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Almost Schouten solitons and contact geometry

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Abstract. The current article is about almost Schouten solitons and gradient Schouten solitons on contact geometry. At first, we demonstrate that if a compact K-contact manifold admits an almost Schouten soliton, then the soliton is shrinking and the manifold is an Einstein manifold. Moreover, we show that if a K-contact manifold admits a gradient Schouten soliton, then the manifold becomes an Einstein manifold. Next, we investigate almost Schouten solitons and gradient Schouten solitons on (k, μ)-contact manifolds. Finally, we show that if a complete H-contact manifold M^{2n+1} satisfying certain restriction on the scalar curvature and the soliton function admits an almost Schouten soliton whose potential vector field V is collinear with ζ, then M^{2n+1} is compact Einstein and Sasakian.

1. Introduction

On a $(2n + 1)$-dimensional Riemannian manifold, Schouten solitons are the self-similar solutions of an intrinsic flow named as a Schouten flow ([7], [9]), that is defined by

$$\pounds_V g + 2S_t + 2\alpha g = 0, \tag{1}$$

where S_t is the Schouten tensor given by

$$S_t = \frac{1}{2n-1}(S - \frac{r}{4n}g), \tag{2}$$

$\alpha \in \mathbb{R}$, S indicates the Ricci tensor and r denotes the scalar curvature. V is called the potential vector field. This soliton is referred to be shrinking, stable or expanding for $\alpha < 0$, $\alpha = 0$ or $\alpha > 0$, respectively. For, $n = 1$ and $r = 0$, this soliton turns into Ricci soliton. The simplest example of Schouten soliton is an Einstein manifold. In [7], the author cited an example of a Schouten soliton in a Riemannian manifold.

We generalize the above notion, named almost Schouten solitons assuming α is a smooth function.

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Catino and Mazzieri[9] introduced the concept of gradient Schouten solitons, defined by

$$S + \nabla^2 f = (\frac{r}{4n} + \alpha)g, \tag{3}$$

f being a smooth function and α a real constant. Gradient Schouten solitons and compact gradient Schouten solitons are both studied in [9]. They proved the triviality of every compact gradient Schouten solitons. They also demonstrated that a complete gradient steady Schouten soliton is trivial and Ricci flat. Moreover, they established that every complete three dimensional gradient shrinking Schouten soliton is isometric to a finite quotient of either S^3 or $\mathbb{R}^3$ or $\mathbb{R} \times S^2$. Complete gradient Schouten solitons have been characterized by Pina and Menezes in [26]. Very recently, one research article based on gradient Schouten solitons has been published by Borges [7]. For $r = 0$, (3) implies that gradient Schouten solitons become gradient Ricci solitons.

A K-contact manifold is an initial idea between a contact and a Sasakian manifold. (As for example, A Sasakian structure is the normal contact metric structure on an odd-dimensional sphere). A K-contact structure is carried by a compact regular contact manifold.

Ricci solitons and almost Ricci solitons have been explored during the past several years by a number of researchers (see [10]-[20], [25]-[29], [31]-[36]). In [29] Sharma started the investigation of Ricci solitons on K-contact manifold and demonstrated that "any complete K-contact metric admitting a Ricci soliton of gradient type is Einstein and Sasakian". Moreover, Cho and Sharma [14] proved that a compact contact metric manifold admitting a Ricci soliton with a non-zero potential vector field which is collinear with ζ at each point is Einstein and Sasakian.

Recently, geometric flows are initiated in the investigation of the cosmological model such as perfect fluid spacetime. In [2] , Blaga studied η-Ricci and η-Einstein soliton in perfect fluid spacetime and obtained the Poisson equation from the soliton equation when the potential vector field ζ is of gradient type. Kumara and Venkatesha [32] analyzed Ricci soliton in perfect fluid spacetime with torse-forming vector field. Very recently, we have studied almost Schouten solitons in spacetimes ([17], [27]). Therefore almost Schouten solitons have applications in theory of relativity.

The above studies encourage us to characterize almost Schouten solitons and gradient Schouten solitons in contact geometry. We specifically achieve the following results:

Theorem 1.1. *If a compact K-contact manifold admits an almost Schouten soliton whose potential vector field is ζ, then the soliton is shrinking and the manifold becomes an Einstein Sasakian.*

Theorem 1.2. *If a K-contact manifold admits a gradient Schouten soliton, then the manifold becomes an Einstein manifold.*

Theorem 1.3. *If a (k, μ)-contact manifold admits an almost Schouten soliton whose potential vector field is ζ, then the soliton becomes Schouten soliton and the soliton is shrinking, steady and expanding for $2(n-1) + (1-8n)k - n\mu < 0$, $2(n-1) + (1-8n)k - n\mu = 0$ and $2(n-1) + (1-8n)k - n\mu > 0$, respectively.*

Theorem 1.4. *Let (M^{2n+1}, g) be a (k, μ)-contact manifold. If M^{2n+1} admits a gradient Schouten soliton, then either (i) for $n = 1$, M^3 is flat and for $n > 1$, M^{2n+1} is locally isometric to $E^{n+1} \times S^n(4)$ or grad f is pointwise collinear with ζ or,*
(ii) M^{2n+1} is an Einstein manifold, provided $k^2 + \mu^2(k-1) \neq 0$.

