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Fiedler linearizations of multivariable state-space systems and its associated system matrix

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Abstract. Linearization is a standard method in the computation of eigenvalues and eigenvectors of matrix polynomials. In the last decade a variety of linearization methods have been developed in order to deal with algebraic structures and in order to construct efficient numerical methods. An important source of linearizations for matrix polynomials are the so called *Fiedler pencils*, which are generalizations of the Frobenius companion form and these linearizations have been extended to regular rational matrix function which is the transfer function of LTI State-space system in [20, 25]. We consider a multivariable state-space system and its associated system matrix $\mathcal{S}(\lambda)$. We introduce Fiedler pencils of $\mathcal{S}(\lambda)$ and describe an algorithm for their construction. We show that Fiedler pencils are linearizations of the system matrix $\mathcal{S}(\lambda)$.

1. Introduction

We denote by $\mathbb{C}[\lambda]$, $\mathbb{C}^{m\times n}$, and $\mathbb{C}[\lambda]^{m\times n}$, the polynomial ring over the complex field $\mathbb{C}$, the vector spaces of $m \times n$ matrices and matrix polynomials over $\mathbb{C}$, respectively.

Consider a matrix polynomial $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$, where $A_j \in \mathbb{C}^{n\times n}$. Then $P(\lambda)$ is said to be regular if $\det P(\lambda)$ is not identically zero. A matrix polynomial $U(\lambda)$ is said to be unimodular if $\det U(\lambda)$ is a nonzero constant, independent of λ. Two matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are said to be equivalent if there exist unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$, such that $Q(\lambda) = U(\lambda)P(\lambda)V(\lambda)$. If $U(\lambda), V(\lambda)$ are constant matrices, then $P(\lambda)$ and $Q(\lambda)$ are said to be strictly equivalent [7]. Let $P(\lambda)$ be an $n \times n$ matrix polynomial (regular or singular) of degree m. Then linearization is a common procedure to solve the polynomial eigenvalue problem $P(\lambda)x = 0$. That is, a matrix pencil $L(\lambda) := X + \lambda Y \in \mathbb{C}[\lambda]^{mn\times mn}$ is said to be a *linearization* [14] of $P(\lambda)$ if there exist unimodular matrix polynomials $U(\lambda)$, $V(\lambda) \in \mathbb{C}[\lambda]^{mn\times mn}$ such that

$$U(\lambda)L(\lambda)V(\lambda) = \mathrm{diag}(I_{(m-1)n},\ P(\lambda))$$

for all $\lambda \in \mathbb{C}$, where I_r denotes the $r \times r$ identity matrix. In [7, 14] and references therein, linearizations of matrix polynomials have been studied extensively and Fiedler linearizations of matrix polynomial have been studied in [8, 9] and references therein. One of the important properties of a Fiedler pencil $L(\lambda)$ of

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the matrix polynomial $P(\lambda)$ is that its construction is operation free, that means, block entries of the Fiedler pencil $L(\lambda)$ are either I_n or the coefficient matrices A_j of $P(\lambda)$ or 0_n. Also, the Fiedler pencil $L(\lambda)$ allows an easy recovery of eigenvectors of $P(\lambda)$ from the eigenvectors of $L(\lambda)$. That is, one can recover the eigenvectos of $P(\lambda)$ from those of Fiedler pencils [8, 9].

There are many different ways to do linearization. Their advantages and disadvantages with respect to backward error (determines the smallest perturbation for which a computed solution is an exact solution of the perturbed problem), conditioning, (sensitivity of eigenvalues under perturbations) have received a lot of attention in recent years, see e.g. [5, 7, 11, 19, 23, 24].

In this paper we extend the concept of Fiedler linearization from LTI state-space system to general multivariable state-space system and associated system matrix. In particular, in this paper we discuss the solution (finding *eigenvalues* $\lambda \in \mathbb{C}$ and *eigenvectors* $v \in \mathbb{C}^n$) of *multivariable state-space system* Σ

$$\begin{aligned} A\left(\frac{d}{dt}\right)x(t) &= Bu(t), \\ y(t) &= Cx(t) + D\left(\frac{d}{dt}\right)u(t) \quad t \geq 0, \end{aligned} \tag{1}$$

such that $\mathcal{S}(\lambda)v = 0$, where $A(\lambda) = \sum_{j=0}^{d_A} \lambda^j A_j \in \mathbb{C}[\lambda]^{n\times n}$ is a regular matrix polynomial of degree d_A, $D(\lambda) = \sum_{j=0}^{d_D} \lambda^j D_j \in \mathbb{C}[\lambda]^{m\times m}$ is a matrix polynomial of degree d_D, and $C \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times m}$, and its associate Rosenbrock system matrix $\mathcal{S}(\lambda)$

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} A(\lambda) & -B \\ \hline C & D(\lambda) \end{array}\right] \in \mathbb{C}[\lambda]^{(n+m)\times(n+m)} \tag{2}$$

and the associated transfer function

$$R(\lambda) = D(\lambda) + CA(\lambda)^{-1}B \in \mathbb{C}(\lambda)^{m\times m}. \tag{3}$$

Now, consider a more general linear multivariable time invariant state-space system Σ_1 on the positive half line $\mathbb{R}_+$ in the representation

$$\begin{aligned} 0 &= A\left(\frac{d}{dt}\right)x(t) + B\left(\frac{d}{dt}\right)u(t), \\ y(t) &= C\left(\frac{d}{dt}\right)x(t) + D\left(\frac{d}{dt}\right)u(t). \end{aligned} \tag{4}$$

The function $u : \mathbb{R}_+ \to \mathbb{R}^m$ is the input vector, $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the state vector, $y : \mathbb{R}_+ \to \mathbb{R}^m$ is the output vector, and for $M(\lambda) = \sum_{i=0}^{d} M_i\lambda^i \in \mathbb{C}[\lambda]^{m\times m}$ we use $M(\frac{d^i}{dt^i})$ to denote the differential operator $\sum_{i=0}^{\ell} M_i \frac{d^i}{dt^i}$, where $\frac{d}{dt}$ denotes time-differentiation. The associated matrix polynomial is

$$\mathcal{S}(\lambda) := \left[\begin{array}{c|c} A(\lambda) & B(\lambda) \\ \hline C(\lambda) & D(\lambda) \end{array}\right] \in \mathbb{C}[\lambda]^{(n+m)\times(n+m)}. \tag{5}$$

The associate transfer function is defined by

$$R(\lambda) := D(\lambda) - C(\lambda)A(\lambda)^{-1}B(\lambda) \in \mathbb{C}(\lambda)^{m\times m}, \tag{6}$$

where, denoting by $\mathbb{C}[\lambda]^{m\times m}$ the vector space of $m \times m$ matrix polynomials, we assume that $A(\lambda) \in \mathbb{C}[\lambda]^{n\times n}$, $B(\lambda) \in \mathbb{C}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{C}[\lambda]^{m\times n}$, and $D(\lambda) \in \mathbb{C}[\lambda]^{m\times m}$. Notice that in (2) B and C are considered to be constant matrices.

Rational eigenvalue problems arise in many applications such as free vibration of plates with elastically attached masses, calculations of quantum dots, vibrations of fluid-solid structures and in control theory, see e.g. [6, 13, 16, 17] and the references therein. Rational matrix value functions of the form (6) arise e.g. in linear system theory, see [3, 29].

If $A(\lambda)$ is regular, i.e., $\det A(\lambda)$ does not vanish identically, then performing a Schur complement, one obtains the rational matrix function (6) which, in frequency domain, describes the *transfer function* from the Laplace transformed input to the Laplace transformed output of the system. In this case $\mathcal{S}(\lambda)$ is called a *Rosenbrock system matrix*, see [12]. Conversely, if one has a given rational matrix function of the form (6), then one can always interpret it as originating from a Rosenbrock system matrix of the form (5). Such rational matrix valued functions arise from realizations of input-output data, see e.g. [4], or in model order reduction, see e.g. [2, 28].

We consider the general square polynomial eigenvalue problem

$$\mathcal{S}(\lambda)\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} := \left[\begin{array}{c|c} A(\lambda) & B(\lambda) \\ \hline C(\lambda) & D(\lambda) \end{array}\right]\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0. \tag{7}$$

If A and D are square and regular, then one can form the rational function $R(\lambda)$ as in (6) and, since $\det\mathcal{S}(\lambda) = \det A(\lambda)\det R(\lambda)$, it is clear that the eigenvalues of $\mathcal{S}(\lambda)$ are the eigenvalues of $A(\lambda)$ and $R(\lambda)$ combined and the eigenvalues of $A(\lambda)$ are the poles of $R(\lambda)$. We restrict ourselves to rational functions of the form (6) with regular $A(\lambda)$ and we assume for simplicity that B, C are constant matrices in λ. All the results can be extended (with a lot of technicalities) to the case that B, C depend on λ.

Note that for the linear time invariant (LTI) system in state-space form [3]

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + D(\lambda)u(t), \end{aligned} \tag{8}$$

where $D(\lambda) \in \mathbb{C}[\lambda]^{m\times m}$ is a matrix polynomial and $A, E \in \mathbb{C}^{n\times n}$ with E being nonsingular, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{m\times n}$ are constant matrices, a framework has been developed in [25] to study the zeros of LTI system in state space form via Fiedler-like pencils and linearizations of the Rosenbrock system polynomial $\mathcal{S}(\lambda)$ associated with the system, see [21, 25–27].

Further, for the higher order linear time invariant (LTI) state-space system

$$\begin{aligned} A\left(\frac{d}{dt}\right)x(t) &= Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{9}$$

where $A(\lambda) = \sum_{j=0}^{d_A} \lambda^j A_j \in \mathbb{C}[\lambda]^{n\times n}$ is regular matrix polynomial of degree d_A and $D \in \mathbb{C}^{m\times m}$, $C \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$, there is a state-space framework developed in [22] to study the zeros of higher order system via Fiedler linearizations, see [22]. Also, the eigenvalues and eigenvectors of the system matrix $\mathcal{S}(\lambda)$ associated with higher order system has been studied in [22].

Multivariable state space system and its associated system matrix play an important role in system theory. Recently, in [1, 10, 15], different linearizations of the $\mathcal{S}(\lambda)$ in (5) were studied. Further, the eigenvector recovery, minimal bases and minimal indices of $\mathcal{S}(\lambda)$ has been analyzed. Consider the system matrix $\mathcal{S}(\lambda)$ given in (2). We wish to study the relationship between the eigenvalues of Rosenbrock system matrix and associated linearizations. For this we develop a framework for construction of Fiedler linearizations of the system matrix $\mathcal{S}(\lambda)$. These linearizations are also helpful to study zeros of the system Σ given in (1). This problem has recently been studied for higher order state-space system in [22] and we will extend these results to the Multivariable state-space case.

The rest of the paper is organized as follows. In section 2 we recall some basic definitions and results on matrix polynomial and rational matrix which we need throughout this paper. In section 3 we extend the results of Fiedler pencils for Rosenbrock system matrix given in [20, 22, 25] to multivariable state space system. In the same section, we define Fiedler pencils for $\mathcal{S}(\lambda)$ given in (2) and present an algorithm for their construction. In Section 4 we prove that Fiedler pencils are linearizations for $\mathcal{S}(\lambda)$.

Notation. An $m \times n$ rational matrix function $R(\lambda)$ is an $m \times n$ matrix whose entries are rational functions of the form $\frac{p(\lambda)}{q(\lambda)}$, where $p(\lambda)$ and $q(\lambda)$ are scalar polynomials in $\mathbb{C}[\lambda]$. We denote the j-th column of the $n \times n$ identity matrix I_n by e_j and the transpose of a matrix A by A^T.

2. Basic Concepts

Definition 2.1. *[30] Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{p\times q}$. Then the Kronecker product (tensor product) of A and B is defined by*

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp\times nq}.$$

One of the properties of Kronecker product is as follows: Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{r\times s}$, $C \in \mathbb{C}^{n\times p}$, and $D \in \mathbb{C}^{s\times t}$. Then $(A \otimes B)(C \otimes D) = (AC \otimes BD) \in \mathbb{C}^{mr\times pt}$.

