{n^2}\frac{(A''(1)+A'(1))_1F_1(a;b;nx)+(B''(1)+B'(1))_1F_1(a;b;-nx)}{A(1)_1F_1(a;b;nx)+B(1)_1F_1(a;b;-nx)}$.

Lemma 3.3. *By using Lemma 3.2 and by the linearty of operators ζ_n, we can compute the following central moments values;*

1. $\zeta_n(t-x;x) = x\frac{a}{b}\left[\frac{A(1)_1F_1(a+1;b+1;nx)-B(1)_1F_1(a+1;b+1;-nx)}{A(1)_1F_1(a;b;nx)+B(1)_1F_1(a;b;-nx)} - 1\right] + \frac{1}{n}\frac{A'(1)_1F_1(a;b;nx)+B'(1)_1F_1(a;b;-nx)}{A(1)_1F_1(a;b;nx)+B(1)_1F_1(a;b;-nx)}$,
2. $\zeta_n((t-x)^2;x) = x^2\left[\frac{a(a+1)}{b(b+1)}\frac{A(1)_1F_1(a+2;b+2;nx)+B(1)_1F_1(a+2;b+2;-nx)}{A(1)_1F_1(a;b;nx)+B(1)_1F_1(a;b;-nx)} - 2\frac{a}{b}\frac{A(1)_1F_1(a+1;b+1;nx)-B(1)_1F_1(a+1;b+1;-nx)}{A(1)_1F_1(a;b;nx)+B(1)_1F_1(a;b;-nx)} + 1\right]$
$+\frac{x}{n}\left[\frac{a}{b}\frac{(2A'(1)+A(1))_1F_1(a+1;b+1;nx)-(2B'(1)+B(1))_1F_1(a+1;b+1;-nx)}{A(1)_1F_1(a;b;nx)+B(1)_1F_1(a;b;-nx)} - 2\frac{(A'(1)+A(1))_1F_1(a;b;nx)+(B'(1)+B(1))_1F_1(a;b;-nx)}{A(1)_1F_1(a;b;nx)+B(1)_1F_1(a;b;-nx)}\right]$
$+\frac{1}{n^2}\frac{(A''(1)+A'(1))_1F_1(a;b;nx)+(B''(1)+B'(1))_1F_1(a;b;-nx)}{A(1)_1F_1(a;b;nx)+B(1)_1F_1(a;b;-nx)}$.

4. Rate of Convergence for Operators ζ_n

In this section, we elucidate the rate of convergence of the operators ζ_n, leveraging the definitions of various tools.

Theorem 4.1. *Let f be continuous on $[0,\infty)$ and*

$$H^* = \left\{f : \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty\right\}.$$

Then, sequence of operators in (5) converges uniformly on the each compact subset on $[0,\infty)$,i.e.

$$\lim_{n\to\infty}\zeta_n(f;x) = f(x).$$

Proof. Now, fix $c > 0$ and consider the lattice homomorphism $T_c : C[0,\infty) \to C[0,c]$ defined by $T_c(f) = f|_{[0,c]}$ for every $f \in C[0,\infty)$. It is apparent from ${}_1F_1(a;b;z) \sim \Gamma(b)\left(\frac{e^z z^{a-b}}{\Gamma(a)} - \frac{(-z)^{-a}}{\Gamma(b-a)}\right)$ for large $|z|$ and $-\frac{3\pi}{2} < arg(z) \le \frac{\pi}{2}$ [1] and Lemma 3.2 that

$$\lim_{n\to\infty} T_c\left(\zeta_n(t^i;x)\right) = T_c\left(t^i\right),\ i = 0,1,2,$$

uniformly on $[0,c]$ for every $f \in C[0,\infty)$. Applying the Korovkin-type property ([2], Theorem 4.1.4 (vi)), the proof is completed.
□

Theorem 4.2. *The operators ζ_n defined in (5) satisfy the following inequality*

$$\left|\zeta_n(f;x) - f(x)\right| \le 2\omega\left(f; \sqrt{\zeta_n((t-x)^2}\right)$$

where $f \in H^*$ and ω is the modulus of continuity of the function f [3] defined by

$$\omega(f;\delta) := \sup_{\substack{x,y\in[0,\infty)\\ |x-y|\le\delta}} \left|f(x) - f(y)\right|.$$

