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Generalized limits and ideal convergence

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Abstract. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, the set of positive integers. Consider the Banach space ℓ^∞ of real bounded sequences x with $\|x\| = \sup_k |x_k|$. A positive linear functional L on ℓ^∞ is called an $S_\mathcal{I}-$limit if $L(\chi_K) = 0$ for every characteristic sequence χ_K of sets $K \subseteq \mathbb{N}$ for which $\mathcal{I} - \lim \chi_K = 0$. We examine regular sublinear functionals that both generate as well as dominate $S_\mathcal{I}-$limits. We also show that these results are closely related to the concept of core and multipliers for bounded sequences.

1. Introduction

Let ℓ^∞ and c be the spaces of all bounded and convergent real sequences $x = (x_k)$ normed by $\|x\| = \sup_k |x_k|$. Let $\mathcal{B}$ be the class of (necessarily continuous) linear functionals β on ℓ^∞ which are nonnegative and regular, that is, if $x \geq 0$ (i.e., $x_k \geq 0$ for all $k \in \mathbb{N} := \{1, 2, ...\}$) then $\beta(x) \geq 0$ and $\beta(x) = \lim_k x_k$ for each $x = (x_k) \in c$. In the paper, we consider some generalized limits such that the space of all bounded $\mathcal{I}-$convergent sequences can be represented as the set of all bounded sequences which have the same value under any such limit. The sublinear functionals that generate or dominate these limits are studied. We show that these results are closely related to the concept of core for bounded sequences. Multipliers for bounded $\mathcal{I}-$convergent sequences are also considered. In proving these results the class $\mathcal{M}(\mathcal{I})$ of nonnegative regular matrices such that $\mathcal{I} \subseteq \mathcal{I}(A)$ plays an important role where $\mathcal{I}(A)$ is the matrix ideal generated by a nonnegative regular matrix A (see e.g. [9, 10]).

We first collect some notation. Let $A = (a_{nk})$ be an infinite matrix. Given a sequence x, the $A-$transform of x, denoted as $Ax := ((Ax)_n)$, is given by $(Ax)_n = \sum_{k=1}^\infty a_{nk} x_k$ provided that the series converges for each n. Let $\lim_A x := \lim_n (Ax)_n$ whenever the limit exists. By c_A we denote the summability domain of A, i.e., $c_A = \{x : \lim_A x \text{ exists}\}$. We say that A is regular [3, 24] if $\lim_n (Ax)_n = \lim_k x_k$ for each $x \in c$. For any nonnegative such matrix A we define the $A-$density of a set $K \subseteq \mathbb{N}$, denoted as $\delta_A(K)$, by

$$\delta_A(K) := \lim_n \sum_{k=1}^\infty a_{nk} \chi_K(k) = \lim_n (A\chi_K)_n$$

provided that the limit exists, where χ_K denotes the characteristic sequence of the set K.

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Let

$$Q_A(x) := \limsup_n \sum_{k=1}^{\infty} a_{nk} x_k \text{ for } x \in \ell^\infty.$$

Let us recall the notion presented by Freedman in [11].

Definition 1.1. *A family $\mathcal{I} \subseteq P(X)$ is called an ideal (or a zero class) on a set X if*
(a) $E, F \in \mathcal{I} \Rightarrow E \cup F \in \mathcal{I}$,
(b) $E \subseteq F \in \mathcal{I} \Rightarrow E \in \mathcal{I}$,
(c) $\mathcal{I}$ contains all finite subsets of X,
(d) $X \notin \mathcal{I}$.

The family $Fin := \{A \subseteq X : A \text{ is finite}\}$ is an ideal. For an ideal $\mathcal{I}$ on X, let $\mathcal{I}^* = \{X \backslash E : E \in \mathcal{I}\}$. Then $\mathcal{I}^*$ is called filter dual to $\mathcal{I}$.

An ideal $\mathcal{I}$ on X is a $P-$ideal if for every countable family $\mathcal{A} \subseteq \mathcal{I}$ there is $F \in \mathcal{I}$ such that $E \backslash F$ is finite for every $E \in \mathcal{A}$.