In order to systematically generate the Fiedler linearizations for Rosenbrock system matrices, we need a few concepts introduced in [9, 12, 29], and [25].

Definition 2.2. *[9] Let small $\sigma : \{0, 1, \ldots, p-1\} \to \{1, 2, \ldots, p\}$ be a bijection.*

(1) For $j = 0, \ldots, p-2$, the bijection σ is said to have a consecution at j if $\sigma(j) < \sigma(j+1)$ and σ has an inversion at j if $\sigma(j) > \sigma(j+1)$.

(2) The tuple $CISS(\sigma) = (c_1, i_1, c_2, i_2, \ldots, c_l, i_l)$ is called the consecution inversion structure sequence of σ, where σ has c_1 consecutive consecutions at $0, 1, \ldots, c_1 - 1$; i_1 consecutive inversions at $c_1, c_1 + 1, \ldots, c_1 + i_1 - 1$ and so on, up to i_l inversions at $p - 1 - i_l, \ldots, p - 2$.

(3) The total number of consecutions and inversions in σ is denoted by $\mathfrak{c}(\sigma)$ and $\mathfrak{i}(\sigma)$, respectively, i.e., $\mathfrak{c}(\sigma) = \sum_{j=1}^{l} c_j$, $\mathfrak{i}(\sigma) = \sum_{j=1}^{l} i_j$, and $\mathfrak{c}(\sigma) + \mathfrak{i}(\sigma) = p - 1$.

Now, consider a rational matrix function $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$. Then the normal rank of $R(\lambda)$ denoted by nrank(R) is defined as nrank$(R) := \max_\lambda \text{rank}(R(\lambda))$, where the maximum is taken over all $\lambda \in \mathbb{C}$ which are not poles of the entries of $R(\lambda)$. If nrank$(R) = n = m$, then $R(\lambda)$ is called regular, otherwise $R(\lambda)$ is singular [12].

Let $\mathcal{S}(\lambda)$ be given in (2). Then $\lambda \in \mathbb{C}$ is said to be an eigenvalue of the system matrix $\mathcal{S}(\lambda)$ if rank$(\mathcal{S}(\lambda)) <$ nrank$(\mathcal{S})$. Note that an eigenvalue λ of $\mathcal{S}(\lambda)$ is called an invariant zero of the system Σ and the set of eigenvalues of $\mathcal{S}(\lambda)$ is denoted by sp$(\mathcal{S})$, see [25].

Let $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ be a rational matrix function with normal rank k. Then the Smith-McMillan form of $R(\lambda)$ is given by [12, 29]

$$\mathbf{SM}(R(\lambda)) = \text{diag}\left(\frac{\phi_1(\lambda)}{\psi_1(\lambda)}, \cdots, \frac{\phi_k(\lambda)}{\psi_k(\lambda)}, 0_{m-k,n-k}\right),$$

where the scalar polynomials $\phi_i(\lambda)$ and $\psi_i(\lambda)$ are monic, pairwise coprime and, satisfy the properties: $\phi_i(\lambda)/\phi_{i+1}(\lambda)$ and $\psi_{i+1}(\lambda)/\psi_i(\lambda)$, for $i = 1, 2, \ldots, k-1$. The polynomials $\phi_i(\lambda)$ and $\psi_i(\lambda)$ are uniquely determined by $R(\lambda)$ and are called elementary divisors of $R(\lambda)$. Also, the polynomials $\phi_1(\lambda), \ldots, \phi_k(\lambda)$ and $\psi_1(\lambda), \ldots, \psi_k(\lambda)$ are called *zero polynomials* and *pole polynomials* of $R(\lambda)$, respectively, see [25]. Define $\phi_R(\lambda) := \prod_{j=1}^{k} \phi_j(\lambda)$ and $\psi_R(\lambda) := \prod_{j=1}^{k} \psi_j(\lambda)$. Then $\mu \in \mathbb{C}$ is said to be a zero of $R(\lambda)$ if $\phi_R(\mu) = 0$ and $\mu \in \mathbb{C}$ is said to be a pole of $R(\lambda)$ if $\psi_R(\mu) = 0$. The **spectrum** sp(R) of $R(\lambda)$ is given by sp$(R) := \{\lambda \in \mathbb{C} : \phi_R(\lambda) = 0\}$. Therefore, sp$(R)$ is the set of zeros of $R(\lambda)$ [25].

3. Fiedler pencils for Rosenbrock system matrix

In this section we define Fiedler pencils for the system polynomial $\mathcal{S}(\lambda)$ and describe an algorithm for their construction. Let us consider a Rosenbrock system matrix of the form (2) with B, C constant in λ,

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} A(\lambda) & -B \\ \hline C & D(\lambda) \end{array}\right] \in \mathbb{C}[\lambda]^{(n+m)\times(n+m)} \tag{10}$$

and the associated transfer function

$$R(\lambda) = D(\lambda) + CA(\lambda)^{-1}B \in \mathbb{C}(\lambda)^{m\times m},$$

where $A(\lambda) = \sum_{i=0}^{d_A} \lambda^i A_i \in \mathbb{C}[\lambda]^{n\times n}$ is regular and $D(\lambda) = \sum_{j=0}^{d_D} \lambda^j D_j \in \mathbb{C}[\lambda]^{m\times m}$. Our aim is to study linearizations of $\mathcal{S}(\lambda)$. The most simple way to perform a direct linearization is to consider a first companion form

$$C_1(\lambda)w := (\lambda X + Y)w = 0, \tag{11}$$

where

$$X = \left[\begin{array}{cccc|cccc} A_{d_A} & & & & & & & \\ & I_n & & & & & & \\ & & \ddots & & & & & \\ & & & I_n & & & & \\ \hline & & & & D_{d_D} & & & \\ & & & & & I_m & & \\ & & & & & & \ddots & \\ & & & & & & & I_m \end{array}\right], Y = \left[\begin{array}{cccc|cccc} A_{d_A-1} & A_{d_A-2} & \cdots & A_0 & 0 & \cdots & 0 & -B \\ -I_n & 0 & \cdots & 0 & & 0 & & 0 \\ & \ddots & \ddots & & & & \ddots & \vdots \\ & & -I_n & 0 & & & & 0 \\ \hline 0 & \cdots & 0 & C & D_{d_D-1} & D_{d_D-2} & \cdots & D_0 \\ & 0 & & 0 & -I_m & 0 & \cdots & 0 \\ & & \ddots & \vdots & & \ddots & \ddots & \vdots \\ & & & 0 & & & -I_m & 0 \end{array}\right],$$

and

$$w = \left[\begin{array}{c} \lambda^{d_A-1}(A(\lambda)^{-1})Bx \\ \lambda^{d_A-2}(A(\lambda)^{-1})Bx \\ \vdots \\ (A(\lambda)^{-1})Bx \\ \hline \lambda^{d_D-1}x \\ \lambda^{d_D-2}x \\ \vdots \\ x \end{array}\right].$$

It is easy to see that if λ is an eigenvalue of $R(\lambda)$ then $R(\lambda)x = 0$ if and only if $C_1(\lambda)w = 0$.

An important class of linearizations (which include the first companion form (11) as special case) that has received a lot of attention are the Fielder pencils, [8, 9, 18]. The *Fiedler matrices* M_i, $i = 0, 1, \ldots, d_A$ associated with $A(\lambda) = \sum_{i=0}^{d_A} \lambda^i A_i \in \mathbb{C}[\lambda]^{n\times n}$ of degree d_A [9], are defined by

$$\begin{aligned} M_{d_A} &:= \begin{bmatrix} A_{d_A} & \\ & I_{(d_A-1)n} \end{bmatrix}, \; M_0 := \begin{bmatrix} I_{(d_A-1)n} & \\ & -A_0 \end{bmatrix}, \\ M_i &:= \begin{bmatrix} I_{(d_A-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, \; i = 1, \ldots, d_A - 1. \end{aligned} \tag{12}$$

If $\sigma : \{0, 1, \ldots, d_A - 1\} \to \{1, 2, \ldots, d_A\}$ is a bijection, then one furthermore defines the products $M_\sigma := M_{\sigma^{-1}(1)} M_{\sigma^{-1}(2)} \cdots M_{\sigma^{-1}(d_A)}$. Note that $\sigma(i)$ describes the position of the factor M_i in the product M_σ; i.e., $\sigma(i) = j$ means that M_i is the jth factor in the product.

Based on the Fiedler matrices, then for a given $A(\lambda) \in \mathbb{C}[\lambda]^{n\times n}$ of degree d_A and a bijection σ, in [9] the associated *Fiedler pencil* is defined as the $d_A n \times d_A n$ matrix pencil

$$L_\sigma(\lambda) := \lambda M_{d_A} - M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(d_A)} = \lambda M_{d_A} - M_\sigma. \tag{13}$$

This concept was extended in [20–22, 25–27] for square Rosenbrock systems of the state-space form (8) and (9). In [25] also a multiplication-free algorithm is presented to construct Fiedler pencils for square system polynomials of the state-space form (8) and it is shown that these Fiedler pencils are linearizations of the system polynomial and as well as of the associated transfer functions under some appropriate conditions.

Extending the definition of [25], based on the idea of the companion like form (11), we define $nd_A \times nd_A$ Fiedler matrices associated with $A(\lambda) \in \mathbb{C}[\lambda]^{n\times n}$ as in (12), and Fiedler matrices associated with the matrix polynomial $D(\lambda) \in \mathbb{C}[\lambda]^{m\times m}$ by

$$\begin{aligned} N_{d_D} &:= \begin{bmatrix} D_{d_D} & \\ & I_{(d_D-1)m} \end{bmatrix}, \; N_0 := \begin{bmatrix} I_{(d_D-1)m} & \\ & -D_0 \end{bmatrix}, \\ N_i &:= \begin{bmatrix} I_{(d_D-i-1)m} & & & \\ & -D_i & I_m & \\ & I_m & 0 & \\ & & & I_{(i-1)m} \end{bmatrix}, \; i = 1, \ldots, d_D - 1. \end{aligned} \tag{14}$$

Based on the Fiedler matrices, then for given $D(\lambda) \in \mathbb{C}[\lambda]^{m\times m}$ of degree d_D and a bijection σ, in [9] the associated *Fiedler pencil* is defined as the $d_D m \times d_D m$ matrix pencil

$$T_\sigma(\lambda) := \lambda N_{d_D} - N_{\sigma^{-1}(1)} \cdots N_{\sigma^{-1}(d_D)} = \lambda N_{d_D} - N_\sigma. \tag{15}$$

Note that $M_iM_j = M_jM_i$, $N_iN_j = N_jN_i$ for $|i-j| > 1$ and except for the terms with index 0, d_A and d_D, respectively, each M_i and N_i is invertible. We then have the following definition of Fiedler matrices for Rosenbrock matrices $\mathcal{S}(\lambda) \in \mathbb{C}[\lambda]^{(n+m)\times(n+m)}$ given in (2).