Theorem 1.5. *Let (M^{2n+1}, g) be a compact contact metric manifold such that g is an almost Schouten soliton with nonzero potential vector field V is collinear with ζ at each point. Then (M^{2n+1}, g) is Einstein and Sasakian, provided $\zeta r + 8n(1-2n)\zeta\alpha = 0$.*

In [24], Perrone introduced H-contact manifold. A contact metric manifold whose Reeb vector field is harmonic is called an H-contact manifold. In [14] Cho and Sharma investigated Ricci solitons in a complete H-contact manifold. Here we intend to study almost Schouten solitons in a complete H-contact manifold, which generalizes the Proposition 1 of [14].

Theorem 1.6. *Let (M^{2n+1}, g) be a complete H-contact manifold. If g is an almost Schouten soliton with nonzero potential vector field V is collinear with ζ, then (M^{2n+1}, g) is compact Einstein and Sasakian, provided $\zeta r + 8n(1-2n)\zeta\boldsymbol{\alpha} = 0$.*

2. Preliminaries

Let M^{2n+1} be an almost contact manifold with almost contact structure $(\varphi, \zeta, \boldsymbol{\eta})$, ζ being a unit vector field, φ is a $(1,1)$-tensor field and $\boldsymbol{\eta}$ being a 1-form satisfying

$$\varphi^2 E_1 = -E_1 + \boldsymbol{\eta}(E_1)\zeta, \quad \boldsymbol{\eta}(\zeta) = 1. \tag{4}$$

A Riemannian metric g on M^{2n+1} is compatible with the almost contact structure if

$$g(\varphi E_1, \varphi E_2) = g(E_1, E_2) - \boldsymbol{\eta}(E_1)\boldsymbol{\eta}(E_2) \tag{5}$$

for any $E_1, E_2 \in \chi(M^{2n+1})$. If the associated complex structure J on $M^{2n+1} \times \mathbb{R}$ is integrable, an almost contact structure $(\varphi, \zeta, \boldsymbol{\eta}, g)$ is said to be normal.

If a Riemannian manifold M^{2n+1} and its almost contact metric structure $(\varphi, \zeta, \boldsymbol{\eta}, g)$ satisfies the condition ([3], p. 47) $d\boldsymbol{\eta}(E_2, E_3) = g(E_2, \varphi E_3)$ for all vector fields E_2, E_3, it is said to be a contact metric manifold. The contact metric structure's associated metric is g. Two self-adjoint operators, $h = \frac{1}{2}\pounds_\zeta\varphi$ and $l = R(., \zeta)\zeta$, are taken into consideration on the contact metric manifold $M^{2n+1}(\varphi, \zeta, \boldsymbol{\eta}, g)$, $\pounds_\zeta$ is the Lie-derivative along ζ. The two operators h and l satisfy ([3], p. 84-85)

$$tr\, h = tr\, h\varphi = 0, \;\; h\zeta = l\zeta = 0, \;\; h\varphi = -\varphi h. \tag{6}$$

Lemma 2.1. *([3]) On a contact metric manifold $M^{2n+1}(\varphi, \zeta, \boldsymbol{\eta}, g)$*

$$\nabla_{E_2}\zeta = -\varphi E_2 - \varphi h E_2, \tag{7}$$

$$S(\zeta, \zeta) = g(Q\zeta, \zeta) = tr\, l = 2n - tr\, h^2, \tag{8}$$

$$(div(h\varphi))E_2 = g(Q\zeta, E_2) - 2n\boldsymbol{\eta}(E_2), \tag{9}$$

where Q is the Ricci operator defined by $g(QE_1, E_2) = S(E_1, E_2)$ for all E_1, E_2.

If ζ is Killing, equivalently $h = 0$ ([3], p. 87), a contact metric manifold is said to be K-contact. Therefore, on a K-contact manifold equation (7) turns into

$$\nabla_{E_2}\zeta = -\varphi E_2. \tag{10}$$

Furthermore, the following formulas are also valid on a K-contact manifold.

Lemma 2.2. *([3], p.113-116) A K-contact manifold $M^{2n+1}(\varphi, \zeta, \boldsymbol{\eta}, g)$ obeys*

$$S(E_2, \zeta) = 2n\boldsymbol{\eta}(E_2), \tag{11}$$

$$Q\zeta = 2n\zeta, \tag{12}$$

$$R(\zeta, E_2)E_3 = (\nabla_{E_2}\varphi)E_3 \tag{13}$$

$$(\nabla_{E_3}\varphi)E_2 + (\nabla_{\varphi E_3}\varphi)\varphi E_2 = 2g(E_2, E_3)\zeta - \boldsymbol{\eta}(E_2)(E_3 + \boldsymbol{\eta}(E_3)\zeta) \tag{14}$$

for all vector fields E_2, E_3.

