Filomat 38:32 (2024), 11227–11238
https://doi.org/10.2298/FIL2432227B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On a parallelism related to the g-angle

Jiaye Bi[a], Huayou Xie[a], Yongjin Li[a,*]

[a]*Department of Mathematics, Sun Yat-sen University, Guangzhou, 510275, P. R. China*

Abstract. In this paper, we introduce the derivative parallelism, a novel concept of parallelism closely related to the g-angle, and demonstrate its fundamental properties of homogeneity, transitivity, and right-sided continuity. Notably, we establish that derivative parallelism coincides with linear dependence and possesses additivity exclusively in strictly convex spaces. Furthermore, we assert that derivative parallelism coincides with normed parallelism precisely when it exhibits symmetry and/or left-sided continuity. Our results reveal that derivative parallelism is in general stronger than normed parallelism. Moreover, we establish several characterizations of derivative parallelism, both in general normed spaces and in Hilbert C^*-modules.

1. Introduction

Throughout this paper, we assume that X is a normed space over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, B_X, S_X denote the closed unit ball and the unit sphere of X, respectively, and $\mathbb{T} := \{\lambda \in \mathbb{F} : |\lambda| = 1\}$.

The concept of *norm parallelism*, initially introduced by Seddik [7], defines an element $x \in X$ as being *norm parallel* to $y \in X$, denoted as $x \parallel y$, if there exists $\lambda \in \mathbb{T}$ such that the condition $\|x+\lambda y\| = \|x\|+\|y\|$ holds. The norm parallelism generalizes the notion of linear dependence, in that linearly dependent elements are necessarily norm parallel. However, the reverse holds only in strictly convex spaces ([4, Theorem 2.8]). Zamani and Moslehian [12, 13] developed several characterizations of the norm parallelism in Hilbert C^*-modules. Furthermore, Zamani [11] investigated the characterization of norm parallelism in certain spaces of continuous functions. Wojcik [8] provided characterizations of the norm parallelism for bounded linear operators between Banach spaces and presented an interesting application to the invariant subspace problem. For a deeper understanding of norm parallelism, readers are advised to refer to the references [6, 9, 10].

The P-angle, introduced in [3], is related to Phythagorean orthogonality and is defined as

$$\angle_P(x, y) = \arccos\left(\frac{\|x\|^2 + \|y\|^2 - \|x - y\|^2}{2\|x\|\|y\|}\right) \quad (x, y \in X \setminus \{0\}).$$

A simple calculation reveals that, for $x, y \in X \setminus \{0\}$ and $\lambda \in \mathbb{T}$, the condition $\|x + \lambda y\| = \|x\| + \|y\|$ is equivalent to $\angle_P(x, -\lambda y) = \pi$. Thus, the P-angle characterizes norm parallelism. Specifically, for $x, y \in X \setminus \{0\}$, $x \parallel y$ is

2020 *Mathematics Subject Classification*. Primary 46B20.
Keywords. parallelism, strictly convex, smooth, g-angle, Hilbert C^*-module.
Received: 12 July 2024; Accepted: 10 September 2024
Communicated by Dragan S. Djordjević
Research supported by the National Natural Science Foundation of P. R. China (No.11971493 and 12071491).
* Corresponding author: Yongjin Li
Email addresses: bijiaye@mail2.sysu.edu.cn (Jiaye Bi), xiehy33@mail2.sysu.edu.cn (Huayou Xie), stslyj@mail.sysu.edu.cn (Yongjin Li)

equivalent to the existence of $\lambda \in \mathbb{T}$ such that $\angle_P(x, -\lambda y) = \pi$. This observation inspires us to introduce a novel concept of parallelism associated with a specific angle.

In Section 2, we introduce the concept of *derivative parallelism* and establish a characterization. The crucial results are some necessary and sufficient conditions for derivative parallelism to exhibit favorable properties.

In Section 3, we delve into the characterization of derivative parallelism in Hilbert C^*-modules. As practical applications, we derive several characterizations of derivative parallelism in certain algebra of continuous functions as well as in the algebra of Hilbert space operators.

2. Derivative parallelism and its basic properties

The g-angle ([3]) is defined as

$$\angle_g(x, y) = \arccos\left(\frac{D_y^+\|x\| + D_y^-\|x\|}{2\|y\|}\right) \quad (x, y \in X \setminus \{0\}),$$

where

$$D_y^{\pm}\|x\| := \lim_{t\to 0^{\pm}} \frac{\|x+ty\| - \|x\|}{t}.$$

The convexity of $t \mapsto \|x + ty\|$ guarantees the existence of these one-sided derivatives and implies that $D_y^+\|x\| \geq D_y^-\|x\|$. Initially, we attempted to define parallelism as $\angle_g(x, y) = 0$ or π. However, in complex normed spaces, this definition faces challenges as linearly dependent elements may not adhere to the parallelism criterion, e.g., $\angle_g(x, ix) = \pi/2$ for any nonzero x. To address this limitation, observe that in real normed spaces, a straightforward calculation shows that $D_{-y}^{\pm}\|x\| = -D_y^{\mp}\|x\|$. Consequently, $\angle_g(x, -y) = 0$ if and only if $\angle_g(x, y) = \pi$, revealing the fact that our initial parallelism definition is equivalent to the existence of $\lambda \in \mathbb{T}$ such that $\angle_g(x, \lambda y) = 0$ holds. Motivated by this observation, we define parallelism by declaring the existence of $\lambda \in \mathbb{T}$ such that $\angle_g(x, \lambda y) = 0$ holds, which is equivalent to $D_{\lambda y}^+\|x\| + D_{\lambda y}^-\|x\| = 2\|y\|$. Since the inequalities $D_{\lambda y}^-\|x\| \leq D_{\lambda y}^+\|x\| \leq \|y\|$ hold, we conclude that $D_{\lambda y}^-\|x\| = \|y\|$. Conversely, this condition implies that $\angle_g(x, \lambda y) = 0$. This leads to our definition.

Definition 2.1. *Let X be a normed space and $x, y \in X$. Then x is said to be* derivative parallel *to y, denoted by $x \parallel_D y$, if either $x = 0$ or there exists $\lambda \in \mathbb{T}$ satisfying the condition:*

$$\lim_{t\to 0^-} \frac{\|x + t\lambda y\| - \|x\|}{t} = \|y\|.$$

The proposition below justifies the rationality of our definition.