Throughout the paper we only consider the ideals on $\mathbb{N}$. The family

$$\mathcal{I}(A) = \{K \subseteq \mathbb{N} : \delta_A(K) = 0\}$$

is a $P-$ideal on $\mathbb{N}$ induced by a nonnegative regular matrix $A = (a_{nk})$ (see e.g. [9, 12]).

The motivation for this paper is interest in classes of subsets of positive integers that properly contain the class of finite sets, but with the property that each set in the family is still, in some sense, small. The class $\mathcal{I}$ defined above fits this general description.

Definition 1.2. *Let $\mathcal{I}$ be an ideal on the set of positive integers. A sequence $x \in \mathbb{R}^{\mathbb{N}}$ is $\mathcal{I}-$convergent if there exists $\ell \in \mathbb{R}$ such that $\{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$. The real number ℓ is called the $\mathcal{I}-$limit of x and we denote it by $\ell = \mathcal{I} - \lim x$ or $\lim^{\mathcal{I}} x = \ell$ (see e.g. [19, 20]).*

Definition 1.3. *Let $\mathcal{I}$ be an ideal. A sequence $x \in \mathbb{R}^{\mathbb{N}}$ is $\mathcal{I}^*-$convergent if there exists $\ell \in \mathbb{R}$ and a set $F \in \mathcal{I}^*$ such that the subsequence $(x_n)_{n \in F}$ is ordinarily convergent (i.e., $\{n \in F : |x_n - \ell| \geq \varepsilon\}$ is finite for every $\varepsilon > 0$). The real number ℓ is called the $\mathcal{I}^*-$limit of x and we denote it by $\ell = \mathcal{I}^* - \lim x$ or $\lim^{\mathcal{I}^*} x = \ell$ (see e.g. [19, 20]).*

By $c(\mathcal{I})$ and $c(\mathcal{I}^*)$ we denote the spaces of all $\mathcal{I}-$convergent and $\mathcal{I}^*-$convergent sequences, respectively.

Recall that $\mathcal{I}^* - \lim x = \ell$ implies that $\mathcal{I} - \lim x = \ell$, but not conversely. If $\mathcal{I}$ is a $P-$ideal then the converse also holds [1, 19].

It is also known that

$$\overline{c(\mathcal{I}^*) \cap \ell^\infty} = c(\mathcal{I}) \cap \ell^\infty \tag{1}$$

where the closure is taken in the sup norm topology of ℓ^∞ (see [20]). A study of $FK-$topology for $\mathcal{I}-$convergent sequences may be found in [22].

Motivated by the results presented by Freedman in [11] and by Yurdakadim et al. in [26] we introduce the following:

Definition 1.4. *Let $\mathcal{I}$ be an ideal and let L be a linear functional on ℓ^∞ that satisfies the following properties:*
1. $L(x) \geq 0$ if $x \geq 0$ (positivity of L),
2. $L(x) = \lim_k x_k$ for $x \in c$ (regularity of L),
3. For every $E \subseteq \mathbb{N}$ such that $E \in \mathcal{I}$ (i.e., $\mathcal{I} - \lim \chi_E = 0$) we have $L(\chi_E) = 0$.

Every such L will be called an $S_{\mathcal{I}}-$limit and we denote their collection by $S\mathcal{I}$.

Combining the Theorem of Freedman [11] with (1) above we conclude that the space of all bounded $\mathcal{I}-$convergent sequences can be represented as the set of all $x \in \ell^\infty$ that have the same value under any $S_{\mathcal{I}}-$limit.

Note that if f is a linear functional and g is a positive functional on ℓ^∞ such that $f(x) \leq g(x) \leq \limsup x$ on ℓ^∞, then we have

$$\liminf x \leq -g(-x) \leq -f(-x) \leq f(x) \leq g(x) \leq \limsup x$$

on ℓ^∞. This necessarily implies that f has to be positive and regular. This fact will be used throughout the paper without citing.

In order to explore further relationship between some generalized limits, we recall the concepts of $\mathcal{I}-$limit superior and $\mathcal{I}-$limit inferior from [5, 7, 13]

$$\mathcal{I} - \limsup x := \begin{cases} \sup B_x & , \quad \text{if } B_x \neq \emptyset \\ -\infty & , \quad \text{if } B_x = \emptyset \end{cases}$$

and

$$\mathcal{I} - \liminf x := \begin{cases} \inf A_x & , \quad \text{if } A_x \neq \emptyset \\ \infty & , \quad \text{if } A_x = \emptyset \end{cases}$$

where $A_x := \{a \in \mathbb{R} : \{k : x_k < a\} \notin \mathcal{I}\}$ and $B_x := \{b \in \mathbb{R} : \{k : x_k > b\} \notin \mathcal{I}\}$.

