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On the (α, β)-Euclidean operator radius and its applications

Messaoud Guesba[a,*], Sid Ahmed Ould Ahmed Mahmoud[b]

[a]*Faculty of Exact Sciences, Department of Mathematics, El Oued University, 39000 Algeria*
[b]*Mathematics Department, College of Science, Jouf University, Sakaka P.O.Box 2014, Saudi Arabia*

Abstract. Our aim in this paper is to introduce a new norm of n-tuple operators which generalizes the (α, β)-norm on the space of all bounded linear operators on a complex Hilbert space due to Sain et al. (Ann. Funct. Anal. 12:51 (2021)). We introduce and study basic properties of this norm. As an application of the present study, we estimate bounds for the Euclidean operator radius (joint numerical radius) of bounded linear operators. Also, we improve on some of the important existing Euclidean operator radius inequalities.

1. Introduction and Preliminaries

Throughout this paper, $\mathcal{H}$ denotes a non trivial complex Hilbert space with inner product $\langle .,. \rangle$ and associated norm $\|.\|$. Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on $\mathcal{H}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and we denote it $A \geq 0$. The absolute value of A is denoted by $|A|$, and let $|A| = (A^*A)^{\frac{1}{2}}$, where A^* stands for the adjoint of A.

For $A \in \mathcal{B}(\mathcal{H})$, the numerical range of A is defined by

$$W(A) = \{\langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1\},$$

while the numerical radius is defined as

$$\omega(A) = \sup\{|\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$

For more facts about the numerical radius of operators, the reader is referred to see [2, 4, 6, 7, 9, 10, 14, 16, 22, 25, 27] and the references therein.

It is well known that $\omega(.)$ defines a norm on $\mathcal{B}(\mathcal{H})$. Moreover, for all $A \in \mathcal{B}(\mathcal{H})$ we have

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{1}$$

If $A, B \in \mathcal{B}(\mathcal{H})$, then

$$\omega(AB) \leq 4\omega(A)\,\omega(B), \tag{2}$$

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* Corresponding author: Messaoud Guesba
Email addresses: guesbamessaoud2@gmail.com, guesba-messaoud@univ-eloued.dz (Messaoud Guesba), sidahmed@ju.edu.sa (Sid Ahmed Ould Ahmed Mahmoud)

(see [14]).
In the case that $AB = BA$, then

$$\omega(AB) \leq 2\omega(A)\omega(B).$$

Some interesting numerical radius inequalities improving inequalities (1) have been obtained by several mathematicians (see, e.g., [1, 11, 19, 20]).
In [18], Kittaneh improved the second inequality in (1), and obtained the following inequality:

$$w(A) \leq \frac{1}{2}\| |A| + |A^*| \| \leq \|A\|.$$

Another refinement of the second inequality in (1) has been established by Kittaneh in [19]. This refinement asserts that if $A \in \mathcal{B}(\mathcal{H})$, then

$$w^2(A) \leq \frac{1}{2}\| |A|^2 + |A^*|^2 \|. \tag{3}$$

After that, in [12] El-Haddad and Kittaneh generalized (3) as follows

$$w^{2r}(A) \leq \frac{1}{2}\| |A|^{2r} + |A^*|^{2r} \|, \text{ for all } r \geq 1. \tag{4}$$

Recently, Abu-Omar et. al. in [1] proved that for $A \in \mathcal{B}(\mathcal{H})$,

$$\omega^2(A) \leq \frac{1}{4}\| |A|^2 + |A^*|^2 \| + \frac{1}{2}\omega\left(A^2\right). \tag{5}$$

In 2021, Bhunia et. al. in [3] proved that for $A \in \mathcal{B}(\mathcal{H})$,

$$w^2(A) \leq \frac{1}{4}\| |A|^2 + |A^*|^2 \| + \frac{1}{2}\omega(|A|\,|A^*|). \tag{6}$$

In [10], Dragomir established an upper bound for the numerical radius of the product of two Hilbert space operators $A, B \in \mathcal{B}(\mathcal{H})$, as follows:

$$w^r(B^*A) \leq \frac{1}{2}\| |A|^{2r} + |B|^{2r} \|, \text{ for all } r \geq 1. \tag{7}$$

In particular, for $r = 1$, we have

$$w(B^*A) \leq \frac{1}{2}\| |A|^2 + |B|^2 \|. \tag{8}$$

For every $n \in \mathbb{N}$, let $\mathcal{B}(\mathcal{H})^n$ denote the product of n-copies of $\mathcal{B}(\mathcal{H})$, i.e.,

$$\mathcal{B}(\mathcal{H})^n := \{\mathbf{T} = (T_1, ..., T_n) : T_1, ..., T_n \in \mathcal{B}(\mathcal{H})\}.$$

The joint operator norm of an n-tuple $\mathbf{T} = (T_1, ..., T_n) \in \mathcal{B}(\mathcal{H})^n$ was defined in [23] by

$$\|\mathbf{T}\| := \sup\left\{\left(\sum_{k=1}^{n} \|T_k x\|^2\right)^{\frac{1}{2}} : x \in \mathcal{H},\ \|x\| = 1\right\}.$$

The Euclidean operator radius of an n-tuple $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ was defined in [6] by

$$\omega_e(\mathbf{T}) := \sup\left\{\left(\sum_{k=1}^{n} |\langle T_k x, x\rangle|^2\right)^{\frac{1}{2}} : x \in \mathcal{H},\ \|x\| = 1\right\}.$$

If $n = 1$, we get the classical numerical radius.

The joint Crawford number of $\mathbf{T} = (T_1, \cdots, T_n)$ is defined [15] as

$$c_e(\mathbf{T}) := \inf\left\{\left(\sum_{k=1}^{n} |\langle T_k x, x\rangle|^2\right)^{\frac{1}{2}} : x \in \mathcal{H},\ \|x\| = 1\right\}.$$

The adjoint of $\mathbf{T}$ is given by $\mathbf{T}^* = \left(T_1^*, \cdots, T_n^*\right)$, where T_j^* is the adjoint of T_j, $j = 1, \cdots, n$. For $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ and $\mathbf{S} = (S_1, \cdots, S_n) \in \mathcal{B}(\mathcal{H})^n$ we denote $\mathbf{T} + \mathbf{S} := (T_1 + S_1, \cdots, T_n + S_n)$, $\mathbf{TS} := (T_1 S_1, \cdots, T_n S_n)$, $\lambda\mathbf{T} := (\lambda T_1, \cdots, \lambda T_n)$ for any scalar $\lambda \in \mathbb{C}$, and $|\mathbf{T}| := (|T_1|, \cdots, |T_n|)$ where $|T_k| = \left(T_k^* T_k\right)^{\frac{1}{2}}$ for each $k = 1, 2, ..., n$.

In [24] Sain et al. introduced a new norm called (α, β)-norm defined on $\mathcal{B}(\mathcal{H})$ as

$$\|T\|_{\alpha,\beta} := \sup\left\{\left(\left(\alpha\,|\langle Tx, x\rangle|^2 + \beta\,\|Tx\|^2\right)\right)^{\frac{1}{2}} : x \in \mathcal{H},\ \|x\| = 1\right\},$$

where $\alpha, \beta \geq 0$ such that $(\alpha, \beta) \neq (0, 0)$.

Our goal in this paper is to define a new norm on $\mathcal{B}(\mathcal{H})^n$ which generalizes (α, β)-norm on $\mathcal{B}(\mathcal{H})$. This work is organized as follows: In Sect. 2, we collect a few lemmas that are required to state and prove the results in the subsequent sections. In Sect. 3, we introduce and study basic properties of this norm. As an application of the present study, we estimate bounds for the Euclidean operator radius of bounded linear operators. Also, we improve on some of the important existing Euclidean operator radius inequalities.

2. Auxiliary lemmas

In this section, we present the following lemmas that will be used to develop new results in this paper.

Lemma 2.1. *[5] Let $x, y, z \in \mathcal{H}$ with $\|z\| = 1$. Then*

$$\left|\langle x, z\rangle\langle z, y\rangle\right| \leq \frac{1}{2}\left(\|x\|\left\|y\right\| + \left|\langle x, y\rangle\right|\right).$$

Lemma 2.2. *[26] Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

(i) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ for all $r \geq 1$.
(ii) $\langle Ax, x\rangle^r \geq \langle A^r x, x\rangle$ for all $0 < r \leq 1$.

Lemma 2.3. *[17] Let $A \in \mathcal{B}(\mathcal{H})$ and let f and g be non-negative continuous functions on $[0, \infty)$ such that $f(t)\,g(t) = t$ for all $t \in [0, \infty)$. Then*

$$\left|\langle Ax, y\rangle\right|^2 \leq \left\langle f^2(|A|)\,x, x\right\rangle\left\langle g^2(|A^*|)\,y, y\right\rangle,$$

for all $x, y \in \mathcal{H}$.

