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Classes of transforms associated with bounded linear operators on abstract Wiener spaces

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Abstract. In this article, we study an algebraic structure of the analytic Fourier–Feynman transform (FFT) associated with bounded linear operators on abstract Wiener space (AWS). It turned out in this paper that a class of the analytic FFTs forms a monoid and that a quotient monoid is isomorphic to the monoid of the FFTs. Additionally, we provide a transformation group freely generated by FFTs. Any transforms in the free group are linear operator isomorphisms.

1. Introduction and preliminaries

The aim of this article is to clarify a class of "analytic FFTs associated with bounded linear operators" for certain bounded functionals on AWS.

The analytic FFT is a well-known transform defined on Wiener spaces. For a historical survey and a rigorous definition of the FFT of functionals on classical and abstract Wiener spaces, we refer the reader to [12] and the references cited therein. In [4], Choi defined an analytic FFT associated with a bounded linear operator on AWS and establish the fact that iterated FFTs can be represented by a single FFT.

We in this article investigate a deep structure of the class of analytic FFTs on AWS. Precisely speaking, we study a new algebraic structure of our FFTs. It turned out in this paper that a class of the analytic FFTs forms a monoid (and hence a semigroup). Moreover, it also turned out that a quotient monoid is isomorphic to the monoid of the FFTs. We also provide a transformation group freely generated by the FFTs.

In order to provide our assertions for the FFTs, we first follow the expositions of [5–9].

1.1. Abstract Wiener space

Let $\mathbb{H}$ be a real separable Hilbert space with norm $|\cdot|$ induced by the inner product $\langle\cdot,\cdot\rangle$, and let $\mathbb{B}$ be a real separable Banach space with norm $\|\cdot\|$. It is assumed that $\mathbb{H}$ is continuously, linearly, and densely embedded in $\mathbb{B}$ by a natural injection. Let ν be a centered Gaussian probability measure on $(\mathbb{B},\mathcal{B}(\mathbb{B}))$, where $\mathcal{B}(\mathbb{B})$ denotes the Borel σ-algebra of $\mathbb{B}$. The triple $(\mathbb{H},\mathbb{B},\nu)$ is called an AWS if

$$\int_{\mathbb{B}} \exp\Big(i(h,x)\Big)d\nu(x) = \exp\Big(-\frac{1}{2}|h|^2\Big)$$

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for any $h \in \mathbb{B}^*$, where $(\cdot,\cdot)$ denotes the $\mathbb{B}^*$–$\mathbb{B}$-pairing, and where $\mathbb{B}^*$ is the topological dual of $\mathbb{B}$. Also, let $\mathbb{H}^*$ be the topological dual of $\mathbb{H}$. Then the space $\mathbb{B}^*$ is identified as a dense subspace of $\mathbb{H}^* \approx \mathbb{H}$ in the sense that, for all $y \in \mathbb{B}^*$ and $x \in \mathbb{H}$, $\langle y, x\rangle = (y, x)$. Thus we have the triple

$$\mathbb{B}^* \subset \mathbb{H}^* \approx \mathbb{H} \subset \mathbb{B}. \tag{1.1}$$

Given a Banach space $\mathbb{X}$, let $\mathcal{L}(\mathbb{X})$ denote the class of bounded linear operators from $\mathbb{X}$ to itself. Then $\mathcal{L}(\mathbb{B}^*)$, $\mathcal{L}(\mathbb{H})$, and $\mathcal{L}(\mathbb{B})$ are Banach spaces. By the concept of the Banach space adjoint operator, given an operator $A \in \mathcal{L}(\mathbb{B})$, there exists an operator A^* in $\mathcal{L}(\mathbb{B}^*)$ such that for all $\theta \in \mathbb{B}^*$ and $x \in \mathbb{B}$,

$$(A^*\theta)x = \theta(Ax). \tag{1.2}$$

By the structure of the $\mathbb{B}^*$–$\mathbb{B}$-pairing and the triple (1.1), equation (1.2) can be rewritten by $(A^*\theta, x) = (\theta, Ax)$.