Proposition 2.2. *Let X be a normed space and $x, y \in X$ be linearly dependent. Then $x \parallel_D y$.*

Proof. Without loss of generality, we can assume that $x \neq 0$. It follows that $y = \alpha x$ for some $\alpha \in \mathbb{F}$. Observe that $\operatorname{sgn}\overline{\alpha} \in \mathbb{T}$ and

$$\lim_{t\to 0^-} \frac{\|x + t(\operatorname{sgn}\overline{\alpha})y\| - \|x\|}{t} = \lim_{t\to 0^-} \frac{\|x + t|\alpha|x\| - \|x\|}{t} = |\alpha|\|x\| = \|y\|.$$

Thus $x \parallel_D y$. $\square$

We now present a characterization of derivative parallelism in terms of *duality mapping*, which is defined as

$$J(x) := \{x^* \in S_{X^*} : x^*(x) = \|x\|\}$$

for any element $x \in X$, where X^* is the dual of X. Recall that a smooth point is defined as a point whose duality mapping is a singleton, and that a normed space is smooth precisely when its unit sphere consists entirely of such smooth points.

Theorem 2.3. *Let X be a normed space and $x, y \in X$. Then $x \parallel_D y$ if and only if either $x = 0$ or there exists $\lambda \in \mathbb{T}$ such that $J(x) \subset \lambda J(y)$.*

Proof. Suppose that $x \parallel_D y$ and $x \neq 0$. Then there exists $\lambda \in \mathbb{T}$ such that

$$\lim_{t\to 0^-} \frac{\|x + t\lambda y\| - \|x\|}{t} = \|y\|.$$

For any $x^* \in J(x)$, we have

$$\begin{aligned}\|y\| = \lim_{t\to 0^-} \frac{\|x + t\lambda y\| - \|x\|}{t} &\le \lim_{t\to 0^-} \frac{\operatorname{Re} x^*(x + t\lambda y) - x^*(x)}{t} \\ &= \operatorname{Re} \lambda x^*(y) \le |x^*(y)| \le \|y\|.\end{aligned}$$

It follows that $\lambda x^*(y) = \|y\|$, leading to $\lambda x^* \in J(y)$. Thus $J(x) \subset \overline{\lambda} J(y)$, which establishes the necessity.

To prove the sufficiency, we assume, without loss of generality, that $x \neq 0$ and there exists $\lambda \in \mathbb{T}$ such that $J(x) \subset \overline{\lambda} J(y)$. For every $n \in \mathbb{Z}_{\ge 0}$, the Hahn-Banach theorem yields $x_n^* \in S_{X^*}$ such that

$$\left\|x - n^{-1}\lambda y\right\| = x_n^*\left(x - n^{-1}\lambda y\right).$$

Then

$$x_n^*(x) = \left\|x - n^{-1}\lambda y\right\| + n^{-1} x_n^*(\lambda y) \to \|x\|$$

and there exists a subsequence $\{x_{n_m}^*\}$ of $\{x_n^*\}$ such that $x_{n_m}^*(\lambda y)$ converges to some $\alpha \in \mathbb{F}$. Since B_{X^*} is weak*-compact, as established in the Banach-Alaoglu theorem, there is a subnet $\left\langle x_{n_{m_i}}^* \right\rangle_{i\in I}$ of the sequence $\{x_{n_m}^*\}$ that converges to some $x^* \in B_{X^*}$ with respect to the weak*-topology. As a result, we have

$$x^*(x) = \|x\|, \quad \lambda x^*(y) = x^*(\lambda y) = \alpha.$$

Hence $x^* \in J(x) \subset \overline{\lambda} J(y)$, which entails that

$$\alpha = \lambda x^*(y) = \|y\|.$$

Observe that

$$\begin{aligned}\|y\| &\ge \lim_{t\to 0^-} \frac{\|x + t\lambda y\| - \|x\|}{t} = \lim_{m\to\infty} \frac{\|x - n_m^{-1}\lambda y\| - \|x\|}{-n_m^{-1}} \\ &= -\lim_{m\to\infty} n_m \left[\operatorname{Re} x_{n_m}^*\left(x - n_m^{-1}\lambda y\right) - \|x\|\right] \\ &= \lim_{m\to\infty} \left[\operatorname{Re} x_{n_m}^*(\lambda y) + n_m\left(\|x\| - \operatorname{Re} x_{n_m}^*(x)\right)\right] \\ &\ge \lim_{m\to\infty} \operatorname{Re} x_{n_m}^*(\lambda y) = \operatorname{Re}\alpha = \|y\|,\end{aligned}$$

which implies that

$$\lim_{t\to 0^-} \frac{\|x + t\lambda y\| - \|x\|}{t} = \|y\|.$$

Consequently, $x \parallel_D y$ holds. □

Based on the previous theorem, some properties of derivative parallelism are derived immediately and listed below without proof.

Corollary 2.4.

(i) *Derivative parallelism possesses homogeneity, which means that if $x, y \in X$ satisfy $x \parallel_D y$, then $\alpha x \parallel_D \beta y$ for all $\alpha, \beta \in \mathbb{F}$.*

(ii) *Derivative parallelism possesses transitivity, that is, if $x, y, z \in X \setminus \{0\}$ satisfy $x \parallel_D y$ and $y \parallel_D z$, then $x \parallel_D z$ holds.*

(iii) *Derivative parallelism is continuous on the right, i.e., if $x \in X$ and the sequence $\{y_n\} \subset X$ satisfy $x \parallel_D y_n$ for all $n \in \mathbb{Z}_{\ge 1}$ and $y_n \to y \in X$, then $x \parallel_D y$.*

Corollary 2.5. *Derivative parallelism implies norm parallelism.*

Proof. Let X be a normed space and $x, y \in X$ satisfy $x \parallel_D y$. To prove $x \parallel y$, we assume $x \neq 0$ without loss of generality. Then Theorem 2.3 ensures the existence of $\lambda \in \mathbb{T}$ such that $J(x) \subset \overline{\lambda} J(y)$. The Hahn-Banach theorem yields $x^* \in J(x)$. Thus $\lambda x^* \in J(y)$. It follows that

$$\|x\| + \|y\| \geq \|x + \lambda y\| \geq x^*(x + \lambda y) = \|x\| + \|y\|,$$

leading to $\|x + \lambda y\| = \|x\| + \|y\|$. Consequently, $x \parallel y$ holds. □

Remark 2.6. *We will see below in Example 2.13 that derivative parallelism may fail to coincide with norm parallelism.*

After establishing the homogeneity, transitivity, and right-sided continuity of the derivative parallelism in Corollary 2.4, we are motivated to investigate its symmetry, additivity, and left-sided continuity. One of the principal results in this section is the characterization of its additivity, which we present as follows.