Definition 3.1. *Consider a system polynomial $\mathcal{S}(\lambda)$ as in (2). Let $d = \max\{d_A, d_D\}$ and $r = \min\{d_A, d_D\}$. Define $(d_A n + d_D m) \times (d_A n + d_D m)$ matrices $\mathbb{M}_0, \ldots, \mathbb{M}_d$ by*

$$\begin{aligned} \mathbb{M}_0 &= \left[\begin{array}{cc|cc} I_{(d_A-1)n} & & & \\ & -A_0 & & (e_{d_A}e_{d_D}^T)\otimes B \\ \hline & & I_{(d_D-1)m} & \\ & -(e_{d_D}e_{d_A}^T)\otimes C & & -D_0 \end{array}\right] = \left[\begin{array}{c|c} M_0 & (e_{d_A}e_{d_D}^T)\otimes B \\ \hline -(e_{d_D}e_{d_A}^T)\otimes C & N_0 \end{array}\right] \\ &= \left[\begin{array}{c|c} M_0 & (e_{d_A}\otimes B)(e_{d_D}^T\otimes I_m) \\ \hline -(e_{d_D}\otimes I_m)(e_{d_A}^T\otimes C) & N_0 \end{array}\right], \end{aligned}$$

$$\mathbb{M}_d := \left[\begin{array}{cc|cc} A_{d_A} & & & \\ & I_{(d_A-1)n} & & \\ \hline & & D_{d_D} & \\ & & & I_{(d_D-1)m} \end{array}\right] = \left[\begin{array}{c|c} M_{d_A} & \\ \hline & N_{d_D} \end{array}\right].$$

For $i = 1, \ldots, r-1$,

$$\mathbb{M}_i := \left[\begin{array}{cccc|cccc} I_{(d_A-i-1)n} & & & & & & & \\ & -A_i & I_n & & & & & \\ & I_n & 0 & & & & & \\ & & & I_{(i-1)n} & & & & \\ \hline & & & & I_{(d_D-i-1)m} & & & \\ & & & & & -D_i & I_m & \\ & & & & & I_m & 0 & \\ & & & & & & & I_{(i-1)m} \end{array}\right] = \left[\begin{array}{c|c} M_i & \\ \hline & N_i \end{array}\right],$$

and if $d_D < d_A$, then for $i = r, r+1, \ldots, d-1$,

$$\mathbb{M}_i := \left[\begin{array}{cccc|c} I_{(d_A-i-1)n} & & & & \\ & -A_i & I_n & & \\ & I_n & 0 & & \\ & & & I_{(i-1)n} & \\ \hline & & & & I_{d_D m} \end{array}\right] = \left[\begin{array}{c|c} M_i & \\ \hline & I_{d_D m} \end{array}\right],$$

and if $d_D > d_A$, then for $i = r, r+1, \ldots, d-1$,

$$\mathbb{M}_i := \left[\begin{array}{c|cccc} I_{d_A n} & & & & \\ \hline & I_{(d_D-i-1)m} & & & \\ & & -D_i & I_m & \\ & & I_m & 0 & \\ & & & & I_{(i-1)m} \end{array}\right] = \left[\begin{array}{c|c} I_{d_A n} & \\ \hline & N_i \end{array}\right],$$

where M_i, $i = 1 : d_A$ and N_i, $i = 1 : d_D$ are Fiedler matrices associated with $A(\lambda)$ and $D(\lambda)$ given in (12) and (14), respectively. We refer to the matrices $\mathbb{M}_0, \ldots, \mathbb{M}_d$ as the Fiedler matrices associated with $\mathcal{S}(\lambda)$.

Observe that as in [25] one has $\mathbb{M}_i\mathbb{M}_j = \mathbb{M}_j\mathbb{M}_i$ for $|i-j| > 1$ and all $\mathbb{M}_i$ (except possibly $\mathbb{M}_0$, $\mathbb{M}_d$) are invertible.

The associated *Fiedler pencils* are then defined as follows.

Definition 3.2. *Consider a system polynomial $\mathcal{S}(\lambda)$ as in (2). Let $d = \max\{d_A, d_D\}$ and $\mathbb{M}_0, \ldots, \mathbb{M}_d$ be Fiedler matrices associated with $\mathcal{S}(\lambda)$ as in Definition 3.1. Given any bijection $\sigma : \{0, 1, \ldots, d-1\} \to \{1, 2, \ldots, d\}$, the matrix pencil*

$$\mathbb{L}_\sigma(\lambda) := \lambda\mathbb{M}_d - \mathbb{M}_{\sigma^{-1}(1)} \cdots \mathbb{M}_{\sigma^{-1}(d)} =: \lambda\mathbb{M}_d - \mathbb{M}_\sigma, \tag{16}$$

is called the Fiedler pencil *of $\mathcal{S}(\lambda)$ associated with σ. We also refer to $\mathbb{L}_\sigma(\lambda)$ as a Fiedler pencil of $R(\lambda)$.*

The companion like form given in (11), then is $C_1(\lambda) = \lambda\mathbb{M}_d - \mathbb{M}_{d-1} \cdots \mathbb{M}_1\mathbb{M}_0$ and the associated second companion form of $\mathcal{S}(\lambda)$ is

$$\begin{aligned} C_2(\lambda) &= \lambda\mathbb{M}_d - \mathbb{M}_0\mathbb{M}_1 \cdots \mathbb{M}_{d-2}\mathbb{M}_{d-1} \\ &= \lambda \left[\begin{array}{cccc|cccc} A_{d_A} & & & & & & & \\ & I_n & & & & & & \\ & & \ddots & & & & & \\ & & & I_n & & & & \\ \hline & & & & D_{d_D} & & & \\ & & & & & I_m & & \\ & & & & & & \ddots & \\ & & & & & & & I_m \end{array}\right] + \left[\begin{array}{cccc|cccc} A_{d_A-1} & -I_n & & & 0 & & & \\ A_{d_A-2} & 0 & \ddots & & \vdots & & & \\ \vdots & \ddots & & -I_n & 0 & & \ddots & \\ A_0 & \cdots & 0 & 0 & -B & 0 & \cdots & 0 \\ \hline 0 & & & & D_{d_D-1} & -I_m & \cdots & 0 \\ \vdots & \ddots & & & D_{d_D-2} & & & \vdots \\ 0 & & \ddots & & \vdots & & & -I_m \\ C & 0 & \cdots & 0 & D_0 & 0 & \cdots & 0 \end{array}\right]. \end{aligned}$$

Example 3.3. *Let $R(\lambda) = D(\lambda) + CA(\lambda)^{-1}B \in \mathbb{C}(\lambda)^{m\times m}$ with $A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3$, $A_i \in \mathbb{C}^{n\times n}$ be a matrix polynomial of degree 3 and $D(\lambda) = D_0 + \lambda D_1 + \lambda^2 D_2, D_i \in \mathbb{C}^{m\times m}$ be a matrix polynomial of degree 2. Here $d_A > d_D$, $r = 2$ and $d = 3$. Let $\sigma_1 = (1,3,2)$ and $\sigma_2 = (2,3,1)$ be bijections from $\{0,1,2\}$ to $\{1,2,3\}$. Then $\mathbb{L}_{\sigma_1}(\lambda) = \lambda\mathbb{M}_3 - \mathbb{M}_0\mathbb{M}_2\mathbb{M}_1$ and $\mathbb{L}_{\sigma_2}(\lambda) = \lambda\mathbb{M}_3 - \mathbb{M}_2\mathbb{M}_0\mathbb{M}_1$. Then the Fiedler matrices for $R(\lambda)$ are given by*

$$\mathbb{M}_0 = \left[\begin{array}{ccc|cc} I_n & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & -A_0 & 0 & B \\ \hline 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & -C & 0 & -D_0 \end{array}\right], \quad \mathbb{M}_1 = \left[\begin{array}{ccc|cc} I_n & 0 & 0 & & \\ 0 & -A_1 & I_n & & \\ 0 & I_n & 0 & & \\ \hline & & & -D_1 & I_m \\ & & & I_m & 0 \end{array}\right],$$

$$\mathbb{M}_2 = \left[\begin{array}{ccc|cc} -A_2 & I_n & 0 & & \\ I_n & 0 & 0 & & \\ 0 & 0 & I_n & & \\ \hline & & & I_m & \\ & & & & I_m \end{array}\right], \quad \mathbb{M}_3 = \left[\begin{array}{ccc|cc} A_3 & & & & \\ & I_n & & & \\ & & I_n & & \\ \hline & & & D_2 & \\ & & & & I_m \end{array}\right].$$

Then

$$\mathbb{M}_{\sigma_1} = \left[\begin{array}{ccc|cc} -A_2 & -A_1 & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 \\ 0 & -A_0 & 0 & B & 0 \\ \hline 0 & 0 & 0 & -D_1 & I_m \\ 0 & -C & 0 & -D_0 & 0 \end{array}\right].$$

By using the commutativity relation it is easy to check that $\mathbb{L}_{\sigma_1}(\lambda) = \mathbb{L}_{\sigma_2}(\lambda)$.

Example 3.4. *Let $R(\lambda) = D(\lambda) + CA(\lambda)^{-1}B \in \mathbb{C}(\lambda)^{m\times m}$ with $A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3$, $A_i \in \mathbb{C}^{n\times n}$ and $D(\lambda) = D_0 + \lambda D_1 + \lambda^2 D_2 + \lambda^3 D_3 + \lambda^4 D_4$, $D_i \in \mathbb{C}^{m\times m}$. Here, $d_A = 3$, $d_D = 4$, $d_A < d_D$ and $r = 3$, $d = 4$. Consider $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_4 - \mathbb{M}_2\mathbb{M}_0\mathbb{M}_1\mathbb{M}_3$. Then the Fiedler matrices for $R(\lambda)$ are given by*

$$\mathbb{M}_0 = \left[\begin{array}{ccc|cccc} I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_0 & 0 & 0 & 0 & B \\ \hline 0 & 0 & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & -C & 0 & 0 & 0 & -D_0 \end{array}\right], \mathbb{M}_1 = \left[\begin{array}{ccc|cccc} I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_1 & I_n & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -D_1 & I_m \\ 0 & 0 & 0 & 0 & 0 & I_m & 0 \end{array}\right],$$

$$\mathbb{M}_2 = \left[\begin{array}{ccc|cccc} -A_2 & I_n & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -D_2 & I_m & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m \end{array}\right], \mathbb{M}_3 = \left[\begin{array}{ccc|cccc} I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -D_3 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m \end{array}\right],$$

$$\mathbb{M}_4 = \left[\begin{array}{ccc|cccc} A_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & D_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m \end{array}\right].$$

Then

$$\mathbb{M}_\sigma = \left[\begin{array}{ccc|cccc} -A_2 & -A_1 & I_n & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_0 & 0 & 0 & 0 & B & 0 \\ \hline 0 & 0 & 0 & -D_3 & I_m & 0 & 0 \\ 0 & 0 & 0 & -D_2 & 0 & -D_1 & I_m \\ 0 & 0 & 0 & I_m & 0 & 0 & 0 \\ 0 & -C & 0 & 0 & 0 & -D_0 & 0 \end{array}\right].$$

Example 3.5. *Let $R(\lambda) = D(\lambda) + CA(\lambda)^{-1}B \in \mathbb{C}(\lambda)^{m\times m}$ with $A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3$, where $A_i \in \mathbb{C}^{n\times n}$ and $D(\lambda) = D_0 + \lambda D_1 + \lambda^2 D_2 + \lambda^3 D_3$, where $D_i \in \mathbb{C}^{m\times m}$. Here, $d_A = 3$, $d_D = 3$, $r = 3$, and $d = 3$. Consider $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_3 - \mathbb{M}_\sigma = \lambda\mathbb{M}_3 - \mathbb{M}_2\mathbb{M}_0\mathbb{M}_1$. Then the Fiedler matrices for $R(\lambda)$ are given by*

$$\mathbb{M}_0 = \left[\begin{array}{ccc|ccc} I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_0 & 0 & 0 & B \\ \hline 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & -C & 0 & 0 & -D_0 \end{array}\right], \quad \mathbb{M}_1 = \left[\begin{array}{ccc|ccc} I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_1 & I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & -D_1 & I_m \\ 0 & 0 & 0 & 0 & I_m & 0 \end{array}\right],$$

$$\mathbb{M}_2 = \left[\begin{array}{ccc|ccc} -A_2 & I_n & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -D_2 & I_m & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m \end{array}\right], \quad \mathbb{M}_3 = \left[\begin{array}{ccc|ccc} A_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & D_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m \end{array}\right].$$

Then

$$\mathbb{M}_\sigma = \left[\begin{array}{ccc|ccc} -A_2 & -A_1 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_0 & 0 & 0 & B & 0 \\ \hline 0 & 0 & 0 & -D_2 & -D_1 & I_m \\ 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & -C & 0 & 0 & -D_0 & 0 \end{array}\right].$$

Having introduced the basic idea of generating Fiedler pencils for Rosenbrock system polynomials given in (2), now we will analyze these constructed pencils.

