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Stretching Double Sequences by "Blocks"

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Abstract. This paper introduces the "Stretching by Blocks" method, a new approach for stretching double sequences that extends Patterson's method by allowing block repetition. We prove several theorems, including the Extended Copy Theorem and the preservation of Pringsheim limit points under RH-regular matrix transformations.

1. Introduction.

Double sequences and their convergence properties have been widely studied, with the notion of Pringsheim convergence being a fundamental concept [6]. Various methods for manipulating double sequences have been proposed, including traditional stretching method [4], which creates a new double sequence by repeating rows and columns of the original sequence according to certain rules. In this article, we introduce a new stretching method called the "Stretching by Blocks" method, which extends the traditional method by providing a more general and flexible approach to stretching double sequences. Our method constructs a new double sequence by repeating blocks of entries from the original sequence, with block sizes determined by two sequences of positive integers.

We investigate the properties of the "Stretching by Blocks" method and prove several important theorems, including the Extended Copy Theorem, preservation of Pringsheim limit points under RH-regular matrix transformations, and the Extended Copy Theorem for subsequences.

The rest of the article is organized as follows: Section 2 defines the "Stretching by Blocks" method and provides a detailed example. We also compare our method with traditional stretching method. Section 3 presents the main theorems related to the properties of the "Stretching by Blocks" method. Finally, Section 4 summarizes the main findings of the article.

2. Stretching by Blocks Method.

We begin by recalling the traditional stretching method for double sequences.

Definition 2.1 ([4]). *The double sequence y is a stretching of x provided that there exist two increasing index sequences $\{R_i\}_{i=0}^{\infty}$ and $\{S_j\}_{j=0}^{\infty}$ of integers such that*

$$R_0 = S_0 = 1,$$
$$y_{n,k} = x_{i,j}, \text{ if } R_{i-1} \le k < R_i \text{ and } S_{j-1} \le n < S_j, \quad i, j = 1, 2, \ldots$$

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Building upon this concept, we now introduce the "Stretching by Blocks" method for constructing new double sequences from existing ones.

Definition 2.2 (Stretching by Blocks). *Let* $x = [x_{i,j}]$ *be a double sequence and* $\{a_i\}_{i=1}^{\infty}$, $\{b_j\}_{j=1}^{\infty}$ *be sequences of positive integers. Define*

$$R_i = a_1 + a_2 + \cdots + a_i \quad (\textit{with } R_0 = 0)$$
$$S_j = b_1 + b_2 + \cdots + b_j \quad (\textit{with } S_0 = 0)$$

The stretched double sequence $y = [y_{n,k}]$ *obtained by stretching* x *by blocks is induced by* $\{a_i\}_{i=1}^{\infty}$ *and* $\{b_j\}_{j=1}^{\infty}$ *and defined as:*

$$y_{n,k} = x_{i,j} \quad \textit{if } R_{i-1} < n \leq R_i \textit{ and } S_{j-1} < k \leq S_j.$$

In other words, the (n,k)-entry of y is equal to the (i,j)-entry of x, where i and j are the unique indices satisfying the above inequalities. The construction of y can be visualized as follows:

1. The first $a_1 \times b_1$ block of y is filled with the value $x_{1,1}$.
2. The next $a_2 \times b_2$ block (starting at position $(R_1 + 1, S_1 + 1)$) is filled with $x_{2,2}$.
3. The $a_1 \times b_2$ block to the right of the first block is filled with $x_{1,2}$.
4. The $a_2 \times b_1$ block below the first block is filled with $x_{2,1}$.

This process continues, with each (i,j)-block of y having dimensions $a_i \times b_j$ and being filled with the value $x_{i,j}$. In this way, the sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ control the sizes of the repeated blocks in the stretched double sequence y, with larger values of a_i or b_j resulting in larger blocks of repeated entries from x.

To find the value of a specific entry $y_{n,k}$ without constructing the entire stretched sequence y, we can use the defining inequalities:

1. Find the unique indices i and j satisfying $R_{i-1} < n \leq R_i$ and $S_{j-1} < k \leq S_j$.
2. The value of $y_{n,k}$ is then equal to $x_{i,j}$.

Let us illustrate this notion using the following example.