1.2. Fourier–Feynman transforms associated with bounded linear operators

In order to define an analytic FFT associated with bounded linear operators on the AWS $(\mathbb{H}, \mathbb{B}, \nu)$, we need the concept of the "scale-invariant measurability".

Let $\mathcal{W}(\mathbb{B})$ be the class of ν-Carathéodory measurable subsets of $\mathbb{B}$. A subset S of $\mathbb{B}$ is said to be scale-invariant measurable (s.i.m.) [5] provided ρS is $\mathcal{W}(\mathbb{B})$-measurable for every $\rho > 0$, and an s.i.m. subset N of $\mathbb{B}$ is said to be scale-invariant null provided $\nu(\rho N) = 0$ for every $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional F on $\mathbb{B}$ is said to be s.i.m. provided F is defined on an s.i.m. set and $F(\rho\,\cdot)$ is $\mathcal{W}(\mathbb{B})$-measurable for every $\rho > 0$. If two functionals F and G on $\mathbb{B}$ are equal s-a.e., we write $F \approx G$. The symbols "$\approx$" is an equivalence relation. For an s.i.m. functional F on $\mathbb{B}$, we denote by $[F]_{\rm s}$ the equivalence class of functionals which are equal to F s-a.e.. For more details, see [5].

The definition of the analytic FFT on AWS $\mathbb{B}$ is based on the analytic Feynman integral and the scale-invariant measurability [1–3]. We now state the definition of the analytic FFT associated with bounded linear operator.

Definition 1.1. *Let $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ and let $\widetilde{\mathbb{C}}_+ := \{\lambda \in \mathbb{C} \setminus \{0\} : \mathrm{Re}(\lambda) \geq 0\}$. Given a bounded linear operator A on $\mathbb{B}$, let $F : \mathbb{B} \to \mathbb{C}$ be an s.i.m. functional such that*

$$J_{F(y+\cdot)}(A;\lambda) := \int_{\mathbb{B}} F(y + \lambda^{-1/2}Ax)d\nu(x)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J^*_{F(y+\cdot)}(A;\lambda)$ analytic on $\mathbb{C}_+$ such that $J^*_{F(y+\cdot)}(A;\lambda) = J_{F(y+\cdot)}(A;\lambda)$ for all $\lambda > 0$, then $J^*_{F(y+\cdot)}(A;\lambda)$ is defined to be the analytic transform (associated with the operator A) of F over $\mathbb{B}$ with parameter λ. For $\lambda \in \mathbb{C}_+$ we write

$$T_{\lambda,A}(F)(y) := J^*_{F(y+\cdot)}(A;\lambda).$$

Let q be a non-zero real number. We define the L_1 analytic FFT associated with the operator A, $T^{(1)}_{q,A}(F)$ of F, by the formula (if it exists)

$$T^{(1)}_{q,A}(F)(y) := \lim_{\substack{\lambda\to -iq \\ \lambda\in\widetilde{\mathbb{C}}_+}} T_{\lambda,A}(F)(y)$$

for s-a.e. $y \in \mathbb{B}$.

Remark 1.2. *If A is the identity operator on $\mathbb{B}$, then this definition agrees with the previous definition of the (ordinary) analytic FFT studied in [1–3]. We note that if $T^{(1)}_{q,A}(F)$ exists and if $F \approx G$, then $T^{(1)}_{q,A}(G)$ exists and $T^{(1)}_{q,A}(G) \approx T^{(1)}_{q,A}(F)$.*