Blair et al. [5] introduced and studied a particular class of contact metric manifolds, called (k, μ)-contact manifolds. In [6], Boeckx properly characterized these manifolds. A (k, μ)-contact manifold is a $M^{2n+1}(\varphi, \zeta, \boldsymbol{\eta}, g)$ contact metric manifold. It's curvature tensor fulfills

$$R(E_2, E_3)\zeta = k[\boldsymbol{\eta}(E_3)E_2 - \boldsymbol{\eta}(E_2)E_3] + \mu[\boldsymbol{\eta}(E_3)hE_2 - \boldsymbol{\eta}(E_2)hE_3] \tag{15}$$

for all E_2, E_3 and k, $\mu \in \mathbb{R}$. A (k, μ)-contact manifold is called an $N(k)$-contact metric manifold if $\mu = 0$([1], [4]).

The preceding formulas are used for non-Sasakian (k, μ)-manifolds [5]:

$$\begin{aligned} S(E_2, E_3) &= [2(n-1) - n\mu]g(E_2, E_3) + [2(n-1) + \mu]g(hE_2, E_3) \\ &\quad +[2(1-n) + n(2k+\mu)]\boldsymbol{\eta}(E_2)\boldsymbol{\eta}(E_3), \end{aligned} \tag{16}$$

$$Q\zeta = 2nk\zeta, \tag{17}$$

$$h^2 = (k-1)\varphi^2, \ k \leq 1, \tag{18}$$

when $k = 1$, M^{2n+1} is Sasakian. The (k, μ)-nullity condition completely determines the curvature of M^{2n+1} in the non-Sasakian case, that is, $k < 1$. Furthermore, the scalar curvature r is obtained by

$$r = 2n(2(n-1) + k - n\mu). \tag{19}$$

Lemma 2.3. *On a* (k, μ)*-contact manfold* $M^{2n+1}(\varphi, \zeta, \boldsymbol{\eta}, g)$ *one has*

$$R(\zeta, E_2)E_3 = k[g(E_2, E_3)\zeta - \boldsymbol{\eta}(E_3)E_2] + \mu[g(hE_2, E_3)\zeta - \boldsymbol{\eta}(E_3)hE_2], \tag{20}$$

$$(\nabla_\zeta Q)E_2 = \mu[2(n-1) + \mu]h\varphi E_2, \tag{21}$$

$$(\nabla_\zeta h)E_2 = \mu h\varphi E_2, \tag{22}$$

$$(\nabla_{E_2} Q)\zeta = (\varphi + \varphi h)QE_2 - 2nk(\varphi + \varphi h)E_2. \tag{23}$$

In the sequel we will use the following results:

Theorem A. ([30]) For dimension ≥ 5, an Einstein $N(k)$-contact metric manifold is necessarily Sasakian.

Theorem B. ([23]) A Ricci soliton on a compact manifold is a gradient Ricci soliton.

Theorem C. ([29]) A compact Ricci soliton of constant scalar curvature is Einstein.

Lemma 2.4. *If a compact K-contact manifold admits an almost Schouten soliton, then the following integral formula*

$$\int_M [\frac{(6n-1)r}{8n(2n-1)} + \alpha(2n+1)]dM = 0 \tag{24}$$

holds where dM stands for M^{2n+1}'s volume form.

Proof. Equations (1) and (2) imply

$$(\pounds_V g)(E_1, E_2) + \frac{2}{2n-1}S(E_1, E_2) + [2\alpha - \frac{r}{4n(2n-1)}]g(E_1, E_2) = 0, \tag{25}$$

which implies

$$g(\nabla_{E_1} V, E_2) + g(E_1, \nabla_{E_2} V) + \frac{2}{2n-1}S(E_1, E_2) + [2\alpha - \frac{r}{4n(2n-1)}]g(E_1, E_2) = 0. \tag{26}$$

Contracting E_1 and E_2 in (26) entails that

$$div\, V = -[\frac{(6n-1)r}{8n(2n-1)} + \alpha(2n+1)]. \tag{27}$$

Integrating (26) and using divergence theorem, we provide

$$0 = \int_M [\frac{(6n-1)r}{8n(2n-1)} + \alpha(2n+1)]dM, \tag{28}$$

where dM stands for M^{2n+1}'s volume form.
Thus the proof is completed.

3. Proof of the Main Results

Proof of Theorem 1.1. Setting $V = \zeta$ in (26) and using (10) gives

$$\frac{2}{2n-1}S(E_1, E_2) + [2\alpha - \frac{r}{4n(2n-1)}]g(E_1, E_2) = 0. \tag{29}$$

Therefore, M^{2n+1} is an Einstein manifold. Contracting (29), we infer

$$\frac{r}{4n(2n-1)} = -2\alpha\frac{(2n+1)}{(6n-1)} \tag{30}$$

Putting $E_1 = \zeta$ in (29), we find

$$\frac{4n}{2n-1} + 2\alpha - \frac{r}{4n(2n-1)} = 0. \tag{31}$$

Using (30) in (29) entails that

$$\alpha = -\frac{6n-1}{4(2n-1)}. \tag{32}$$

Therefore, $\alpha < 0$, hence the soliton is shrinking.

It is widely known that an Einstein compact K-contact manifold is a Sasakian manifold [8]. Hence, the manifold becomes a Sasakian manifold.
Hence the result follows.