Proof. The modulus of continuity of function $f \in H^*$ satisfies the below inequality in [2]

$$\left|f(x) - f(t)\right| \leq \omega(f;\delta)\left(1 + \frac{|t-x|}{\delta}\right). \tag{6}$$

From the inequality (6), we get

$$\left|\zeta_n(f;x) - f(x)\right| \leq \zeta_n(\left|f(t) - f(x)\right|;x) \leq \left(1 + \frac{1}{\delta}\zeta_n(|t-x|;x)\right)\omega(f;\delta). \tag{7}$$

Using the Cauchy-Schwarz inequality leads us to

$$\begin{aligned}
\zeta_n(|t-x|;x) &= \sum_{k=0}^{\infty}\sqrt{\left(\frac{1}{A(1)\,{}_1F_1(a;b;nx) + B(1)\,{}_1F_1(a;b;-nx)}\right)^2\left(\frac{p_k^{(a,b)}(nx)}{k!}\right)^2\left(\frac{k}{n} - x\right)^2} \\
&\leq \left[\sqrt{\sum_{k=0}^{\infty}\left(\frac{1}{A(1)\,{}_1F_1(a;b;nx) + B(1)\,{}_1F_1(a;b;-nx)}\right)\left(\frac{p_k^{(a,b)}(nx)}{k!}\right)\left(\frac{k}{n} - x\right)^2}\right] \\
&\quad \times\left[\sqrt{\sum_{k=0}^{\infty}\left(\frac{1}{A(1)\,{}_1F_1(a;b;nx) + B(1)\,{}_1F_1(a;b;-nx)}\right)\left(\frac{p_k^{(a,b)}(nx)}{k!}\right)}\right].
\end{aligned}$$

We can write the following inequality,

$$\zeta_n(|t-x|;x) \leq \sqrt{\zeta_n((t-x)^2;x)}\sqrt{\zeta_n(1;x)} = \sqrt{\zeta_n((t-x)^2;x)} \tag{8}$$

Using the inequality (8) in (7), we obtain

$$\left|\zeta_n(f;x) - f(x)\right| \leq \left(1 + \frac{1}{\delta}\sqrt{\zeta_n((t-x)^2;x)}\right)\omega(f;\delta). \tag{9}$$

Here, by choosing $\delta(t,x) = \sqrt{\zeta_n((t-x)^2;x)}$ in inequality (9), the proof is completed. □

Now, for $0 < \beta \leq 1$ and $\eta_1, \eta_2 \in [0,\infty)$, let us introduce the following class of functions [6]:

$$Lip_M^{(\beta)} := \left\{f \in C[0,\infty) : \left|f(\eta_1) - f(\eta_2)\right| \leq M\left|\eta_1 - \eta_2\right|^{\beta},\ t,x \in [0,\infty)\right\}. \tag{10}$$

Theorem 4.3. *Let ζ_n be operator defined in (5). Then for each $f \in Lip_M^{(\beta)}$ $(M > 0, 0 < \beta \leq 1)$ satisfy (10). We have*

$$\left|\zeta_n(f;x) - f(x)\right| \leq M\left(\zeta_n((t-x)^2)\right)^{\frac{\beta}{2}}.$$

Proof. We prove it by using (10) and Hölder's inequality. First, as in the proof of Theorem 4.2, we have

$$\left|\zeta_n(f;x) - f(x)\right| \leq M\zeta_n\left(|t-x|^{\beta};x\right). \tag{11}$$

Then, we can use Hölder's inequality ar the right-hand side of the inequality in (11), we get

$$\begin{aligned}
\left|\zeta_n(f;x) - f(x)\right| &\leq M\zeta_n\left(|t-x|^{\beta};x\right) \\
&\leq M\left(\zeta_n\left((t-x)^2;x\right)\right)^{\frac{\beta}{2}}(\zeta_n(1;x))^{\frac{2-\beta}{2}} \\
&\leq M\left[\zeta_n((t-x)^2\right]^{\frac{\beta}{2}}.
\end{aligned}$$