The next section provides results concerning the properties of these generalized limits.

2. Existence of $S_\mathcal{I}-$limits

In this section we show that $S_\mathcal{I}-$limits exist and then study the sublinear functionals that generate and/or dominate $S_\mathcal{I}-$limits. This section is largely influenced by the results presented by Yurdakadim et al. in [26].

First, we present some basic properties of $\mathcal{I} - \limsup$.

Proposition 2.1. *Let $P_\mathcal{I}(x) := \mathcal{I} - \limsup x$. Then the following results hold.*
(a) $-P_\mathcal{I}(-x) = \mathcal{I} - \liminf x$ for all $x \in \ell^\infty$.
(b) $P_\mathcal{I}(x + y) \leq P_\mathcal{I}(x) + P_\mathcal{I}(y)$ for any $x, y \in \ell^\infty$.
(c) $P_\mathcal{I}(\alpha x) = \alpha P_\mathcal{I}(x)$ for any $\alpha \geq 0$ and $x \in \ell^\infty$.

Proof. Part *(a):*

$$\begin{aligned} P_\mathcal{I}(-x) &= \mathcal{I} - \limsup (-x) \\ &= \sup B_{-x} \\ &= \sup \{b \in \mathbb{R} : \{k : -x_k > b\} \notin \mathcal{I}\} \\ &= \sup \{b \in \mathbb{R} : \{k : x_k < -b\} \notin \mathcal{I}\} \\ &= -\inf \{-b \in \mathbb{R} : \{k : x_k < -b\} \notin \mathcal{I}\} \\ &= -\inf A_x \\ &= -(\mathcal{I} - \liminf x). \end{aligned}$$

Part *(b)* is proved in [21].

Part *(c):* For $\alpha > 0$, one can get

$$\begin{aligned}
P_{\mathcal{I}}(\alpha x) &= \mathcal{I} - \limsup (\alpha x) \\
&= \sup B_{\alpha x} \\
&= \sup \{b \in \mathbb{R} : \{k : \alpha x_k > b\} \notin \mathcal{I}\} \\
&= \sup \left\{ b \in \mathbb{R} : \left\{ k : x_k > \frac{b}{\alpha} \right\} \notin \mathcal{I} \right\} \\
&= \alpha \sup \left\{ \frac{b}{\alpha} \in \mathbb{R} : \left\{ k : x_k > \frac{b}{\alpha} \right\} \notin \mathcal{I} \right\} \\
&= \alpha \sup B_x \\
&= \alpha P_{\mathcal{I}}(x).
\end{aligned}$$

The case in which $\alpha = 0$ is obvious. □

Theorem 2.2. *$S_{\mathcal{I}}-$limits exist.*

Proof. Consider the sublinear functional

$$P_{\mathcal{I}}(x) := \mathcal{I} - \limsup x, \, x \in \ell^\infty.$$

If $x \in c$, then we immediately see that $P_{\mathcal{I}}(x) = \lim_k x_k$. By the Hahn-Banach theorem, there exists a bounded linear functional T over ℓ^∞ such that

$$-P_{\mathcal{I}}(-x) \le T(x) \le P_{\mathcal{I}}(x), \, x \in \ell^\infty. \tag{2}$$

We claim that T is an $S_{\mathcal{I}}-$limit. To see this, observe that $T(x) \ge 0$ for every $x \ge 0$, and $T(x) = \lim_k x_k$ for every $x \in c$. Next, if $E \in \mathcal{I}$, then $\mathcal{I} - \lim \chi_E = 0$, and hence by (2) we get that $0 \le T(\chi_E) \le \mathcal{I} - \lim \chi_E = 0$. Hence T is an $S_{\mathcal{I}}-$limit. □

The referee has also suggested the following alternate proof of Theorem 2.2 which is much quicker.