In particular, if $f(t) = g(t) = \sqrt{t}$, then we have

$$\left|\langle Ax, y\rangle\right|^2 \leq \langle |A|\,x, x\rangle\langle |A^*|\,y, y\rangle.$$

Lemma 2.4. *[21] Let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint operators. Then*

$$\omega^2(A + iB) \leq \left\|A^2 + B^2\right\|.$$

Lemma 2.5. *[23] Let $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ be an n-tuple of operators. Then*

$$\frac{1}{2\sqrt{n}} \|\mathbf{T}\| \leq \omega_e(\mathbf{T}) \leq \|\mathbf{T}\|.$$

Lemma 2.6. *[15] Let $\mathbf{T} = (T_1, \cdots, T_n), \mathbf{S} = (S_1, \cdots, S_n) \in \mathcal{B}(\mathcal{H})^n$. Then*

$$\|\mathbf{TS}\| \leq \|\mathbf{T}\| \|\mathbf{S}\|.$$

Lemma 2.7. *[15] Let $\mathbf{T} = (T_1, \cdots, T_n), \mathbf{S} = (S_1, \cdots, S_n) \in \mathcal{B}(\mathcal{H})^n$. Then*

$$\omega_e(\mathbf{TS}) \leq 4n\omega_e(\mathbf{T})\omega_e(\mathbf{S}).$$

Lemma 2.8. *[15] Let $\mathbf{T} = (T_1, \cdots, T_n), \mathbf{S} = (S_1, \cdots, S_n) \in \mathcal{B}(\mathcal{H})^n$. If $\mathbf{TS} = \mathbf{ST}$ (i.e, $T_kS_k = S_kT_k$ for each $k = 1, 2, ..., n$), then*

$$\omega_e(\mathbf{TS}) \leq 2\sqrt{n}\omega_e(\mathbf{T})\omega_e(\mathbf{S}).$$

We give now the following definition.

Definition 2.9. *An n-tuple of operators $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ is said to be isometric if $T_k^*T_k = I$ for each $k = 1, 2, ..., n$.*

Next, we state the following lemma which can be easily verified.

Lemma 2.10. *Let $\mathbf{T} = (T_1, \cdots, T_n), \mathbf{S} = (S_1, \cdots, S_n) \in \mathcal{B}(\mathcal{H})^n$. If $\mathbf{T}$ is an n-tuple isometry operator , then*

(i) $\omega_e(\mathbf{TS}) \leq \|\mathbf{S}\|$.
(ii) $\|\mathbf{TS}\| \leq \|\mathbf{S}\|$.

3. Main results

In this section, we present our results.

Let $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ be an n-tuple of operators and let $\alpha, \beta \geq 0$ such that $(\alpha, \beta) \neq (0, 0)$. Consider a mapping $\|\cdot\|_{\alpha,\beta} : \mathcal{B}(\mathcal{H})^n \to \mathbb{R}^+$ defined as follows:

$$\|\mathbf{T}\|_{\alpha,\beta} := \sup\left\{\left(\sum_{k=1}^{n}\left(\alpha\left|\langle T_kx, x\rangle\right|^2 + \beta\|T_kx\|^2\right)\right)^{\frac{1}{2}} : x \in \mathcal{H},\ \|x\| = 1\right\}.$$

Remark 3.1. *(i) If $\alpha = 0, \beta = 1$, then $\|\mathbf{T}\|_{\alpha,\beta} = \|\mathbf{T}\|$.*

(ii) If $\alpha = 1, \beta = 0$, then $\|\mathbf{T}\|_{\alpha,\beta} = \omega_e(\mathbf{T})$.

In the following proposition, we show that $\|\cdot\|_{\alpha,\beta}$ defnes a norm on $\mathcal{B}(\mathcal{H})^n$.

Proposition 3.2. *Let $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ and $\mathbf{S} = (S_1, \cdots, S_n) \in \mathcal{B}(\mathcal{H})^n$. Then, the following properties hold:*

(1) $\|\mathbf{T}\|_{\alpha,\beta} = 0$ if and only if $\mathbf{T} = 0$, i.e, $T_k = 0$ for each $k = 1, 2, ..., n$.
(2) $\|\lambda\mathbf{T}\|_{\alpha,\beta} = |\lambda| \|\mathbf{T}\|_{\alpha,\beta}$, for all $\lambda \in \mathbb{C}$.
(3) $\|\mathbf{T} + \mathbf{S}\|_{\alpha,\beta} \leq \|\mathbf{T}\|_{\alpha,\beta} + \|\mathbf{S}\|_{\alpha,\beta}$.

Proof. (1) and (2) obvious.

(3) Let $x \in \mathcal{H}$ with $\|x\| = 1$. We have

$$\begin{aligned}
& \sum_{k=1}^{n} \left(\alpha \left| \langle (T_k + S_k) x, x \rangle \right|^2 + \beta \left\| (T_k + S_k) x \right\|^2 \right) \\
\leq & \sum_{k=1}^{n} \left(\alpha \left(|\langle T_k x, x \rangle| + |\langle S_k x, x \rangle| \right)^2 + \beta \left(\|T_k x\| + \|S_k x\| \right)^2 \right) \\
= & \sum_{k=1}^{n} \alpha \left(|\langle T_k x, x \rangle|^2 + |\langle S_k x, x \rangle|^2 + 2 |\langle T_k x, x \rangle| |\langle S_k x, x \rangle| \right) + \sum_{k=1}^{n} \beta \left(\|T_k x\|^2 + \|S_k x\|^2 + 2 \|T_k x\| \|S_k x\| \right) \\
= & \sum_{k=1}^{n} \left(\alpha |\langle T_k x, x \rangle|^2 + \beta \|T_k x\|^2 \right) + \sum_{k=1}^{n} \left(\alpha |\langle S_k x, x \rangle|^2 + \beta \|S_k x\|^2 \right) + 2 \sum_{k=1}^{n} \left(\alpha |\langle T_k x, x \rangle| |\langle S_k x, x \rangle| + \beta \|T_k x\| \|S_k x\| \right) \\
= & \sum_{k=1}^{n} \left(\alpha |\langle T_k x, x \rangle|^2 + \beta \|T_k x\|^2 \right) + \sum_{k=1}^{n} \left(\alpha |\langle S_k x, x \rangle|^2 + \beta \|S_k x\|^2 \right) \\
& + 2 \sum_{k=1}^{n} \left(\left| \left\langle \sqrt{\alpha} T_k x, x \right\rangle \right| \left| \left\langle \sqrt{\alpha} S_k x, x \right\rangle \right| + \left\| \sqrt{\beta} T_k x \right\| \left\| \sqrt{\beta} S_k x \right\| \right) \\
\leq & \sum_{k=1}^{n} \left(\alpha |\langle T_k x, x \rangle|^2 + \beta \|T_k x\|^2 \right) + \sum_{k=1}^{n} \left(\alpha |\langle S_k x, x \rangle|^2 + \beta \|S_k x\|^2 \right) \\
& + 2 \sum_{k=1}^{n} \sqrt{\alpha |\langle T_k x, x \rangle|^2 + \beta \|T_k x\|^2} \sqrt{\alpha |\langle S_k x, x \rangle|^2 + \beta \|S_k x\|^2} \\
\leq & \sum_{k=1}^{n} \left(\alpha |\langle T_k x, x \rangle|^2 + \beta \|T_k x\|^2 \right) + \sum_{k=1}^{n} \left(\alpha |\langle S_k x, x \rangle|^2 + \beta \|S_k x\|^2 \right) \\
& + 2 \left(\sum_{k=1}^{n} \left(\alpha |\langle T_k x, x \rangle|^2 + \beta \|T_k x\|^2 \right) \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \left(\alpha |\langle S_k x, x \rangle|^2 + \beta \|S_k x\|^2 \right) \right)^{\frac{1}{2}} \\
\leq & \|\mathbf{T}\|_{\alpha,\beta}^2 + \|\mathbf{S}\|_{\alpha,\beta}^2 + 2 \|\mathbf{T}\|_{\alpha,\beta} \|\mathbf{S}\|_{\alpha,\beta} \\
= & \left(\|\mathbf{T}\|_{\alpha,\beta} + \|\mathbf{S}\|_{\alpha,\beta} \right)^2 .
\end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$\|\mathbf{T} + \mathbf{S}\|_{\alpha,\beta}^2 \leq \left(\|\mathbf{T}\|_{\alpha,\beta} + \|\mathbf{S}\|_{\alpha,\beta} \right)^2 .$$

Thus,

$$\|\mathbf{T} + \mathbf{S}\|_{\alpha,\beta} \leq \|\mathbf{T}\|_{\alpha,\beta} + \|\mathbf{S}\|_{\alpha,\beta} .$$

□

Now, we state the following result, proof of which follows easily.

Proposition 3.3. *Let $S \in \mathcal{B}(\mathcal{H})$ and consider the n-tuple $\mathbf{T} = (S, \cdots, S) \in \mathcal{B}(\mathcal{H})^n$. Then*

$$\|\mathbf{T}\|_{\alpha,\beta} = \sqrt{n} \|S\|_{\alpha,\beta} .$$

The next theorem shows that $\|\cdot\|_{\alpha,\beta}$ is equivalent to the Euclidean operator radius and the joint operator norm on $\mathcal{B}(\mathcal{H})^n$ satisfying the inequalities (i) and (ii).

Theorem 3.4. *Let* $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$. *Then*

(i) $\sqrt{\alpha+\beta}\omega_e(\mathbf{T}) \leq \|\mathbf{T}\|_{\alpha,\beta} \leq \sqrt{\alpha+4\beta n}\omega_e(\mathbf{T})$.

(ii) $\max\left\{\sqrt{\beta}, \frac{1}{2}\sqrt{\frac{\alpha+\beta}{n}}\right\}\|\mathbf{T}\| \leq \|\mathbf{T}\|_{\alpha,\beta} \leq \sqrt{\alpha+\beta}\|\mathbf{T}\|$.