Theorem 2.7. *Let X be a normed space. The following statements are equivalent:*
(i) *X is strictly convex.*
(ii) *Derivative parallelism in X coincides with the linear dependence, that is, $x \parallel_D y$ if and only if x and y are linearly dependent.*
(iii) *Derivative parallelism in X is additive on the left, which means that if $x, y, z \in X$ satisfy $x \parallel_D z$ and $y \parallel_D z$, then $x + y \parallel_D z$ holds.*
(iv) *Derivative parallelism in X is additive on the right, which means that if $x, y, z \in X$ satisfy $x \parallel_D y$ and $x \parallel_D z$, then $x \parallel_D y + z$ holds.*

To establish this theorem, we need to perform some preliminary work.

Lemma 2.8. *Let X be a normed space. If $x_0, x_1 \in S_X$ are distinct elements with the line segment between them entirely in S_X, define $x_t := (1-t)x_0 + tx_1$ for all $t \in (0, 1)$. The conclusion is, for all $t \in (0, 1)$, $x_t \parallel_D x_1$ holds, and neither $x_1 - x_0 \parallel_D x_t$ nor $x_t \parallel_D x_1 - x_0$ holds.*

Proof. Fix $t \in (0, 1)$. For any $x^* \in J(x_t)$, we have

$$(1-t)x^*(x_0) + tx^*(x_1) = x^*(x_t) = 1.$$

This leads to $x^*(x_0) = x^*(x_1) = 1$ and hence $x^*(x_1 - x_0) = 0$. Thus $J(x_t) \subset J(x_1)$, and $J(x_t) \cap (\lambda J(x_1 - x_0)) = \emptyset$ for all $\lambda \in \mathbb{T}$. Consequently, we can invoke Theorem 2.3 to obtain the desired conclusion. □

The subsequent proposition focus on *rotund points*, defined as elements of S_X that do not lie within any line segment contained in S_X. Recall that X is strictly convex precisely when S_X consists entirely of rotund points.

Proposition 2.9. *Let X be a normed space and $x \in S_X$. Then the following conditions are equivalent:*
(i) *x is a rotund point.*
(ii) *If $y \in X$ satisfies $x \parallel_D y$ or $y \parallel_D x$, then x and y are linearly dependent.*
(iii) *If $y \in X$ satisfies $y \parallel_D x$, then x and y are linearly dependent.*

Proof. (i) ⇒ (ii): Without loss of generality, we assume that $y \neq 0$. Let

$$u = \begin{cases} x, & x \parallel_D y, \\ y/\|y\|, & y \parallel_D x, \end{cases} \qquad v = \begin{cases} y/\|y\|, & x \parallel_D y, \\ x, & y \parallel_D x. \end{cases}$$

Then by the homogeneity of derivative parallelism, we have $u \parallel_D v$. It suffices to show that u and v are linearly dependent. By Theorem 2.3, there exists $\lambda \in \mathbb{T}$ such that $J(u) \subset \overline{\lambda} J(v)$. The Hahn-Banach theorem yields $u^* \in J(u)$. Thus $\lambda u^* \in J(v)$, leading to

$$2 \geq \|u + \lambda v\| \geq u^*(u + \lambda v) = \|u\| + \|v\| = 2.$$

It follows that

$$\left\|\frac{1}{2}\overline{\lambda}u + \frac{1}{2}v\right\| = \left\|\frac{1}{2}u + \frac{1}{2}\lambda v\right\| = 1.$$

Since x is a rotund point, either u or v must also be a rotund point. Consequently, $u = \lambda v$ holds, as desired.

(ii) ⇒ (iii): Clearly.

(iii) ⇒ (i): Suppose for contrary that x is not a rotund point. then there exists a distinct $z \in S_X$ with the line segment between x and z entirely in S_X. Let $y = (x+z)/2$, then x and y are linearly independent. Moreover, Lemma 2.8 asserts that $y \parallel_D x$, which contradicts (iii). □

Now it is time to prove Theorem 2.7.

Proof. (i) ⇒ (ii): Since linear dependence implies derivative parallelism, as established in Proposition 2.2, it suffices to prove the converse implication. Suppose that $x, y \in X$ satisfy $x \parallel_D y$. By the homogeneity of derivative parallelism, we can assume, without loss of generality, that $y \in S_X$. Then y is a rotund point. Applying Proposition 2.9, we conclude that x and y are linearly dependent.

(ii) ⇒ (iii) and (ii) ⇒ (iv): These implications are evident.

(iii) ⇒ (i) and (iv) ⇒ (i): Suppose for contrary that X is not strictly convex. Then there exists $x \in S_X$ that is not a rotund point. As a result, we deduce the existence of a distinct $y \in S_X$ with the line segment joining x and y lies entirely in S_X. Let

$$z = \frac{1}{3}x + \frac{2}{3}y, \quad w = \frac{2}{3}x + \frac{1}{3}y.$$

Then $z = w/2 + y/2$, $w = z/2 + x/2$, $z - w = (y-x)/3$. By Lemma 2.8, we arrive at $z \parallel_D w$ and $w \parallel_D z$ hold, and neither $y - x \parallel_D w$ nor $w \parallel_D y - x$ holds. We can invoke the homogeneity of derivative parallelism to derive that neither $z - w \parallel_D w$ nor $w \parallel_D z - w$ holds, which contradicts (iii) and (iv). □

Next, we introduce another key result characterizing the symmetry and left-side continuity of derivative parallelism.

Theorem 2.10. *Let X be a normed space. The following statements are equivalent:*

(i) *Derivative parallelism in X coincides with norm parallelism.*

(ii) *Derivative parallelism in X possesses symmetry, which means that $y \parallel_D x$ holds whenever $x, y \in X$ satisfy $x \parallel_D y$.*

(iii) *Derivative parallelism in X is continuous on the left, i.e., if $y \in X$ and the sequence $\{x_n\} \subset X$ satisfy $x_n \parallel_D y$ for all $n \in \mathbb{Z}_{\geq 1}$ and $x_n \to x \in X$, then $x \parallel_D y$.*

(iv) *Every point $x \in S_X$ is a rotund point or a smooth point in any 2-dimensional subspace including x.*

Proof. (i) ⇒ (ii): It follows directly form the symmetry of norm parallelism.