In particular, for $n = 1$ and $r = 0$, equation (30) implies $\boldsymbol{\alpha} = 0$. Therefore, from (29) we get $S(E_1, E_2) = 0$ and hence the manifold is flat, since $R(E_1, E_2)E_3 = 0$ for 3-dimension. Hence we have:

Corollary 3.1. *If a 3-dimensional K-contact manifold admits a Ricci soliton, then the soliton is steady and the manifold is flat.*

Proof of Theorem 1.2. From (3), we get

$$\nabla_{E_1} Df + QE_1 = (\frac{r}{4n} + \boldsymbol{\alpha})E_1. \tag{33}$$

Above equation implies

$$\nabla_{E_2}\nabla_{E_1} Df + \nabla_{E_2} QE_1 = \frac{E_2 r}{4n} E_1 + (\frac{r}{4n} + \boldsymbol{\alpha})\nabla_{E_2} E_1. \tag{34}$$

Interchanging E_1 and E_2 in (34) gives

$$\nabla_{E_1}\nabla_{E_2} Df + \nabla_{E_1} QE_2 = \frac{E_1 r}{4n} E_2 + (\frac{r}{4n} + \boldsymbol{\alpha})\nabla_{E_1} E_2. \tag{35}$$

From (33), we get

$$\nabla_{[E_1,E_2]} Df + Q([E_1, E_2]) = (\frac{r}{4n} + \boldsymbol{\alpha})([E_1, E_2]). \tag{36}$$

With the help of (34)-(36), we find

$$R(E_1, E_2)Df = \frac{E_1 r}{4n} E_2 - \frac{E_2 r}{4n} E_1 - (\nabla_{E_1} Q)E_2 + (\nabla_{E_2} Q)E_1. \tag{37}$$

In a K-contact manifold, we have [21]

$$(\nabla_{E_1} Q)\zeta = Q\varphi E_1 - 2n\varphi E_1 \ \ \textit{and} \tag{38}$$

$$(\nabla_{\zeta} Q)E_1 = Q\varphi E_1 - \varphi QE_1. \tag{39}$$

Putting $E_1 = \zeta$ in (37) and using (38) and (39), we provide

$$\begin{aligned} R(\zeta, E_2)Df &= -[2n\varphi E_2 - \varphi QE_2] \\ &\quad + \frac{(\zeta r)}{4n} Y - \frac{(E_2 r)}{4n}\zeta. \end{aligned} \tag{40}$$

Consider the inner product of (40) and we know ζ is Killing in K-cm ($\zeta r = 0$), hence we get

$$\begin{aligned} & g(R(\zeta, E_2)E_3, Df) - [2ng(\varphi E_2, E_3) - g(\varphi QE_2, E_3)] \\ & -\frac{(E_2 r)}{4n}\boldsymbol{\eta}(E_3) = 0. \end{aligned} \tag{41}$$

Using (13) in (41) gives

$$g((\nabla_{E_2}\varphi)E_3, Df) - [2ng(\varphi E_2, E_3) - g(\varphi QE_2, E_3)] \\ -\frac{(E_2 r)}{4n}\boldsymbol{\eta}(E_3) = 0. \tag{42}$$

Replacing E_2 by φE_2 and E_3 by φE_3 in (42), we get

$$g((\nabla_{\varphi E_2}\varphi)\varphi E_3, Df) - [2ng(\varphi E_2, E_3) - g(Q\varphi E_2, E_3)] = 0. \tag{43}$$

In view of (14), (42) and (43), we find

$$(\zeta f)g(E_2, E_3) - (E_2 f)\boldsymbol{\eta}(E_3) - (\zeta f)\boldsymbol{\eta}(E_2)\boldsymbol{\eta}(E_3) \\ -[2ng(\varphi E_2, E_3) - g(\varphi QE_2, E_3)] - [2ng(\varphi E_2, E_3) - g(Q\varphi E_2, E_3)] \\ -\frac{(E_2 r)}{4n}\boldsymbol{\eta}(E_3) = 0. \tag{44}$$

Replacing E_2 by φE_2 and E_3 by φE_3 in (44), we provide

$$(\zeta f)g(\varphi E_2, \varphi E_3) - 4ng(\varphi E_2, E_3) + g(Q\varphi E_2, E_3) - g(QE_2, \varphi E_3) = 0. \tag{45}$$

