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On r-ideals of $\mathcal{R}(L)$

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Abstract. In this paper, we study the concept of r-ideal (a proper ideal I in a ring R is said to be an r-ideal if $ra \in I$ with $\mathrm{Ann}(r) = (0)$, implies that $a \in I$ for each $a, r \in R$) in the ring $\mathcal{R}(L)$, as the point-free counterpart of $C(X)$ and a reduced commutative ring. We investigate the behavior of this type of ideal in the ring $\mathcal{R}(L)$ for cozero complemented frames, P-frames, almost P-frames, and weakly almost P-frames. We prove the characterization of these frames via the concept of r-ideal in the ring $\mathcal{R}(L)$.

We examine other groups of ideals, namely z_r-ideal and s_r-ideal in the ring $\mathcal{R}(L)$, by combining the concept of r-ideal with z-ideal and also with the semiprime ideal. We show that the sum of the z_r-ideals in the ring $\mathcal{R}(L)$ has the same behavior as the z^0-ideals in this ring in a simple way: The sum of every two z_r-ideals in $\mathcal{R}(L)$ is a z_r-ideal or all of $\mathcal{R}(L)$ if and only if L is a quasi-F-frame. Here, this fact is also proved for s_r-ideals.

1. Introduction

The abstract lattice of open sets can contain a lot of information about a topological space. By this fact, the point-free topology provides a good constructive foundation for topological theories, as argued by Ball and Walters-Wayland [9]: "... what the point-free formulation adds to the classical theory is a remarkable combination of elegance of statement, simplicity of proof, and increase of extent." In an overview of the historical development of this theory, it can be seen the works of [9, 10, 20, 22, 23, 29], as some of the pioneers that made a point-free approach to $C(X)$, the ring of real-valued continuous functions on a completely regular Hausdorff space X.

Dube is one who played an effective role in extending the study of ring $\mathcal{R}(L)$. He introduced and characterized some frames related to $\mathcal{R}(L)$ and determined their properties, especially the cozero complemented frames and weakly almost P-frames [11–17].

Ideals play a fundamental role in studying the structure of $C(X)$. In this paper, we consider $\mathcal{R}(L)$, with a completely regular frame L and study some types of the ideals in it. One of these is r-ideal, introduced in the context of the theory of commutative ring by Mohamadian [26] in 2015. He investigated generally the behavior of r-ideals in commutative rings. Also, as a significant result, he considered $C(X)$ and proved that every ideal in $C(X)$ is an r-ideal if and only if X is almost P-space. Moreover, he showed that in cozero complemented spaces (m-spaces), every prime r-ideal of $C(X)$ is a z^0-ideal. Inspired by it, we determine the

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r-ideals in $\mathcal{R}(L)$ and their properties. We establish similar results, as in $C(X)$, and we characterize the frame L with respect to the behavior of r-ideal in it.

In 2021, Azarpanah, Mohamadian, and Monjezi [8] introduced another class of ideals based on the r-ideal concept called z_r-ideal and s_r-ideal in a ring $C(X)$. The class of z_r-ideals can be considered between the two classes of z^0-ideals and z-ideals. They showed that the sum of z_r-ideals in the ring $C(X)$ behaves similarly to the sum of z^0-ideals in the ring $C(X)$. They investigated the properties of z_r-ideals and s_r-ideals in the ring $C(X)$ and obtained interesting results. Also, they showed that a space is the cozero complemented space if and only if every z_r-ideal of $C(X)$ is an s_r-ideal of $C(X)$.

The plan of this paper is as follows:

In Section 2, we present the basic concepts of frames and ring $\mathcal{R}(L)$, which are needed in this paper.

In Section 3, we examine r-ideals in cozero complemented frames. We express and prove equivalences for these frames based on the concept of r-ideals. Also, we show that in cozero complemented frames, every r-ideal is a z-ideal and every prime r-ideal is a z^0-ideal of the ring $\mathcal{R}(L)$. We discuss the above in Proposition 3.2 and in Corollaries 3.3 and 3.6.