Theorem 3.6. *Let $\mathcal{S}(\lambda)$ be given in (2). Let $d = \max\{d_A, d_D\}$ and $\sigma : \{0, 1, \ldots, d-1\} \to \{1, 2, \ldots, d\}$ be a bijection. Let $L_\sigma(\lambda)$, $T_\sigma(\lambda)$, and $\mathbb{L}_\sigma(\lambda)$ be the Fiedler pencils of $A(\lambda)$ of degree d_A, $D(\lambda)$ of degree d_D, and $\mathcal{S}(\lambda)$, respectively, associated with σ, that is, $L_\sigma(\lambda) := \lambda M_{d_A} - M_\sigma$, $T_\sigma(\lambda) := \lambda N_{d_D} - N_\sigma$, and $\mathbb{L}_\sigma(\lambda) := \lambda\mathbb{M}_d - \mathbb{M}_\sigma$. If $\sigma^{-1} = (\sigma_1^{-1}, 0, \sigma_2^{-1})$ for some bijections σ_1 and σ_2, then*

$$\mathbb{L}_\sigma(\lambda) = \left[\begin{array}{c|c} L_\sigma(\lambda) & -M_{\sigma_1}(e_{d_A}e_{d_D}^T \otimes B)N_{\sigma_2} \\ \hline N_{\sigma_1}(e_{d_D}e_{d_A}^T \otimes C)M_{\sigma_2} & T_\sigma(\lambda) \end{array}\right].$$

Further, if $CISS(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$, then

$$\mathbb{L}_\sigma(\lambda) = \left[\begin{array}{c|c} L_\sigma(\lambda) & -e_{d_A}e_{d_D-c_1}^T \otimes B \\ \hline e_{d_D}e_{d_A-c_1}^T \otimes C & T_\sigma(\lambda) \end{array}\right], \qquad \text{if } c_1 > 0$$

and

$$\mathbb{L}_\sigma(\lambda) = \left[\begin{array}{c|c} L_\sigma(\lambda) & -e_{(d_A-i_1)}e_{d_D}^T \otimes B \\ \hline e_{(d_D-i_1)}e_{d_A}^T \otimes C & T_\sigma(\lambda) \end{array}\right], \qquad \text{if } c_1 = 0.$$

Proof. We have

$$
\begin{aligned}
\mathbb{L}_\sigma(\lambda) &= \lambda\mathbb{M}_d - \mathbb{M}_\sigma = \lambda\mathbb{M}_d - \mathbb{M}_{\sigma_1}\mathbb{M}_0\mathbb{M}_{\sigma_2}\\
&= \lambda\left[\begin{array}{c|c} M_{d_A} & 0 \\ \hline 0 & N_{d_D}\end{array}\right] - \left[\begin{array}{c|c} M_{\sigma_1} & 0 \\ \hline 0 & N_{\sigma_1}\end{array}\right]\left[\begin{array}{c|c} M_0 & (e_{d_A}e_{d_D}^T)\otimes B \\ \hline -(e_{d_D}e_{d_A}^T)\otimes C & N_0\end{array}\right]\left[\begin{array}{c|c} M_{\sigma_2} & 0 \\ \hline 0 & N_{\sigma_2}\end{array}\right]\\
&= \lambda\left[\begin{array}{c|c} M_{d_A} & 0 \\ \hline 0 & N_{d_D}\end{array}\right] - \left[\begin{array}{c|c} M_{\sigma_1}M_0M_{\sigma_2} & M_{\sigma_1}(e_{d_A}e_{d_D}^T\otimes B)N_{\sigma_2} \\ \hline -N_{\sigma_1}(e_{d_D}e_{d_A}^T\otimes C)M_{\sigma_2} & N_{\sigma_1}N_0N_{\sigma_2}\end{array}\right]\\
&= \left[\begin{array}{c|c} L_\sigma(\lambda) & -M_{\sigma_1}(e_{d_A}e_{d_D}^T\otimes B)N_{\sigma_2} \\ \hline N_{\sigma_1}(e_{d_D}e_{d_A}^T\otimes C)M_{\sigma_2} & T_\sigma(\lambda)\end{array}\right].
\end{aligned}
$$

Now, suppose that $\mathrm{CISS}(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$.

Case I : Suppose that $c_1 > 0$. Then by commutativity relation we have $\mathbb{M}_\sigma = \mathbb{M}_{\sigma_1}\mathbb{M}_0\mathbb{M}_1\cdots\mathbb{M}_{c_1}$ with $c_1 + 1 \in \sigma_1$. Thus $\mathbb{M}_\sigma = \mathbb{M}_{\sigma_1}\mathbb{M}_0\mathbb{M}_{\sigma_2}$, where $\mathbb{M}_{\sigma_2} = \mathbb{M}_1\cdots\mathbb{M}_{c_1}$. Hence

$$
\begin{aligned}
\mathbb{M}_\sigma &= \left[\begin{array}{c|c} M_{\sigma_1} & \\ \hline & N_{\sigma_1}\end{array}\right]\left[\begin{array}{c|c} M_0 & (e_{d_A}e_{d_D}^T)\otimes B \\ \hline -(e_{d_D}e_{d_A}^T)\otimes C & N_0\end{array}\right]\left[\begin{array}{c|c} M_{\sigma_2} & 0 \\ \hline 0 & N_{\sigma_2}\end{array}\right]\\
&= \left[\begin{array}{c|c} M_{\sigma_1}M_0M_{\sigma_2} & M_{\sigma_1}(e_{d_A}e_{d_D}^T\otimes B)N_{\sigma_2} \\ \hline -N_{\sigma_1}(e_{d_D}e_{d_A}^T\otimes C)M_{\sigma_2} & N_{\sigma_1}N_0N_{\sigma_2}\end{array}\right].
\end{aligned}
$$

Since $j \in \sigma_1$ implies that $j \geq c_1 + 1$, we have $M_{\sigma_1} = \left[\begin{array}{c|c} * & \\ \hline & I_{c_1n}\end{array}\right]$ and $N_{\sigma_1} = \left[\begin{array}{c|c} * & \\ \hline & I_{c_1m}\end{array}\right]$. This shows that $M_{\sigma_1}(e_{d_A}\otimes I_n) = e_{d_A}\otimes I_n$ and $N_{\sigma_1}(e_{d_D}\otimes I_m) = e_{d_D}\otimes I_m$. So, we have $M_{\sigma_1}(e_{d_A}\otimes B) = e_{d_A}\otimes B$. Next, we have $N_{\sigma_1}(e_{d_D}e_{d_A}^T\otimes C)M_{\sigma_2} = N_{\sigma_1}(e_{d_D}\otimes I_m)(e_{d_A}^T\otimes C)M_{\sigma_2}$ and $M_{\sigma_1}(e_{d_A}e_{d_D}^T\otimes B)N_{\sigma_2} = M_{\sigma_1}(e_{d_A}\otimes B)(e_{d_D}^T\otimes I_m)N_{\sigma_2}$. Now, we have

$$
(e_{d_A}^T\otimes I_n)M_1 = (e_{d_A}^T\otimes I_n)\begin{bmatrix} I_{(d_A-2)n} & & \\ & -A_1 & I_n \\ & I_n & 0\end{bmatrix} = (e_{d_A-1}^T\otimes I_n),
$$

$$
(e_{d_A}^T\otimes I_n)M_1M_2 = (e_{d_A-1}^T\otimes I_n)\begin{bmatrix} I_{(d_A-3)n} & & & \\ & -A_2 & I_n & \\ & I_n & 0 & \\ & & & I_n\end{bmatrix} = (e_{d_A-2}^T\otimes I_n),
$$

and so on. Thus $(e_{d_A}^T\otimes I_n)M_1M_2\cdots M_{c_1} = (e_{d_A-c_1}^T\otimes I_n)$. Hence $(e_{d_A}^T\otimes I_n)M_{\sigma_2} = (e_{d_A-c_1}^T\otimes I_n)$ and $(-(e_{d_A-c_1}^T\otimes C)M_{\sigma_2} = -(e_{d_A-c_1}^T\otimes C)$. Similarly, we have

$$
(e_{d_D}^T\otimes I_m)N_1 = (e_{d_D}^T\otimes I_m)\begin{bmatrix} I_{(d_D-2)m} & & \\ & -D_1 & I_m \\ & I_m & 0\end{bmatrix} = (e_{d_D-1}^T\otimes I_m),
$$

$$
(e_{d_D}^T\otimes I_m)N_1N_2 = (e_{d_D-1}^T\otimes I_m)\begin{bmatrix} I_{(d_D-3)m} & & & \\ & -D_2 & I_m & \\ & I_m & 0 & \\ & & & I_m\end{bmatrix} = (e_{d_D-2}^T\otimes I_m),
$$

and so on. Thus $(e_{d_D}^T\otimes I_m)N_1N_2\cdots N_{c_1} = (e_{d_D-c_1}^T\otimes I_m)$. Hence $(e_{d_D}^T\otimes I_m)N_{\sigma_2} = (e_{d_D-c_1}^T\otimes I_m)$. Now, we have $N_{\sigma_1}(e_{d_D}e_{d_A}^T\otimes I_n)M_{\sigma_2} = N_{\sigma_1}(e_{d_D}\otimes I_m)(e_{d_A}^T\otimes C)M_{\sigma_2} = (e_{d_D}e_{d_A-c_1}^T\otimes I_n)$ and $-(N_{\sigma_1}e_{d_D}e_{d_A}^T\otimes C)M_{\sigma_2} = -(e_{d_D}e_{d_A-c_1}^T\otimes C)$.

Similarly, $M_{\sigma_1}(e_{d_A}e_{d_D}^T \otimes B)N_{\sigma_2} = (e_{d_A}e_{d_D-c_1}^T \otimes B)$. Consequently, we have

$$\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_d - \mathbb{M}_\sigma = \left[\begin{array}{c|c} L_\sigma(\lambda) & -e_{d_A}e_{d_D-c_1}^T \otimes B \\ \hline e_{d_D}e_{d_A-c_1}^T \otimes C & T_\sigma(\lambda) \end{array}\right].$$

Case II : Suppose that $c_1 = 0$. Then σ has i_1 inversions at 0. Hence by commutativity relations we have $\mathbb{M}_\sigma = \mathbb{M}_{i_1} \cdots \mathbb{M}_1\mathbb{M}_0\mathbb{M}_{\sigma_2} =: \mathbb{M}_{\sigma_1}\mathbb{M}_0 M_{\sigma_2}$ with $i_1 + 1 \in \sigma_2$. Hence

$$\begin{aligned}\mathbb{M}_\sigma &= \left[\begin{array}{c|c} M_{\sigma_1} & \\ \hline & N_{\sigma_1} \end{array}\right]\left[\begin{array}{c|c} M_0 & (e_{d_A}e_{d_D}^T) \otimes B \\ \hline -(e_{d_D}e_{d_A}^T) \otimes C & N_0 \end{array}\right]\left[\begin{array}{c|c} M_{\sigma_2} & 0 \\ \hline 0 & N_{\sigma_2} \end{array}\right] \\ &= \left[\begin{array}{c|c} M_{\sigma_1}M_0M_{\sigma_2} & M_{\sigma_1}(e_{d_A}e_{d_D}^T \otimes B)N_{\sigma_2} \\ \hline -N_{\sigma_1}(e_{d_D}e_{d_A}^T \otimes C)M_{\sigma_2} & N_{\sigma_1}N_0N_{\sigma_2} \end{array}\right].\end{aligned}$$

Since $j \in \sigma_2$ implies that $j \geq i_1 + 1$, we have $M_{\sigma_2} = \left[\begin{array}{c|c} * & \\ \hline & I_{i_1 n} \end{array}\right]$ and $N_{\sigma_2} = \left[\begin{array}{c|c} * & \\ \hline & I_{i_1 m} \end{array}\right]$. This shows that $(e_{d_A}^T \otimes I_n)M_{\sigma_2} = e_{d_A}^T \otimes I_n$ and $(e_{d_D}^T \otimes I_m)N_{\sigma_2} = e_{d_D}^T \otimes I_m$. Hence $(-e_{d_A}^T \otimes C)M_{\sigma_2} = -e_{d_A}^T \otimes C$. Next, we have

$$M_1(e_{d_A} \otimes I_n) = \begin{bmatrix} I_{(m-2)n} & & \\ & -A_1 & I_n \\ & I_n & 0 \end{bmatrix}(e_{d_A} \otimes I_n) = (e_{d_A-1} \otimes I_n),$$

$$M_2M_1(e_{d_A} \otimes I_n) = \begin{bmatrix} I_{(d_A-3)n} & & & \\ & -A_2 & I_n & \\ & I_n & 0 & \\ & & & I_n \end{bmatrix}(e_{d_A-1} \otimes I_n) = (e_{d_A-2} \otimes I_n).$$