Alternate proof: Using Zorn's lemma, it follows that there exists a maximal ideal $\mathcal{J}$ such that $\mathcal{J} \supseteq \mathcal{I}$. By Lemma 5.2 in [19] for any $x \in \ell^\infty$ there exists $\ell \in \mathbb{R}$ such that $\ell = \mathcal{J} - \lim x$.

We claim that $\mathcal{J} - \lim$ is an $S_{\mathcal{I}}-$limit. We know that $\mathcal{J} - \lim$ is a linear functional on ℓ^∞ (see e.g. [20]). Observe that $\mathcal{J} - \lim$ is positive and regular. Next, if $E \in \mathcal{I}$ then $\mathcal{J} - \lim \chi_E = 0$. This proves that $\mathcal{J} - \lim$ is an $S_{\mathcal{I}}-$limit. □

We denote by $(\ell^\infty)^*$ the algebraic dual of ℓ^∞.

Definition 2.3 ([23, 26]). *Let R and T be sublinear functionals on ℓ^∞.*
(i) We say that R generates $S\mathcal{I}$ if for any $L \in (\ell^\infty)^*$ such that $L(x) \le R(x)$ for all $x \in \ell^\infty$ we have $L \in S\mathcal{I}$.
(ii) We say that T dominates $S\mathcal{I}$ if for every $L \in S\mathcal{I}$ we have $L(x) \le T(x)$ for all $x \in \ell^\infty$.

A sublinear functional R on ℓ^∞ generates $S\mathcal{I}$ if and only if $R(x) \le W(x)$ for all $x \in \ell^\infty$, where

$$W(x) := \sup \{L(x) : L \in S\mathcal{I}\} \text{ for all } x \in \ell^\infty.$$

Evidently a sublinear functional R dominates $S\mathcal{I}$ if and only if $W(x) \le R(x)$ for all $x \in \ell^\infty$. Hence a sublinear functional R on ℓ^∞ generates as well as dominates $S\mathcal{I}-$limits if and only if it equals W.

The next theorem shows that $P_{\mathcal{I}}$ both generates and dominates $S_{\mathcal{I}}-$limits.

Theorem 2.4. *(i) $P_{\mathcal{I}}$ both generates and dominates $S\mathcal{I}$. Therefore,*
$P_{\mathcal{I}}(x) = \sup \{L(x) : L \in S\mathcal{I}\}$, for all $x \in \ell^\infty$.
(ii) Let $\mathcal{I}$ be an ideal such that $\mathcal{I} \supsetneqq Fin$ and let $Q(x) := \limsup x$. Then Q dominates $S\mathcal{I}$ but cannot generate $S\mathcal{I}$.

Proof. *(i)* The fact that $P_{\mathcal{I}}$ generates $S\mathcal{I}$ follows by an identical proof of Theorem 2.2. To prove that $P_{\mathcal{I}}$ dominates $S\mathcal{I}$, let $L \in S\mathcal{I}$. If there exists a sequence $x \in \ell^\infty$ such that $L(x) > P_{\mathcal{I}}(x)$ then take $p \in (P_{\mathcal{I}}(x), L(x))$ and let $E := \{k : x_k > p\}$. By the properties of $\mathcal{I}-\limsup x$ (see [7], Theorem 1) we observe that $\mathcal{I}-\limsup \chi_E = 0$. Hence $L(\chi_E) = 0$. So we have

$$\begin{aligned} L(x) &= L(x\chi_E) + L(x\chi_{E^c}) \\ &\le |L(x\chi_E)| + L(x\chi_{E^c}) \\ &\le \|x\|\, L(\chi_E) + pL(\chi_{E^c}) \\ &= 0 + p \\ &= p \\ &< L(x) \end{aligned}$$

where $e = (1, 1, ...)$. This contradiction proves that $L(x) \le P_{\mathcal{I}}(x)$ for all $x \in \ell^\infty$. Hence $P_{\mathcal{I}}$ dominates $S\mathcal{I}$.

(ii) Since $P_{\mathcal{I}}(x) \le Q(x)$ for all $x \in \ell^\infty$, and $P_{\mathcal{I}}$ dominates $S\mathcal{I}$, we must have that Q dominates $S\mathcal{I}$.