Proof. (i) Let $x \in \mathcal{H}$ with $\|x\| = 1$. We have

$$\begin{aligned}
\|\mathbf{T}\|^2_{\alpha,\beta} &= \sup_{\|x\|=1}\sum_{k=1}^{n}\left(\alpha|\langle T_kx, x\rangle|^2 + \beta\|T_kx\|^2\right)\\
&\leq \sup_{\|x\|=1}\left(\sum_{k=1}^{n}\alpha|\langle T_kx, x\rangle|^2 + \sum_{k=1}^{n}\beta\|T_kx\|^2\right)\\
&\leq \sup_{\|x\|=1}\left(\sum_{k=1}^{n}\alpha|\langle T_kx, x\rangle|^2\right) + \sup_{\|x\|=1}\left(\sum_{k=1}^{n}\beta\|T_kx\|^2\right)\\
&= \alpha\omega_e^2(\mathbf{T}) + \beta\|\mathbf{T}\|^2\\
&\leq \alpha\omega_e^2(\mathbf{T}) + 4\beta n\omega_e^2(\mathbf{T})\\
&\quad \text{(by Lemma 2.5)}\\
&= (\alpha+4\beta n)\,\omega_e^2(\mathbf{T}).
\end{aligned}$$

Thus,

$$\|\mathbf{T}\|_{\alpha,\beta} \leq \sqrt{\alpha+4\beta n}\omega_e(\mathbf{T}).$$

Now, we have

$$\begin{aligned}
\|\mathbf{T}\|_{\alpha,\beta} &= \sup_{\|x\|=1}\left(\sum_{k=1}^{n}\alpha|\langle T_kx, x\rangle|^2 + \beta\|T_kx\|^2\right)^{\frac{1}{2}}\\
&\geq \sup_{\|x\|=1}\left(\sum_{k=1}^{n}\alpha|\langle T_kx, x\rangle|^2 + \beta|\langle T_kx, x\rangle|^2\right)^{\frac{1}{2}}\\
&= \sup_{\|x\|=1}\left(\sum_{k=1}^{n}(\alpha+\beta)|\langle T_kx, x\rangle|^2\right)^{\frac{1}{2}}\\
&= \sqrt{\alpha+\beta}\omega_e(\mathbf{T}).
\end{aligned}$$

Therefore,

$$\sqrt{\alpha+\beta}\omega_e(\mathbf{T}) \leq \|\mathbf{T}\|_{\alpha,\beta} \leq \sqrt{\alpha+4\beta n}\omega_e(\mathbf{T}).$$

(ii) Let $x \in \mathcal{H}$ with $\|x\| = 1$. We have

$$\begin{aligned}
\|\mathbf{T}\|_{\alpha,\beta} &= \sup_{\|x\|=1}\left(\sum_{k=1}^{n}\left(\alpha|\langle T_kx, x\rangle|^2 + \beta\|T_kx\|^2\right)\right)^{\frac{1}{2}}\\
&\leq \sup_{\|x\|=1}\left(\sum_{k=1}^{n}\left(\alpha\|T_kx\|^2 + \beta\|T_kx\|^2\right)\right)^{\frac{1}{2}}\\
&= \sup_{\|x\|=1}\left(\sum_{k=1}^{n}(\alpha+\beta)\|T_kx\|^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$= \sqrt{\alpha+\beta}\,\|\mathbf{T}\|.$$

Thus,

$$\|\mathbf{T}\|_{\alpha,\beta} \leq \sqrt{\alpha+\beta}\,\|\mathbf{T}\|.$$

Moreover, we have

$$\begin{aligned}\|\mathbf{T}\|_{\alpha,\beta} &\geq \sqrt{\alpha+\beta}\,\omega_e(\mathbf{T}) \\ &\geq \frac{1}{2\sqrt{n}}\sqrt{\alpha+\beta}\,\|\mathbf{T}\| \\ &= \frac{1}{2}\sqrt{\frac{\alpha+\beta}{n}}\,\|\mathbf{T}\|.\end{aligned}$$

Thus,

$$\|\mathbf{T}\|_{\alpha,\beta} \geq \frac{1}{2}\sqrt{\frac{\alpha+\beta}{n}}\,\|\mathbf{T}\|. \tag{9}$$

Also, it is easy to proof

$$\|\mathbf{T}\|_{\alpha,\beta} \geq \sqrt{\beta}\,\|\mathbf{T}\|. \tag{10}$$

From the inequalities (9) and (10), we obtain

$$\max\left\{\sqrt{\beta}, \frac{1}{2}\sqrt{\frac{\alpha+\beta}{n}}\right\}\|\mathbf{T}\| \leq \|\mathbf{T}\|_{\alpha,\beta}.$$

Therefore,

$$\max\left\{\sqrt{\beta}, \frac{1}{2}\sqrt{\frac{\alpha+\beta}{n}}\right\}\|\mathbf{T}\| \leq \|\mathbf{T}\|_{\alpha,\beta} \leq \sqrt{\alpha+\beta}\,\|\mathbf{T}\|.$$

This completes the proof. □

Remark 3.5. *Taking $n = 1$ in Theorem 3.4, we recapture [24, Theorem 2.1].*

The following proposition proves that the (α,β)-Euclidean operator radius is weakly unitarily invariant on $\mathcal{B}(\mathcal{H})^n$. It can be easily proved.

Proposition 3.6. *Let $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ be an n-tuple of operators and let $\mathbf{U} = (U_1, \cdots, U_n) \in \mathcal{B}(\mathcal{H})^n$ be a unitary of n-tuple of operators, i.e, U_k is a unitary for $k = 1, 2, ..., n$. Then, the (α,β)-Euclidean operator radius is weakly unitarily invariant i.e.,*

$$\|\mathbf{U}^*\mathbf{T}\mathbf{U}\|_{\alpha,\beta} = \|\mathbf{T}\|_{\alpha,\beta}.$$

Next, we obtain an upper bound for the (α,β)-norm of the product of two n-tuple operators.

Theorem 3.7. *Let $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$, $\mathbf{S} = (S_1, \cdots, S_n) \in \mathcal{B}(\mathcal{H})^n$ and let $\beta \neq 0$. Then*

$$\|\mathbf{TS}\|_{\alpha,\beta} \leq \min\left\{2\sqrt{\frac{n}{\beta}}, \frac{\sqrt{\alpha+\beta}}{\beta}, \frac{4n}{\sqrt{\alpha+\beta}}\right\}\|\mathbf{T}\|_{\alpha,\beta}\,\|\mathbf{S}\|_{\alpha,\beta}.$$

Proof. By using Theorem 3.4, it follows that

$$\begin{aligned}\|\mathbf{TS}\|_{\alpha,\beta} &\leq \sqrt{\alpha+\beta}\,\|\mathbf{TS}\| \\ &\leq \sqrt{\alpha+\beta}\,\|\mathbf{T}\|\,\|\mathbf{S}\| \\ &\quad \text{(by Lemma 2.6)} \\ &\leq 2\sqrt{n}\,\sqrt{\alpha+\beta}\,\omega_e(\mathbf{T})\,\|\mathbf{S}\| \\ &\quad \text{(by Lemma 2.5)} \\ &\leq 2\sqrt{\frac{n}{\beta}}\,\|\mathbf{T}\|_{\alpha,\beta}\,\|\mathbf{S}\|_{\alpha,\beta}.\end{aligned}$$

Thus,

$$\|\mathbf{TS}\|_{\alpha,\beta} \leq 2\sqrt{\frac{n}{\beta}}\,\|\mathbf{T}\|_{\alpha,\beta}\,\|\mathbf{S}\|_{\alpha,\beta}. \tag{11}$$

Since $\sqrt{\beta}\,\|\mathbf{T}\| \leq \|\mathbf{T}\|_{\alpha,\beta}$ and $\sqrt{\beta}\,\|\mathbf{S}\| \leq \|\mathbf{S}\|_{\alpha,\beta}$. Then

$$\begin{aligned}\|\mathbf{TS}\|_{\alpha,\beta} &\leq \sqrt{\alpha+\beta}\,\|\mathbf{TS}\| \\ &\leq \sqrt{\alpha+\beta}\,\|\mathbf{T}\|\,\|\mathbf{S}\| \\ &\quad \text{(by Lemma 2.6)} \\ &\leq \frac{\sqrt{\alpha+\beta}}{\beta}\,\|\mathbf{T}\|_{\alpha,\beta}\,\|\mathbf{S}\|_{\alpha,\beta}.\end{aligned}$$

Thus,

$$\|\mathbf{TS}\|_{\alpha,\beta} \leq \frac{\sqrt{\alpha+\beta}}{\beta}\,\|\mathbf{T}\|_{\alpha,\beta}\,\|\mathbf{S}\|_{\alpha,\beta}. \tag{12}$$

Now, since $\|\mathbf{T}\| \leq 2\sqrt{n}\omega_e(\mathbf{T})$ and $\|\mathbf{S}\| \leq 2\sqrt{n}\omega_e(\mathbf{S})$. Then

$$\begin{aligned}\|\mathbf{TS}\|_{\alpha,\beta} &\leq \sqrt{\alpha+\beta}\,\|\mathbf{TS}\| \\ &\leq \sqrt{\alpha+\beta}\,\|\mathbf{T}\|\,\|\mathbf{S}\| \\ &\leq 4n\sqrt{\alpha+\beta}\,\omega_e(\mathbf{T})\,\omega_e(\mathbf{S}) \\ &\leq \frac{4n}{\sqrt{\alpha+\beta}}\,\|\mathbf{T}\|_{\alpha,\beta}\,\|\mathbf{S}\|_{\alpha,\beta}.\end{aligned}$$

Thus,

$$\|\mathbf{TS}\|_{\alpha,\beta} \leq \frac{4n}{\sqrt{\alpha+\beta}}\,\|\mathbf{T}\|_{\alpha,\beta}\,\|\mathbf{S}\|_{\alpha,\beta}. \tag{13}$$

Combining (11), (12) and (13), we get the desired inequality. □

Remark 3.8. *If we take $n = 1$ in Theorem 3.7, then we obtain*

$$\|TS\|_{\alpha,\beta} \leq \min\left\{2\sqrt{\frac{1}{\beta}}, \frac{\sqrt{\alpha+\beta}}{\beta}, \frac{4}{\sqrt{\alpha+\beta}}\right\}\|T\|_{\alpha,\beta}\,\|S\|_{\alpha,\beta},$$

which has been given in [24, Theorem 2.9].