Thus $M_{i_1} \cdots M_2M_1(e_{d_A} \otimes I_n) = (e_{d_A-i_1} \otimes I_n)$. Hence $M_{\sigma_1}(e_{d_A} \otimes I_n) = (e_{(d_A-i_1)} \otimes I_n)$ and $M_{\sigma_1}(e_{d_A} \otimes B) = (e_{(d_A-i_1)} \otimes B)$. Similarly, we have

$$N_1(e_{d_D} \otimes I_m) = \begin{bmatrix} I_{(d_D-2)m} & & \\ & -D_1 & I_m \\ & I_m & 0 \end{bmatrix}(e_{d_D} \otimes I_m) = (e_{d_D-1} \otimes I_m),$$

$$N_2N_1(e_{d_D} \otimes I_m) = \begin{bmatrix} I_{(d_D-3)m} & & & \\ & -D_2 & I_m & \\ & I_m & 0 & \\ & & & I_m \end{bmatrix}(e_{d_D-1} \otimes I_m) = (e_{d_D-2} \otimes I_m).$$

Thus $N_{i_1} \cdots N_2N_1(e_{d_D} \otimes I_m) = (e_{d_D-i_1} \otimes I_m)$. Hence $N_{\sigma_1}(e_{d_D} \otimes I_m) = (e_{(d_D-i_1)} \otimes I_m)$. Now, we have $N_{\sigma_1}(e_{d_D}e_{d_A}^T \otimes I_n)M_{\sigma_2} = N_{\sigma_1}(e_{d_D} \otimes I_m)(e_{d_A}^T \otimes C)M_{\sigma_2} = (e_{d_D-i_1}e_{d_A}^T \otimes I_n)$ and $-(N_{\sigma_1}e_{d_D}e_{d_A}^T \otimes C)M_{\sigma_2} = -(e_{d_D-i_1}e_{d_A}^T \otimes C)$. Similarly, $M_{\sigma_1}(e_{d_A}e_{d_D}^T \otimes B)N_{\sigma_2} = (e_{d_A-i_1}e_{d_D}^T \otimes B)$. Consequently, we have

$$\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_d - \mathbb{M}_\sigma = \left[\begin{array}{c|c} L_\sigma(\lambda) & -e_{(d_A-i_1)}e_{d_D}^T \otimes B \\ \hline e_{(d_D-i_1)}e_{d_A}^T \otimes C & T_\sigma(\lambda) \end{array}\right].$$

Note that for each $i, j \in \sigma$, we have $M_iM_j = M_jM_i$, $N_iN_j = N_jN_i \Leftrightarrow \mathbb{M}_i\mathbb{M}_j = \mathbb{M}_j\mathbb{M}_i$. This completes the proof. □

Theorem 3.7. *Let $\mathcal{S}(\lambda)$ be in (2) with $A(\lambda) = \sum_{i=0}^{d_A} \lambda^i A_i$, $A_i \in \mathbb{C}^{n\times n}$, $D(\lambda) = \sum_{i=0}^{d_D} \lambda^i D_i$, $D_i \in \mathbb{C}^{m\times m}$. Suppose that $d_A > d_D$. Let $\sigma : \{0, 1, \ldots, d_A - 1\} \to \{1, 2, \ldots, d_A\}$ be a bijection. The following algorithm constructs a sequence of matrices $\{\mathbb{W}_0, \mathbb{W}_1, \ldots, \mathbb{W}_{d_A-2}\}$, where each matrix $\mathbb{W}_i$ for $i = 1, 2, \ldots, d_A - 2$ is partitioned into blocks in such a way that the blocks of $\mathbb{W}_{i-1}$ are blocks of $\mathbb{W}_i$.*

Algorithm 1 Construction of $\mathbb{M}_\sigma$ for $\mathbb{L}_\sigma(\lambda) := \lambda\mathbb{M}_{d_A} - \mathbb{M}_\sigma$.

Input: $\mathcal{S}(\lambda) = \left[\begin{array}{c|c} \sum_{i=0}^{d_A} \lambda^i A_i & -B \\ \hline C & \sum_{i=0}^{d_D} \lambda^i D_i \end{array}\right]$ and a bijection $\sigma : \{0, 1, \ldots, d_A - 1\} \to \{1, 2, \ldots, d_A\}$, and $d_A > d_D$.

Output: $\mathbb{M}_\sigma$

if σ has a consecution at 0 **then**

$$\mathbb{W}_0 := \left[\begin{array}{cc|cc} -A_1 & I_n & 0 & 0 \\ -A_0 & 0 & B & 0 \\ \hline 0 & 0 & -D_1 & I_m \\ -C & 0 & -D_0 & 0 \end{array}\right]$$

else

$$\mathbb{W}_0 := \left[\begin{array}{cc|cc} -A_1 & -A_0 & 0 & B \\ I_n & 0 & 0 & 0 \\ \hline 0 & -C & -D_1 & -D_0 \\ 0 & 0 & I_m & 0 \end{array}\right]$$

end if

for $i = 1 : d_D - 2$ **do**

if σ has a consecution at i **then**

$$\mathbb{W}_i := \left[\begin{array}{ccc|c} -A_{i+1} & I_n & 0 & \\ \mathbb{W}_{i-1}(1 : i+1, 1) & 0 & \mathbb{W}_{i-1}(1 : i+1, 2 : i+1) & W_{12} \\ \hline 0 & 0 & 0 & \\ \mathbb{W}_{i-1}(2+i : 2i+2, 1) & 0 & \mathbb{W}_{i-1}(2+i : 2i+2, 2 : i+1) & W_{22} \end{array}\right], \text{ where}$$

$$W_{12} = \begin{bmatrix} 0 & 0 & 0 \\ \mathbb{W}_{i-1}(1 : i+1, i+2) & 0 & \mathbb{W}_{i-1}(1 : i+1, i+3 : 2i+2) \end{bmatrix},$$

$$W_{22} = \begin{bmatrix} -D_{i+1} & I_m & 0 \\ \mathbb{W}_{i-1}(2+i : 2i+2, i+2) & 0 & \mathbb{W}_{i-1}(2+i : 2i+2, i+3 : 2i+2) \end{bmatrix}.$$

else

$$\mathbb{W}_i := \left[\begin{array}{cc|cc} -A_{i+1} & \mathbb{W}_{i-1}(1, 1 : i+1) & 0 & \mathbb{W}_{i-1}(1, 2+i : 2i+2) \\ I_n & 0 & 0 & 0 \\ 0 & \mathbb{W}_{i-1}(2 : i+1, 1 : i+1) & 0 & \mathbb{W}_{i-1}(2 : i+1, 2+i : 2i+2) \\ \hline 0 & \mathbb{W}_{i-1}(i+2, 1 : i+1) & -D_{i+1} & \mathbb{W}_{i-1}(i+2, 2+i : 2i+2) \\ 0 & 0 & I_m & 0 \\ 0 & \mathbb{W}_{i-1}(i+3 : 2i+2, 1 : i+1) & 0 & \mathbb{W}_{i-1}(i+3 : 2i+2, 2+i : 2i+2) \end{array}\right]$$

end if

end for

for $i = d_D - 1 : d_A - 2$ **do**

if σ has a consecution at i **then**

$$\mathbb{W}_i := \begin{bmatrix} -A_{i+1} & I_n & 0 & 0 \\ \mathbb{W}_{i-1}(:, 1) & 0 & \mathbb{W}_{i-1}(:, 2 : i+1) & \mathbb{W}_{i-1}(:, i+2 : d_D + i + 1) \end{bmatrix}$$

else

$$\mathbb{W}_i := \begin{bmatrix} -A_{i+1} & \mathbb{W}_{i-1}(1, :) \\ I_n & 0 \\ 0 & \mathbb{W}_{i-1}(2 : i+1, :) \\ 0 & \mathbb{W}_{i-1}(i+2 : d_D + i + 1, :) \end{bmatrix}$$

end if

end for

$\mathbb{M}_\sigma := \mathbb{W}_{d_A - 2}$

Proof. Using induction on the degree $d_A = \max\{d_A, d_D\}$ and the idea of proof of Theorem 3.11 in [25] one can conclude the result. □

Theorem 3.8. *Let $\mathcal{S}(\lambda)$ be in (2) with $A(\lambda) = \sum_{i=0}^{d_A} \lambda^i A_i, A_i \in \mathbb{C}^{n\times n}$, $D(\lambda) = \sum_{i=0}^{d_D} \lambda^i D_i, D_i \in \mathbb{C}^{m\times m}$. Suppose that $d_A < d_D$. Let $\sigma : \{0, 1, \dots, d_D - 1\} \to \{1, 2, \dots, d_D\}$ be a bijection. The following algorithm constructs a sequence of matrices $\{\mathbb{W}_0, \mathbb{W}_1, \dots, \mathbb{W}_{d_D-2}\}$, where each matrix $\mathbb{W}_i$ for $i = 1, 2, \dots, d_D - 2$ is partitioned into blocks in such a way that the blocks of $\mathbb{W}_{i-1}$ are blocks of $\mathbb{W}_i$.*

Algorithm 2 Construction of $\mathbb{M}_\sigma$ for $\mathbb{L}_\sigma(\lambda) := \lambda\mathbb{M}_{d_D} - \mathbb{M}_\sigma$.

Input: $\mathcal{S}(\lambda) = \left[\begin{array}{c|c} \sum_{i=0}^{d_A} \lambda^i A_i & -B \\ \hline C & \sum_{i=0}^{d_D} \lambda^i D_i \end{array}\right]$ and a bijection $\sigma : \{0, 1, \dots, d_D - 1\} \to \{1, 2, \dots, d_D\}$, and $d_A < d_D$.

Output: $\mathbb{M}_\sigma$

if σ has a consecution at 0 **then**

$$\mathbb{W}_0 = \left[\begin{array}{cc|cc} -A_1 & I_n & 0 & 0 \\ -A_0 & 0 & B & 0 \\ \hline 0 & 0 & -D_1 & I_m \\ -C & 0 & I_m & 0 \end{array}\right]$$

else

$$\mathbb{W}_0 = \left[\begin{array}{cc|cc} -A_1 & A_0 & 0 & B \\ I_n & 0 & 0 & 0 \\ \hline 0 & -C & -D_1 & -D_0 \\ 0 & 0 & I_m & 0 \end{array}\right]$$

end if

for $i = 1 : d_A - 2$ **do**

if σ has a consecution at i **then**

$$\mathbb{W}_i := \left[\begin{array}{ccc|c} -A_{i+1} & I_n & 0 & \\ \mathbb{W}_{i-1}(1 : i+1, 1) & 0 & \mathbb{W}_{i-1}(1 : i+1, 2 : i+1) & W_{12} \\ \hline 0 & 0 & 0 & \\ \mathbb{W}_{i-1}(2+i : 2i+2, 1) & 0 & \mathbb{W}_{i-1}(2+i : 2i+2, 2 : i+1) & W_{22} \end{array}\right], \text{ where}$$

$$W_{12} = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \mathbb{W}_{i-1}(1 : i+1, i+2) & 0 & \mathbb{W}_{i-1}(1 : i+1, i+3 : 2i+2) \end{array}\right]$$

$$W_{22} = \left[\begin{array}{ccc} -D_{i+1} & I_m & 0 \\ \mathbb{W}_{i-1}(2+i : 2i+2, i+2) & 0 & \mathbb{W}_{i-1}(2+i : 2i+2, i+3 : 2i+2) \end{array}\right]$$