Example 2.3. *Let* $x = [x_{i,j}] = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \\ x_{3,1} & x_{3,2} & x_{3,3} & \\ \vdots & & & \ddots \end{bmatrix}$, $\{a_i\}_{i=1}^{\infty} = \{1,3,2,\ldots\}$ *and* $\{b_j\}_{j=1}^{\infty} = \{2,1,4,\ldots\}$. *Then* $\{R_i\}_{i=0}^{\infty} = \{0,1,4,6,\ldots\}$ *and* $\{S_j\}_{j=0}^{\infty} = \{0,2,3,7,\ldots\}$.

To construct the stretched matrix $y = [y_{n,k}]$, *we follow these steps:*

Step 1: Start with the first diagonal entry $x_{1,1}$. *Create a block of size* $a_1 \times b_1 = 1 \times 2$, *filled with the value* $x_{1,1}$.

$$y = \left[\begin{array}{|cc|} \hline x_{1,1} & x_{1,1} \\ \hline \end{array} \right]$$

Step 2: Move to the next diagonal entry $x_{2,2}$ *of the original sequence. Create a block of size* $a_2 \times b_2 = 3 \times 1$, *filled with the value* $x_{2,2}$. *Place this block diagonally adjacent to the previous block.*

$$y = \left[\begin{array}{|cc|c|} \hline x_{1,1} & x_{1,1} & \\ \hline & & x_{2,2} \\ & & x_{2,2} \\ & & x_{2,2} \\ \cline{3-3} \end{array} \right]$$

Step 3: Fill the gap to the right of the first block with a block of size $a_1 \times b_2 = 1 \times 1$, *using the entry* $x_{1,2}$ *from the original sequence.*

$$y = \left[\begin{array}{|cc|c|} \hline x_{1,1} & x_{1,1} & x_{1,2} \\ \hline & & x_{2,2} \\ & & x_{2,2} \\ & & x_{2,2} \\ \cline{3-3} \end{array} \right]$$

Step 4: Fill the gap below the first block with a block of size $a_2 \times b_1 = 3 \times 2$*, using the entry* $x_{2,1}$ *from the original sequence.*

$$y = \left[\begin{array}{cc|c} x_{1,1} & x_{1,1} & x_{1,2} \\ \hline x_{2,1} & x_{2,1} & x_{2,2} \\ x_{2,1} & x_{2,1} & x_{2,2} \\ x_{2,1} & x_{2,1} & x_{2,2} \end{array}\right]$$

Step 5: Move to the next diagonal entry $x_{3,3}$ *of the original sequence. Create a block of size* $a_3 \times b_3 = 2 \times 4$*, filled with the value* $x_{3,3}$*. Place this block diagonally adjacent to the previous diagonal block.*

$$y = \left[\begin{array}{cc|c|cccc} x_{1,1} & x_{1,1} & x_{1,2} & & & & \\ \hline x_{2,1} & x_{2,1} & x_{2,2} & & & & \\ x_{2,1} & x_{2,1} & x_{2,2} & & & & \\ x_{2,1} & x_{2,1} & x_{2,2} & & & & \\ \hline & & & x_{3,3} & x_{3,3} & x_{3,3} & x_{3,3} \\ & & & x_{3,3} & x_{3,3} & x_{3,3} & x_{3,3} \end{array}\right]$$

Step 6: Fill the gap to the right of the second diagonal block with a block of size $a_2 \times b_3 = 3 \times 4$*, using the entry* $x_{2,3}$ *from the original sequence.*

$$y = \left[\begin{array}{cc|c|cccc} x_{1,1} & x_{1,1} & x_{1,2} & & & & \\ \hline x_{2,1} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & x_{2,3} \\ x_{2,1} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & x_{2,3} \\ x_{2,1} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & x_{2,3} \\ \hline & & & x_{3,3} & x_{3,3} & x_{3,3} & x_{3,3} \\ & & & x_{3,3} & x_{3,3} & x_{3,3} & x_{3,3} \end{array}\right]$$

Step 7: Fill the gap to the right of the first block with a block of size $a_1 \times b_3 = 1 \times 4$*, using the entry* $x_{1,3}$ *from the original sequence*

$$y = \left[\begin{array}{cc|c|cccc} x_{1,1} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,3} & x_{1,3} & x_{1,3} \\ \hline x_{2,1} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & x_{2,3} \\ x_{2,1} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & x_{2,3} \\ x_{2,1} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & x_{2,3} \\ \hline & & & x_{3,3} & x_{3,3} & x_{3,3} & x_{3,3} \\ & & & x_{3,3} & x_{3,3} & x_{3,3} & x_{3,3} \end{array}\right]$$