In the following theorem we obtain upper bounds for the (α,β)-norm of the product of two n-tuple operators under some assumptions.

Theorem 3.9. *Let* $\mathbf{T}=(T_1,\cdots,T_n)\in\mathcal{B}(\mathcal{H})^n$, $\mathbf{S}=(S_1,\cdots,S_n)\in\mathcal{B}(\mathcal{H})^n$. *Then*

(i) If $\mathbf{TS}=\mathbf{ST}$ (i.e, $T_kS_k=S_kT_k$ for each $k=1,2,...,n$) and $\beta\neq 0$, then

$$\|\mathbf{TS}\|_{\alpha,\beta}\leq\sqrt{\frac{4\alpha n}{(\alpha+\beta)^2}+\frac{1}{\beta}}\,\|\mathbf{T}\|_{\alpha,\beta}\,\|\mathbf{S}\|_{\alpha,\beta}\,.$$

(ii) If $\mathbf{T}$ is an n-tuple isometry operator (i.e., $T_k^*T_k=I$ for each $k=1,2,...,n$), then

$$\|\mathbf{TS}\|_{\alpha,\beta}\leq\sqrt{\frac{4\alpha n}{\alpha+\beta}+1}\,\|\mathbf{S}\|_{\alpha,\beta}\,.$$

Proof. (i) Let $x\in\mathcal{H}$ with $\|x\|=1$. From the defnition of (α,β)-norm on $\mathcal{B}(\mathcal{H})^n$, we have

$$\begin{aligned}
\|\mathbf{TS}\|_{\alpha,\beta}^2 &= \sup_{\|x\|=1}\sum_{k=1}^{n}\left(\alpha\,|\langle T_kS_kx,x\rangle|^2+\beta\,\|T_kS_kx\|^2\right)\\
&\leq \alpha\omega_e^2(\mathbf{TS})+\beta\,\|\mathbf{TS}\|^2\\
&\leq 4\alpha n\omega_e^2(\mathbf{T})\,\omega_e^2(\mathbf{S})+\beta\,\|\mathbf{T}\|^2\,\|\mathbf{S}\|^2\\
&\quad\text{(by Lemma 2.8)}\\
&\leq \frac{4\alpha n}{(\alpha+\beta)^2}\,\|\mathbf{T}\|_{\alpha,\beta}^2\,\|\mathbf{S}\|_{\alpha,\beta}^2+\frac{1}{\beta}\,\|\mathbf{T}\|_{\alpha,\beta}^2\,\|\mathbf{S}\|_{\alpha,\beta}^2\\
&= \left(\frac{4\alpha n}{(\alpha+\beta)^2}+\frac{1}{\beta}\right)\|\mathbf{T}\|_{\alpha,\beta}^2\,\|\mathbf{S}\|_{\alpha,\beta}^2\,.
\end{aligned}$$

(ii) Let $x\in\mathcal{H}$ with $\|x\|=1$. We have

$$\begin{aligned}
\|\mathbf{TS}\|_{\alpha,\beta}^2 &= \sup_{\|x\|=1}\sum_{k=1}^{n}\left(\alpha\,|\langle T_kS_kx,x\rangle|^2+\beta\,\|T_kS_kx\|^2\right)\\
&\leq \alpha\omega_e^2(\mathbf{TS})+\beta\,\|\mathbf{TS}\|^2\\
&\leq \alpha\,\|\mathbf{S}\|^2+\beta\,\|\mathbf{S}\|^2\\
&\quad\text{(by Lemma 2.10)}\\
&\leq 4\alpha n\omega_e^2(\mathbf{S})+\beta\,\|\mathbf{S}\|^2\\
&\leq \frac{4\alpha n}{\alpha+\beta}\,\|\mathbf{S}\|_{\alpha,\beta}^2+\|\mathbf{S}\|_{\alpha,\beta}^2\\
&= \left(\frac{4\alpha n}{\alpha+\beta}+1\right)\|\mathbf{S}\|_{\alpha,\beta}^2\,.
\end{aligned}$$

□

Next, we obtain a lower bound for the (α,β)-norm on $\mathcal{B}(\mathcal{H})^n$ which generalizes [24, Theorem 2.7].

Theorem 3.10. *Let* $\mathbf{T}=(T_1,\cdots,T_n)\in\mathcal{B}(\mathcal{H})^n$*be an n-tuple of operators. Then*

$$\|\mathbf{T}\|_{\alpha,\beta}^2\geq\max\left\{\alpha\omega_e^2(\mathbf{T})+\beta c_e^2(|\mathbf{T}|)\,,\alpha c_e^2(\mathbf{T})+\beta\,\|\mathbf{T}\|^2\right\}.$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then, we have

$$\begin{aligned}
\|\mathbf{T}\|_{\alpha,\beta}^2 &\geq \sum_{k=1}^{n} \left(\alpha \left|\langle T_k x, x\rangle\right|^2 + \beta \|T_k x\|^2\right) \\
&= \sum_{k=1}^{n} \alpha \left|\langle T_k x, x\rangle\right|^2 + \sum_{k=1}^{n} \beta \left\langle T_k^* T_k x, x\right\rangle \\
&= \alpha \sum_{k=1}^{n} \left|\langle T_k x, x\rangle\right|^2 + \beta \sum_{k=1}^{n} \left\langle T_k^* T_k x, x\right\rangle^{\frac{1}{2}\times 2} \\
&\geq \alpha \sum_{k=1}^{n} \left|\langle T_k x, x\rangle\right|^2 + \beta \sum_{k=1}^{n} \left\langle \left(T_k^* T_k\right)^{\frac{1}{2}} x, x\right\rangle^2 \\
&\quad \text{(by Lemma 2.2 (ii))} \\
&= \alpha \sum_{k=1}^{n} \left|\langle T_k x, x\rangle\right|^2 + \beta \sum_{k=1}^{n} \langle |T_k| \, x, x\rangle^2 \\
&\geq \alpha \sum_{k=1}^{n} \left|\langle T_k x, x\rangle\right|^2 + \beta c_e^2 \left(|\mathbf{T}|\right).
\end{aligned}$$

Taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$\|\mathbf{T}\|_{\alpha,\beta}^2 \geq \alpha \omega_e^2 \left(\mathbf{T}\right) + \beta c_e^2 \left(|\mathbf{T}|\right). \tag{14}$$

Also, we have

$$\begin{aligned}
\|\mathbf{T}\|_{\alpha,\beta}^2 &\geq \sum_{k=1}^{n} \left(\alpha \left|\langle T_k x, x\rangle\right|^2 + \beta \|T_k x\|^2\right) \\
&= \alpha \sum_{k=1}^{n} \left|\langle T_k x, x\rangle\right|^2 + \beta \sum_{k=1}^{n} \|T_k x\|^2 \\
&\geq \alpha c_e^2 \left(\mathbf{T}\right) + \beta \sum_{k=1}^{n} \|T_k x\|^2.
\end{aligned}$$

Taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$\|\mathbf{T}\|_{\alpha,\beta}^2 \geq \alpha c_e^2 \left(\mathbf{T}\right) + \beta \|\mathbf{T}\|^2. \tag{15}$$

Combining (14) and (15), we get

$$\|\mathbf{T}\|_{\alpha,\beta}^2 \geq \max \left\{\alpha \omega_e^2 \left(\mathbf{T}\right) + \beta c_e^2 \left(|\mathbf{T}|\right), \alpha c_e^2 \left(\mathbf{T}\right) + \beta \|\mathbf{T}\|^2\right\},$$

as desired. □

The following result reads as follows.

Theorem 3.11. *Let $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ be an n-tuple of operators. Then*

$$\|\mathbf{T}\|_{\alpha,\beta}^2 \leq \sqrt{n}\, \omega_e \left(\alpha \left|\mathbf{T}^*\right|^2 + \beta \left|\mathbf{T}\right|^2\right).$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. By using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\|\mathbf{T}\|_{\alpha,\beta}^2 &= \sup_{\|x\|=1} \sum_{k=1}^{n} \left(\alpha\left|\langle T_k x, x\rangle\right|^2 + \beta \|T_k x\|^2\right) \\
&= \sup_{\|x\|=1} \left(\sum_{k=1}^{n} \left(\alpha\left|\left\langle x, T_k^* x\right\rangle\right|^2 + \beta \|T_k x\|^2\right)\right) \\
&\le \sup_{\|x\|=1} \left(\sum_{k=1}^{n} \left(\alpha\left\|T_k^* x\right\|^2 + \beta \|T_k x\|^2\right)\right) \\
&= \sup_{\|x\|=1} \left(\sum_{k=1}^{n} \left(\alpha\left\langle \left|T_k^*\right|^2 x, x\right\rangle + \beta \left\langle |T_k|^2 x, x\right\rangle\right)\right) \\
&= \sup_{\|x\|=1} \left(\sum_{k=1}^{n} \left\langle \left(\alpha\left|T_k^*\right|^2 + \beta |T_k|^2\right) x, x\right\rangle\right) \\
&\le \sqrt{n} \sup_{\|x\|=1} \left(\sum_{k=1}^{n} \left|\left\langle \left(\alpha\left|T_k^*\right|^2 + \beta |T_k|^2\right) x, x\right\rangle\right|^2\right)^{\frac{1}{2}} \\
&= \sqrt{n}\,\omega_e\left(\alpha\left|\mathbf{T}^*\right|^2 + \beta |\mathbf{T}|^2\right).
\end{aligned}$$

Hence, the desired inequality is proved. □

As a consequence of Theorem 3.11, we have the following two corollaries.