else

$$\mathbb{W}_i = \left[\begin{array}{cc|cc} -A_{i+1} & \mathbb{W}_{i-1}(1, 1 : i+1) & 0 & \mathbb{W}_{i-1}(1, 2+i : 2i+2) \\ I_n & 0 & 0 & 0 \\ 0 & \mathbb{W}_{i-1}(2 : i+1, 1 : i+1) & 0 & \mathbb{W}_{i-1}(2 : i+1, 2+i : 2i+2) \\ \hline 0 & \mathbb{W}_{i-1}(i+2, 1 : i+1) & -D_{i+1} & \mathbb{W}_{i-1}(i+2, 2+i : 2i+2) \\ 0 & 0 & I_m & 0 \\ 0 & \mathbb{W}_{i-1}(i+3 : 2i+2, 1 : i+1) & 0 & \mathbb{W}_{i-1}(i+3 : 2i+2, 2+i : 2i+2) \end{array}\right]$$

end if

end for

for $i = d_A - 1 : d_D - 2$ **do**

if σ has a consecution at i **then**

$$\mathbb{W}_i = \left[\begin{array}{c|ccc} \mathbb{W}_{i-1}(1 : d_A, 1 : d_A) & \mathbb{W}_{i-1}(1 : d_A, d_A+1) & 0 & \mathbb{W}_{i-1}(1 : d_A, d_A+2 : d_A+i+1) \\ \hline 0 & -D_{i+1} & I_m & 0 \\ \mathbb{W}_{i-1}(d_A+1 : d_A+i+1, 1 : d_A) & \mathbb{W}_{i-1}(d_A+1 : d_A+i+1, d_A+1) & 0 & \mathbb{W}_{i-1}(d_A+1 : d_A+i+1, d_A+1 : d_A+i+1) \end{array}\right]$$

else

$$\mathbb{W}_i = \left[\begin{array}{c|cc} \mathbb{W}_{i-1}(1 : d_A, 1 : d_A) & 0 & \mathbb{W}_{i-1}(1 : d_A, d_A+1 : d_A+i+1) \\ \hline \mathbb{W}_{i-1}(d_A+1, 1 : d_A & -D_{i+1} & \mathbb{W}_{i-1}(d_A+1, d_A+1 : d_A+i+1) \\ 0 & I_m & 0 \\ \mathbb{W}_{i-1}(d_A+2 : d_A+i+1, 1 : d_A) & 0 & \mathbb{W}_{i-1}(d_A+2 : d_A+i+1, d_A+1 : d_A+i+1) \end{array}\right]$$

end if

end for

$\mathbb{M}_\sigma := \mathbb{W}_{d_D-2}$

Proof. Using induction on the degree $d_D = \max\{d_A, d_D\}$ and the idea of proof of Theorem 3.11 in [25] one can conclude the result. $\square$

4. Fiedler linearizations of Rosenbrock system matrix

In this section we show that the constructed Fiedler pencils associated with Rosenbrock systems are indeed linearizations. To do this we have to recall a few basic facts.

Definition 4.1 (System equivalence). *Let $\mathcal{S}_1(\lambda)$ and $\mathcal{S}_2(\lambda)$ be $(n+m)\times(n+m)$ Rosenbrock system polynomials of the form (5), partitioned conformably. Then $\mathcal{S}_1(\lambda)$ is said to be* system equivalent *to $\mathcal{S}_2(\lambda)$ (denoted as $\mathcal{S}_1(\lambda) \sim_{se} \mathcal{S}_2(\lambda)$), if there exist unimodular matrix polynomials $U(\lambda), V(\lambda) \in \mathbb{C}[\lambda]^{n\times n}$ and $\widetilde{U}(\lambda), \widetilde{V}(\lambda) \in \mathbb{C}[\lambda]^{m\times m}$ such that for all $\lambda \in \mathbb{C}$ we have*

$$\left[\begin{array}{c|c} U(\lambda) & 0 \\ \hline 0 & \widetilde{U}(\lambda) \end{array}\right] \mathcal{S}_1(\lambda) \left[\begin{array}{c|c} V(\lambda) & 0 \\ \hline 0 & \widetilde{V}(\lambda) \end{array}\right] = \mathcal{S}_2(\lambda). \tag{17}$$

Definition 4.2 (Rosenbrock linearization). *Let $\mathcal{S}(\lambda)$ be an $(n+m)\times(n+m)$ system polynomial of the form (5) with degree $d = \max\{d_A, d_D\}$. A linear matrix polynomial $\mathbb{L}(\lambda)$ is called a* Rosenbrock linearization *of $\mathcal{S}(\lambda)$, if it has the form*

$$\mathbb{L}(\lambda) := \left[\begin{array}{c|c} \mathcal{A}(\lambda) & \mathcal{B} \\ \hline C & \mathcal{D}(\lambda) \end{array}\right],$$

with matrix polynomials $\mathcal{A}(\lambda), \mathcal{D}(\lambda)$ of degree less than or equal to 1, constant matrices $\mathcal{B}, C$, and $\mathbb{L}(\lambda)$ is system equivalent to

$$\tilde{\mathcal{S}}(\lambda) := \left[\begin{array}{c|c} U(\lambda) & 0 \\ \hline 0 & \widetilde{U}(\lambda) \end{array}\right] \mathbb{L}(\lambda) \left[\begin{array}{c|c} V(\lambda) & 0 \\ \hline 0 & \widetilde{V}(\lambda) \end{array}\right] = \left[\begin{array}{c|c|c} I_{(d_A-1)n} & & \\ \hline & \mathcal{S}(\lambda) & \\ \hline & & I_{(d_D-1)m} \end{array}\right], \tag{18}$$

where $U(\lambda)$, $V(\lambda)$, $\tilde{U}(\lambda)$, and $\tilde{V}(\lambda)$ are unimodular matrix polynomials. If, in addition, $U(\lambda)$, $V(\lambda)$, $\tilde{U}(\lambda)$, and $\tilde{V}(\lambda)$ in (18) are constant matrices, then $\mathbb{L}(\lambda)$ is said to be a strict Rosenbrock linearization *of $\mathcal{S}(\lambda)$.*

Let $E := (E_{ij})$ be a block $m \times n$ matrix with $p \times q$ blocks E_{ij}. The *block transpose* of E, see [9], denoted by $E^{\mathcal{B}}$, is the block $n \times m$ matrix with $p \times q$ blocks defined by $(E^{\mathcal{B}})_{ij} := E_{ji}$. We slightly modify this definition for the special structure of Rosenbrock linearizations.

Definition 4.3 (Rosenbrock block transpose). *Let $\mathbb{S}$ be a $(d_A n + d_D m) \times (d_A n + d_D m)$ system matrix given by*

$$\mathbb{S} = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline C & \mathcal{D} \end{array}\right],$$

where $\mathcal{B} := -(e_i e_j^T) \otimes B$ with $B \in \mathbb{C}^{n\times m}$, $C := (e_k e_\ell^T) \otimes C$ with $C \in \mathbb{C}^{m\times n}$, $\mathcal{A} := [\mathcal{A}_{ij}]$ is an $d_A \times d_A$ block matrix with $\mathcal{A}_{ij} \in \mathbb{C}^{n\times n}$, and $\mathcal{D} = [\mathcal{D}_{ij}]$ is a $d_D \times d_D$ block matrix with $\mathcal{D}_{ij} \in \mathbb{C}^{m\times m}$, and e_k is the kth column of I_{d_A}. The Rosenbrock block transpose *of $\mathbb{S}$, denoted by $\mathbb{S}^{\mathbb{B}}$ is defined by*

$$\mathbb{S}^{\mathbb{B}} := \left[\begin{array}{c|c} \mathcal{A}^{\mathcal{B}} & -(e_\ell e_k^T) \otimes B \\ \hline (e_j e_i^T) \otimes C & \mathcal{D}^{\mathcal{B}} \end{array}\right],$$

where $\mathcal{A}^{\mathcal{B}}$ is the block transpose of $\mathcal{A}$.

For $C_1(\lambda)$ and $C_2(\lambda)$ given in (11) and (17), respectively, we have $C_2(\lambda) = C_1(\lambda)^{\mathbb{B}}$.

Definition 4.4. *[9] Let $P(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^m A_m$ be a matrix polynomial of degree m. For $k = 0, \ldots m$, the degree k Horner shift of $P(\lambda)$ is the matrix polynomial $P_k(\lambda) := A_{m-k} + \lambda A_{m-k+1} + \cdots + \lambda^k A_m$. These Horner shifts satisfy the following:*

$$P_0(\lambda) = A_m, \;\; P_{k+1}(\lambda) = \lambda P_k(\lambda) + A_{m-k-1}, \text{ for } 0 \leq k \leq m-1, \;\; P_m(\lambda) = P(\lambda).$$

Extending [[9], Definition 4.2] we define auxiliary matrix polynomials associated with Horner shifts for system polynomials.

Definition 4.5. *Let $A(\lambda) = \sum_{i=0}^{d_A} \lambda^i A_i \in \mathbb{C}[\lambda]^{n\times n}$ be of degree d_A and let $P_i(\lambda)$ be the degree i Horner shift of $A(\lambda)$. For $1 \leq i \leq d_A - 1$, define the matrix polynomials*

$$Q_i(\lambda) := \begin{bmatrix} I_{(i-1)n} & & & \\ & I_n & \lambda I_n & \\ & 0_n & I_n & \\ & & & I_{(d_A-i-1)n} \end{bmatrix}, \quad R_i(\lambda) := \begin{bmatrix} I_{(i-1)n} & & & \\ & 0_n & I_n & \\ & -I_n & P_i(\lambda) & \\ & & & I_{(d_A-i-1)n} \end{bmatrix},$$

$$T_i(\lambda) := \begin{bmatrix} 0_{(i-1)n} & & & \\ & 0_n & \lambda P_{i-1}(\lambda) & \\ & \lambda I_n & \lambda^2 P_{i-1}(\lambda) & \\ & & & 0_{(d_A-i-1)n} \end{bmatrix}, \quad C_i(\lambda) := \begin{bmatrix} 0_{(i-1)n} & & & \\ & P_{i-1}(\lambda) & 0_n & \\ & 0_n & I_n & \\ & & & I_{(d_A-i-1)n} \end{bmatrix},$$

and $C_{d_A}(\lambda) := \operatorname{diag}\left[0_{(d_A-1)n}, P_{d_A-1}(\lambda)\right]$.

For simplicity, we often write Q_i, R_i, T_i, C_i in place of $Q_i(\lambda)$, $R_i(\lambda)$, $T_i(\lambda)$, $C_i(\lambda)$. Note that $C_1(\lambda) = N_{d_A}$, and $Q_i(\lambda)$, $R_i(\lambda)$ are unimodular for all $i = 1, \ldots, d_A - 1$. Also note that $R_i^{\mathcal{B}}(\lambda) = R_i(\lambda)$. The auxiliary matrices satisfy the following relations.

Lemma 4.6 ([9], Lemma 4.3). *Let Q_i, R_i, T_i, C_i be as in Definition 4.5 and M_i's be Fiedler matrices associated with $A(\lambda)$. Then the following relations hold for $i = 1, \ldots, d_A - 1$.*

(a) $Q_i^{\mathcal{B}}(\lambda C_i)R_i = \lambda C_{i+1} + T_i$, *and* $Q_i^{\mathcal{B}}(M_{d_A-(i+1)}M_{d_A-i})R_i = M_{d_A-(i+1)} + T_i$.

(b) $R_i^{\mathcal{B}}(\lambda C_i)Q_i = \lambda C_{i+1} + T_i^{\mathcal{B}}$, *and* $R_i^{\mathcal{B}}(M_{d_A-i}M_{d_A-(i+1)})Q_i = M_{d_A-(i+1)} + T_i^{\mathcal{B}}$.

(c) $T_iM_j = M_jT_i = T_i$ *and* $T_i^{\mathcal{B}}M_j = M_jT_i^{\mathcal{B}} = T_i^{\mathcal{B}}$ *for all* $j \leq d_A - i - 2$.