(ii) ⇒ (iii): Suppose that the derivative parallelism in X possesses symmetry. Since the derivative parallelism is continuous on the right, as established in Proposition 2.2, it must be continuous on the left.

(iii) ⇒ (iv): It follows immediately form the proposition that follows.

(iv) ⇒ (i): Since derivative parallelism implies norm parallelism as established in Corollary 2.5, it suffices to show the converse implication. Suppose that $x, y \in X$ satisfy $x \parallel y$. By the homogeneity of norm parallelism and derivative parallelism, we can assume, without loss of generality, that $x \in S_X$. If x, y are linearly dependent, then Proposition 2.2 ensures that $x \parallel_D y$. If x, y are linearly independent, then x is a rotund point or a smooth point in $\operatorname{span}\{x, y\}$. Applying the proposition that follows, we arrive at $x \parallel_D y$, as desired. □

Proposition 2.11. *Let X be a 2-dimensional normed space and $x \in S_X$. Then the following statements are equivalent:*

(i) *x is a rotund point or a smooth point.*

(ii) *If $y \in X$ satisfies $x \parallel y$, then $x \parallel_D y$ holds.*

(iii) *If $y \in X$ satisfies $y \parallel_D x$, then $x \parallel_D y$ holds.*

(iv) *If $y \in X$ and the sequence $\{x_n\} \subset X$ satisfy $x_n \parallel_D y$ for all $n \in \mathbb{Z}_{\geq 1}$ and $x_n \to x$, then $x \parallel_D y$ holds.*

Proof. (i) $\Rightarrow$ (ii): Suppose that $y \in X$ satisfies $x \parallel y$. To prove $x \parallel_D y$, we assume, without loss of generality, that x and y are linearly independent. Then there exists $\lambda \in \mathbb{T}$ such that

$$\|x + \lambda y\| = \|x\| + \|y\| = 1 + \|y\|,$$

i.e.,

$$\left\| \frac{1}{1+\|y\|} x + \frac{\|y\|}{1+\|y\|} \frac{\lambda y}{\|y\|} \right\| = 1. \tag{1}$$

If x is a rotund point, then $x = \lambda y/\|y\|$. By Proposition 2.2, we deduce that $x \parallel_D y$. If x is a smooth point, then $J(x)$ is a singleton. The Hahn-Banach theorem yields $x^* \in J(x + \lambda y)$. Then

$$x^*(x) + \lambda x^*(y) = x^*(x + \lambda y) = \|x + \lambda y\| = 1 + \|y\|,$$

leading to $x^*(x) = 1$ and $\lambda x^*(y) = \|y\|$. Thus $x^* \in J(x) \cap \left(\overline{\lambda} J(y)\right)$. Since $J(x)$ is a singleton, $J(x) \subset \overline{\lambda} J(y)$. Applying Theorem 2.3, we conclude that $x \parallel_D y$.

(ii) $\Rightarrow$ (iii): Suppose that $y \in X$ satisfies $y \parallel_D x$. Then by Corollary 2.5, we have $y \parallel x$. Since the norm parallelism is symmetry, $x \parallel y$ holds, which implies that $x \parallel_D y$.

(iii) $\Rightarrow$ (i): If $x \in S_X$ is not a rotund point, then Proposition 2.9 guarantees the existence of $y \in X$ such that $y \parallel_D x$ and x, y are linearly independent. As a result, Theorem 2.3 ensures the existence of $\lambda \in \mathbb{T}$ such that every $y^* \in J(y)$ satisfies $y^*(y) = \|y\|$ and $y^*(x) = \lambda\|x\|$. Since x and y are linearly independent, $J(y)$ must be a singleton. If, in addition, x is not a smooth point, then $J(x)$ is not a singleton. In view of Theorem 2.3, x is not derivative parallel to y, which contradicts (iii).

(ii) $\Rightarrow$ (iv): Suppose that $y \in X$ and the sequence $\{x_n\} \subset X$ satisfy $x_n \parallel_D y$ for all $n \in \mathbb{Z}_{\geq 1}$ and $x_n \to x$. Then by Corollary 2.5, we have $y \parallel x_n$ for all $n \in \mathbb{Z}_{\geq 1}$. In other words, for every $n \in \mathbb{Z}_{\geq 1}$, there exists $\lambda_n \in \mathbb{T}$ such that $\|y + \lambda_n x_n\| = \|y\| + \|x_n\|$. By passing to a subsequence, we may assume that $\{\lambda_n\}$ converges to some $\lambda \in \mathbb{T}$. As a consequence,

$$\|y + \lambda x\| = \lim_{n\to\infty} \|y + \lambda_n x_n\| = \|y\| + \lim_{n\to\infty} \|x_n\| = \|y\| + \|x\|,$$

leading to $x \parallel y$ and hence $x \parallel_D y$.

(iv) $\Rightarrow$ (i): Suppose for contrary that x is neither a rotund point nor a smooth point. Then the implication (ii) $\Rightarrow$ (i) yields $y \in X$ such that $x \parallel y$ and x fails to be derivative parallel to y. Thus x, y are linearly independent and there exists $\lambda \in \mathbb{T}$ such that (1) holds. In other words, the line segment between x and $\lambda y/\|y\|$ is entirely in S_X. For each $n \in \mathbb{Z}_{\geq 1}$, let

$$x_n = \frac{n-1}{n} x + \frac{\lambda y}{n\|y\|}.$$

Applying Lemma 2.8 and the homogeneity of derivative parallelism, we have $x_n \parallel_D y$. However, $x_n \to x$ holds and x is not derivative parallel to y, a contradiction. □

Corollary 2.12. *In both strictly convex spaces and smooth spaces, derivative parallelism coincides with norm parallelism, thereby possesses symmetry and left-side continuity.*

Example 2.13. *Let X be the $\mathbb{F}$-vector space $\mathbb{F}^2$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, equipped with the norm of element (x, y) defined by $\|(x, y)\| = |x| + |y|$. As is known, $(1, 0)$ is neither a rotund point nor a smooth point. Consequently, derivative parallelism in X is incompatible with the properties stated in Theorem 2.10* (i) – (iii).