Hereby, the proof is done. □

5. Approximation Properties in Weighted Space

Gadjiev has extended Korovkin's theorem, a pivotal result in approximation theory, to an unbounded interval within weighted function spaces [4]. Let function f be a monotone increased function, $\lim_{x\to\infty} f(x)=\infty$, $\rho(x)=1+x^2$ is a weighted function and M_f and γ_f are a positive constants that depend to function f. Accordingly, we recall the following weighted space of functions defined on $[0,\infty)$,

$$\begin{aligned} B_\rho[0,\infty) &: =\left\{f\in[0,\infty):\left|f(x)\right|\leq M_f\cdot\rho(x)\right\},\\ C_\rho[0,\infty) &: =\left\{f\in B_\rho[0,\infty): f \text{ is continuous}\right\},\\ C_\rho^\gamma[0,\infty) &: =\left\{f\in C_\rho[0,\infty):\lim_{n\to\infty}\frac{f(x)}{\rho(x)}=\gamma_f<\infty\right\}. \end{aligned}$$

It obvious that $C_\rho^\gamma[0,\infty)\subset C_\rho[0,\infty)\subset B_\rho[0,\infty)$. $B_\rho[0,\infty)$ is a normed space with the following norm:

$$\left\|f\right\|_\rho=\sup_{x\in[0,\infty)}\frac{\left|f(x)\right|}{\rho(x)}.$$

Theorem 5.1. *[4] Let $(T_n)_{n\geq1}$ be a sequence of linear positive operators. If $(T_n)_{n\geq1}$ satisfy two conditions:*
i) The operators T_n act from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$,
ii) $\lim_{n\to\infty}\left\|T_n\left(t^i;x\right)-x^i\right\|_\rho=0$, $i=0,1,2$, then for any function $f\in C_\rho^\gamma[0,\infty)$

$$\lim_{n\to\infty}\left\|T_n(f;x)-f\right\|_\rho=0.$$

Lemma 5.2. *The operators ζ_n defined in (5) satisfy the following inequality*

$$\zeta_n(\rho,x)\leq C\rho(x),\ C>0$$

where $\rho(x)=1+x^2$.

Theorem 5.3. *The operators ζ_n give in (5) confirm the following equality*

$$\lim_{n\to\infty}\left\|\zeta_n(f;x)-f\right\|_\rho=0$$

for $f\in C_\rho^\gamma[0,\infty)$ where $\rho(x)=1+x^2$.

Proof. i) Let $f\in C_\rho[0,\infty)$, from Lemma 5.2, we obtain

$$\zeta_n(f,x)=\zeta_n\left(\frac{f}{\rho}\rho,x\right)\leq\left\|f\right\|_\rho\zeta_n(\rho,x)\leq\left\|f\right\|_\rho C\cdot\rho(x)\leq M_f\cdot\rho(x) \tag{12}$$

where $M_f>0$. From the inequality (12), it is $\zeta_n\in B_\rho[0,\infty)$. So, we get that the operators ζ_n act from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$.

ii) From Lemma 3.2 , it is clear that

$$\lim_{n\to\infty}||\zeta_n(1;x)-1||_\rho=0.$$

Also, by using Lemma 3.3 and from ${}_1F_1(a;b;z)\sim\Gamma(b)\left(\frac{e^z z^{a-b}}{\Gamma(a)}-\frac{(-z)^{-a}}{\Gamma(b-a)}\right)$ for large $|z|$ and $-\frac{3\pi}{2}<arg(z)\leq\frac{\pi}{2}$ [1] ,we can write

$$\begin{aligned} ||\zeta_n(t;x)-x||_\rho &\leq \sup_{x\in[0,\infty)}\frac{x}{1+x^2}\left|\frac{a}{b}\frac{A(1)\,{}_1F_1(a+1;b+1;nx)-B(1)\,{}_1F_1(a+1;b+1;-nx)}{A(1)\,{}_1F_1(a;b;nx)+B(1)\,{}_1F_1(a;b;-nx)}-1\right|\\ &+\sup_{x\in[0,\infty)}\frac{\left|\frac{1}{n}\frac{A'(1)_1F_1(a;b;nx)+B'(1)_1F_1(a;b;-nx)}{A(1)_1F_1(a;b;nx)+B(1)_1F_1(a;b;-nx)}\right|}{1+x^2}\\ &\leq\frac{1}{2}\left|\frac{2B(1)}{e^{2n}A(1)+(-1)^{a-b}B(1)}\right|+\frac{1}{n}\left|\frac{A'(1)e^{2n}+(-1)^{a-b}B'(1)}{A(1)e^{2n}+(-1)^{a-b}B(1)}\right| \end{aligned}$$