Interchanging E_1 and E_2 in (45) and then substructing from (45), we get

$$4ng(\varphi E_2, E_3) = g(Q\varphi E_2, E_3) + g(\varphi QE_2, E_3), \tag{46}$$

which implies

$$Q\varphi E_2 + \varphi QE_2 = 4n\varphi E_2. \tag{47}$$

Let $(u_i, \varphi u_i, \zeta)$, $i = 1, 2, ..., n$ be a φ-basis of M^{2n+1} such that $Qu_i = w_i u_i$. From which we deduce that $\varphi Qu_i = w_i \varphi u_i$. Setting u_i for E_2 in (47), we get $Q\varphi u_i = (4n - w_i)\varphi u_i$. Hence we have

$$r = g(Q\zeta, \zeta) + \sum_{i=1}^{n}[g(Qu_i, u_i) + g(Q\varphi u_i, \varphi u_i)] \tag{48}$$

$$= g(Q\zeta, \zeta) + \sum_{i=1}^{n}[w_i g(u_i, u_i) + (4n - w_i)g(\varphi u_i, \varphi u_i)] \tag{49}$$

$$= 2n(2n+1) = constant. \tag{50}$$

Contracting E_1 in (37) and using (48) entails that

$$S(E_2, Df) = 0. \tag{51}$$

In view of (13), (40) and (48), we obtain

$$(E_2 f)\zeta - (\zeta f)E_2 = -[2n\varphi E_2 - \varphi QE_2]. \tag{52}$$

From the above equation, we infer

$$(E_2 f)\eta(E_4) - (\zeta f)g(E_2, E_4) = [g(\varphi QE_2, E_4) - 2ng(\varphi E_2, E_4)]. \tag{53}$$

Interchanging E_2 and E_4 in (53) and adding with (53), we get

$$(E_2 f)\eta(E_4) + (E_4 f)\eta(E_2) - 2(\zeta f)g(E_2, E_4) \\ = [g(\varphi QE_2, E_4) + g(\varphi QE_4, E_2)]. \tag{54}$$

Using (47) in the foregoing equation, we get

$$(E_2 f)\eta(E_4) + (E_4 f)\eta(E_2) - 2(\zeta f)g(E_2, E_4) = 4ng(\varphi E_2, E_4). \tag{55}$$

Contracting E_2 and E_4 from the above equation, we get

$$\zeta f = 0. \tag{56}$$

Puting $E_4 = \zeta$ and using (56) in (55), we get

$$E_2 f = 0, \tag{57}$$

which implies f = constant. Hence equation (33) implies

$$S(E_1, E_2) = (\frac{r}{4n} + \alpha)g(E_1, E_2). \tag{58}$$

Thus, the manifold is an Einstein manifold.
Hence the result follows.

It is widely circulated that an Einstein compact K-contact manifold is a Sasakian manifold [8]. As a result, we have:

Corollary 3.2. *A compact Einstein K-contact manifold obeying gradient Schouten soliton is Sasakian.*

Proof of Theorem 1.3. From (1) and (2), we get

$$(\pounds_V g)(E_1, E_2) + \frac{2}{2n-1}S(E_1, E_2) + [2\alpha - \frac{r}{4n(2n-1)}]g(E_1, E_2) = 0, \tag{59}$$

which implies

$$g(\nabla_{E_1} V, E_2) + g(E_1, \nabla_{E_2} V) + \frac{2}{2n-1}S(E_1, E_2) + [2\alpha - \frac{r}{4n(2n-1)}]g(E_1, E_2) = 0. \tag{60}$$

Putting $V = \zeta$ and using (7) yields

$$2g(\varphi h E_1, E_2) - \frac{2}{2n-1}S(E_1, E_2) - [2\alpha - \frac{r}{4n(2n-1)}]g(E_1, E_2) = 0. \tag{61}$$

Setting $E_1 = E_2 = \zeta$ in the above equation provides

$$\alpha = \frac{2(n-1) + (1-8n)k - n\mu}{2(2n-1)}, \tag{62}$$

which is a constnat and the solitons are shrinking, steady and expanding for $2(n-1) + (1-8n)k - n\mu < 0$, $2(n-1) + (1-8n)k - n\mu = 0$ and $2(n-1) + (1-8n)k - n\mu > 0$, respectively.
Hence completes the proof.

Contracting (61), we get

$$\frac{2r}{2n-1} + (2n+1)[2\alpha - \frac{r}{4n(2n-1)}]. \tag{63}$$

Taking $n = 1$ and $r = 0$ in (63), we get $\alpha = 0$. Hence the solioton is steady. As a result, we have:

Corollary 3.3. *If a 3-dimensional (k, μ)-contact manifold admits a Ricci soliton, then the soliton is steady.*

Again, if g is a compact Ricci soliton, then from Theorem C, we get it is an Einstein manifold. It is well known that a non-Sasakian Einstein (k,μ)-manifold is 3-dimensional and flat. Hence, we have:

Corollary 3.4. *If a 3-dimensional compact non-Sasakian (k,μ)-contact manifold admits a Ricci soliton, then the manifold is flat.*

In particular, for $\mu = 0$, equation (62) implies

$$\alpha = \frac{2(n-1)+(1-8n)k}{2(2n-1)}. \tag{64}$$

Since $k < 1$, for non-Sasakian (k,μ)-manifold, then (64) implies $\alpha < 0$. Hence the soliton is shrinking. Therefore we have:

Corollary 3.5. *If an $N(k)$-contact metric manifold admits an almost Schouten soliton, then the soliton is shrinking.*

Proof of Theorem 1.4. From (3), we get

$$\nabla_{E_1}Df + QE_1 = (\frac{r}{4n} + \alpha)E_1. \tag{65}$$

As similar to the proof of Theorem 2, we get from the above equation

$$R(E_1,E_2)Df = \frac{E_1 r}{4n}E_2 - \frac{E_2 r}{4n}E_1 - (\nabla_{E_1}Q)E_2 + (\nabla_{E_2}Q)E_1. \tag{66}$$