Step 8: Fill the gap below the second diagonal block with a block of size $a_3 \times b_2 = 2 \times 1$*, using the entry* $x_{3,2}$ *from the original sequence.*

$$y = \left[\begin{array}{cc|c|cccc} x_{1,1} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,3} & x_{1,3} & x_{1,3} \\ \hline x_{2,1} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & x_{2,3} \\ x_{2,1} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & x_{2,3} \\ x_{2,1} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & x_{2,3} \\ \hline & & x_{3,2} & x_{3,3} & x_{3,3} & x_{3,3} & x_{3,3} \\ & & x_{3,2} & x_{3,3} & x_{3,3} & x_{3,3} & x_{3,3} \end{array}\right]$$

Step 9: Fill the gap below the first diagonal block with a block of size $a_3 \times b_1 = 2 \times 2$*, using the entry* $x_{3,1}$ *from the original sequence.*

$$y = \left[\begin{array}{cc|c|cccc} x_{1,1} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,3} & x_{1,3} & x_{1,3} \\ \hline x_{2,1} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & x_{2,3} \\ x_{2,1} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & x_{2,3} \\ x_{2,1} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & x_{2,3} \\ \hline x_{3,1} & x_{3,1} & x_{3,2} & x_{3,3} & x_{3,3} & x_{3,3} & x_{3,3} \\ x_{3,1} & x_{3,1} & x_{3,2} & x_{3,3} & x_{3,3} & x_{3,3} & x_{3,3} \end{array}\right]$$

Now, let's find, for example, the value of $y_{6,2}$:

Step 1: Find i and j such that $R_{i-1} < 6 \leq R_i$ and $S_{j-1} < 2 \leq S_j$. For i: $R_0 = 0 < 6$; $R_1 = 1 < 6$; $R_2 = 4 < 6$; $R_3 = 6 \geq 6$. So, $R_2 < 6 \leq R_3$, which means $i = 3$. For j: $S_0 = 0 < 2$; $S_1 = 2 \geq 2$. So, $S_0 < 2 \leq S_1$, which means $j = 1$.

Step 2: The value of $y_{6,2}$ is equal to $x_{3,1}$, which can be verified from the stretched matrix y.

The "Stretching by Blocks" method introduced in this article is an extension of the traditional stretching method. While both methods aim to create a new double sequence by stretching an original sequence according to certain rules, there are some key differences between them:

1. **Construction**: In traditional method, the stretching is performed by repeating rows and columns of the original sequence x according to the increasing index sequences $\{R_i\}_{i=0}^{\infty}$ and $\{S_j\}_{j=0}^{\infty}$. The stretched sequence y is constructed by placing the repeated rows and columns in a specific order, as described in Remark 3 of [5]. In contrast, our new method constructs the stretched sequence y by repeating blocks of entries from x, with the block sizes determined by the sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$.
2. **Index sequences**: The traditional method uses two increasing index sequences $\{R_i\}_{i=0}^{\infty}$ and $\{S_j\}_{j=0}^{\infty}$ to control the stretching process, with the condition that $R_0 = S_0 = 1$. Our method uses two sequences of positive integers $\{a_i\}_{i=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$, which may or may not be increasing, to determine the block sizes. We define the cumulative sums R_i and S_j based on these sequences, with $R_0 = S_0 = 0$.
3. **Entry mapping**: In the traditional method, the entry $y_{n,k}$ of the stretched sequence is equal to $x_{i,j}$ if $R_{i-1} \leq k < R_i$ and $S_{j-1} \leq n < S_j$. In our method, $y_{n,k} = x_{i,j}$ if $R_{i-1} < n \leq R_i$ and $S_{j-1} < k \leq S_j$. The difference in the inequalities arises from the different constructions of the stretched sequences.

The block repetition method extends the traditional method by providing a more general way to stretch double sequences. By allowing the block sizes to be determined by arbitrary sequences of positive integers, rather than just increasing index sequences, which offers more flexibility in constructing stretched sequences. Despite these differences, both methods share the same goal of creating a new double sequence by stretching an original sequence.

3. Properties of the Stretching by Blocks Method.

In this section, we present and prove theorems analogous to those in [3], but applied to the "Stretching by Blocks" method introduced in this article.