2. Fourier–Feynman transforms of bounded functionals

We now introduce the class $\mathcal{F}(\mathbb{B}^*)$ of functionals on $\mathbb{B}$, which forms a Banach algebra. Let $\mathcal{M}(\mathbb{B}^*)$ denote the class of complex-valued Borel measures on $\mathbb{B}^*$. Under total variation norm $\|\cdot\|$ with convolution as multiplication, $\mathcal{M}(\mathbb{B}^*)$ is a commutative Banach algebra with identity. The class $\mathcal{F}(\mathbb{B}^*)$ is defined as the space of all s-equivalence classes of stochastic Fourier transforms of elements of $\mathcal{M}(\mathbb{B}^*)$, that is,

$$\mathcal{F}(\mathbb{B}^*) := \Big\{[F_\sigma]_s : F_\sigma(x) = \int_{\mathbb{B}^*} \exp\{i(g,x)\}d\sigma^*(g),\ x \in \mathbb{B},\ \sigma^* \in \mathcal{M}(\mathbb{B}^*)\Big\}.$$

We will identify a functional with its s-equivalence class and think of $\mathcal{F}(\mathbb{B}^*)$ as a collection of functionals on $\mathbb{B}^*$ rather than as a collection of s-equivalence classes. The class $\mathcal{F}(\mathbb{B}^*)$ is a Banach algebra with the norm

$$\|F_\sigma\| := \|\sigma^*\| = \int_{\mathbb{B}^*} d|\sigma^*|(g)$$

and the mapping $\sigma^* \mapsto F_\sigma$ is a Banach algebra isomorphism where $\sigma^* \in \mathcal{M}(\mathbb{B}^*)$ is related to F_σ by

$$F_\sigma(x) = \int_{\mathbb{B}^*} \exp\{i(g,x)\}d\sigma^*(g) \tag{2.3}$$

for s-a.e. $x \in \mathbb{B}$. For more details, see [4].

In [4], Choi established the existence of the L_1 analytic FFT of functionals in $\mathcal{F}(\mathbb{B}^*)$ as follows.

Theorem 2.1. *Let F_σ be a functional in $\mathcal{F}(\mathbb{B}^*)$, and let A be an operator in $\mathcal{L}(\mathbb{B})$. Then the following assertions hold true:*

(i) The L_1 analytic FFT associated with the operator A, $T^{(1)}_{q,A}(F_\sigma)$ exists for each non-zero real q, and is given by

$$T^{(1)}_{q,A}(F_\sigma)(y) = \int_{\mathbb{B}^*} \exp\{i(g,y)\}d(\sigma^*)^A_t(g)$$

for s-a.e. $y \in \mathbb{B}$, where $(\sigma^*)^A_t$ is the complex measure on $\mathbb{B}^*$ defined by

$$(\sigma^*)^A_t(U) = \int_U \exp\Big\{-\frac{i}{2q}(g, AA^*g)\Big\}d\sigma^*(g)$$

for $U \in \mathcal{B}(\mathbb{B}^*)$. Thus $T^{(1)}_{q,A}(F_\sigma)$ is an element of $\mathcal{F}(\mathbb{B}^*)$.

(ii) For all non-zero real q,

$$T^{(1)}_{-q,A}\big(T^{(1)}_{q,A}(F_\sigma)\big) \approx F_\sigma. \tag{2.4}$$

That is, the analytic FFT, $T^{(1)}_{q,A} : \mathcal{F}(\mathbb{B}^*) \to \mathcal{F}(\mathbb{B}^*)$ has the inverse transform $\{T^{(1)}_{q,A}\}^{-1} = T^{(1)}_{-q,A}$.

In order to provide a transform monoid and a free group of the FFTs, we quote the following expositions from [4].

(O1) Let A be an operator in $\mathcal{L}(\mathbb{B})$ such that $A(\mathbb{H}) \subseteq \mathbb{H}$. Then A is an element of $\mathcal{L}(\mathbb{H})$. Let

$$\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H}) := \{A \in \mathcal{L}(\mathbb{B}) : A(\mathbb{H}) \subseteq \mathbb{H}\}.$$

Then the class $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$ is a linear space. For any A in $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$, AA^* is positive definite on $\mathbb{H}$. Thus, by the square root lemma [10], there exists a positive operator $|A|$ on $\mathbb{H}$ such that $|A| = \sqrt{AA^*}$.