Corollary 3.12. *Let $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ be an n-tuple of operators. Then*

$$\omega_e^2(\mathbf{T}) \le \inf_{\alpha,\beta} \frac{\sqrt{n}}{\alpha+\beta} \omega_e\left(\alpha\left|\mathbf{T}^*\right|^2 + \beta |\mathbf{T}|^2\right).$$

Proof. Using Theorem 3.4 (i), namely, $\omega_e(\mathbf{T}) \le \frac{1}{\sqrt{\alpha+\beta}} \|\mathbf{T}\|_{\alpha,\beta}$, we obtain

$$\omega_e^2(\mathbf{T}) \le \frac{\sqrt{n}}{\alpha+\beta} \omega_e\left(\alpha\left|\mathbf{T}^*\right|^2 + \beta |\mathbf{T}|^2\right).$$

Now, taking infimum over α, β, we get the required inequality. □

For the case $n = 1$, we get the following corollary which is a new refinement of the inequality (3).

Corollary 3.13. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}
\omega^2(T) &\le \inf_{\alpha,\beta} \frac{1}{\alpha+\beta} \left\|\alpha\left|T^*\right|^2 + \beta |T|^2\right\| \\
&\le \frac{1}{2} \left\||T|^2 + \left|T^*\right|^2\right\|.
\end{aligned}$$

Theorem 3.14. *Let $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ be an n-tuple of operators. If f and g are two non-negative continuous functions on $[0, \infty)$ satisfying $f(t)\,g(t) = t, \forall t \ge 0$, then*

$$\|\mathbf{T}\|_{\alpha,\beta}^2 \le \left\|\sum_{k=1}^{n} \left(\frac{\alpha}{2}\left(f^4(|T_k|) + g^4\left(\left|T_k^*\right|\right)\right) + \beta |T_k|^2\right)\right\|.$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. By using Lemma 2.3, it follows that

$$\begin{aligned}
|\langle T_k x, x\rangle|^2 &\leq \left\langle f^2\left(|T_k|\right)x, x\right\rangle \left\langle g^2\left(\left|T_k^*\right|\right)x, x\right\rangle \\
&\leq \frac{1}{2}\left(\left\langle f^2\left(|T_k|\right)x, x\right\rangle^2 + \left\langle g^2\left(\left|T_k^*\right|\right)x, x\right\rangle^2\right) \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&\leq \frac{1}{2}\left(\left\langle f^4\left(|T_k|\right)x, x\right\rangle + \left\langle g^4\left(\left|T_k^*\right|\right)x, x\right\rangle\right) \\
&\quad \text{(by Lemma 2.2)} \\
&= \frac{1}{2}\left\langle \left(f^4\left(|T_k|\right) + g^4\left(\left|T_k^*\right|\right)\right)x, x\right\rangle.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\|T_k x\|^2 &= \langle T_k x, T_k x\rangle \\
&= \left\langle T_k^* T_k x, x\right\rangle \\
&= \left\langle |T_k|^2 x, x\right\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathbf{T}\|_{\alpha,\beta}^2 &= \sup_{\|x\|=1}\sum_{k=1}^{n}\left(\alpha\left|\langle T_k x, x\rangle\right|^2 + \beta\|T_k x\|^2\right) \\
&= \sup_{\|x\|=1}\sum_{k=1}^{n}\left(\frac{\alpha}{2}\left\langle\left(f^4\left(|T_k|\right) + g^4\left(\left|T_k^*\right|\right)\right)x, x\right\rangle + \beta\left\langle |T_k|^2 x, x\right\rangle\right) \\
&= \sup_{\|x\|=1}\sum_{k=1}^{n}\left(\left\langle\left(\frac{\alpha}{2}\left(f^4\left(|T_k|\right) + g^4\left(\left|T_k^*\right|\right)\right) + \beta|T_k|^2\right)x, x\right\rangle\right) \\
&= \omega\left(\sum_{k=1}^{n}\left(\frac{\alpha}{2}\left(f^4\left(|T_k|\right) + g^4\left(\left|T_k^*\right|\right)\right) + \beta|T_k|^2\right)\right) \\
&= \left\|\sum_{k=1}^{n}\left(\frac{\alpha}{2}\left(f^4\left(|T_k|\right) + g^4\left(\left|T_k^*\right|\right)\right) + \beta|T_k|^2\right)\right\|.
\end{aligned}$$

□

Choosing $f(t) = g(t) = \sqrt{t}$ in Theorem 3.14, we get the following inequality.

Corollary 3.15. *Let $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ be an n-tuple of operators. Then*

$$\|\mathbf{T}\|_{\alpha,\beta}^2 \leq \left\|\sum_{k=1}^{n}\left(\left(\frac{\alpha}{2} + \beta\right)|T_k|^2 + \frac{\alpha}{2}\left|T_k^*\right|^2\right)\right\|.$$

The following corollary gives a refinement of [20, Corollary 3.6].

Corollary 3.16. *Let $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ be an n-tuple of operators. Then*

$$\begin{aligned}
\omega_e(\mathbf{T}) &\leq \inf_{\alpha,\beta}\frac{1}{\sqrt{\alpha+\beta}}\left\|\sum_{k=1}^{n}\left(\left(\frac{\alpha}{2} + \beta\right)|T_k|^2 + \frac{\alpha}{2}\left|T_k^*\right|^2\right)\right\|^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2}}\left\|\sum_{k=1}^{n}\left(|T_k|^2 + \left|T_k^*\right|^2\right)\right\|^{\frac{1}{2}}.
\end{aligned}$$

Proof. By using Theorem 3.4 (i), it follows that

$$\begin{aligned}\omega_e(\mathbf{T}) &\leq \frac{1}{\alpha+\beta}\|\mathbf{T}\|_{\alpha,\beta}\\ &\leq \frac{1}{\alpha+\beta}\left\|\sum_{k=1}^{n}\left(\left(\frac{\alpha}{2}+\beta\right)|T_k|^2+\frac{\alpha}{2}\left|T_k^*\right|^2\right)\right\|^{\frac{1}{2}}.\end{aligned}$$

Taking infimum over all α,β, we get

$$\omega_e(\mathbf{T}) \leq \inf_{\alpha,\beta}\frac{1}{\sqrt{\alpha+\beta}}\left\|\sum_{k=1}^{n}\left(\left(\frac{\alpha}{2}+\beta\right)|T_k|^2+\frac{\alpha}{2}\left|T_k^*\right|^2\right)\right\|^{\frac{1}{2}}.$$

The remaining inequality follows from the case $\alpha=1,\beta=0$. □

Remark 3.17. *For the case $n=1$, we get*

$$\begin{aligned}\omega^2(T) &\leq \inf_{\alpha,\beta}\frac{1}{\alpha+\beta}\left\|\left(\left(\frac{\alpha}{2}+\beta\right)|T|^2+\frac{\alpha}{2}|T^*|^2\right)\right\|\\ &\leq \frac{1}{2}\left\||T|^2+|T^*|^2\right\|.\end{aligned}$$

Thus, this inequality improves on the inequality (3).

Theorem 3.18. *Let $\mathbf{T}=(T_1,\cdots,T_n)\in\mathcal{B}(\mathcal{H})^n$ be an n-tuple of operators. Then*

$$\|\mathbf{T}\|_{\alpha,\beta}^2 \leq \sqrt{n}\left\{\omega_e\left(\left(\frac{\alpha}{4}+\beta\right)\mathbf{T}^*\mathbf{T}+\frac{\alpha}{4}\mathbf{T}\mathbf{T}^*\right)+\frac{\alpha}{2}\omega_e\left(\mathbf{T}^2\right)\right\}.$$