Thus, we get

$$\lim_{n\to\infty}\|\zeta_n(t;x)-x\|_\rho=0.$$

Then,

$$\begin{aligned}\left\|\zeta_n\left(t^2;x\right)-x^2\right\|_\rho &\le \sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}\left|\frac{a(a+1)}{b(b+1)}\frac{A(1)\,{}_1F_1(a+2;b+2;nx)+B(1)\,{}_1F_1(a+2;b+2;-nx)}{A(1)\,{}_1F_1(a;b;nx)+B(1)\,{}_1F_1(a;b;-nx)}\right.\\ &\quad\left.-2\frac{a}{b}\frac{A(1)\,{}_1F_1(a+1;b+1;nx)-B(1)\,{}_1F_1(a+1;b+1;-nx)}{A(1)\,{}_1F_1(a;b;nx)+B(1)\,{}_1F_1(a;b;-nx)}+1\right|\\ &+\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\left|\frac{a}{b}\frac{(2A'(1)+A(1))\,{}_1F_1(a+1;b+1;nx)-(2B'(1)+B(1))\,{}_1F_1(a+1;b+1;-nx)}{n(A(1)\,{}_1F_1(a;b;nx)+B(1)\,{}_1F_1(a;b;-nx))}\right.\\ &\quad\left.-2\frac{(A'(1)+A(1))\,{}_1F_1(a;b;nx)+(B'(1)+B(1))\,{}_1F_1(a;b;-nx)}{n(A(1)\,{}_1F_1(a;b;nx)+B(1)\,{}_1F_1(a;b;-nx))}\right|\\ &+\sup_{x\in[0,\infty)}\frac{1}{1+x^2}\left|\frac{(A''(1)+A(1))\,{}_1F_1(a;b;nx)+(B''(1)+B(1))\,{}_1F_1(a;b;-nx)}{n^2(A(1)\,{}_1F_1(a;b;nx)+B(1)\,{}_1F_1(a;b;-nx))}\right|\\ &\le \left|\frac{4B(1)}{e^{2n}A(1)+(-1)^{a-b}B(1)}\right|+\frac{1}{n}\left|\frac{A(1)e^{2n}-(-1)^{a-b}(4B'(1)+B(1))}{A(1)e^{2n}+(-1)^{a-b}B(1)}\right|\\ &\quad+\frac{1}{n^2}\left|\frac{(A''(1)+A')(1)e^{2n}+(-1)^{a-b}(B''(1)+B'(1))}{A(1)e^{2n}+(-1)^{a-b}B(1)}\right|\end{aligned}$$

Thus, we get

$$\lim_{n\to\infty}\left\|\zeta_n(t^2;x)-x^2\right\|_\rho=0.$$

As a result, we obtain

$$\lim_{n\to\infty}\left\|\zeta_n(t^k;x)-x^k\right\|_\rho=0,\ k=0,1,2.$$

If we apply the Theorem 5.1, we obtain the desired results. □

6. Graphical Results

Finally, in this section, we present graphical examples illustrating the convergence of Szász operators, including the confluent Appell polynomials of class $A^{(2)}$. These graphical examples provide a clearer understanding of how our operators converge to specific functions.

Example 6.1. *The first illustration demonstrates the convergence of the operators ζ_n depending on n. Here, taken $a=\frac{1}{2}$ and $b=\frac{4}{5}$ values and approximating function $f(x)=\frac{\cos(7x)}{2+\cos(x)}$, we see the operators respectively for $n=50, n=100$ and $n=200$ values in Figure 1.*

Example 6.2. *Another example is illustrated in Figure 2 to show impact of shape parameters b. For $a=1$ and $n=100$ are fixed and $b=2$, $b=4$ and $b=8$ approximation ζ_n convergence to $f(x)=\frac{\cos(7x)}{2+\cos(x)}$.*

Example 6.3. *The last example is illustrated in Figure 3 to demonstrate the impact of the shape parameters a. For $b=10$ and $n=100$ are fixed and $a=1$, $a=3$ and $a=9$ approximation ζ_n convergence to $f(x)=\frac{\cos(7x)}{2+\cos(x)}$.*