(O2) Given operators A_1 and A_2 in $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$, it follows that the operator $A_1A_1^* + A_2A_2^*$ is positive definite on $\mathbb{H}$. Thus, by the square root lemma, there is an operator $\sqrt{A_1A_1^* + A_2A_2^*}$, uniquely, in $\mathcal{L}(\mathbb{H})$. It is clear that the operator $\sqrt{A_1A_1^* + A_2A_2^*}$ is in $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$.

In order to identify these operators, we consider the relation $\overset{\text{op}}{\sim}$ on $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$ given by

$$A_1 \overset{\text{op}}{\sim} A_2 \iff A_1A_1^* = A_2A_2^* \text{ on } \mathbb{H}.$$

Then $\overset{\text{op}}{\sim}$ is an equivalence relation. Let $[A]$ denote the equivalence class of an operator A in $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$. In view of the observation (O1), it follows that there exists a positive definite operator $\mathfrak{S}(A)$ such that $A \overset{\text{op}}{\sim} \mathfrak{S}(A)$.

Given two operators A_1 and A_2 in $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$, we will use the symbol '$\mathfrak{S}(A_1, A_2)$' to indicate the representative element of the equivalence class

$$\left[\mathfrak{S}(A_1, A_2)\right] = \left\{\mathfrak{S} \in \mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H}) : \mathfrak{S} \overset{\text{op}}{\sim} \sqrt{A_1A_1^* + A_2A_2^*} \text{ on } \mathbb{H}\right\}.$$

Then, in view of (O2), we see that for any $\mathfrak{S}$ in $[\mathfrak{S}(A_1, A_2)]$ and all $g \in \mathbb{B}^*$,

$$|\mathfrak{S}^* g|^2 = (\mathfrak{S}^* g, \mathfrak{S}^* g) = (g, \mathfrak{S}\mathfrak{S}^* g) = (g, (A_1A_1^* + A_2A_2^*)g).$$

For a notational convenience, we will regard $[\mathfrak{S}(A_1, A_2)] \equiv \mathfrak{S}(A_1, A_2)$ as an operator in $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$. Then we see that

$$\mathfrak{S}(A_1, A_2)\mathfrak{S}(A_1, A_2)^* = A_1A_1^* + A_2A_2^*.$$

(O3) Given a finite sequence $O = (A_1, \ldots, A_n)$ of operators in $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$, let $\mathfrak{S}(O) \equiv \mathfrak{S}(A_1, A_2, \ldots, A_n)$ be the positive operators $\mathfrak{S}$ which satisfy the relation

$$\mathfrak{S}\mathfrak{S}^* = A_1A_1^* + \cdots + A_nA_n^* \text{ on } \mathbb{H}. \tag{2.5}$$

By an induction argument, it follows that

$$\mathfrak{S}(\mathfrak{S}(A_1, A_2, \ldots, A_{k-1}), A_k) = \mathfrak{S}(A_1, A_2, \ldots, A_k) \tag{2.6}$$

for all $k \in \{2, \ldots, n\}$. Also, for any permutation π of $\{1, \ldots, n\}$, we also see that

$$\mathfrak{S}(A_1, A_2, \ldots, A_n) = \mathfrak{S}(A_{\pi(1)}, A_{\pi(2)}, \ldots, A_{\pi(n)}). \tag{2.7}$$

Under these observations, Choi established the following theorem in [4].

Theorem 2.2. *Let F_σ be a functional in $\mathcal{F}(\mathbb{B}^*)$, let $\{q_1, q_2, \ldots, q_n\}$ be a set of non-zero real numbers with $\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_k} \neq 0$ for each $k \in \{2, \ldots, n\}$, and let $O = \{A_1, \ldots, A_n\}$ be a finite set of operators in $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$. Then for all non-zero real q,*

$$T^{(1)}_{q,A_n}\Big(T^{(1)}_{q,A_{n-1}}\Big(\cdots\Big(T^{(1)}_{q,A_2}\Big(T^{(1)}_{q,A_1}(F_\sigma)\Big)\Big)\cdots\Big)\Big)(y) \approx T^{(1)}_{q,\mathfrak{S}(O)}(F_\sigma), \tag{2.8}$$

where $\mathfrak{S}(O)$ is an operator in $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$ which satisfies the relation (2.5).