Proof. Let $x\in\mathcal{H}$ with $\|x\|=1$. Using Lemma 2.3, we have

$$\begin{aligned}|\langle T_kx,x\rangle|^2 &= |\langle T_kx,x\rangle|\left|\left\langle x,T_k^*x\right\rangle\right|\\ &\leq \frac{1}{2}\left(\|T_kx\|\left\|T_k^*x\right\|+\left|\left\langle T_kx,T_k^*x\right\rangle\right|\right)\\ &\quad \text{(by Lemma 2.1)}\\ &\leq \frac{1}{4}\left(\|T_kx\|^2+\left\|T_k^*x\right\|^2\right)+\frac{1}{2}\left|\left\langle T_k^2x,x\right\rangle\right|\\ &\quad \text{(by the arithmetic-geometric mean inequality)}\\ &= \frac{1}{4}\left\langle\left(T_k^*T_k+T_kT_k^*\right)x,x\right\rangle+\frac{1}{2}\left|\left\langle T_k^2x,x\right\rangle\right|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{k=1}^{n}\left(\alpha|\langle T_kx,x\rangle|^2+\beta\|T_kx\|^2\right)\\ &\leq \sum_{k=1}^{n}\left(\left\langle\frac{\alpha}{4}\left(T_k^*T_k+T_kT_k^*\right)x,x\right\rangle+\frac{\alpha}{2}\left|\left\langle T_k^2x,x\right\rangle\right|+\beta\left\langle T_k^*T_kx,x\right\rangle\right)\\ &= \sum_{k=1}^{n}\left\langle\left(\frac{\alpha}{4}\left(T_k^*T_k+T_kT_k^*\right)+\beta T_k^*T_k\right)x,x\right\rangle+\frac{\alpha}{2}\sum_{k=1}^{n}\left|\left\langle T_k^2x,x\right\rangle\right|\\ &= \sum_{k=1}^{n}\left\langle\left(\left(\frac{\alpha}{4}+\beta\right)T_k^*T_k+\frac{\alpha}{4}T_kT_k^*\right)x,x\right\rangle+\frac{\alpha}{2}\sum_{k=1}^{n}\left|\left\langle T_k^2x,x\right\rangle\right|\end{aligned}$$

$$\leq \sqrt{n}\left(\left(\sum_{k=1}^{n}\left|\left\langle\left(\left(\frac{\alpha}{4}+\beta\right)T_k^*T_k+\frac{\alpha}{4}T_kT_k^*\right)x,x\right\rangle\right|^2\right)^{\frac{1}{2}}+\frac{\alpha}{2}\left(\sum_{k=1}^{n}\left|\left\langle T_k^2x,x\right\rangle\right|^2\right)^{\frac{1}{2}}\right)$$

(by the Cauchy-Schwarz inequality)

$$\leq \sqrt{n}\left\{\omega_e\left(\left(\frac{\alpha}{4}+\beta\right)\mathbf{T}^*\mathbf{T}+\frac{\alpha}{4}\mathbf{T}\mathbf{T}^*\right)+\frac{\alpha}{2}\omega_e\left(\mathbf{T}^2\right)\right\}.$$

Taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$\|\mathbf{T}\|_{\alpha,\beta}^2 \leq \sqrt{n}\left\{\omega_e\left(\left(\frac{\alpha}{4}+\beta\right)\mathbf{T}^*\mathbf{T}+\frac{\alpha}{4}\mathbf{T}\mathbf{T}^*\right)+\frac{\alpha}{2}\omega_e\left(\mathbf{T}^2\right)\right\}.$$

This completes the proof. □

Applying Theorem 3.18, we derive the following corollary.

Corollary 3.19. *Let $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ be an n-tuple of operators. Then*

$$\begin{aligned}\omega_e^2(\mathbf{T}) &\leq \inf_{\alpha,\beta}\frac{\sqrt{n}}{\alpha+\beta}\left\{\omega_e\left(\left(\frac{\alpha}{4}+\beta\right)\mathbf{T}^*\mathbf{T}+\frac{\alpha}{4}\mathbf{T}\mathbf{T}^*\right)+\frac{\alpha}{2}\omega_e\left(\mathbf{T}^2\right)\right\}\\ &\leq \sqrt{n}\left(\frac{1}{4}\omega_e\left(\mathbf{T}^*\mathbf{T}+\mathbf{T}\mathbf{T}^*\right)+\frac{1}{2}\omega_e\left(\mathbf{T}^2\right)\right).\end{aligned}$$

Remark 3.20. *If $\mathbf{T}^2 = 0$ (i.e., $T_k^2 = 0$ for each $k = 1, 2, ..., n$), then it follows from Corollary 3.19 that*

$$\omega_e^2(\mathbf{T}) \leq \frac{\sqrt{n}}{4}\omega_e\left(\mathbf{T}^*\mathbf{T}+\mathbf{T}\mathbf{T}^*\right).$$

For the case $n = 1$, we have the following corollary, which has also been given in [24, Theorem 2.26].

Corollary 3.21. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega^2(T) \leq \inf_{\alpha,\beta}\frac{1}{\alpha+\beta}\left\{\left\|\left(\frac{\alpha}{4}+\beta\right)T^*T+\frac{\alpha}{4}TT^*\right\|+\frac{\alpha}{2}\omega\left(T^2\right)\right\}$$

Remark 3.22. *We observe that the inequality obtained in Corollary 3.21 refines the inequality (5). Indeed, we have*

$$\begin{aligned}\omega^2(T) &\leq \inf_{\alpha,\beta}\frac{1}{\alpha+\beta}\left\{\left\|\left(\frac{\alpha}{4}+\beta\right)T^*T+\frac{\alpha}{4}TT^*\right\|+\frac{\alpha}{2}\omega\left(T^2\right)\right\}\\ &\leq \frac{1}{4}\|T^*T+TT^*\|+\frac{1}{2}\omega\left(T^2\right).\end{aligned}$$

Theorem 3.23. *Let $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ be an n-tuple of operators. Then*

$$\begin{aligned}\|\mathbf{T}\|_{\alpha,\beta}^2 &\leq \frac{\alpha}{4}\omega_e^2\left(|\mathbf{T}|+i|\mathbf{T}^*|\right)+\left\|\sum_{k=1}^{n}\left(\left(\frac{\alpha}{8}+\beta\right)|T_k|^2+\frac{\alpha}{8}\left|T_k^*\right|^2\right)\right\|\\ &\quad+\frac{\alpha}{4}\left\|\sum_{k=1}^{n}\left|T_k^*\right||T_k|\right\|.\end{aligned}$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. By Lemma 2.3, we have

$$\begin{aligned}
|\langle T_k x, x\rangle|^2 &\leq \langle |T_k| x, x\rangle \left\langle \left|T_k^*\right| x, x\right\rangle \\
&\leq \frac{1}{4}\left(\langle |T_k| x, x\rangle + \left\langle \left|T_k^*\right| x, x\right\rangle\right)^2 \\
&= \frac{1}{4}\left(\langle |T_k| x, x\rangle^2 + \left\langle \left|T_k^*\right| x, x\right\rangle^2\right) + \frac{1}{2}\langle |T_k| x, x\rangle \left\langle \left|T_k^*\right| x, x\right\rangle \\
&= \frac{1}{4}\left|\langle |T_k| x, x\rangle + i\left\langle \left|T_k^*\right| x, x\right\rangle\right|^2 + \frac{1}{2}\langle |T_k| x, x\rangle \left\langle \left|T_k^*\right| x, x\right\rangle \\
&\leq \frac{1}{4}\left|\left\langle\left(|T_k| + i\left|T_k^*\right|\right)x, x\right\rangle\right|^2 + \frac{1}{4}\||T_k| x\| \left\|\left|T_k^*\right| x\right\| + \frac{1}{4}\left\langle |T_k| x, \left|T_k^*\right| x\right\rangle \\
&\quad \text{(by Lemma 2.1)} \\
&\leq \frac{1}{4}\left|\left\langle\left(|T_k| + i\left|T_k^*\right|\right)x, x\right\rangle\right|^2 + \frac{1}{8}\left(\||T_k| x\|^2 + \left\|\left|T_k^*\right| x\right\|^2\right) + \frac{1}{4}\left\langle \left|T_k^*\right| |T_k| x, x\right\rangle \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{1}{4}\left|\left\langle\left(|T_k| + i\left|T_k^*\right|\right)x, x\right\rangle\right|^2 + \frac{1}{8}\left\langle\left(|T_k|^2 + \left|T_k^*\right|^2\right)x, x\right\rangle + \frac{1}{4}\left\langle \left|T_k^*\right| |T_k| x, x\right\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{k=1}^{n}\left(\alpha\,|\langle T_k x, x\rangle|^2 + \beta\,\|T_k x\|^2\right) \\
\leq\ & \sum_{k=1}^{n}\left(\frac{\alpha}{4}\left|\left\langle\left(|T_k| + i\left|T_k^*\right|\right)x, x\right\rangle\right|^2 + \frac{\alpha}{8}\left\langle\left(|T_k|^2 + \left|T_k^*\right|^2\right)x, x\right\rangle\right) \\
&+ \sum_{k=1}^{n}\frac{\alpha}{4}\left\langle \left|T_k^*\right| |T_k| x, x\right\rangle + \sum_{k=1}^{n}\beta\left\langle |T_k|^2 x, x\right\rangle \\
=\ & \sum_{k=1}^{n}\left(\frac{\alpha}{4}\left|\left\langle\left(|T_k| + i\left|T_k^*\right|\right)x, x\right\rangle\right|^2 + \left\langle\left(\left(\frac{\alpha}{8} + \beta\right)|T_k|^2 + \frac{\alpha}{8}\left|T_k^*\right|^2\right)x, x\right\rangle\right) \\
&+ \sum_{k=1}^{n}\frac{\alpha}{4}\left\langle \left|T_k^*\right| |T_k| x, x\right\rangle \\
\leq\ & \frac{\alpha}{4}\omega_e^2\left(|\mathbf{T}| + i\,|\mathbf{T}^*|\right) + \omega\left(\sum_{k=1}^{n}\left(\left(\frac{\alpha}{8} + \beta\right)|T_k|^2 + \frac{\alpha}{8}\left|T_k^*\right|^2\right)\right) + \frac{\alpha}{4}\omega\left(\sum_{k=1}^{n}\left|T_k^*\right| |T_k|\right) \\
=\ & \frac{\alpha}{4}\omega_e^2\left(|\mathbf{T}| + i\,|\mathbf{T}^*|\right) + \left\|\sum_{k=1}^{n}\left(\left(\frac{\alpha}{8} + \beta\right)|T_k|^2 + \frac{\alpha}{8}\left|T_k^*\right|^2\right)\right\| + \frac{\alpha}{4}\left\|\sum_{k=1}^{n}\left|T_k^*\right| |T_k|\right\|.
\end{aligned}$$

Taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$\begin{aligned}
\|\mathbf{T}\|_{\alpha,\beta}^2 \leq\ & \frac{\alpha}{4}\omega_e^2\left(|\mathbf{T}| + i\,|\mathbf{T}^*|\right) + \left\|\sum_{k=1}^{n}\left(\left(\frac{\alpha}{8} + \beta\right)|T_k|^2 + \frac{\alpha}{8}\left|T_k^*\right|^2\right)\right\| \\
&+ \frac{\alpha}{4}\left\|\sum_{k=1}^{n}\left|T_k^*\right| |T_k|\right\|.
\end{aligned}$$

This completes the proof. □

Letting $n = 1$ in Theorem 3.23, gives the following corollary.