To show that Q cannot generate $S\mathcal{I}$, we will find a positive regular functional T such that $T(x) \le Q(x)$ for all $x \in \ell^\infty$ but $T \notin S\mathcal{I}$. To do that let $E = \{j_n : j_1 < j_2 < ...\} \subseteq \mathbb{N}$ be an infinite set such that $E \in \mathcal{I}$. We can find such an infinite set since $\mathcal{I} \supsetneq Fin$. Hence $\mathcal{I} - \lim \chi_E = 0$.

Now define a nonnegative regular matrix $B = (b_{nk})$ where $b_{nk} = 1$ when $k = j_n$ and $b_{nk} = 0$ for $k \ne j_n$ $(n = 1, 2, ...)$. Using the resulting $Q_B(x) := \limsup_n \sum_k b_{nk}x_k$ and the linear functional $\lim_B$ on $\ell^\infty \cap c_B$, by the Hanh-Banach theorem we get a bounded linear functional T on ℓ^∞ such that $T(x) = \lim_B x$ on $\ell^\infty \cap c_B$ and $T(x) \le Q_B(x)$ for all $x \in \ell^\infty$. Certainly, $Q_B(x) \le Q(x)$, and hence $T(x) \le Q(x)$ for all $x \in \ell^\infty$. Observe that $T(\chi_E) = \lim_B \chi_E = 1$ by the construction of the matrix B. On the other hand $\mathcal{I} - \lim \chi_E = 0$ which implies that for every $L \in S\mathcal{I}$ we must have $L(\chi_E) = 0$. Hence $T \notin S\mathcal{I}$. □

As in [10], let $\mathcal{NRM}$ be the family of all nonnegative regular matrices. For an ideal $\mathcal{I}$ on $\mathbb{N}$, Filipów and Tryba [9, 10] introduced the following class:

$$\mathcal{M}(\mathcal{I}) = \{A \in \mathcal{NRM} : \mathcal{I} \subseteq \mathcal{I}(A)\}$$

where $\mathcal{I}(A) = \{E \subseteq \mathbb{N} : \delta_A(E) = 0\}$.

In the remainder of the paper the class $\mathcal{M}(\mathcal{I})$ will play a very important role.

An ideal $\mathcal{I}$ has the property *GMV* if for every $x \in \ell^\infty$, $\lim^{\mathcal{I}} x = L \Leftrightarrow \lim_A x = L$ for every $A \in \mathcal{M}(\mathcal{I})$. This is equivalent to the fact that [9, 10]

$$\mathcal{I} = \cap \{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I})\}$$

provided that $\mathcal{M}(\mathcal{I}) \ne \emptyset$.

With this notation we have the following:

Theorem 2.5. *Let $A = (a_{nk})$ be a nonnegative infinite matrix with $\sup_n \sum_k a_{nk} < \infty$. Then*

(a) Q_A generates $S\mathcal{I}$ if and only if $A \in \mathcal{M}(\mathcal{I})$.

(b) If Q_A dominates $S\mathcal{I}$ then $\liminf_n \sum_k a_{nk} \le 1 \le \limsup_n \sum_k a_{nk}$.

Proof. *(a)* Suppose that $A \in \mathcal{M}(\mathcal{I})$, and let $L \in (\ell^\infty)^*$ be such that $L(x) \le Q_A(x)$ for all $x \in \ell^\infty$. Since A is nonnegative, L is positive. Since $Q_A(e) = 1$, we have $L(e) = 1$. Since $A \in \mathcal{M}(\mathcal{I})$, for any $E \subseteq \mathbb{N}$ for which $\mathcal{I} - \lim \chi_E = 0$ we must have that $\delta_A(E) = \lim_A \chi_E = Q_A(\chi_E) = 0$. This implies that $L(\chi_E) = 0$. As A is regular we see that L is regular, hence $L(x) = \lim x_k$ for $x \in c$. Therefore Q_A generates $S\mathcal{I}$. Conversely assume that Q_A generates $S\mathcal{I}$. Hence it must be that $Q_A(x) \le P_{\mathcal{I}}(x)$ for all $x \in \ell^\infty$. Taking $e = (1, 1, ...)$ we immediately observe that $\lim_n \sum_k a_{nk} = 1$. Also, if $E \subseteq \mathbb{N}$ such that $\mathcal{I} - \lim \chi_E = 0$, then