3. Monoids of Fourier–Feynman transforms

We in this section will provide a deep algebraic structure of classes of the FFTs. To do this, for any $A \in \mathcal{L}(\mathbb{B})$, let $T^{(1)}_{0,A}$ denote the identity transform on $\mathcal{F}(\mathbb{B}^*)$.

Firstly, for $q \in \mathbb{R}$, let

$$\mathsf{T}(q; \mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})) := \Big\{ T^{(1)}_{q,A} : A \in \mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H}) \Big\}.$$

By (i) of Theorem 2.1 and Theorem 2.2, it follows that for all $A_1, A_2 \in \mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$ and all $F_\sigma \in \mathcal{F}(\mathbb{B}^*)$,

$$\Big(T^{(1)}_{q,A_2} \circ T^{(1)}_{q,A_1}\Big)(F_\sigma) = T^{(1)}_{q,A_2}\Big(T^{(1)}_{q,A_1}(F_\sigma)\Big) \approx T^{(1)}_{q,\mathfrak{S}(A_1,A_2)}(F_\sigma)$$

is in $\mathcal{F}(\mathbb{B}^*)$. One can see that the composition $\circ$ of FFTs is associative, because for all $A_1, A_2, A_3 \in \mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$,

$$\mathfrak{S}(\mathfrak{S}(A_1, A_2), A_3) = \mathfrak{S}(A_1, A_2, A_3) = \mathfrak{S}(A_1, \mathfrak{S}(A_2, A_3)).$$

Also, one can see that

$$\Big(T^{(1)}_{q,A_1} \circ T^{(1)}_{q,A_2}\Big)(F_\sigma) \approx \Big(T^{(1)}_{q,A_2} \circ T^{(1)}_{q,A_1}\Big)(F_\sigma),$$

for any A_1 and A_2 in $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$ and every $F_\sigma \in \mathcal{F}(\mathbb{B}^*)$, because $\mathfrak{S}(A_1, A_2) = \mathfrak{S}(A_2, A_1)$. Clearly,

$$(T^{(1)}_{q,O} \circ T^{(1)}_{q,A})(F_\sigma) \equiv T^{(1)}_{q,A}(F_\sigma)$$

for any A in $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$, where O indicates the trivial operator in $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$. Thus we have the following theorem.

Theorem 3.1. *For any non-zero real q, the space $(\mathsf{T}(q; \mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})), \circ)$ forms a commutative monoid (and hence semigroup). Indeed, the monoid $\mathsf{T}(q; \mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H}))$ acts on the Banach space $\mathcal{F}(\mathbb{B}^*)$ in the sense that $(T^{(1)}_{q,A}, F_\sigma) \mapsto T^{(1)}_{q,A}(F_\sigma)$.*

Let S_f denote the set of all finite sequences in $\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})$, and let

$$M^{(1)}_{q,\mathcal{O}_n} \equiv T^{(1)}_{q,A_n} \circ \cdots \circ T^{(1)}_{q,A_1} \tag{3.9}$$

for any real q and any $\mathcal{O} = (A_1, \ldots, A_n) \in \mathsf{S}_\mathrm{f}$. Next, for $q \in \mathbb{R}$, let

$$\mathsf{M}(q; \mathsf{S}_\mathrm{f}) := \Big\{ M^{(1)}_{q,\mathcal{O}} : \mathcal{O} \in \mathsf{S}_\mathrm{f} \Big\}.$$