3. Derivative parallelism in Hilbert C^*-modules

Let A be a nonzero C^*-algebra (not necessarily unital). An element $a \in A$ is designated as *positive*, denoted $a \geq 0$, if there exists an element $b \in A$ such that $a = b^*b$. A linear functional $\varphi : A \to \mathbb{C}$ is considered *positive* if $\varphi(a) \geq 0$ for any positive $a \in A$. Such a positive linear functional φ is referred to as a *state* if $\|\varphi\| = 1$. For a more profound exploration of C^*-algebras, we recommend readers refer to [1].

Let A be a C^*-algebra, an *inner product A-module* is a $(\mathbb{C}, A)$-bimodule M, equipped with an A-valued inner product $\langle\cdot,\cdot\rangle : M \times M \to A$ satisfying the following axioms:

(i) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$ for any $x, y, z \in M$.

(ii) $\langle x, \alpha y a\rangle = \alpha \langle x, y\rangle a$ for any $x, y \in M, \alpha \in \mathbb{C}, a \in A$.

(iii) $\langle y, x\rangle = \langle x, y\rangle^*$ for any $x, y \in M$.

(iv) $\langle x, x\rangle \geq 0$ for any $x \in M$, and $\langle x, x\rangle = 0$ if and only if $x = 0$.

It is not hard to verify that $\|x\| := \|\langle x, x\rangle\|^{1/2}$ for $x \in M$ defines a norm on M and that $\|\langle x, y\rangle\| \leq \|x\|\|y\|$ holds for all $x, y \in M$. When M is complete with respect to this norm, it is termed a *Hilbert A-module* or a *Hilbert C^*-modules over A*. As an illustrative example, A itself can be equipped with the structure of a Hilbert A-module, where the inner product of elements $a, b \in A$ is given by $\langle a, b\rangle := a^*b$. For a deeper dive into the theory of Hilbert C^*-modules, we suggest readers refer to [5].

In this section, we denote the C^*-algebra of bounded linear operators on a Hilbert space H by $\mathbb{B}(H)$, and its C^*-subalgebra of compact operators is denoted as $\mathbb{K}(H)$.

Zamani and Moslehian, in their works [12, Theorem 4.1] and [13, Theorem 2.3], have offered several characterizations of the norm-parallelism in Hilbert C^*-modules. Based on the Gelfand-Naimark theorem, which establishes that every C^*-algebra can be identified with a C^*-subalgebra of $\mathbb{B}(H)$ for a suitable complex Hilbert space H, we introduce a characterization of derivative parallelism in Hilbert C^*-modules.

Theorem 3.1. *Let $(H, (\cdot,\cdot))$ be a complex Hilbert space, A a nonzero C^*-subalgebra of $\mathbb{B}(H)$, M a Hilbert A-module, and $x, y \in M$. Then the following statements are equivalent:*

(i) *$x \parallel_D y$ holds.*

(ii) *There exist $\lambda \in \mathbb{T}$ and a sequence $\{v_n\} \subset S_H$ such that*

$$\left(\left\langle \lambda y, x - n^{-1}\lambda y\right\rangle v_n, v_n\right) = \|y\|\|x - n^{-1}\lambda y\| + o\left(n^{-1}\right) \quad as \quad n \to \infty, \tag{2}$$

where $o\left(n^{-1}\right)$ is an infinitesimal of higher order than n^{-1} as $n \to \infty$.

(iii) *There exist a state φ on A and $\lambda \in \mathbb{T}$ such that*

$$\varphi\left(\langle \lambda y, x + t\lambda y\rangle\right) = \|y\|\|x + t\lambda y\| + o\left(t\right) \quad as \quad t \to 0^-, \tag{3}$$

where $o(t)$ is an infinitesimal of higher order than t as $t \to 0^-$.

Proof. (i) ⇒ (ii): If either $x = 0$ or $y = 0$, then the proof will be trivial. Suppose that $x \parallel_D y$ and that $\|x\|\|y\| \neq 0$. Then there exists $\lambda \in \mathbb{T}$ such that

$$\lim_{t\to 0^-} \frac{\|x + t\lambda y\| - \|x\|}{t} = \|y\| \tag{4}$$

holds, which is equivalent to

$$\|x\| - \|x - n^{-1}\lambda y\| = n^{-1}\|y\| + o\left(n^{-1}\right) \quad \text{as} \quad n \to \infty. \tag{5}$$

Since $\langle x, x\rangle \geq 0$, we have

$$\|\langle x, x\rangle\| = \sup\left\{\left(\langle x, x\rangle v, v\right) : v \in S_H\right\}.$$

Then there exists a sequence $\{v_n\} \subset S_H$ such that

$$\left(\langle x, x\rangle v_n, v_n\right) = \|\langle x, x\rangle\| + o\left(n^{-1}\right) = \|x\|^2 + o\left(n^{-1}\right)$$

as $n \to \infty$. It follows that

$$
\begin{aligned}
&(\langle x,x\rangle v_n,v_n) - n^{-2}\left\|y\right\|^2 - \left\|x-n^{-1}\lambda y\right\|^2 \\
=&\|x\|^2 - n^{-2}\left\|y\right\|^2 - \left\|x-n^{-1}\lambda y\right\|^2 + o\left(n^{-1}\right) \\
=&2n^{-1}\|y\|\left\|x-n^{-1}\lambda y\right\| + o\left(n^{-1}\right)
\end{aligned}
$$

as $n \to \infty$. Observe that

$$
\begin{aligned}
&(\langle x,x\rangle v_n,v_n) - n^{-2}\left\|y\right\|^2 - \left\|x-n^{-1}\lambda y\right\|^2 \\
\leq &(\langle x,x\rangle v_n,v_n) - \left(\left\langle n^{-1}\lambda y, n^{-1}\lambda y\right\rangle v_n,v_n\right) - \left(\left\langle x-n^{-1}\lambda y, x-n^{-1}\lambda y\right\rangle v_n,v_n\right) \\
= &\left(\left\langle n^{-1}\lambda y, x-n^{-1}\lambda y\right\rangle v_n,v_n\right) + \left(\left\langle x-n^{-1}\lambda y, n^{-1}\lambda y\right\rangle v_n,v_n\right) \\
= &\left(\left\langle n^{-1}\lambda y, x-n^{-1}\lambda y\right\rangle v_n,v_n\right) + \left(\left\langle n^{-1}\lambda y, x-n^{-1}\lambda y\right\rangle^* v_n,v_n\right) \\
=&2\mathrm{Re}\left(\left\langle n^{-1}\lambda y, x-n^{-1}\lambda y\right\rangle v_n,v_n\right) \\
\leq&2\left|\left(\left\langle n^{-1}\lambda y, x-n^{-1}\lambda y\right\rangle v_n,v_n\right)\right| \leq 2n^{-1}\|y\|\left\|x-n^{-1}\lambda y\right\|
\end{aligned}
$$