$$0 \le Q_A(\chi_E) \le P_{\mathcal{I}}(\chi_E) = 0.$$

That is $Q_A(\chi_E) = 0$, i.e., $\delta_A(E) = 0$. Hence $A \in \mathcal{NRM}$, and also $\mathcal{I} \subseteq \mathcal{I}(A)$. This implies that $A \in \mathcal{M}(\mathcal{I})$.
(b) If Q_A dominates $S\mathcal{I}$, then by Theorem 2.4 we have $P_{\mathcal{I}}(x) \leq Q_A(x)$ for all $x \in \ell^\infty$. This implies that

$$-Q_A(-x) \leq -P_{\mathcal{I}}(-x) \leq P_{\mathcal{I}}(x) \leq Q_A(x).$$

Since $P_{\mathcal{I}}(e) = 1$ we get the result. □

3. Core for Bounded Sequences

We have proved in Theorem 2.5 that if A is a nonnegative matrix that satisfies $\sup_n \sum_k a_{nk} < \infty$ then Q_A generates $S\mathcal{I}$ if and only if $A \in \mathcal{M}(\mathcal{I})$. On the other hand, $P_{\mathcal{I}}$ both generates and dominates $S\mathcal{I}$ (see Theorem 2.4). Hence it must be that

$$Q_A(x) \leq P_{\mathcal{I}}(x) \text{ for all } x \in \ell^\infty \text{and for all } A \in \mathcal{M}(\mathcal{I}). \tag{3}$$

This inequality suggests to look at core type results for bounded sequences. So this section is devoted to the core of bounded sequences.

Recall that the Knopp core of $x \in \ell^\infty$ [18], denoted $K - core\{x\}$, is the closed interval $[\liminf x, \limsup x]$. Knopp proved that if T is a nonnegative regular matrix, then $K - core\{Tx\} \subseteq K - core\{x\}$, whenever $x \in \ell^\infty$.

Following Knopp, the $\mathcal{I} - core$ of a real-valued $\mathcal{I}-$bounded sequence denoted $\mathcal{I} - core\{x\}$, is defined to be the closed interval $[-P_{\mathcal{I}}(-x), P_{\mathcal{I}}(x)]$ [7, 13].

Hence we have

Proposition 3.1. *Let A be a nonnegative infinite matrix and let* $\sup_n \sum_k a_{nk} < \infty$*. Then Q_A generates $S_{\mathcal{I}}-$limits if and only if*

$$K - core\{Ax\} \subseteq \mathcal{I} - core\{x\} \textit{ for all } x \in \ell^\infty. \tag{4}$$

Proof. If Q_A generates $S_{\mathcal{I}}-$limits, then (3) holds which yields that $K - core\{Ax\} \subseteq \mathcal{I} - core\{x\}$ for all $x \in \ell^\infty$. Conversely assume that (4) holds. We now claim that $A \in \mathcal{M}(\mathcal{I})$. To see this, first we show that A is regular. Let $x = (x_k) \in c$ and $\lim_k x_k = \ell$. Hence $\mathcal{I} - core\{x\} = \{\ell\}$. Observe that Ax exists and it is a bounded sequence. Hence $K - core\{Ax\} \neq \emptyset$. So we must have $K - core\{Ax\} = \{\ell\}$. This implies $\lim Ax = \ell$. Hence A is regular. Now let $E \in \mathcal{I}$. Then $\mathcal{I} - \lim \chi_E = 0$. Hence $\mathcal{I} - core\{\chi_E\} = \{0\}$. Since $\chi_E \in \ell^\infty$ and $\sup_n \sum_k a_{nk} < \infty$, we get that $((A\chi_E)_n)$ is a bounded sequence. So $K - core\{A\chi_E\} \neq \emptyset$. Thus we must have $K - core\{A\chi_E\} = \{0\}$ which necessarily implies that $\lim_n \sum_{k \in E} a_{nk} = 0$. Hence $A \in \mathcal{M}(\mathcal{I})$. By Theorem 2.5, the result follows. □

4. Bounded Multipliers

Assume that two sequence spaces, X and Y, are given. A multiplier from X into Y is a sequence u such that $u \cdot x = (u_n x_n) \in Y$ whenever $x \in X$. The linear space of all such multipliers will be denoted by $\mathfrak{m}(X, Y)$. Bounded multipliers will be denoted by $M(X, Y)$. Hence $\mathrm{M}(X, Y) = \ell^\infty \cap \mathfrak{m}(X, Y)$. If $X = Y$, then we write $\mathfrak{m}(X)$ and $M(X)$ instead of $\mathfrak{m}(X, X)$ and $M(X, X)$, respectively.