First, we recall some necessary definitions and results.

Definition 3.1 ([6]). *A double sequence $x = [x_{k,l}]$ is said to converge in the Pringsheim sense (or P-convergent) to a limit L if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. We write $P\text{-}\lim x = L$.*

Definition 3.2 ([4]). *A double sequence y is a double subsequence of the sequence x provided that there exist two increasing double index sequences $\{n_j\}$ and $\{k_j\}$ such that if $z_j = x_{n_j,k_j}$, then y is formed by*

$$\begin{matrix} z_1 & z_2 & z_5 & z_{10} \\ z_4 & z_3 & z_6 & - \\ z_9 & z_8 & z_7 & - \\ - & - & - & - \end{matrix}$$

Definition 3.3 ([7]). *The four-dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.*

Theorem 3.4 ([2, 7]). *The four-dimensional matrix A is RH-regular if and only if*

$$RH_1 : P\text{-}\lim_{m,n} a_{m,n,k,l} = 0 \text{ for each } k \text{ and } l;$$

$$RH_2 : P\text{-}\lim_{m,n} \sum_{k,l=1}^{\infty} a_{m,n,k,l} = 1;$$

$$RH_3 : P\text{-}\lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0 \text{ for each } l;$$

$$RH_4 : P\text{-}\lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0 \text{ for each } k;$$

$$RH_5 : \sum_{k,l=1}^{\infty} |a_{m,n,k,l}| \text{ is } P\text{-convergent; and}$$

$$RH_6 : \text{there exist finite positive integers } A \text{ and } B \text{ such that } \sum_{k,l>B} |a_{m,n,k,l}| \leq A.$$

Definition 3.5 ([4]). *A double sequence y contains an ε-Pringsheim-copy of x if there exists a subsequence y_{n_i,k_j} of y such that $|y_{n_i,k_j} - x_{i,j}| < \varepsilon_{i,j}$, for all $i, j \in \mathbb{N}$, where $\{\varepsilon_{i,j}\}$ is a double sequence converging to zero in the Pringsheim sense.*

Theorem 3.6 (Copy Theorem [1]). *If each of T and A is a regular matrix, x is any complex sequence (bounded or not), and ε is any positive term null sequence, then there exists a stretching u of x such that $T(Ay)$ exists and contains an ε-copy of x.*

The following theorem, which we call the Extended Copy Theorem for the "Stretching by Blocks" method, is inspired by the Copy Theorem in [1]. While the original Copy Theorem deals with the preservation of sequence structure under matrix transformations, our Extended Copy Theorem specifically addresses the preservation of sequence structure when using the "Stretching by Blocks" method in conjunction with RH-regular matrices.

Theorem 3.7. *If A and T are RH-regular matrices, x is a bounded double complex sequence, and ε is a bounded positive term double sequence with $P - \lim_{i,j} \varepsilon_{i,j} = 0$, then there exists a block stretching y of x such that $T(Ay)$ exists and contains an ε-Pringsheim-copy of x.*

Proof. Let x be a bounded double complex sequence, and let ε be a bounded positive term double sequence with $P - \lim_{i,j} \varepsilon_{i,j} = 0$. Define the sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ as follows: For each $i, j \in \mathbb{N}$, choose a_i and b_j large enough so that

$$\sum_{p>a_i} \sum_{q=1}^{\infty} |t_{m,n,p,q}| < \frac{\varepsilon_{i,j}}{8M} \quad \text{and} \quad \sum_{p=1}^{\infty} \sum_{q>b_j} |t_{m,n,p,q}| < \frac{\varepsilon_{i,j}}{8M},$$

where $M = \sup_{i,j} |x_{i,j}|$ and $T = [t_{m,n,p,q}]$.

Now, construct the stretching y of x using the "Stretching by Blocks" method with the sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$. Let $R_i = a_1 + a_2 + \cdots + a_i$ (with $R_0 = 0$) and $S_j = b_1 + b_2 + \cdots + b_j$ (with $S_0 = 0$). The stretched double sequence $[y] = [y_{n,k}]$ is defined by:

$$y_{n,k} = x_{i,j}, \quad \text{if } R_{i-1} < n \leq R_i \text{ and } S_{j-1} < k \leq S_j.$$