Using (19) in (66) yields

$$R(E_1,E_2)Df = -(\nabla_{E_1}Q)E_2 + (\nabla_{E_2}Q)E_1. \tag{67}$$

Putting $E_1 = \zeta$ in (67) and using (21) and (23) provides

$$\begin{aligned} R(\zeta,E_2)Df &= -\mu[2(n-1)+\mu]h\varphi E_2 \\ &\quad + Q(\varphi+\varphi h)E_2 - 2nk(\varphi+\varphi h)E_2. \end{aligned} \tag{68}$$

Using (20) in (68) entails that

$$\begin{aligned} &k[(E_2 f)\zeta - (\zeta f)E_2] + \mu[g(hE_2,Df)\zeta - (\zeta f)hE_2] \\ &= -\mu[2(n-1)+\mu]h\varphi E_2 + Q(\varphi+\varphi h)E_2 - 2nk(\varphi+\varphi h)E_2. \end{aligned} \tag{69}$$

Consider the innerproduct of (69) with E_3, we deduce

$$\begin{aligned} &k[(E_2 f)\eta(E_3) - (\zeta f)g(E_2,E_3)] \\ &+\mu[g(hE_2,Df)\eta(E_3) - (\zeta f)g(hE_2,E_3)] \\ &= -\mu[2(n-1)+\mu]g(h\varphi E_2,E_3) \\ &+g(Q\varphi E_2,E_3) + g(Q\varphi hY,E_3) - 2nkg(\varphi E_2,E_3) - 2nkg(\varphi hE_2,E_3). \end{aligned} \tag{70}$$

Replacing E_2 by φE_2 and E_3 by φE_3 in (70), we obtain

$$\begin{aligned} &-k(\zeta f)g(\varphi E_2,\varphi E_3) - \mu(\zeta f)g(h\varphi E_2,E_3) \\ &= \mu[2(n-1)+\mu]g(E_2,\varphi E_3) - g(QE_2,\varphi E_3) + g(QhE_2,\varphi E_3) \\ &+2nkg(E_2,\varphi E_3) - 2nkg(hE_2,\varphi E_3). \end{aligned} \tag{71}$$

Interchanging E_2 and E_3 in (71) and substructing from (71) gives

$$2\mu[2(n-1)+\mu]g(E_2,\varphi E_3)+4nkg(E_2,\varphi E_3)$$
$$-S(E_2,\varphi E_3)+S(E_3,\varphi E_2)+S(hE_2,\varphi E_3)-S(hE_3,\varphi E_2)=0. \tag{72}$$

Using (16) in (72) yields

$$[2\mu-2(k-1)][2(n-1)+\mu]-2[2(n-1)-n\mu]+4nk=0. \tag{73}$$

Putting $E_3=\zeta$ in (70) entails that

$$k[E_2 f-(\zeta f)\boldsymbol{\eta}(E_2)]+\mu g(hE_2,Df)=0. \tag{74}$$

Replacing E_2 by hE_2 in (74), we infer

$$kg(hE_2,Df)-\mu(k-1)g(E_2,Df)+\mu(k-1)(\zeta f)\boldsymbol{\eta}(E_2)=0. \tag{75}$$

In view of (74) and (73), we find

$$[k^2+\mu^2(k-1)][E_2 f-(\zeta f)\boldsymbol{\eta}(E_2)]=0. \tag{76}$$

Contracting E_1 in (67) and using (19) gives

$$S(E_2,Df)=0. \tag{77}$$

Replacing E_3 by Df in (16) and comparing with (77), we get

$$[2(n-1)-n\mu]E_2 f+[2(n-1)+\mu]g(hE_2,Df)$$
$$+[2(1-n)+n(2k+\mu)](\zeta f)\boldsymbol{\eta}(E_2)=0. \tag{78}$$

Putting $E_2=\zeta$ in (78) yields

$$k(\zeta f)=0, \tag{79}$$

which implies either $k=0$ or, $k\neq 0$.

Case I: If $k=0$, then (76) implies
(i) either $\mu=0$ which means the manifold is flat for $n=1$ and $n>1$ it is locally isometric to $E^{n+1}\times S^n(4)$
(ii) or, $E_2 f-(\zeta f)\boldsymbol{\eta}(E_2)=0$, that is, *grad f* is pointwise collinear with ζ.

Case II: If $k\neq 0$, then (79) implies $\zeta f=0$. Hence (76) implies

$$E_2 f=0 \ \ for\ k^2+\mu^2(k-1)\neq 0. \tag{80}$$

The above equation implies $f=$ constant. Hence (65), we get

$$S(E_1,E_2)=(\frac{r}{4n}+\boldsymbol{\alpha})g(E_1,E_2), \tag{81}$$

which is an Einstein manifold.
Hence the proof is completed.