In this section we deal with multipliers for bounded $\mathcal{I}-$convergent sequences. In particular we show that bounded multipliers of $c(\mathcal{I}(b)) := c(\mathcal{I}) \cap \ell^\infty$ into itself may be described as the intersection of multipliers of a family of convergence domains whose intersection is $c(\mathcal{I}(b))$.

We first recall some terminology. An ideal $\mathcal{I}$ has the property *GMV* (see [9]) if

$$\lim^{\mathcal{I}} \restriction \ell^\infty = \cap \{\lim_A \restriction \ell^\infty : A \in \mathcal{M}(\mathcal{I})\}.$$

Hence $c(\mathcal{I}(b)) = \cap \{c_A(b) : A \in \mathcal{M}(\mathcal{I})\}$ where $c_A(b) := c_A \cap \ell^\infty$.

This is equivalent (see [10]) to the fact that

$$\mathcal{I} = \cap \{\mathcal{I}(A) : A \in \mathcal{M}(\mathcal{I})\}.$$

Recall also that the bounded strong summability field of a nonnegative matrix A is the space

$$W_b(A) := \left\{ x \in \ell^\infty : \lim_n \sum_{k=1}^{\infty} a_{nk} |x_k - \alpha| = 0 \text{ for some } \alpha \right\}.$$

It is well known that

$$M(c_A(b)) = W_b(A) \tag{5}$$

provided that $a_{nk} \geq 0$ for all n and k (see e.g. [15, 17]).

We will also need the following result of Hill and Sleed [15].

Theorem 4.1. *If T is a regular matrix, then the bounded sequence x is strongly $T-$summable to α if and only if there exists a subset Z of $\mathbb{N}$ such that $\chi_{\mathbb{N}\setminus Z}$ is strongly $T-$summable to zero and $\lim_{n\in Z} x_n = \alpha$.*

The following result may be found in [14, 25].

Theorem 4.2. *$x \in M(c(\mathcal{I}(b)))$ if and only if $x \in c(\mathcal{I}(b))$.*

Some results on multipliers may be found in [4, 25].

The next result is an analog of Theorem 6 in [6].

Theorem 4.3. *Assume that $\mathcal{I}$ has the property GMV. Then*

$$M(c(\mathcal{I}(b))) = \bigcap \{M(c_T(b)) : T \in \mathcal{M}(\mathcal{I})\}.$$

Proof. Let $x \in \bigcap_{T\in\mathcal{M}(\mathcal{I})} M(c_T(b))$. We show that $x \cdot y \in c(\mathcal{I}(b))$ for any $y \in c(\mathcal{I}(b))$. Let $y \in c(\mathcal{I}(b))$. By the property *GMV*, we know that $c(\mathcal{I}(b)) \subseteq c_T(b)$ for every $T \in \mathcal{M}(\mathcal{I})$ which yields that $y \in c_T(b)$. Since $x \in \bigcap_{T\in\mathcal{M}(\mathcal{I})} M(c_T(b))$, we have $x \cdot y \in \bigcap_{T\in\mathcal{M}(\mathcal{I})} c_T(b)$. By the property *GMV* we get that $x \cdot y \in c(\mathcal{I}(b))$.