holds for all $n \in \mathbb{Z}_{\geq 1}$. Thus

$$
\begin{aligned}
\mathrm{Re}\left(\left\langle n^{-1}\lambda y, x-n^{-1}\lambda y\right\rangle v_n,v_n\right) =&\left|\left(\left\langle n^{-1}\lambda y, x-n^{-1}\lambda y\right\rangle v_n,v_n\right)\right| + o\left(n^{-1}\right) \\
=&n^{-1}\|y\|\left\|x-n^{-1}\lambda y\right\| + o\left(n^{-1}\right)
\end{aligned}
$$

as $n \to \infty$, which entails that

$$\left(\left\langle \lambda n^{-1} y, x-n^{-1}\lambda y\right\rangle v_n,v_n\right) = n^{-1}\|y\|\left\|x-n^{-1}\lambda y\right\| + o\left(n^{-1}\right),$$

i.e.,

$$\left(\left\langle \lambda y, x-n^{-1}\lambda y\right\rangle v_n,v_n\right) = \|y\|\left\|x-n^{-1}\lambda y\right\| + o\left(1\right) = \|y\|\|x\| + o\left(1\right)$$

as $n \to \infty$. This leads to

$$
\begin{aligned}
(\langle \lambda y, x\rangle v_n,v_n) =&\left(\left\langle \lambda y, x-n^{-1}\lambda y\right\rangle v_n,v_n\right) - n^{-1}\left(\langle \lambda y,\lambda y\rangle v_n,v_n\right) \\
=&\|y\|\|x\| + o\left(1\right) \to \|y\|\|x\|
\end{aligned}
$$

as $n \to \infty$. By passing to a subsequence, we can assume that

$$(\langle \lambda y, x\rangle v_n,v_n) = \|y\|\|x\| + o\left(n^{-1}\right) \quad \text{as} \quad n \to \infty. \tag{6}$$

Moreover, by the Schwarz inequality for the (degenerate) inner product $(\langle\cdot,\cdot\rangle v_n, v_n)$ on M, we have

$$
\begin{aligned}
&\|y\|^2\|x\|^2 \geq \left(\langle \lambda y,\lambda y\rangle v_n,v_n\right)\|x\|^2 \\
\geq &\left(\langle \lambda y,\lambda y\rangle v_n,v_n\right)\left(\langle x,x\rangle v_n,v_n\right) \\
\geq&\left|\left(\langle \lambda y,x\rangle v_n,v_n\right)\right|^2 \to \|y\|^2\|x\|^2.
\end{aligned}
$$

Thus

$$\left(\langle \lambda y,\lambda y\rangle v_n,v_n\right) = \|y\|^2 + o\left(1\right) \quad \text{as} \quad n \to \infty. \tag{7}$$

By (5)-(7), we arrive at

$$
\begin{aligned}
\left(\left\langle \lambda y, x-n^{-1}\lambda y\right\rangle v_n,v_n\right) &= \left(\langle \lambda y, x\rangle v_n,v_n\right) - n^{-1}\left(\langle \lambda y,\lambda y\rangle v_n,v_n\right) \\
&= \|y\|\|x\| - n^{-1}\|y\|^2 + o\left(n^{-1}\right) \\
&= \|y\|\left\|x-n^{-1}\lambda y\right\| + o\left(n^{-1}\right)
\end{aligned}
$$

as $n \to \infty$.

(ii) ⇒ (iii): Without loss of generality, we may assume that $\|x\|\|y\| \neq 0$. Suppose that there exist $\lambda \in \mathbb{T}$ and a sequence $\{v_n\} \subset S_H$ such that (2) holds. By a similar argument as in the proof of the implication (i) ⇒ (ii), we can show that (7) holds. By (2) and (7), we have

$$\begin{aligned}\|y\|\|x\| &\geq \left\|\langle \lambda y, x\rangle\right\| \geq \left|(\langle \lambda y, x\rangle v_n, v_n)\right| \geq \operatorname{Re}(\langle \lambda y, x\rangle v_n, v_n) \\ &= \operatorname{Re}\left(\left\langle \lambda y, x - n^{-1}\lambda y\right\rangle v_n, v_n\right) + n^{-1}\operatorname{Re}(\langle \lambda y, \lambda y\rangle v_n, v_n) \\ &= \|y\|\|x - n^{-1}\lambda y\| + n^{-1}\|y\|^2 + o\left(n^{-1}\right) \\ &\geq \|y\|\|x\| + o\left(n^{-1}\right)\end{aligned}$$

as $n \to \infty$. As a consequence,

$$(\langle \lambda y, x\rangle v_n, v_n) = \|y\|\|x\| + o\left(n^{-1}\right) \quad \text{as} \quad n \to \infty. \tag{8}$$

For each $n \in \mathbb{Z}_{\geq 1}$, let

$$\varphi_n : A \to \mathbb{C},\ a \mapsto (av_n, v_n).$$

Then φ_n is a positive functional on A with $\|\varphi_n\| \leq 1$. Since B_{A^*} is weak*-compact, as established in the Banach-Alaoglu theorem, there is a subnet $\langle \varphi_{n_i} \rangle_{i \in I}$ of the sequence $\{\varphi_n\}$ that converges to some $\varphi \in B_{A^*}$ with respect to the weak*-topology. As a result, φ is a positive functional that satisfies

$$\varphi(\langle \lambda y, \lambda y\rangle) = \|y\|^2, \quad \varphi(\langle \lambda y, x\rangle) = \|y\|\|x\| \geq \left\|\langle \lambda y, x\rangle\right\|.$$