We will now show that $T(Ay)$ exists and contains an ε-Pringsheim-copy of x. Let $M = \sup_{i,j} |x_{i,j}|$, and let $N_1 \in \mathbb{N}$ be such that

$$\sum_{p,q>N_1} |a_{m,n,p,q}| < \frac{\varepsilon_{i,j}}{8M}$$

for all $m, n \in \mathbb{N}$. The existence of such an N_1 is guaranteed by the RH-regularity of the matrix A. For each $i, j \in \mathbb{N}$, let $N_{i,j} = \max\{R_i, S_j, N_1\}$. We will show that for all $m, n > N_{i,j}$,

$$\left|\sum_{p,q=1}^{\infty} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}y_{k,l}\right) - x_{i,j}\right| < \varepsilon_{i,j}.$$

Consider the following:

$$\begin{aligned}
&\left|\sum_{p,q=1}^{\infty} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}y_{k,l}\right) - x_{i,j}\right| \\
&\leq \left|\sum_{p=1}^{R_i}\sum_{q=1}^{S_j} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}y_{k,l}\right) - x_{i,j}\right| + \left|\sum_{p>R_i}\sum_{q=1}^{\infty} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}y_{k,l}\right)\right| + \left|\sum_{p=1}^{R_i}\sum_{q>S_j} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}y_{k,l}\right)\right|.
\end{aligned}$$

We will now estimate each of the three terms on the right-hand side of the inequality.
For the first term, we have:

$$\begin{aligned}
&\left|\sum_{p=1}^{R_i}\sum_{q=1}^{S_j} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}y_{k,l}\right) - x_{i,j}\right| \\
&= \left|\sum_{p=1}^{R_i}\sum_{q=1}^{S_j} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}x_{i,j}\right) - x_{i,j}\right| \\
&= \left|\sum_{p=1}^{R_i}\sum_{q=1}^{S_j} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}\right)x_{i,j} - x_{i,j}\right| \\
&\leq \left|\sum_{p=1}^{R_i}\sum_{q=1}^{S_j} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}\right) - 1\right| |x_{i,j}| \\
&\leq \left|\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}\right) - 1\right| M.
\end{aligned}$$

By the RH-regularity of the matrices A and T, we have $P-\lim_{m,n} \sum_{p,q=1}^{\infty} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}\right) = 1$. Therefore, there exists $N_2 \in \mathbb{N}$ such that for all $m, n > N_2$,

$$\left|\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}\right) - 1\right| < \frac{\varepsilon_{i,j}}{4M}.$$

Hence, for all $m, n > \max\{N_{i,j}, N_2\}$,

$$\left|\sum_{p=1}^{R_i}\sum_{q=1}^{S_j} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}y_{k,l}\right) - x_{i,j}\right| < \frac{\varepsilon_{i,j}}{4}.$$

For the second term, we have:

$$\begin{aligned}
&\left|\sum_{p>R_i}\sum_{q=1}^{\infty} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}y_{k,l}\right)\right| \\
&\leq \sum_{p>R_i}\sum_{q=1}^{\infty} |t_{m,n,p,q}|\left|\sum_{k,l=1}^{\infty} a_{p,q,k,l}y_{k,l}\right| \\
&\leq M\sum_{p>R_i}\sum_{q=1}^{\infty} |t_{m,n,p,q}|\sum_{k,l=1}^{\infty} |a_{p,q,k,l}| \\
&\leq M\sum_{p>R_i}\sum_{q=1}^{\infty} |t_{m,n,p,q}|.
\end{aligned}$$

By the choice of a_i, we have $\sum_{p>R_i}\sum_{q=1}^{\infty} |t_{m,n,p,q}| < \frac{\varepsilon_{i,j}}{8M}$. Hence, for all $m, n > N_{i,j}$,

$$\left|\sum_{p>R_i}\sum_{q=1}^{\infty} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}y_{k,l}\right)\right| < \frac{\varepsilon_{i,j}}{8}.$$

For the third term, we have:

$$\begin{aligned}
&\left|\sum_{p=1}^{R_i}\sum_{q>S_j} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}y_{k,l}\right)\right| \\
&\leq \sum_{p=1}^{R_i}\sum_{q>S_j} |t_{m,n,p,q}|\left|\sum_{k,l=1}^{\infty} a_{p,q,k,l}y_{k,l}\right| \\
&\leq M\sum_{p=1}^{R_i}\sum_{q>S_j} |t_{m,n,p,q}|\sum_{k,l=1}^{\infty} |a_{p,q,k,l}| \\
&\leq M\sum_{p=1}^{R_i}\sum_{q>S_j} |t_{m,n,p,q}|.
\end{aligned}$$