Now let $x \in M(c(\mathcal{I}(b)))$ and $T \in \mathcal{M}(\mathcal{I})$. By Theorem 4.2, we have $x \in c(\mathcal{I}(b))$, and hence $x \in c(\mathcal{I}(T)(b))$. Recall that $\mathcal{I}(T)$ is a $P-$ideal, thus $\mathcal{I}(T)-$convergence is equivalent to $\mathcal{I}(T)^*-$convergence (see e.g. [1, 19]). Hence for every $\mathcal{I}(T)-$convergent sequence (x_n) there is a set $K \in \mathcal{I}(T)^*$ such that $(x_n)_{n\in K}$ is ordinarily convergent, i.e.

$$\lim_{n\in K} x_n = \alpha. \tag{6}$$

On the other hand we can write

$$\sum_{j=1}^{\infty} t_{nj} = \sum_{j\in K} t_{nj} + \sum_{j\in\mathbb{N}\setminus K} t_{nj}.$$

Since T is regular and $\lim_n (T\chi_K)_n = 1$, we have $\lim_n \sum_{j\in\mathbb{N}\setminus K} t_{nj} = 0$, i.e., $\chi_{\mathbb{N}\setminus K}$ is strongly $T-$summable to 0. Combining this with (6), it follows from Theorem 4.1 that x is strongly $T-$summable to α. Now (5) implies that $x \in \bigcap_{T\in\mathcal{M}(\mathcal{I})} M(c_T(b))$, and this completes the proof. □

The referee has also suggested the following alternate proof of Theorem 4.3 which is much quicker.

Alternate proof:

$$\begin{aligned} M(c(\mathcal{I}(b))) &= c(\mathcal{I}(b)), \text{ by Theorem 4.2} \\ &= \bigcap_{T\in\mathcal{M}(\mathcal{I})} c(\mathcal{I}(T)(b)), \text{ for } \mathcal{I} \text{ with } GMV \\ &= \bigcap_{T\in\mathcal{M}(\mathcal{I})} W_b(T) \\ &= \bigcap_{T\in\mathcal{M}(\mathcal{I})} M(c_T(b)), \text{ by (5).} \quad \square \end{aligned}$$

5. Concluding Remarks and Open Questions

1. It should be noted that if Q_A dominates $S\mathcal{I}$ for some $A \in \mathcal{M}(\mathcal{I})$, then by Theorem 2.4 *(i)* and Theorem 2.5 *(a)*, it will be that

$$Q_A(x) = P_{\mathcal{I}}(x) \text{ for all } x \in \ell^\infty. \tag{7}$$

This raise the question of whether or not there is an ideal $\mathcal{I} \supsetneq Fin$ such that (7) holds.

The referee suggested the following: Regarding the first open question, note that (7) holds for $\mathcal{I} = Fin$ and A being the identity matrix, which belongs to $\mathcal{M}(\mathcal{I})$. (7) holds also for the ideal $Fin \oplus \mathcal{P}(\mathbb{N})$. Using Theorem 2.1 from [5], it seems that one could show that there are no other ideals for which (7) holds.

2. Recall that a regular matrix $A = (a_{nk})$ is called strongly regular [8] if $\lim_n \sum_{k=1}^{\infty} \left|a_{nk} - a_{n,k+1}\right| = 0$. Duran [8] proved that, given $x \in \ell^\infty$, there exist strongly regular matrices A and B such that

$$\sup\{T(x) : T \in BL\} = \lim_n (Ax)_n$$

and

$$\inf\{T(x) : T \in BL\} = \lim_n (Bx)_n$$

where BL denotes the set of all Banach limits [2, 16].

One may now raise the question if one could find matrices B_1 and B_2 in $\mathcal{M}(\mathcal{I})$ such that

$$\sup\{G(x) : G \in S\mathcal{I}\} = \lim_n (B_1x)_n \tag{8}$$

and

$$\inf\{G(x) : G \in S\mathcal{I}\} = \lim_n (B_2x)_n. \tag{9}$$

It is shown in [26] that there are no matrices B_1 and B_2 in $\mathcal{M}(\mathcal{I}(C_1))$ such that (8) and (9) hold where C_1 is the Cesàro matrix of order 1. What about the other ideals?

The referee suggested the following: Regarding the second open question, it is easy to see that GMV property is necessary for (8) and (9) to hold. By adding to GMV the condition that $\mathcal{I}^+$is a P^+−coideal (i.e. for any sequence $A_1 \supseteq A_2 \supseteq ...$ with $A_n \notin \mathcal{I}$ there exists $A \notin \mathcal{I}$ such that $A\backslash A_n$ is finite for all $n \in \mathbb{N}$) one could obtain a sufficient condition, though mayhap not necessary.

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