Thus $\|\varphi\| \geq 1$. Since the opposite inequality holds, we show that φ is a state on A. Furthermore, by (2), (7) and (8), we obtain

$$\begin{aligned}\varphi\left(\left\langle \lambda y, x - n^{-1}\lambda y\right\rangle\right) &= \varphi(\langle \lambda y, x\rangle) - n^{-1}\varphi(\langle \lambda y, \lambda y\rangle) = \|y\|\|x\| - n^{-1}\|y\|^2 \\ &= (\langle \lambda y, x\rangle v_n, v_n) - n^{-1}(\langle \lambda y, \lambda y\rangle v_n, v_n) + o\left(n^{-1}\right) \\ &= \left(\left\langle \lambda y, x - n^{-1}\lambda y\right\rangle v_n, v_n\right) + o\left(n^{-1}\right) \\ &= \|y\|\|x - n^{-1}\lambda y\| + o\left(n^{-1}\right)\end{aligned}$$

as $n \to \infty$. Notice that the map

$$f : \mathbb{R} \to \mathbb{R},\ t \mapsto \|y\|\|x + t\lambda y\| - \operatorname{Re}\varphi(\langle \lambda y, x + t\lambda y\rangle)$$

is convex as $t \mapsto \|x + t\lambda y\|$ is convex and $t \mapsto \operatorname{Re}\varphi(\langle \lambda y, x + t\lambda y\rangle)$ is affine. We conclude that

$$\lim_{t \to 0^-} \frac{f(t) - f(0)}{t}$$

exists. Thus

$$\lim_{t \to 0^-} \frac{f(t) - f(0)}{t} = \lim_{n \to \infty} \frac{f(-n^{-1}) - f(0)}{-n^{-1}},$$

i.e.,

$$\begin{aligned}&\lim_{t \to 0^-} \frac{\|y\|\|x + t\lambda y\| - \operatorname{Re}\varphi(\langle \lambda y, x + t\lambda y\rangle)}{t} \\ &= \lim_{n \to \infty} \frac{\|y\|\|x - n^{-1}\lambda y\| - \operatorname{Re}\varphi\left(\left\langle \lambda y, x - n^{-1}\lambda y\right\rangle\right)}{-n^{-1}} = \lim_{n \to \infty} \frac{o\left(n^{-1}\right)}{n^{-1}} = 0,\end{aligned}$$

which implies that

$$\operatorname{Re}\varphi\left(\langle\lambda y, x + t\lambda y\rangle\right) = \|y\|\|x + t\lambda y\| + o\left(t\right)$$

as $t \to 0^-$. Observe that

$$\begin{aligned}\|y\|\|x + t\lambda y\| &\geq \left\|\langle\lambda y, x + t\lambda y\rangle\right\| \geq \left|\varphi\left(\langle\lambda y, x + t\lambda y\rangle\right)\right| \\ &\geq \operatorname{Re}\varphi\left(\langle\lambda y, x + t\lambda y\rangle\right) = \|y\|\|x + t\lambda y\| + o\left(t\right)\end{aligned}$$

as $t \to 0^-$. Therefore, (3) holds.

(iii) ⇒ (i): Without loss of generality, we may assume that $\|x\|\|y\| \neq 0$. By (3), we have $\varphi\left(\langle\lambda y, x\rangle\right) = \|y\|\|x\|$. By the Schwarz inequality for the (degenerate) inner product $\varphi\left(\langle\cdot,\cdot\rangle\right)$ on M, we obtain

$$\|y\|^2\|x\|^2 = \varphi\left(\langle\lambda y, x\rangle\right)^2 \leq \varphi\left(\langle\lambda y, \lambda y\rangle\right)\varphi\left(\langle x, x\rangle\right) \leq \|y\|^2\|x\|^2.$$

Since the equality is satisfied in the aforementioned inequalities, we deduce that

$$\varphi\left(\langle\lambda y, \lambda y\rangle\right) = \|y\|^2.$$

It follows that

$$\begin{aligned}\|y\|\|x + t\lambda y\| - \|y\|\|x\| &= \varphi\left(\langle\lambda y, x + t\lambda y\rangle\right) - \varphi\left(\langle\lambda y, x\rangle\right) + o\left(t\right) \\ &= t\varphi\left(\langle\lambda y, \lambda y\rangle\right) + o\left(t\right) = t\|y\|^2 + o\left(t\right),\end{aligned}$$

i.e., $\|x + t\lambda y\| - \|x\| = t\|y\| + o\left(t\right)$ as $t \to 0^-$. Consequently, (4) holds, yielding $x \parallel_D y$. □

Drawing upon the preceding theorem, our subsequent result, analogous to [11, Theorem 2.5], provides a characterization of derivative parallelism in the space $C_0(\Omega)$ of continuous functions $f : \Omega \to \mathbb{C}$ that vanish at infinity (i.e., for every $\epsilon > 0$ there exists a compact set K such that $|f(x)| \leq \epsilon$ for all $x \in K$), equipped with the uniform norm, where Ω is a nonempty locally compact Hausdorff space.

Corollary 3.2. *Let Ω be a nonempty locally compact Hausdorff space and $f, g \in C_0(\Omega)$. Then $f \parallel_D g$ if and only if there exist a Radon probability measure μ on Ω and $\lambda \in \mathbb{T}$ such that the following condition holds:*

$$\int \overline{\lambda g}\left(f + t\lambda g\right)\, d\mu = \|g\|\|f + t\lambda g\| + o\left(t\right) \quad as \quad t \to 0^-.$$

Proof. By observing the involution $h \mapsto \overline{h}$ on $C_0(\Omega)$, we discern that $C_0(\Omega)$ constitutes a C^*-algebra and thereby a Hilbert $C_0(\Omega)$-module, equipped with the $C_0(\Omega)$-valued inner product defined as $\langle x, y\rangle := \overline{x}y$ for $x, y \in C_0(\Omega)$. According to the Riesz representation theorem (cf. [2, 7.17]), a state on $C_0(\Omega)$ is characterized by

$$h \mapsto \int h\, d\mu,$$

where μ represents a Radon probability measure on Ω. Consequently, the conclusion follows directly from the previous theorem. □

Applying Theorem 3.1, we can establish the following characterization of derivative parallelism in $\mathbb{B}(H)$.