Corollary 3.24. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} \|T\|_{\alpha,\beta}^2 &\leq \frac{\alpha}{4}\omega^2(|T|+i|T^*|)+\left\|\left(\frac{\alpha}{8}+\beta\right)|T|^2+\frac{\alpha}{8}|T^*|^2\right\| \\ &+\frac{\alpha}{4}\||T^*||T|\|. \end{aligned}$$

Corollary 3.25. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} \omega^2(T) &\leq \inf_{\alpha,\beta}\frac{1}{\alpha+\beta}\left\{\frac{\alpha}{4}\omega^2(|T|+i|T^*|)+\left\|\left(\frac{\alpha}{8}+\beta\right)|T|^2+\frac{\alpha}{8}|T^*|^2\right\|\right. \\ &\left.+\frac{\alpha}{4}\||T^*||T|\|\right\}. \end{aligned}$$

Proof. Using Theorem 3.4 (i) for $n=1$, namely, $\omega(T) \leq \frac{1}{\sqrt{\alpha+\beta}}\|T\|_{\alpha,\beta}$, we obtain

$$\begin{aligned} \omega^2(T) &\leq \frac{1}{\alpha+\beta}\left\{\frac{\alpha}{4}\omega^2(|T|+i|T^*|)+\left\|\left(\frac{\alpha}{8}+\beta\right)|T|^2+\frac{\alpha}{8}|T^*|^2\right\|\right. \\ &\left.+\frac{\alpha}{4}\||T^*||T|\|\right\}. \end{aligned}$$

Taking infimum over α, β, we get the desired inequality. □

Remark 3.26. *We note that the inequality obtained in Corollary 3.25 refines the inequality (3). Indeed, we have*

$$\begin{aligned} \omega^2(T) &\leq \inf_{\alpha,\beta}\frac{1}{\alpha+\beta}\left\{\frac{\alpha}{4}\omega^2(|T|+i|T^*|)+\left\|\left(\frac{\alpha}{8}+\beta\right)|T|^2+\frac{\alpha}{8}|T^*|^2\right\|\right. \\ &\left.+\frac{\alpha}{4}\||T^*||T|\|\right\} \\ &\leq \frac{1}{4}\omega^2(|T|+i|T^*|)+\frac{1}{8}\left\||T|^2+|T^*|^2\right\|+\frac{1}{4}\||T^*||T|\| \\ &\leq \frac{1}{4}\left\||T|^2+|T^*|^2\right\|+\frac{1}{8}\left\||T|^2+|T^*|^2\right\|+\frac{1}{4}\||T^*||T|\| \\ &\quad \text{(by Lemma 2.4)} \\ &= \frac{1}{4}\left\||T|^2+|T^*|^2\right\|+\frac{1}{8}\left\||T|^2+|T^*|^2\right\|+\frac{1}{4}\omega(|T^*||T|) \\ &\leq \frac{1}{4}\left\||T|^2+|T^*|^2\right\|+\frac{1}{8}\left\||T|^2+|T^*|^2\right\|+\frac{1}{8}\left\||T|^2+|T^*|^2\right\| \\ &\quad \text{(by the inequality (8))} \\ &= \frac{1}{2}\left\||T|^2+|T^*|^2\right\|. \end{aligned}$$

Theorem 3.27. *Let $\mathbf{T}=(T_1,\cdots,T_n) \in \mathcal{B}(\mathcal{H})^n$ be an n-tuple of operators. Then*

$$\|\mathbf{T}\|_{\alpha,\beta}^2 \leq \left\|\sum_{k=1}^n\left(\left(\frac{3\alpha}{8}+\beta\right)|T_k|^2+\frac{3\alpha}{8}\left|T_k^*\right|^2\right)\right\|+\frac{\alpha}{4}\left\|\sum_{k=1}^n\left|T_k^*\right||T_k|\right\|.$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|=1$. In view of Lemma 2.3, we have

$$\begin{aligned} |\langle T_k x,x\rangle|^2 &\leq \langle|T_k|x,x\rangle\left\langle\left|T_k^*\right|x,x\right\rangle \\ &\leq \frac{1}{4}\left(\langle|T_k|x,x\rangle+\left\langle\left|T_k^*\right|x,x\right\rangle\right)^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\left(\langle |T_k|\, x, x\rangle^2 + \left\langle \left|T_k^*\right| x, x\right\rangle^2\right) + \frac{1}{2}\langle |T_k|\, x, x\rangle \left\langle \left|T_k^*\right| x, x\right\rangle \\
&= \frac{1}{4}\left(\left\langle |T_k|^2 x, x\right\rangle + \left\langle \left|T_k^*\right|^2 x, x\right\rangle\right) + \frac{1}{2}\langle |T_k|\, x, x\rangle \left\langle \left|T_k^*\right| x, x\right\rangle \\
&\leq \frac{1}{4}\left\langle \left(|T_k|^2 + \left|T_k^*\right|^2\right) x, x\right\rangle + \frac{1}{4}\, \| |T_k|\, x\| \left\| \left|T_k^*\right| x\right\| + \frac{1}{4}\left\langle |T_k|\, x, \left|T_k^*\right| x\right\rangle \\
&\quad \text{(by Lemma 2.1)} \\
&\leq \frac{1}{4}\left\langle \left(|T_k|^2 + \left|T_k^*\right|^2\right) x, x\right\rangle + \frac{1}{8}\left(\| |T_k|\, x\|^2 + \left\| \left|T_k^*\right| x\right\|^2\right) + \frac{1}{4}\left\langle \left|T_k^*\right| |T_k|\, x, x\right\rangle \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{1}{4}\left\langle \left(|T_k|^2 + \left|T_k^*\right|^2\right) x, x\right\rangle + \frac{1}{8}\left\langle \left(|T_k|^2 + \left|T_k^*\right|^2\right) x, x\right\rangle + \frac{1}{4}\left\langle \left|T_k^*\right| |T_k|\, x, x\right\rangle \\
&= \frac{3}{8}\left\langle \left(|T_k|^2 + \left|T_k^*\right|^2\right) x, x\right\rangle + \frac{1}{4}\left\langle \left|T_k^*\right| |T_k|\, x, x\right\rangle .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{k=1}^{n}\left(\alpha\, |\langle T_k x, x\rangle|^2 + \beta\, \|T_k x\|^2\right) \\
\leq\ & \sum_{k=1}^{n}\left(\frac{3\alpha}{8}\left\langle \left(|T_k|^2 + \left|T_k^*\right|^2\right) x, x\right\rangle + \frac{\alpha}{4}\left\langle \left|T_k^*\right| |T_k|\, x, x\right\rangle + \beta\left\langle |T_k|^2 x, x\right\rangle\right) \\
=\ & \sum_{k=1}^{n}\left\langle \left(\left(\frac{3\alpha}{8} + \beta\right)|T_k|^2 + \frac{3\alpha}{8}\left|T_k^*\right|^2\right) x, x\right\rangle + \frac{\alpha}{4}\sum_{k=1}^{n}\left\langle \left|T_k^*\right| |T_k|\, x, x\right\rangle \\
\leq\ & \left\| \sum_{k=1}^{n}\left(\left(\frac{3\alpha}{8} + \beta\right)|T_k|^2 + \frac{3\alpha}{8}\left|T_k^*\right|^2\right)\right\| + \frac{\alpha}{4}\left\| \sum_{k=1}^{n}\left|T_k^*\right| |T_k|\right\| .
\end{aligned}
$$

Taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$
\|\mathbf{T}\|_{\alpha,\beta}^2 \leq \left\| \sum_{k=1}^{n}\left(\left(\frac{3\alpha}{8} + \beta\right)|T_k|^2 + \frac{3\alpha}{8}\left|T_k^*\right|^2\right)\right\| + \frac{\alpha}{4}\left\| \sum_{k=1}^{n}\left|T_k^*\right| |T_k|\right\| .
$$

□

The following corollary follows immediately for the case $n = 1$ in Theorem 3.27.