Definition 4.7. *Let $D(\lambda) = \sum_{i=0}^{d_D} \lambda^i D_i$ be an $m \times m$ matrix polynomial, and let $X_i(\lambda)$ be the degree i Horner shift of $D(\lambda)$. For $1 \leq i \leq d_D - 1$, define the following $md_D \times md_D$ matrix polynomials:*

$$Z_i(\lambda) := \begin{bmatrix} I_{(i-1)m} & & & \\ & I_m & \lambda I_m & \\ & 0_m & I_m & \\ & & & I_{(d_D-i-1)m} \end{bmatrix}, \quad J_i(\lambda) := \begin{bmatrix} I_{(i-1)m} & & & \\ & 0_m & I_m & \\ & I_m & X_i(\lambda) & \\ & & & I_{(d_D-i-1)m} \end{bmatrix},$$

$$H_i(\lambda) := \begin{bmatrix} 0_{(i-1)m} & & & \\ & 0_m & \lambda X_{i-1}(\lambda) & \\ & \lambda I_m & \lambda^2 X_{i-1}(\lambda) & \\ & & & 0_{(d_D-i-1)m} \end{bmatrix}, \quad E_i(\lambda) := \begin{bmatrix} 0_{(i-1)m} & & & \\ & X_{i-1}(\lambda) & 0_m & \\ & 0_m & I_m & \\ & & & I_{(d_D-i-1)m} \end{bmatrix},$$

and $E_{d_D}(\lambda) := \operatorname{diag}\left[0_{(d_D-1)m}, X_{d_D-1}(\lambda)\right]$.

For simplicity, we often write Z_i, J_i, H_i, E_i in place of $Z_i(\lambda)$, $J_i(\lambda)$, $H_i(\lambda)$, $E_i(\lambda)$. Note that $E_1(\lambda) = M_{d_D}$, and $Z_i(\lambda)$, $J_i(\lambda)$ are unimodular for all $i = 1, \ldots, d_D - 1$. Also note that $J_i^{\mathcal{B}}(\lambda) = J_i(\lambda)$.

Remark 4.8. *Consider the auxiliary matrices $Z_i(\lambda), J_i(\lambda), H_i(\lambda)$, and $E_i(\lambda)$ given in Definition 4.7. Then the Lemma 4.6 also holds for $Z_i(\lambda), J_i(\lambda), H_i(\lambda)$, and $E_i(\lambda)$.*

Definition 4.9 (Auxiliary system polynomials). *Let $Q_i(\lambda)$, $R_i(\lambda)$, $T_i(\lambda)$, and $D_i(\lambda)$ be as in Definition 4.5. Let $Z_i(\lambda)$, $J_i(\lambda)$, $H_i(\lambda)$, and $E_i(\lambda)$ be as in Definition 4.7. Let $d = \max\{d_A, d_D\}$, and $r = \min\{d_A, d_D\}$. For $i = 1, \ldots, d-1$, define $(nd_A + md_D) \times (nd_A + md_D)$ system polynomials:*

$$\mathcal{Q}_i(\lambda) = \begin{cases} \left[\begin{array}{c|c} Q_i(\lambda) & 0 \\ \hline 0 & Z_i(\lambda) \end{array}\right], & \text{for } 1 \le i \le r-1 \\ \left[\begin{array}{c|c} Q_i(\lambda) & 0 \\ \hline 0 & I_{d_D m} \end{array}\right], & \text{for } r \le i \le d-1 \text{ and } d_A > d_D \\ \left[\begin{array}{c|c} I_{d_A n} & 0 \\ \hline 0 & Z_i(\lambda) \end{array}\right], & \text{for } r \le i \le d-1 \text{ and } d_A < d_D, \end{cases}$$

$$\mathcal{R}_i(\lambda) = \begin{cases} \left[\begin{array}{c|c} R_i(\lambda) & 0 \\ \hline 0 & J_i(\lambda) \end{array}\right], & \text{for } 1 \le i \le r-1 \\ \left[\begin{array}{c|c} R_i(\lambda) & 0 \\ \hline 0 & I_{d_D m} \end{array}\right], & \text{for } r \le i \le d-1 \text{ and } d_A > d_D \\ \left[\begin{array}{c|c} I_{d_A n} & 0 \\ \hline 0 & J_i(\lambda) \end{array}\right], & \text{for } r \le i \le d-1 \text{ and } d_A < d_D, \end{cases}$$

$$\mathcal{T}_i(\lambda) = \begin{cases} \left[\begin{array}{c|c} T_i(\lambda) & 0 \\ \hline 0 & H_i(\lambda) \end{array}\right], & \text{for } 1 \le i \le r-1 \\ \left[\begin{array}{c|c} T_i(\lambda) & 0 \\ \hline 0 & I_{d_D m} \end{array}\right], & \text{for } r \le i \le d-1 \text{ and } d_A > d_D \\ \left[\begin{array}{c|c} I_{d_A n} & 0 \\ \hline 0 & H_i(\lambda) \end{array}\right], & \text{for } r \le i \le d-1 \text{ and } d_A < d_D, \end{cases}$$

$$\mathcal{D}_i(\lambda) = \begin{cases} \left[\begin{array}{c|c} C_i(\lambda) & 0 \\ \hline 0 & E_i(\lambda) \end{array}\right], & \text{for } 1 \le i \le r-1 \\ \left[\begin{array}{c|c} C_i(\lambda) & 0 \\ \hline 0 & I_{d_D m} \end{array}\right], & \text{for } \le i \le d-1 \text{ and } d_A > d_D \\ \left[\begin{array}{c|c} I_{d_A n} & 0 \\ \hline 0 & E_i(\lambda) \end{array}\right], & \text{for } r \le i \le d-1 \text{ and } d_A < d_D, \end{cases}$$

$$\text{and } \mathcal{D}_d(\lambda) := \left[\begin{array}{c|c} C_{d_A}(\lambda) & 0 \\ \hline 0 & E_{d_D}(\lambda) \end{array}\right], \text{ where } d = \max\{d_A, d_D\}.$$

Note that $\mathcal{D}_1(\lambda) = \left[\begin{array}{c|c} C_1(\lambda) & 0 \\ \hline 0 & E_1(\lambda) \end{array}\right] = \left[\begin{array}{c|c} M_{d_A} & 0 \\ \hline 0 & N_{d_D} \end{array}\right] = \mathbb{M}_d$ and that $\mathcal{Q}_i(\lambda)$ and $\mathcal{R}_i(\lambda)$ are unimodular matrix polynomials for $i = 1, \ldots, d-1$. Also, note that $\mathcal{R}_i^{\mathbb{B}}(\lambda) = \mathcal{R}_i(\lambda)$ for $i = 1, \ldots, d-1$.

The auxiliary system polynomials satisfy the following relations.

Lemma 4.10. *Let $\mathcal{Q}_i, \mathcal{R}_i, \mathcal{T}_i, \mathcal{D}_i$ be the system polynomials given in Definition 4.9 and $\mathbb{M}_i$'s be Fiedler matrices associated with $\mathcal{S}(\lambda)$. Then the following system equivalence relations hold for $i = 1, \ldots, d-1$.*

(a) $\mathcal{Q}_i^{\mathbb{B}}(\lambda\mathcal{D}_i)\mathcal{R}_i = \lambda\mathcal{D}_{i+1} + \mathcal{T}_i$, *and* $\mathcal{Q}_i^{\mathbb{B}}(\mathbb{M}_{d-(i+1)}\mathbb{M}_{d-i})\mathcal{R}_i = \mathbb{M}_{d-(i+1)} + \mathcal{T}_i$.

(b) $\mathcal{R}_i^{\mathbb{B}}(\lambda\mathcal{D}_i)\mathcal{Q}_i = \lambda\mathcal{D}_{i+1} + \mathcal{T}_i^{\mathbb{B}}$, *and* $\mathcal{R}_i^{\mathbb{B}}(\mathbb{M}_{d-i}\mathbb{M}_{d-(i+1)})\mathcal{Q}_i = \mathbb{M}_{d-(i+1)} + \mathcal{T}_i^{\mathbb{B}}$.

(c) $\mathcal{T}_i\mathbb{M}_j = \mathbb{M}_j\mathcal{T}_i = \mathcal{T}_i$ *and* $\mathcal{T}_i^{\mathbb{B}}\mathbb{M}_j = \mathbb{M}_j\mathcal{T}_i^{\mathbb{B}} = \mathcal{T}_i^{\mathbb{B}}$ *for all* $j \leq d - i - 2$.

Proof. (a) We have

$$\begin{aligned}
\mathcal{Q}_i^{\mathbb{B}}(\lambda\mathcal{D}_i)\mathcal{R}_i &= \left[\begin{array}{c|c} Q_i^{\mathcal{B}} & 0 \\ \hline 0 & Z_i^{\mathcal{B}} \end{array}\right]\left[\begin{array}{c|c} \lambda C_i & \\ \hline & \lambda E_i \end{array}\right]\left[\begin{array}{c|c} R_i & 0 \\ \hline 0 & J_i \end{array}\right] = \left[\begin{array}{c|c} Q_i^{\mathcal{B}}(\lambda C_i)R_i & \\ \hline & Z_i^{\mathcal{B}}(\lambda E_i)J_i \end{array}\right] \\
&= \left[\begin{array}{c|c} \lambda C_{i+1} + T_i & \\ \hline & \lambda E_{i+1} + H_i \end{array}\right] \text{(By Lemma 4.6(a) and Remark 4.8)} \\
&= \left[\begin{array}{c|c} \lambda C_{i+1} & \\ \hline & \lambda E_{i+1} \end{array}\right] + \left[\begin{array}{c|c} T_i & \\ \hline & H_i \end{array}\right] = \lambda\mathcal{D}_{i+1} + \mathcal{T}_i \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_i^{\mathbb{B}}\mathbb{M}_{d-(i+1)}\mathbb{M}_{d-i}\mathcal{R}_i &= \left[\begin{array}{c|c} Q_i^{\mathcal{B}} & 0 \\ \hline 0 & Z_i^{\mathcal{B}} \end{array}\right]\left[\begin{array}{c|c} M_{d-(i+1)} & \\ \hline & N_{d-(i+1)} \end{array}\right]\left[\begin{array}{c|c} M_{d-i} & \\ \hline & N_{d-i} \end{array}\right]\left[\begin{array}{c|c} R_i & 0 \\ \hline 0 & J_i \end{array}\right] \\
&= \left[\begin{array}{c|c} Q_i^{\mathcal{B}}M_{d-(i+1)}M_{d-i}R_i & \\ \hline & Z_i^{\mathcal{B}}N_{m-(i+1)}N_{m-1}J_i \end{array}\right] \\
&= \left[\begin{array}{c|c} M_{d-(i+1)} + T_i & \\ \hline & N_{d-(i+1)} + H_i \end{array}\right] \text{(Lemma 4.6(a) and Remark 4.8)} \\
&= \left[\begin{array}{c|c} M_{d-(i+1)} & \\ \hline & N_{d-(i+1)} \end{array}\right] + \left[\begin{array}{c|c} T_i & \\ \hline & H_i \end{array}\right] = \mathbb{M}_{d-(i+1)} + \mathcal{T}_i.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\mathcal{R}_i^{\mathbb{B}}(\lambda\mathcal{D}_i)\mathcal{Q}_i &= \left[\begin{array}{c|c} R_i^{\mathcal{B}} & 0 \\ \hline 0 & J_i^{\mathcal{B}} \end{array}\right]\left[\begin{array}{c|c} \lambda C_i & \\ \hline & \lambda E_i \end{array}\right]\left[\begin{array}{c|c} Q_i & 0 \\ \hline 0 & Z_i \end{array}\right] = \left[\begin{array}{c|c} R_i^{\mathcal{B}}(\lambda C_i)Q_i & \\ \hline & J_i^{\mathcal{B}}(\lambda E_i)Z_i \end{array}\right] \\
&= \left[\begin{array}{c|c} \lambda C_{i+1} + T_i^{\mathcal{B}} & \\ \hline & \lambda E_{i+1} + H_i^{\mathcal{B}} \end{array}\right] \text{(From Lemma 4.6(b) and Remark 4.8)} \\
&= \left[\begin{array}{c|c} \lambda C_{i+1} & \\ \hline & \lambda E_{i+1} \end{array}\right] + \left[\begin{array}{c|c} T_i^{\mathcal{B}} & \\ \hline & H_i^{\mathcal{B}} \end{array}\right] = \lambda\mathcal{D}_{i+1} + \mathcal{T}_i^{\mathbb{B}} \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_i^{\mathbb{B}}\mathbb{M}_{d-i}\mathbb{M}_{d-(i+1)}\mathcal{Q}_i &= \left[\begin{array}{c|c} R_i^{\mathcal{B}} & 0 \\ \hline 0 & J_i^{\mathcal{B}} \end{array}\right]\left[\begin{array}{c|c} M_{d-i} & \\ \hline & N_{d-i} \end{array}\right]\left[\begin{array}{c|c} M_{d-(i+1)} & \\ \hline & N_{d-(i+1)} \end{array}\right]\left[\begin{array}{c|c} Q_i & 0 \\ \hline 0 & Z_i \end{array}\right] \\
&= \left[\begin{array}{c|c} R_i^{\mathcal{B}}M_{d-i}M_{d-(i+1)}Q_i & \\ \hline & J_i^{\mathcal{B}}N_{d-i}N_{d-(i+1)}Z_i \end{array}\right] \\
&= \left[\begin{array}{c|c} M_{d-(i+1)} + T_i^{\mathcal{B}} & \\ \hline & N_{d-(i+1)} + H_i^{\mathcal{B}} \end{array}\right] \text{(By Lemma 4.6(b) and Remark 4.8)} \\
&= \left[\begin{array}{c|c} M_{d-(i+1)} & \\ \hline & N_{d-(i+1)} \end{array}\right] + \left[\begin{array}{c|c} T_i^{\mathcal{B}} & \\ \hline & H_i^{\mathcal{B}} \end{array}\right] = \mathbb{M}_{d-(i+1)} + \mathcal{T}_i^{\mathbb{B}}.
\end{aligned}$$