In particular, for $\mu=0$ and $k\neq 0$, (76) implies

$$E_2 f-(\zeta f)\boldsymbol{\eta}(E_2)=0. \tag{82}$$

Again, from (79) we get

$$\zeta f = 0. \tag{83}$$

Above two equations together imply $E_2 f = 0$, which means that f = constant. Hence, (65) implies

$$S(E_1, E_2) = (\frac{r}{4n} + \boldsymbol{\alpha})g(E_1, E_2), \tag{84}$$

which is an Einstein manifold. Hence from Theorem A, we get it is a Sasakian manifold. Therefore we have:

Corollary 3.6. *If an $N(k)$-contact metric manifold admits a gradient Schouten soliton, then the manifold becomes an Einstein and Sasakian.*

Again, if we put $r = 0$ in (84), we get

$$S(E_1, E_2) = \boldsymbol{\alpha} g(E_1, E_2), \tag{85}$$

which is an Einstein manifold and after contracting the above equation, we get $\boldsymbol{\alpha} = 0$. Hence the soliton is steady. Therefore, from Theorem A, we have:

Corollary 3.7. *If an $N(k)$-contact metric manifold admits a gradient Ricci soliton, then the soliton is steady and the manifold is an Einstein and Sasakian.*

From Theorem B and the above corollary, we have:

Corollary 3.8. *If a compact $N(k)$-contact metric manifold admits a Ricci soliton, then the soliton is steady and the manifold is an Einstein Sasakian.*

Proof of Theorem 1.5. From equations (1) and (2), we get

$$(\pounds_V g)(E_1, E_2) + \frac{2}{2n-1}S(E_1, E_2) + [2\boldsymbol{\alpha} - \frac{r}{4n(2n-1)}]g(E_1, E_2) = 0, \tag{86}$$

which implies

$$g(\nabla_{E_1} V, E_2) + g(E_1, \nabla_{E_2} V) + \frac{2}{2n-1}S(E_1, E_2) + [2\boldsymbol{\alpha} - \frac{r}{4n(2n-1)}]g(E_1, E_2) = 0. \tag{87}$$

Putting $V = b\zeta$ in the foregoing equation gives

$$\begin{aligned}&(E_1 b)\boldsymbol{\eta}(E_2) + (E_2 b)\boldsymbol{\eta}(E_1) + 2bg(h\varphi E_1, E_2) + \frac{2}{2n-1}S(E_1, E_2)\\&+[2\boldsymbol{\alpha} - \frac{r}{4n(2n-1)}]g(E_1, E_2) = 0.\end{aligned} \tag{88}$$

Above equation implies

$$\begin{aligned}&(E_1 b)\zeta + (Db)\boldsymbol{\eta}(E_1) + 2bh\varphi E_1 + \frac{2}{2n-1}QE_1\\&+[2\boldsymbol{\alpha} - \frac{r}{4n(2n-1)}]E_1 = 0.\end{aligned} \tag{89}$$

Contracting (88), we infer

$$\zeta b = -\boldsymbol{\alpha}(2n+1) - \frac{(6n-1)r}{8n(2n-1)}. \tag{90}$$

Differentiating (89), we obtain

$$
\begin{aligned}
&(E_2(E_1b))\zeta - (\varphi E_2 + \varphi hE_2)E_1b - [g(\varphi E_2, E_1) + g(\varphi hE_2, E_1)]Db \\
&+\boldsymbol{\eta}(E_1)\nabla_{E_2}Db + 2(E_2b)\varphi hE_1 + 2b(\nabla_{E_2}\varphi h)E_1 \\
&+\frac{2}{2n-1}(\nabla_{E_2}Q)E_1 + [2\boldsymbol{\alpha} - \frac{r}{4n(2n-1)}]\nabla_{E_2}E_1 \\
&-\frac{(E_2r)}{4n(2n-1)}E_1 + 2(E_2\boldsymbol{\alpha})E_1 = 0.
\end{aligned} \tag{91}
$$

Contracting E_2 from (91) and using (9) entails that

$$
\begin{aligned}
&\zeta(E_1b) + [g(\varphi E_1, Db) - g(\varphi hE_1, Db)] - \boldsymbol{\eta}(E_1)\triangle b \\
&+2bg(\varphi E_1, Db) + 2b[g(Q\zeta, E_1) - 2n\boldsymbol{\eta}(E_1)] + \frac{4n-1}{4n(2n-1)}E_1r + 2(E_1\boldsymbol{\alpha}) = 0,
\end{aligned} \tag{92}
$$

where $\triangle b = -div(Db)$.
Putting $E_1 = \zeta$ in (92) and using (8), we find

$$\zeta(\zeta b) - \triangle b - 2b(||h||^2) + \frac{4n-1}{4n(2n-1)}(\zeta r) + 2(\zeta\boldsymbol{\alpha}) = 0. \tag{93}$$

From (90), we get

$$\zeta(\zeta b) = -(\zeta\boldsymbol{\alpha})(2n+1) - \frac{6n-1}{8n(2n-1)}(\zeta r). \tag{94}$$

The above two equations together imply

$$\triangle b = -2b(||h||^2), \tag{95}$$

where we take $\zeta r + 8n(1-2n)\zeta\boldsymbol{\alpha} = 0$.