Then, by Theorem 2.2, it follows that

$$\mathsf{M}(q; \mathsf{S}_\mathrm{f}) = \Big\{ T^{(1)}_{q,\mathfrak{S}(\mathcal{O})} : \mathcal{O} \in \mathsf{S}_\mathrm{f} \Big\}.$$

Thus we have the inclusions

$$\mathsf{T}(q; \mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})) \subset \mathsf{M}(q; \mathsf{S}_\mathrm{f}) \subset \mathsf{T}(q; \mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})).$$

From this, one can see that $\mathsf{T}(q; \mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H}))$ and $\mathsf{M}(q; \mathsf{S}_\mathrm{f})$ coincide as sets. However, we will consider other operation on $\mathsf{M}(q; \mathsf{S}_\mathrm{f})$ defined as follows: for $\mathcal{O}_1 = (A_{11}, \ldots, A_{1n_1})$ and $\mathcal{O}_2 = (A_{21}, \ldots, A_{2n_2})$ in S_f, let

$$\mathcal{O}_1 \wedge \mathcal{O}_2 \equiv (A_{11}, \ldots, A_{1n_1}) \wedge (A_{21}, \ldots, A_{2n_2}) := (A_{11}, \ldots, A_{1n_1}, A_{21}, \ldots, A_{2n_2})$$

and for $M^{(1)}_{q,\mathcal{O}_1}$ and $M^{(1)}_{q,\mathcal{O}_2}$ in $\mathsf{M}(q; \mathsf{S}_\mathrm{f})$, let

$$M^{(1)}_{q,\mathcal{O}_1} \circledcirc M^{(1)}_{q,\mathcal{O}_2} := M^{(1)}_{q,\mathcal{O}_1 \wedge \mathcal{O}_2}.$$

In view of Theorem 2.2 and the observation (O3), we see that for a permutation π of $\{1, \ldots, n\}$,

$$M^{(1)}_{q,(A_1,A_2,\ldots,A_n)} = M^{(1)}_{q,(A_{\pi(1)},A_{\pi(2)},\ldots,A_{\pi(n)})}.$$

Thus we have

$$M^{(1)}_{q,O_1} \circledcirc M^{(1)}_{q,O_2} = M^{(1)}_{q,O_2} \circledcirc M^{(1)}_{q,O_1}$$

for all $M^{(1)}_{q,O_1}$ and $M^{(1)}_{q,O_2}$ in $\mathsf{M}(q;\mathsf{S}_{\mathrm{f}})$, and so we conclude that the operation $\circledcirc$ is well defined and is commutative.

Clearly, $M^{(1)}_{q,(O)} = T^{(1)}_{q,O}$ gives an identity transform. Next, note that, by (2.6),

$$\mathfrak{S}\big(\mathfrak{S}(O_1), \mathfrak{S}(O_2)\big) = \mathfrak{S}(O_1 \wedge O_2). \tag{3.10}$$

From this, we also see that for all $O_1, O_2, O_3 \in \mathsf{S}_{\mathrm{f}}$,

$$\mathfrak{S}\big(\mathfrak{S}(O_1 \wedge O_2), \mathfrak{S}(O_3)\big) = \mathfrak{S}\big(O_1 \wedge O_2 \wedge O_3\big) = \mathfrak{S}\big(\mathfrak{S}(O_1), \mathfrak{S}(O_2 \wedge O_3)\big),$$

and so the operation $\circledcirc$ is associative. In view of these observations, we get the following theorem.

Theorem 3.2. *Given any real q, the space $(\mathsf{M}(q;\mathsf{S}_{\mathrm{f}}), \circledcirc)$ is a commutative monoid. Indeed, the monoid $\mathsf{M}(q;\mathsf{S}_{\mathrm{f}})$ acts on the space $\mathcal{F}(\mathbb{B}^*)$ in the sense that*

$$(M^{(1)}_{q,O}, F_\sigma) \mapsto M^{(1)}_{q,O}(F_\sigma) \equiv T^{(1)}_{q,\mathfrak{S}(O)}(F_\sigma).$$