By the choice of b_j, we have $\sum_{p=1}^{R_i}\sum_{q>S_j} |t_{m,n,p,q}| < \frac{\varepsilon_{i,j}}{8M}$. Hence, for all $m, n > N_{i,j}$,

$$\left|\sum_{p=1}^{R_i}\sum_{q>S_j} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}y_{k,l}\right)\right| < \frac{\varepsilon_{i,j}}{8}.$$

Combining the estimates for the three terms, we have for all $m, n > \max\{N_{i,j}, N_2\}$,

$$\left|\sum_{p,q=1}^{\infty} t_{m,n,p,q}\left(\sum_{k,l=1}^{\infty} a_{p,q,k,l}y_{k,l}\right) - x_{i,j}\right| < \frac{\varepsilon_{i,j}}{4} + \frac{\varepsilon_{i,j}}{8} + \frac{\varepsilon_{i,j}}{8} < \varepsilon_{i,j}.$$

Thus, $T(Ay)$ contains an ε-Pringsheim-copy of x. □

Using the Extended Copy Theorem, we can prove the following result about the preservation of Pringsheim limit points under the action of RH-regular matrices and the "Stretching by Blocks" method.

Theorem 3.8. *Let T and A be RH-regular matrices, x a bounded double complex sequence, and ε a bounded positive term double sequence with $P\text{-}\lim_{n,k} \varepsilon_{n,k} = 0$. Then there exists a block stretching y of x such that $T(Ay)$ exists and each P-limit point of x is a P-limit point of $T(Ay)$.*

Proof. We first construct a double sequence containing all finite P-limit points of x. By the separability of the metric space of complex numbers, we can write the finite P-limit points of x in a double sequence u such that each P-limit point of x is either a term of u or a P-limit of u. We can achieve this by writing u as a sequence of blocks, where each block consists of repeated entries of a specific P-limit point of x.

$$u = \begin{bmatrix} u_{1,1} & u_{1,1} & u_{1,2} & u_{1,2} & u_{1,2} & u_{1,2} & \cdots & \cdots & & & & \\ u_{2,1} & u_{2,1} & u_{2,2} & u_{2,2} & u_{2,2} & u_{2,2} & \cdots & \cdots & & & & \\ u_{2,1} & u_{2,1} & u_{2,2} & u_{2,2} & u_{2,2} & u_{2,2} & \cdots & \cdots & & & & \\ u_{2,1} & u_{2,1} & u_{2,2} & u_{2,2} & u_{1,1} & u_{1,1} & u_{1,2} & u_{1,2} & u_{1,3} & u_{1,3} & u_{1,3} & \cdots \\ u_{2,1} & u_{2,1} & u_{2,2} & u_{2,2} & u_{1,1} & u_{1,1} & u_{1,2} & u_{1,2} & u_{1,3} & u_{1,3} & u_{1,3} & \cdots \\ \vdots & \vdots & u_{2,3} & u_{2,3} & u_{2,3} & u_{2,3} & u_{1,1} & u_{1,1} & u_{1,1} & \cdots & & \\ \vdots & \vdots & u_{2,3} & u_{2,3} & u_{2,3} & u_{2,3} & u_{1,1} & u_{1,1} & u_{1,1} & \cdots & & \\ \vdots & \vdots & u_{3,1} & u_{3,1} & u_{3,2} & u_{3,2} & u_{3,3} & u_{3,3} & u_{3,3} & \cdots & & \\ \vdots & \vdots & u_{3,1} & u_{3,1} & u_{3,2} & u_{3,2} & u_{3,3} & u_{3,3} & u_{3,3} & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & u_{3,1} & u_{3,1} & u_{3,2} & \cdots & & & & \\ \vdots & \vdots & \vdots & \vdots & u_{3,1} & u_{3,1} & u_{3,2} & \cdots & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & & \end{bmatrix}$$