Now,

$$
\begin{aligned}
\triangle(b^2) &= -\sum_{i=1}^{2n+1} g(\nabla_{ei}Db^2, u_i) \\
&= -2(||Db^2||) + 2b\triangle b.
\end{aligned} \tag{96}
$$

From the above two equations we find

$$\triangle(b^2) = -2(||Db^2||) - 4(||h||^2)b^2. \tag{97}$$

The divergence theorem and integration of the preceding equation over the compact M^{2n+1} lead us to the conclusions that b is a nonzero constant and $h = 0$, that is, M^{2n+1} is K-contact. As $V = b\zeta$ is Killing, equation (86) implies $\frac{2}{2n-1}S(E_1, E_2) = -[2\boldsymbol{\alpha} - \frac{r}{2n(2n-1)}]g(E_1, E_2)$, which is an Einstein manifold. From [8], we know that a compact Einstein K-contact manifold is Sasakian. Hence M^{2n+1} is Sasakian. Therefore, the proof is completed.

Finally, we investigate almost Schouten solitons on H-contact manifolds. A H-contact manifold [24] is a contact manifold with harmonic Reeb vector field.

Proof of Theorem 1.6. Let M^{2n+1} be H-contact, then $Q\zeta = (tr.l)\zeta$ [24]. Substituting $E_1 = E_2 = \zeta$ in (88), we get

$$\zeta b = -\frac{1}{2n-1}(2n - ||h||^2) - \frac{1}{2}[2\boldsymbol{\alpha} - \frac{r}{4n(2n-1)}]. \tag{98}$$

Putting $E_1 = \zeta$ in (89) and using (98) entails that

$$E_1 b = -[\frac{1}{2n-1}(2n - ||h||^2) + \frac{1}{2}(2\boldsymbol{\alpha} - \frac{r}{4n(2n-1)})]\boldsymbol{\eta}(E_1), \tag{99}$$

which implies

$$db = -[\frac{1}{2n-1}(2n - ||h||^2) + \frac{1}{2}(2\boldsymbol{\alpha} - \frac{r}{4n(2n-1)})]\eta. \tag{100}$$

Applying d on (100) and using ($d^2 = 0$) provides

$$d[\frac{1}{2n-1}(2n - ||h||^2) + \frac{1}{2}(2\boldsymbol{\alpha} - \frac{r}{4n(2n-1)})] \wedge \eta \tag{101}$$

$$+[\frac{1}{2n-1}(2n - ||h||^2) + \frac{1}{2}(2\boldsymbol{\alpha} - \frac{r}{4n(2n-1)})]d\eta = 0. \tag{102}$$

Taking exterior product of (101) with η, we find

$$[\frac{1}{2n-1}(2n - ||h||^2) + \frac{1}{2}(2\boldsymbol{\alpha} - \frac{r}{4n(2n-1)})]\eta \wedge d\eta = 0. \tag{103}$$

By definition of contact manifold $\eta \wedge d\eta \neq 0$, hence (103) implies

$$\frac{1}{2n-1}(2n - ||h||^2) + \frac{1}{2}(2\boldsymbol{\alpha} - \frac{r}{4n(2n-1)}) = 0. \tag{104}$$

Using (104) in (99), we get $E_1 b = 0$, which implies $b =$ constant. Since b is nonzero constant function, then (95) implies $h = 0$. Hence it is K-contact. Also, since $V = b\zeta$ is Killing, equation (86) reduces to Einstein manifold. From [18], it is known that a complete Einstein K-contact manifold is compact. Hence M^{2n+1} is compact Einstein and Sasakian. Therefore, the result follows.

Conclusions

A K-contact manifold is an initial idea between a contact and a Sasakian manifold. As for example, A Sasakian structure is the normal contact metric structure on an odd-dimensional sphere. A K-contact structure is carried by a compact regular contact manifold.

In their purest form, solitons are nothing more than waves. After colliding with another wave of the same kind, waves physically propagate with the least amount of energy loss and maintain their speed and shape. Solitons play a key role in the resolution of initial-value problems for wave propagation-related nonlinear PDEs. Furthermore, it explains the Fermi-Pasta-Ulam system's [22] recurrence.

The several researchers have studied different types of solitons in contact geometry. In this study, we investigate the almost Schouten solitons and gradient Schouten solitons in contact geometry.

Here we show that if a compact K-contact manifold admits an almost Schouten soliton, then the soliton is shrinking and the manifold becomes an Einstein manifold. Amongothers, if a K-contact manifold admits a gradient Schouten soliton, then the manifold becomes an Einstein manifold. Next, we prove that if a (k, μ)-contact manifold admits an almost Schouten soliton, then the soliton becomes Schouten soliton and the soliton is shrinking, steady and expanding for $2(n-1) + (1-8n)k - n\mu < 0$, $2(n-1) + (1-8n)k - n\mu = 0$ and $2(n-1) + (1-8n)k - n\mu > 0$, respectively. Lastly, it is prove that if a complete H-contact manifold admits an almost Schouten soliton with nonzero potential vector field V collinear with ζ, then the manifold is compact Einstein and Sasakian under a restriction of potential function and scalar curvature.

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