Remark 3.3. *The operation $\circledcirc$ is a semigroup action of $\mathsf{M}(q;\mathsf{S}_{\mathrm{f}})$, i.e., $\mathsf{M}(q;\mathsf{S}_{\mathrm{f}})$ is a transform semigroup.*

Define a relation $\overset{\mathfrak{S}}{\sim}$ on S_{f} by

$$O_1 \overset{\mathfrak{S}}{\sim} O_2 \quad \text{if and only if} \quad \mathfrak{S}(O_1) = \mathfrak{S}(O_2). \tag{3.11}$$

Then $\overset{\mathfrak{S}}{\sim}$ is an equivalence relation on S_{f}. Next, define a relation $\overset{\mathbf{M}}{\sim}$ on $\mathsf{M}(q;\mathsf{S}_{\mathrm{f}})$ by

$$M^{(1)}_{q,O_1} \overset{\mathbf{M}}{\sim} M^{(1)}_{q,O_2} \quad \text{if and only if} \quad O_1 \overset{\mathfrak{S}}{\sim} O_2.$$

From (3.9), (2.8), (2.7), and (3.11), we see that the relation $\overset{\mathbf{M}}{\sim}$ is a well-defined equivalence relation. Consequently, we can obtain the quotient space

$$\mathfrak{G}(q;\mathsf{S}_{\mathrm{f}}) := \mathsf{M}(q;\mathsf{S}_{\mathrm{f}})/\overset{\mathbf{M}}{\sim}$$

with the operation

$$[M^{(1)}_{q,O_1}] \circledcirc_{\mathbf{M}} [M^{(1)}_{q,O_2}] := [M^{(1)}_{q,O_1\wedge O_2}]. \tag{3.12}$$

These settings yield the result as the main theorem of this paper.

Theorem 3.4. *Define a map $\mathfrak{P} : (\mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}}), \circledcirc_{\mathbf{M}}) \to (\mathsf{T}^{(1)}(q;\mathcal{L}(\mathbb{B}) \Cap \mathcal{L}(\mathbb{H})), \circ)$ by*

$$\mathfrak{P}([M^{(1)}_{q,O}]) := T^{(1)}_{q,\mathfrak{S}(O)}. \tag{3.13}$$

Then $\mathfrak{P}$ is a monoid isomorphism.

Proof. Clearly, the space $(\mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}}), \circledcirc_{\mathbf{M}})$ is a commutative monoid and the map $\mathfrak{P}$ given by equation (3.13) is bijective. Next, applying (3.12) and (3.10), we obtain

$$\begin{aligned}
\mathfrak{P}([M^{(1)}_{q,O_1}] \circledcirc_{\mathbf{M}} [M^{(1)}_{q,O_2}]) &= \mathfrak{P}([M^{(1)}_{q,O_1\wedge O_2}]) \\
&= T^{(1)}_{q,\mathfrak{S}(O_1\wedge O_2)} \\
&= T^{(1)}_{q,\mathfrak{S}(\mathfrak{S}(O_1),\mathfrak{S}(O_2))} \\
&= T_{q,\mathfrak{S}(O_1)} \circ T_{q,\mathfrak{S}(O_2)} \\
&= \mathfrak{P}([M^{(1)}_{q,O_1}]) \circ \mathfrak{P}([M^{(1)}_{q,O_2}])
\end{aligned}$$

as desired. □

4. A free group

In view of equation (2.4), the class $\mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}})$ is not a group if $q \neq 0$. In this section, we will clarify a transformation group freely generated by $\mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}})$.