Corollary 3.3. *Let $(H, (\cdot,\cdot))$ be a complex Hilbert space and $T, S \in \mathbb{B}(H)$. Then $T \parallel_D S$ if and only if there exist a sequence $\{v_n\} \subset S_H$ and $\lambda \in \mathbb{T}$ such that the following condition holds:*

$$\left(\left(T - n^{-1}\lambda S\right)v_n, \lambda S v_n\right) = \left\|T - n^{-1}\lambda S\right\|\|S\| + o\left(n^{-1}\right) \quad as \quad n \to \infty.$$

Proof. Observe that $\mathbb{B}(H)$ is a C^*-algebra, and thus it naturally forms a Hilbert $\mathbb{B}(H)$-module equipped with the $\mathbb{B}(H)$-valued inner product defined as $\langle T, S\rangle := T^*S$ for $T, S \in \mathbb{B}(H)$. As a consequence, the conclusion follows immediately from Theorem 3.1. □

Finally, we derive a necessary and sufficient condition for a bounded operator on a Hilbert space H to be derivative parallel to a compact operator.

Proposition 3.4. *Let $(H, (\cdot,\cdot))$ be a complex Hilbert space, $T \in \mathbb{B}(H)$ and $S \in \mathbb{K}(H)$. Then $T \parallel_D S$ if and only if there exist $v \in S_H$ and $\lambda \in \mathbb{T}$ such that the following condition holds:*

$$((T + t\lambda S)\, v, \lambda S v) = \|T + t\lambda S\| \, \|S\| + o\,(t) \quad \text{as} \quad t \to 0^-.$$

Proof. Without loss of generality, we may assume that $\|T\|\|S\| \neq 0$. Suppose that $T \parallel_D S$. Then there exists $\lambda \in \mathbb{T}$ such that

$$\lim_{t \to 0^-} \frac{\|T + t\lambda S\| - \|T\|}{t} = \|S\|$$

holds, which is equivalent to

$$\|T + t\lambda S\| - \|T\| = t\|S\| + o\,(t) \quad \text{as} \quad t \to 0^-. \tag{9}$$

We can proceed as in the proof of Theorem 3.1 to show that there exists a sequence $\{v_n\} \subset S_H$ such that

$$\left(\overline{\lambda} S^* T v_n, v_n\right) \to \|S\|\|T\|, \quad (S^* S v_n, v_n) \to \|S\|^2.$$

Since B_H is weakly sequentially compact, by passing to a subsequence, we can assume that $\{v_n\}$ converges weakly to some $v \in B_H$. Since S is compact, $\overline{\lambda} S^* T$ and $S^* S$ must also be compact. Thus $\overline{\lambda} S^* T v_n \to \overline{\lambda} S^* T v$ and $S^* S v_n \to S^* S v$ hold, leading to

$$\begin{aligned}
&\left|\left(\overline{\lambda} S^* T v_n, v_n\right) - \left(\overline{\lambda} S^* T v, v\right)\right| \\
\leq &\left|\left(\overline{\lambda} S^* T v_n - \overline{\lambda} S^* T v, v_n\right)\right| + \left|\left(\overline{\lambda} S^* T v, v_n - v\right)\right| \to 0, \\
&\left|(S^* S v_n, v_n) - (S^* S v, v)\right| \\
\leq &\left|(S^* S v_n - S^* S v, v_n)\right| + \left|(S^* S v, v_n - v)\right| \to 0.
\end{aligned}$$

It follows that $\left(\overline{\lambda} S^* T v, v\right) = \|S\|\|T\|$ and $(S^* S v, v) = \|S\|^2$. As a result,

$$\|S^* S\| = \|S\|^2 = (S^* S v, v) \leq \|S^* S\|\|v\|^2,$$

which implies that $\|v\| \geq 1$. Since the opposite inequality holds, we conclude that $v \in S_H$. Moreover, by (9), we deduce that

$$\begin{aligned}
((T + t\lambda S)\, v, \lambda S v) &= (Tv, \lambda S v) + t\,(\lambda S v, \lambda S v) = \left(\overline{\lambda} S^* T v, v\right) + t\,(S^* S v, v) \\
&= \|S\|\|T\| + t\|S\|^2 = \|T + t\lambda S\|\|S\| + o\,(t)
\end{aligned}$$

as $t \to 0^-$. This establishes the necessity. The sufficiency, on the other hand, is an immediate consequence of the preceding corollary. □

4. Acknowledgements

We gratefully acknowledge the detailed and constructive comments provided by the referee, which have significantly improved the quality of our manuscript, particularly in enhancing the introduction to better engage the readers.

References

[1] B. Blackadar. *Operator algebras, Springer-Verlag, Berlin, 2006.*

[2] G.B. Folland. Real analysis, Modern techniques and their applications, (2nd edition), A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1999.

[3] H. Gunawan, J. Lindiarni, and O. Neswan, P-, I-, g-, and D-angles in normed spaces, J. Math. Fundam., **40** *(2008), 24–32.*

[4] A. Mal, D. Sain, and K. Paul, On some geometric properties of operator spaces, Banach J. Math. Anal., **13** *(2019), 174–191.*

[5] V.M. Manuilov and E.V. Troitsky, Hilbert C-modules, American Mathematical Society, Providence, 2005.*

[6] M. Mehrazin, M. Amyari, and A. Zamani, Numerical radius parallelism of Hilbert space operators, Bull. Iranian Math. Soc., **46***(2020), 821–829.*

[7] A. Seddik, Rank one operators and norm of elementary operators, Linear Algebra Appl., **424***(2007), 177–183.*

[8] P. Wojcik, Norm-parallelism in classical M-ideals, Indag. Math. (N.S.), **28***(2017), 287–293.*

[9] P. Wojcik and A. Zamani, From norm derivatives to orthogonalities in Hilbert C-modules, Linear Multilinear A,* **71***(2023), 875–888.*

[10] A. Zamani, The operator-valued parallelism, Linear Algebra Appl., **505***(2016), 282–295.*

[11] A. Zamani, Characterizations of norm-parallelism in spaces of continuous functions, Bull. Iranian Math. Soc., **45***(2018), 557–567.*

[12] A. Zamani and M. S. Moslehian, Exact and approximate operator parallelism, Canad. Math. Bull., **58***(2015), 207–224.*

[13] A. Zamani and M. S. Moslehian, Norm-parallelism in the geometry of Hilbert C-modules, Indag. Math. (N.S.),* **27***(2016), 266–281.*