We then relabel this sequence as

$$v = [v_{i,j}] = \begin{bmatrix} v_{1,1} & v_{1,2} & \cdots \\ v_{2,1} & v_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Now we apply the "Stretching by Blocks" method to v. Let $\varepsilon = [\varepsilon_{k,l}]$ be a bounded positive term double sequence with $P\text{-}\lim_{k,l} \varepsilon_{k,l} = 0$. Choose sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ such that for each $i, j \in \mathbb{N}$,

$$\sum_{p>a_i} \sum_{q=1}^{\infty} |t_{m,n,p,q}| < \frac{\varepsilon_{i,j}}{8M} \quad \text{and} \quad \sum_{p=1}^{\infty} \sum_{q>b_j} |t_{m,n,p,q}| < \frac{\varepsilon_{i,j}}{8M},$$

where $M = \sup_{i,j} |v_{i,j}|$ and $T = [t_{m,n,k,l}]$. Let $R_i = \sum_{k=1}^{i} a_k$ and $S_j = \sum_{k=1}^{j} b_k$. Define the "block stretching" $z = [z_{r,s}]$ by:

$$z_{r,s} = v_{i,j}, \text{ if } R_{i-1} < r \le R_i \text{ and } S_{j-1} < s \le S_j.$$

Using this newly constructed double sequence we now build a stretching y of x. For each block in z determined by the indices (i, j), select a subsequence of x that converges to the P-limit point $v_{i,j}$ in the

Pringsheim sense. Denote this subsequence by $x^{(i,j)} = [x^{(i,j)}_{n'_k,m'_k}]_{k=1}^{\infty}$, where $\{n'_k\}_{k=1}^{\infty}$ and $\{m'_k\}_{k=1}^{\infty}$ are increasing sequences of positive integers. Define $y = [y_{\alpha,\beta}]$ as follows:

$$y_{\alpha,\beta} = x^{(i,j)}_{n'_{k+p-1},m'_{l+q-1}}, \text{ if } R_{i-1} + p < \alpha \leq R_i \text{ and } S_{j-1} + q < \beta \leq S_j,$$

where, $1 \leq p \leq a_i$ and $1 \leq q \leq b_j$ determine the position within the block, k and l are chosen such that $n'_k > N_{i,j}$ and $m'_l > K_{i,j}$ where $N_{i,j}$ and $K_{i,j}$ are indices that ensure the convergence of $x^{(i,j)}$ to $v_{i,j}$ in the Pringsheim sense. This construction ensures that:

(1) each entry in y comes from the "tail" of the subsequence $x^{(i,j)}$, guaranteeing convergence to the P-limit point $v_{i,j}$.

(2) The entire block corresponding to (i, j) is filled, with size determined by a_i and b_j.

(3) The structure of y mirrors that of z, preserving the arrangement of P-limit points.

We will now show that $z - y$ is a Pringsheim null sequence. Let $\varepsilon > 0$ be given. Since each subsequence $x^{(i,j)}$ converges to $v_{i,j}$ in the Pringsheim sense, for each (i, j) there exist indices $N_{i,j}$ and $K_{i,j}$ such that

$$|x^{(i,j)}_{n'_k,m'_l} - v_{i,j}| < \varepsilon \text{ for all } k > N_{i,j} \text{ and } l > K_{i,j}.$$

Let $N = \max_{i,j}\{N_{i,j}, K_{i,j}\}$, where the maximum is taken over all (i, j) corresponding to the blocks in z that have been defined up to this point in the construction. This N ensures that for any block (i, j), if we choose $k, l > N$, then $x^{(i,j)}_{n'_k,m'_l}$ is within ε of $v_{i,j}$. Now, let $M = \max\{R_I, S_J\}$, where I and J are the largest indices such that $R_{I-1} < N$ and $S_{J-1} < N$. This ensures that for any $\alpha, \beta > M$, we are considering blocks in z and y where the convergence condition is satisfied. Then for any $\alpha, \beta > M$, there exist unique indices $i > I$ and $j > J$ such that $R_{i-1} < \alpha \leq R_i$ and $S_{j-1} < \beta \leq S_j$. By the construction of y, we have

$$y_{\alpha,\beta} = x^{(i,j)}_{n'_{k+p-1},m'_{l+q-1}}$$

where $1 \leq p \leq a_i$, $1 \leq q \leq b_j$, and k, l are chosen such that $n'_k > N_{i,j}$ and $m'_l > K_{i,j}$. Therefore, for $\alpha, \beta > M$, we have:

$$|z_{\alpha,\beta} - y_{\alpha,\beta}| = |v_{i,j} - x^{(i,j)}_{n'_{k+p-1},m'_{l+q-1}}| < \varepsilon.$$