We recall the fact that the class $\mathcal{F}(\mathbb{B}^*)$ is a Banach algebra with the norm $\|F_\sigma\| = \|\sigma^*\| = \int_{\mathbb{B}^*} d|\sigma^*|(g)$ and the mapping $\sigma^* \mapsto F_\sigma$ by (2.3) is a Banach algebra isomorphism between $\mathcal{M}(\mathbb{B})^*$ and $\mathcal{F}(\mathbb{B}^*)$. In view of these facts and the assertion (i) of Theorem 2.1, one can see that

$$\|F_\sigma\| = \|\sigma^*\| = \int_{\mathbb{B}^*} d|\sigma^*|(g) = \int_{\mathbb{B}^*} d|(\sigma^*)_t^A|(g) = \|(\sigma^*)_t^A\| = \|T_{q,A}^{(1)}(F)\|$$

for any $A \in \mathcal{L}(\mathbb{B})$, and every $F_\sigma \in \mathcal{F}(\mathbb{B}^*)$. Thus we can assert the following theorem.

Theorem 4.1. *For any $q \in \mathbb{R} \setminus \{0\}$ and let $A \in \mathcal{L}(\mathbb{B})$, the L_1 analytic FFT associated with A, $T_{q,A}^{(1)} : \mathcal{F}(\mathbb{B}^*) \to \mathcal{F}(\mathbb{B}^*)$ is a linear operator isomorphism. Furthermore, $\|T_{q,A}^{(1)}\|_{\mathrm{o}} = 1$, where $\|\cdot\|_{\mathrm{o}}$ means the operator norm.*

For any non-zero real q, let $\mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}})^* := \mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}}) \setminus \{[M_{q,(O)}^{(1)}]\}$. Define a map

$$\mathcal{W} : \mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}})^* \longrightarrow \mathfrak{G}^{(1)}(-q;\mathsf{S}_{\mathrm{f}})^*$$

by $\mathcal{W}([M_{q,\mathfrak{S}(O)}^{(1)}]) = [M_{-q,\mathfrak{S}(O)}^{(1)}]$. Then, the mapping $\mathcal{W}$ is an one-to-one correspondence. Thus, by the usual argument in the free group theory [11], one can obtain the group $\mathsf{F}(\mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}}))$ freely generated by $\mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}})^*$.

Note that

$$[M_{q,O_1}^{(1)}] \circledcirc_{\mathbf{M}} [M_{q,O_2}^{(1)}] = [M_{q,O_1\wedge O_2}^{(1)}] = [T_{q,\mathfrak{S}(O_1\wedge O_2)}]$$

for $[M_{q,O_1}^{(1)}]$ and $[M_{q,O_2}^{(1)}]$ in $\mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}})$. Given two transforms $\mathcal{T}_1$ and $\mathcal{T}_2$ in $\mathsf{F}(\mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}}))$, let the group operation between $\mathcal{T}_1$ and $\mathcal{T}_2$ be given by

$$(\mathcal{T}_1 \circ \mathcal{T}_2)(F_\sigma) \equiv \mathcal{T}_1(\mathcal{T}_2(F_\sigma)), \quad F_\sigma \in \mathcal{F}(\mathbb{B}).$$

For an element $\mathcal{T}$ of $\mathsf{F}(\mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}}))$, let $\mathsf{l}_w(\mathcal{T})$ denote the length of the word $\mathcal{T}$. Given $\mathcal{T} \in \mathsf{F}(\mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}}))$, assume that $\mathcal{T}$ is not the empty word (i.e., it is not the identity transform $[M_{q,(O)}^{(1)}]$). In the case that $\mathsf{l}_w(\mathcal{T}) = 1$, the transform $\mathcal{T}$ is a member of the set

$$\mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}}) \,\dot{\cup}\, \mathfrak{G}^{(1)}(-q;\mathsf{S}_{\mathrm{f}}).$$

Alternatively, in the case that $\mathsf{l}_w(\mathcal{T}) > 1$, $\mathcal{T}$ may not be expressed as an equivalence class of a single FFT. However, in view of Theorem 4.1, we can assert the fact that for any $\mathcal{T} \in \mathsf{F}(\mathfrak{G}^{(1)}(q;\mathsf{S}_{\mathrm{f}}))$, $\mathcal{T}$ is a linear operator isomorphism from $\mathcal{F}(\mathbb{B}^*)$ into itself.

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