Corollary 3.28. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$
\|T\|_{\alpha,\beta}^2 \leq \left\| \left(\frac{3\alpha}{8} + \beta\right)|T|^2 + \frac{3\alpha}{8}|T^*|^2\right\| + \frac{\alpha}{4}\, \| |T^*|\, |T| \| .
$$

Corollary 3.29. *Let $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ be an n-tuple of operators. Then*

$$
\begin{aligned}
\omega_e^2(\mathbf{T}) \leq\ & \inf_{\alpha,\beta} \frac{1}{\alpha + \beta}\left\{ \left\| \sum_{k=1}^{n}\left(\left(\frac{3\alpha}{8} + \beta\right)|T_k|^2 + \frac{3\alpha}{8}\left|T_k^*\right|^2\right)\right\| \right. \\
& \left. + \frac{\alpha}{4}\left\| \sum_{k=1}^{n}\left|T_k^*\right| |T_k|\right\| \right\} .
\end{aligned}
$$

Proof. Using Theorem 3.4 (i), namely, $\omega_e(\mathbf{T}) \leq \frac{1}{\sqrt{\alpha+\beta}} \|\mathbf{T}\|_{\alpha,\beta}$, we obtain

$$\begin{aligned}\omega_e^2(\mathbf{T}) &\leq \frac{1}{\alpha+\beta}\left\{\left\|\sum_{k=1}^{n}\left\{\left(\frac{3\alpha}{8}+\beta\right)|T_k|^2+\frac{3\alpha}{8}\left|T_k^*\right|^2\right\}\right\|\right.\\ &\quad \left.+\frac{\alpha}{4}\left\|\sum_{k=1}^{n}\left|T_k^*\right|\left|T_k\right|\right\|\right\}.\end{aligned}$$

Now, taking infimum over α, β, we get the required inequality. □

For the case $n = 1$, in Corollary 3.29, we get the following inequality.

Corollary 3.30. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega^2(T) \leq \inf_{\alpha,\beta} \frac{1}{\alpha+\beta}\left\{\left\|\left(\frac{3\alpha}{8}+\beta\right)|T|^2+\frac{3\alpha}{8}|T^*|^2\right\| + \frac{\alpha}{4}\||T^*||T|\|\right\}.$$

Remark 3.31. *We note that the inequality obtained in Corollary 3.30 refines the inequality (3) and (6). Indeed, if we take $\alpha = \frac{2}{3}$ and $\beta = 0$, then*

$$\begin{aligned}\omega^2(T) &\leq \inf_{\alpha,\beta} \frac{1}{\alpha+\beta}\left\{\left\|\left(\frac{3\alpha}{8}+\beta\right)|T|^2+\frac{3\alpha}{8}|T^*|^2\right\| + \frac{\alpha}{4}\||T^*||T|\|\right\}\\ &\leq \frac{1}{4}\left\||T|^2+|T^*|^2\right\| + \frac{1}{6}\||T^*||T|\|\\ &= \frac{1}{4}\left\||T|^2+|T^*|^2\right\| + \frac{1}{6}\omega(|T^*||T|)\\ &\leq \frac{1}{4}\left\||T|^2+|T^*|^2\right\| + \frac{1}{2}\omega(|T^*||T|)\\ &\leq \frac{1}{4}\left\||T|^2+|T^*|^2\right\| + \frac{1}{4}\left\||T|^2+|T^*|^2\right\|\\ &\quad \text{(by the inequality (8))}\\ &= \frac{1}{2}\left\||T|^2+|T^*|^2\right\|.\end{aligned}$$

Theorem 3.32. *Let $\mathbf{T} = (T_1, \cdots, T_n)$, $\mathbf{S} = (S_1, \cdots, S_n) \in \mathcal{B}(\mathcal{H})^n$. Then*

$$\|\mathbf{S}^*\mathbf{T}\|_{\alpha,\beta}^2 \leq \frac{\alpha}{2}\left\|\sum_{k=1}^{n}\left(|T_k|^4+|S_k|^4\right)\right\| + \beta\left\|\sum_{k=1}^{n}\left|S_k^*T_k\right|^2\right\|.$$

Proof. Let $x \in \mathcal{H}$ be an unit vector. Then, we have

$$\begin{aligned}\sum_{k=1}^{n}\left|\left\langle S_k^*T_kx, x\right\rangle\right|^2 &= \sum_{k=1}^{n}|\langle T_kx, S_kx\rangle|^2\\ &\leq \sum_{k=1}^{n}\|T_kx\|^2\|S_kx\|^2\\ &\quad \text{(by the Cauchy-Schwarz inequality)}\\ &= \sum_{k=1}^{n}\langle T_kx, T_kx\rangle\langle S_kx, S_kx\rangle\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \left\langle T_k^* T_k x, x \right\rangle \left\langle S_k^* S_k x, x \right\rangle \\
&\leq \sum_{k=1}^{n} \frac{1}{2} \left(\left\langle T_k^* T_k x, x \right\rangle^2 + \left\langle S_k^* S_k x, x \right\rangle^2 \right) \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&\leq \sum_{k=1}^{n} \frac{1}{2} \left(\left\langle \left(T_k^* T_k \right)^2 x, x \right\rangle + \left\langle \left(S_k^* S_k \right)^2 x, x \right\rangle \right) \\
&\quad \text{(by Lemma 2.2)} \\
&= \sum_{k=1}^{n} \frac{1}{2} \left\langle \left(|T_k|^4 + |S_k|^4 \right) x, x \right\rangle \\
&\leq \frac{1}{2} \left\| \sum_{k=1}^{n} \left(|T_k|^4 + |S_k|^4 \right) \right\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{k=1}^{n} \left(\alpha \left| \left\langle S_k^* T_k x, x \right\rangle \right|^2 + \beta \left\| S_k^* T_k x \right\|^2 \right) \\
\leq\ & \sum_{k=1}^{n} \left(\frac{\alpha}{2} \left\langle \left(|T_k|^4 + |S_k|^4 \right) x, x \right\rangle + \beta \left\langle S_k^* T_k x, S_k^* T_k x \right\rangle \right) \\
\leq\ & \sum_{k=1}^{n} \left(\frac{\alpha}{2} \left\langle \left(|T_k|^4 + |S_k|^4 \right) x, x \right\rangle + \beta \left\langle \left| S_k^* T_k \right|^2 x, x \right\rangle \right) \\
\leq\ & \frac{\alpha}{2} \left\| \sum_{k=1}^{n} \left(|T_k|^4 + |S_k|^4 \right) \right\| + \beta \left\| \sum_{k=1}^{n} \left| S_k^* T_k \right|^2 \right\|.
\end{aligned}
$$

Taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$
\|\mathbf{S}^*\mathbf{T}\|_{\alpha,\beta}^2 \leq \frac{\alpha}{2} \left\| \sum_{k=1}^{n} \left(|T_k|^4 + |S_k|^4 \right) \right\| + \beta \left\| \sum_{k=1}^{n} \left| S_k^* T_k \right|^2 \right\|.
$$

This completes the proof of the theorem. □

Corollary 3.33. *Let* $\mathbf{T} = (T_1, \cdots, T_n)$, $\mathbf{S} = (S_1, \cdots, S_n) \in \mathcal{B}(\mathcal{H})^n$. *Then*

$$
\begin{aligned}
\omega_e^2 (\mathbf{S}^*\mathbf{T}) &\leq \inf_{\alpha,\beta} \frac{1}{\alpha + \beta} \left\{ \frac{\alpha}{2} \left\| \sum_{k=1}^{n} \left(|T_k|^4 + |S_k|^4 \right) \right\| + \beta \left\| \sum_{k=1}^{n} \left| S_k^* T_k \right|^2 \right\| \right\} \\
&\leq \frac{1}{2} \left\| \sum_{k=1}^{n} \left(|T_k|^4 + |S_k|^4 \right) \right\|.
\end{aligned}
$$

Proof. By using Theorem 3.4 (i), it follows that

$$
\begin{aligned}
\omega_e^2 (\mathbf{S}^*\mathbf{T}) &\leq \frac{1}{\alpha + \beta} \|\mathbf{S}^*\mathbf{T}\|_{\alpha,\beta}^2 \\
&\leq \frac{1}{\alpha + \beta} \left\{ \frac{\alpha}{2} \left\| \sum_{k=1}^{n} \left(|T_k|^4 + |S_k|^4 \right) \right\| + \beta \left\| \sum_{k=1}^{n} \left| S_k^* T_k \right|^2 \right\| \right\}.
\end{aligned}
$$

Taking infimum over all α, β, we get

$$\omega_e^2(\mathbf{S}^*\mathbf{T}) \leq \inf_{\alpha,\beta} \frac{1}{\alpha+\beta}\left\{\frac{\alpha}{2}\left\|\sum_{k=1}^{n}\left(|T_k|^4+|S_k|^4\right)\right\| + \beta\left\|\sum_{k=1}^{n}\left|S_k^*T_k\right|^2\right\|\right\}.$$

The remaining inequality follows from the case $\alpha = 1, \beta = 0$. □

Corollary 3.34. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$\|S^*T\|_{\alpha,\beta}^2 \leq \frac{\alpha}{2}\left\||T|^4+|S|^4\right\| + \beta\left\||S^*T|^2\right\|.$$

Proof. By taking $n = 1$ in Theorem 3.32, the inequality follows directly. □

The following corollary gives a new refinement of the inequality (7) for $r = 2$.

Corollary 3.35. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}\omega^2(S^*T) &\leq \inf_{\alpha,\beta}\frac{1}{\alpha+\beta}\left\{\frac{\alpha}{2}\left\||T|^4+|S|^4\right\| + \beta\left\||S^*T|^2\right\|\right\} \\ &\leq \frac{1}{2}\left\||T|^4+|S|^4\right\|.\end{aligned}$$

Proof. By taking $n = 1$ in Corollary 3.33, the result follows immediately. □

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