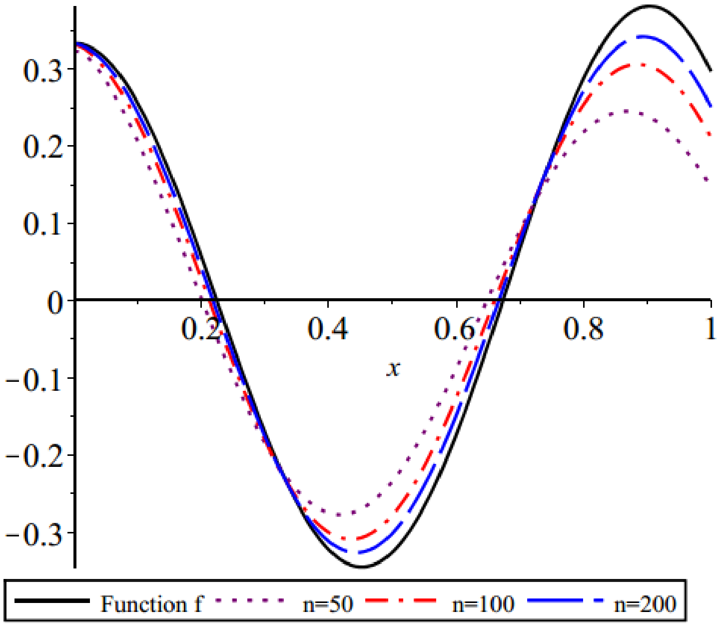

Figure 1:

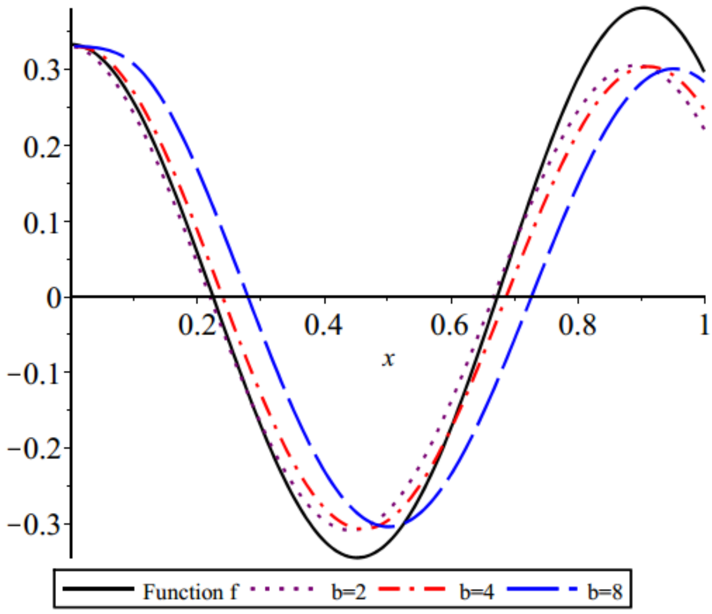

Figure 2:

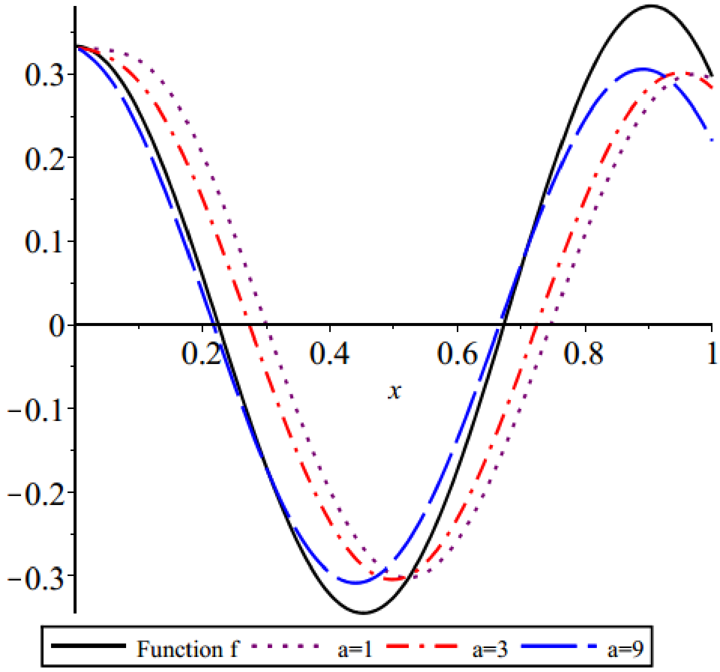

Figure 3:

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