Thus, $z - y$ is a Pringsheim null sequence. With this P-null sequence, we will show that each P-limit point of x is a P-limit point of $T(Ay)$. We have:

$$T(Ay) = T(A(y - z)) + T(Az).$$

Since A and T are RH-regular matrices and $z - y$ is a Pringsheim null sequence, $T(A(y - z))$ is also a Pringsheim null sequence. Moreover, $T(Az)$ contains an ε-Pringsheim-copy of v, which means that for each P-limit point $v_{i,j}$ of x, there exists a subsequence of $T(Az)$ that converges to $v_{i,j}$ in the Pringsheim sense. Therefore, each P-limit point of x is a P-limit point of $T(Ay)$. □

As a corollary, we obtain an Extended Copy Theorem for subsequences.

Theorem 3.9. *Let T and A be RH-regular matrices, x a bounded double complex sequence, and ε a bounded positive term double sequence with $P\text{-}\lim_{n,k} \varepsilon_{n,k} = 0$. Then there exists a subsequence y of x such that $T(Ay)$ exists and each P-limit point of x is a P-limit of $T(Ay)$.*

Proof. The proof is similar to that of the previous theorem, with the difference that we choose y to be a subsequence of x such that $z - y$ is a Pringsheim null sequence, where z is the stretching constructed in the proof of the previous theorem. Let $T(Ay) = T(A(y - z)) + T(Az)$. Note that $T(Ay)$ is the sum of a Pringsheim null sequence and a sequence that contains an ε-Pringsheim-copy of v. Therefore, each P-limit point of x is a P-limit of $T(Ay)$. □

These results demonstrate that the main properties and theorems related to the preservation of P-limit points under RH-regular matrix transformations can be extended to the "Stretching by Blocks" method introduced in this article.
Here are some examples to support the theorems and illustrate their implications:

Example 3.10. *Let $x = [x_{i,j}]$ be a bounded double sequence defined by:*

$$x_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\varepsilon = [\varepsilon_{i,j}]$ be a bounded positive term double sequence with P-$\lim_{i,j} \varepsilon_{i,j} = 0$, defined by:

$$\varepsilon_{i,j} = \frac{1}{i+j}.$$

Consider the stretching y of x using the "Stretching by Blocks" method with the sequences $\{a_i\}_{i=1}^{\infty} = \{1, 2, 3, \ldots\}$ and $\{b_j\}_{j=1}^{\infty} = \{1, 2, 3, \ldots\}$. Then, according to the Extended Copy Theorem, for any RH-regular matrices A and T, $T(Ay)$ exists and contains an ε-Pringsheim-copy of x.

The implication of this example is that even if the original sequence x has a simple structure (e.g., only the diagonal entries are non-zero), its stretching y can have a more complex structure while still preserving the essential properties of x under RH-regular matrix transformations.

Example 3.11. *Let $x = [x_{i,j}]$ be a bounded double sequence with P-limit points 0 and 1, defined by:*

$$x_{i,j} = \begin{cases} 1, & \text{if } i + j \text{ is even}, \\ 0, & \text{if } i + j \text{ is odd}. \end{cases}$$

Let $\varepsilon = [\varepsilon_{i,j}]$ be a bounded positive term double sequence with P-$\lim_{i,j} \varepsilon_{i,j} = 0$, defined by:

$$\varepsilon_{i,j} = \frac{1}{2^{i+j}}.$$

Consider any stretching y of x using the "Stretching by Blocks" method. According to the theorem on the preservation of Pringsheim limit points, for any RH-regular matrices A and T, $T(Ay)$ exists and each P-limit point of x (i.e., 0 and 1) is a P-limit point of $T(Ay)$.

This example illustrates that the "Stretching by Blocks" method, combined with RH-regular matrix transformations, preserves the P-limit points of the original sequence, even if the original sequence has a more complex structure (e.g., alternating entries).

4. Conclusion.

In this article, we introduced the "Stretching by Blocks" method, a new approach for stretching double sequences. This method constructs a new double sequence by repeating blocks of entries from the original sequence, with block sizes determined by two sequences of positive integers. We compared this method with traditional existing stretching method [4] and highlighted its advantages in terms of generality and flexibility. We proved several important theorems, including the Extended Copy Theorem, preservation of Pringsheim limit points under RH-regular matrix transformations, and the Extended Copy Theorem for subsequences.

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