(c) We have

$$\begin{aligned}\mathcal{T}_i\mathbb{M}_j &= \left[\begin{array}{c|c} T_i & \\ \hline & H_i\end{array}\right]\left[\begin{array}{c|c} M_j & \\ \hline & N_j\end{array}\right] = \left[\begin{array}{c|c} T_iM_j & \\ \hline & H_iN_j\end{array}\right]\\ &= \left[\begin{array}{c|c} M_jT_i & \\ \hline & N_jH_i\end{array}\right] \text{ (by Lemma 4.6(c) and Remark 4.8)}\\ &= \left[\begin{array}{c|c} M_j & \\ \hline & N_j\end{array}\right]\left[\begin{array}{c|c} T_i & \\ \hline & H_i\end{array}\right] = \mathbb{M}_j\mathcal{T}_i \quad \text{and}\end{aligned}$$

$$\begin{aligned}\mathcal{T}_i^{\mathbb{B}}\mathbb{M}_j &= \left[\begin{array}{c|c} T_i^{\mathcal{B}} & \\ \hline & H_i^{\mathcal{B}}\end{array}\right]\left[\begin{array}{c|c} M_j & \\ \hline & N_j\end{array}\right] = \left[\begin{array}{c|c} T_i^{\mathcal{B}}M_j & \\ \hline & H_i^{\mathcal{B}}N_j\end{array}\right]\\ &= \left[\begin{array}{c|c} M_jT_i^{\mathcal{B}} & \\ \hline & N_jH_i^{\mathcal{B}}\end{array}\right] \text{ (by Lemma 4.6(c) and Remark 4.8)}\\ &= \left[\begin{array}{c|c} M_j & \\ \hline & N_j\end{array}\right]\left[\begin{array}{c|c} T_i^{\mathcal{B}} & \\ \hline & H_i^{\mathcal{B}}\end{array}\right] = \mathbb{M}_j\mathcal{T}_i^{\mathbb{B}}.\end{aligned}$$

□

Definition 4.11. *Let $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_d - \mathbb{M}_\sigma$ be the Fiedler pencil of $\mathcal{S}(\lambda)$ given in (2) associated with a bijection σ. For $j = 1, 2, \ldots, d$, define*

$$\mathbb{M}_\sigma^{(j)} := \prod_{\sigma^{-1}(i)\leq d-j} \mathbb{M}_{\sigma^{-1}(i)},$$

where the factors $\mathbb{M}_{\sigma^{-1}(i)}$ are in the same relative order as they are in $\mathbb{M}_\sigma$. Note that $\mathbb{M}_\sigma^{(1)} = \prod_{\sigma^{-1}(i)\leq d-1}\mathbb{M}_{\sigma^{-1}(i)} = \mathbb{M}_\sigma$ and that $\mathbb{M}_\sigma^{(d)} = \mathbb{M}_0$. Also for $j = 1, 2, \ldots, d$, define the $(nd_A + md_D) \times (nd_A + md_D)$ system pencils $\mathbb{L}_\sigma^{(j)}(\lambda) := \lambda\mathcal{D}_j(\lambda) - \mathbb{M}_\sigma^{(j)}$. Observe that $\mathbb{L}_\sigma^{(1)}(\lambda) = \lambda\mathcal{D}_1 - \mathbb{M}_\sigma^{(1)} = \lambda\mathbb{M}_d - \mathbb{M}_\sigma(\lambda) = \mathbb{L}_\sigma$ and that

$$\mathbb{L}_\sigma^{(d)}(\lambda) = \lambda\mathcal{D}_d - \mathbb{M}_\sigma^{(d)} = \lambda\left[\begin{array}{c|c} D_{d_A} & 0 \\ \hline 0 & -E_{d_D}\end{array}\right] - \mathbb{M}_0 = \left[\begin{array}{c|c} \begin{matrix} -I_{(d_A-1)n} & \\ & A(\lambda)\end{matrix} & -(e_{d_A}e_{d_D}^T)\otimes B \\ \hline (e_{d_D}e_{d_A}^T)\otimes C & \begin{matrix} -I_{(d_D-1)m} & \\ & D(\lambda)\end{matrix}\end{array}\right].$$

The next result shows that $\mathbb{L}_\sigma^{(i)}(\lambda) \sim_{se} \mathbb{L}_\sigma^{(i+1)}(\lambda)$ for $i = 1, 2, \ldots, d-1$.

Lemma 4.12. *We have $\mathbb{L}_\sigma^{(i)}(\lambda) \sim_{se} \mathbb{L}_\sigma^{(i+1)}(\lambda)$ for $i = 1, 2, \ldots, d-1$. More precisely, if $\mathcal{Q}_i$ and $\mathcal{R}_i$ are the system polynomials given in Definition 4.9, then*

$$\mathbb{L}_\sigma^{(i+1)}(\lambda) = \begin{cases} \mathcal{Q}_i^{\mathbb{B}}\mathbb{L}_\sigma^{(i)}(\lambda)\mathcal{R}_i, & \text{if } \sigma \text{ has a consecution at } d-i-1,\\ \mathcal{R}_i^{\mathbb{B}}\mathbb{L}_\sigma^{(i)}\mathcal{Q}_i, & \text{if } \sigma \text{ has an inversion at } d-i-1.\end{cases}$$

Proof. The proof is exactly the same as that of Lemma 4.5 in [9]. □

It is now immediate that a Fiedler pencil is a Rosenbrock linearization of $\mathcal{S}(\lambda)$.

Theorem 4.13 (Rosenbrock linearization). *Let $\mathcal{S}(\lambda)$ be an $(n+m)\times(n+m)$ system matrix (regular or singular) given in (2). If $\sigma : \{0, 1, \ldots, d_A - 1\} \to \{1, 2, \ldots, d_A\}$ is a bijection, then the Fiedler pencil $\mathbb{L}_\sigma(\lambda)$ of $\mathcal{S}(\lambda)$ is a Rosenbrock linearization for $\mathcal{S}(\lambda)$.*

Proof. By Lemma 4.12, we have $d-1$ system equivalences

$$\mathbb{L}_\sigma(\lambda) = \mathbb{L}_\sigma^{(1)}(\lambda) \sim_{se} \mathbb{L}_\sigma^{(2)}(\lambda) \sim_{se} \cdots \sim_{se} \mathbb{L}_\sigma^{(d)}(\lambda)$$
$$= \left[\begin{array}{c|c} \begin{matrix} -I_{(d_A-1)n} & \\ & A(\lambda) \end{matrix} & -(e_{d_A}e_{d_D}^T)\otimes B \\ \hline (e_{d_D}e_{d_A}^T)\otimes C & \begin{matrix} -I_{(d_D-1)m} & \\ & D(\lambda) \end{matrix} \end{array}\right],$$

where $\mathbb{L}_\sigma^{(i)}(\lambda)$ is as in Lemma 4.12. This shows that $\mathbb{L}_\sigma(\lambda) \sim_{se} I_{(d_A-1)n} \oplus \mathcal{S}(\lambda) \oplus I_{(d_D-1)m}$. □

Corollary 4.14. *Let $\mathbb{L}_\sigma(\lambda)$ be the Fiedler pencil of $\mathcal{S}(\lambda)$ given in (2) associated with a bijection σ, and $Q_i, \mathcal{R}_i$ for $i = 1, 2, \dots d-1$, be as in Definition 4.9. Then*

$$\mathcal{U}(\lambda)\mathbb{L}_\sigma(\lambda)\mathcal{V}(\lambda) = \left[\begin{array}{c|c} \begin{matrix} -I_{(d_A-1)n} & \\ & A(\lambda) \end{matrix} & -(e_{d_A}e_{d_D}^T)\otimes B \\ \hline (e_{d_D}e_{d_A}^T)\otimes C & \begin{matrix} -I_{(d_D-1)m} & \\ & D(\lambda) \end{matrix} \end{array}\right] \sim_{se} I_{(d_A-1)n} \oplus \mathcal{S}(\lambda) \oplus I_{(d_D-1)m},$$

where $\mathcal{U}(\lambda)$ and $\mathcal{V}(\lambda)$ are $(nd_A + md_D) \times (nd_A + md_D)$ unimodular system polynomials given by

$$\mathcal{U}(\lambda) := \mathcal{U}_0\mathcal{U}_1 \cdots \mathcal{U}_{d-3}\mathcal{U}_{d-2}, \text{ with } \mathcal{U}_i = \begin{cases} Q_{d-(i+1)}^{\mathbb{B}}, & \text{if } \sigma \text{ has a consecution at } i, \\ \mathcal{R}_{d-(i+1)}^{\mathbb{B}}, & \text{if } \sigma \text{ has an inversion at } i, \end{cases}$$

$$\mathcal{V}(\lambda) := \mathcal{V}_{d-2}\mathcal{V}_{d-3} \cdots \mathcal{V}_1\mathcal{V}_0, \text{ with } \mathcal{V}_i = \begin{cases} \mathcal{R}_{d-(i+1)}, & \text{if } \sigma \text{ has a consecution at } i, \\ Q_{d-(i+1)}, & \text{if } \sigma \text{ has an inversion at } i. \end{cases}$$

The indexing of $\mathcal{U}_i$ and $\mathcal{V}_i$ factors in $\mathcal{U}(\lambda)$ and $\mathcal{V}(\lambda)$, respectively, in Corollary 4.14 has been chosen for simplification of notation and has no other special significance.

Remark 4.15. *If we consider $D(\lambda)$ is a matrix polynomial of degree 1 then the Fiedler pencils $\mathbb{L}_\sigma(\lambda)$ are linearizations of the system matrix of LTI state-space system, see [25].*

Remark 4.16. *Consider the system matrix $\mathcal{S}(\lambda)$ and associated transfer function $R(\lambda)$ given in (2) and (3), respectively. Given an eigenvector x of $\mathbb{L}_\sigma(\lambda)$ one can determine an eigenvector of $\mathcal{S}(\lambda)$ from x. That is, one can recover eigenvectors of $R(\lambda)$ and $\mathcal{S}(\lambda)$ from those of the Fiedler pencils of $R(\lambda)$. It directly follows from the Theorem 4.10 and Theorem 4.11 in [22].*

5. Conclusions

We have constructed an algorithm for construction of Fiedler pencils of system matrix $\mathcal{S}(\lambda)$ associated to a multivariable state-space system and shown that these Fiedler pencils are linearizations of $\mathcal{S}(\lambda)$.

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