In Section 4, we examine r-ideals in P-frames and almost P-frames. To learn about these frames, see [9, 12, 13, 15, 16]. We express and prove equivalences for almost P-frames based on the concept of r-ideal. We show that in P-frames, the set of all r-ideals coincides with the set of all z-ideals of $\mathcal{R}(L)$. We discuss the above in Proposition 4.1 and in Corollary 4.3.

In the last section, we define the concept of z_r-ideals and s_r-ideals in the ring $\mathcal{R}(L)$ and examine the characteristics of these types of ideals. We show in Theorem 5.10 that a frame L is a quasi-F-frame if and only if the sum of both z_r-ideals in $\mathcal{R}(L)$ is a z_r-ideal or the whole ring. We also propose and prove this statement about s_r-ideals in Theorem 5.24. To learn about these frames, see [14]. Also, after examining the relationship of r-ideals, z_r-ideals, and s_r-ideals with each other as well as with other known ideals in the ring $\mathcal{R}(L)$, we present a regular chain of these ideals in Corollaries 5.22, 5.25, and 5.26.

2. Preliminaries

2.1. Ring

A ring R is reduced if it has no nonzero nilpotent elements. The principal ideal of a ring generated by an element a in R is denoted by (a), and for $S \subseteq R$, the set $\{x \in R\colon xs = 0 \text{ for each } s \in S\}$ is the annihilator of S, which is denoted by $\mathrm{Ann}(S)$.

From [26], we recall that a proper ideal I in a commutative ring R is said to be an **r-ideal** if $ra \in I$ with $r \in \mathrm{r}(R) := \{x \in R\colon \mathrm{Ann}(x) = (0)\}$ implies that $a \in I$ for each $a, r \in R$.

Also, we recall from [27] that for any multiplicative closed set S of a ring R, the S-component of an ideal I is defined by $I_S := \{x \in R\colon \text{There exists } s \in S \text{ for some } xs \in I\}$. Since the set $r(R)$ is a multiplicative closed set, similarly it is defined the set $I_r := \{a \in R\colon \text{There exists } r \in r(R) \text{ for some } ra \in I\}$ of I.

Clearly, if $I \cap r(R) \neq \emptyset$, then $I_r = R$. In [8, Lemma 2.2], it was shown, for an ideal I of a reduced ring R with $I \cap r(R) = \emptyset$, that the set I_r is the smallest r-ideal containing I. Also, they showed in the same lemma that I is an r-ideal if and only if $I = I_r$.

2.2. Frame L and the ring $\mathcal{R}(L)$

For a general theory of frames, we refer to [22]. Also, for more information about frames and ring $\mathcal{R}(L)$, refer to [29]. Here we collect a few facts that will be relevant for our discussion.

Recall that a **frame (locale)** is a complete lattice L in which the distributive law $a \wedge \bigvee S = \bigvee\{a \wedge x | x \in S\}$ holds for all $a \times S \in L \times \mathcal{P}(L)$. We denote the top element and the bottom element of L by $\top$ and $\bot$, respectively.

The **pseudocomplement** of an element a in a frame L is the element a^* that is

$$a^* = \bigvee\{x \in L \,|\, x \wedge a = \bot\}.$$

An element a of frame L is **complemented** if $a \vee a^* = \top$, and it is **dense** if $a^* = \bot$.

A **frame homomorphism** is a map between frames that preserves finite meets including the top element, and arbitrary joins including the bottom element.

Regarding the frame of reals $\mathcal{L}(\mathbb{R})$ and the f-ring $\mathcal{R}(L)$ of continuous real-valued functions on L, we use the notation of [10]. A **continuous real function** on a frame is a homomorphism $\mathcal{L}(\mathbb{R}) \to L$. The set of all continuous real functions on a frame L is denoted by $\mathcal{R}(L)$.

It is known that the mapping $\mathrm{coz} : \mathcal{R}(L) \longrightarrow L$ is given by

$$\mathrm{coz}(\alpha) = \bigvee \Big\{\alpha(p, 0) \vee \alpha(0, q) \,|\, p, q \in \mathbb{Q}\Big\}.$$

A **cozero element** of L is an element of the form $\mathrm{coz}(\alpha)$ for some $\alpha \in \mathcal{R}(L)$. The cozero part of L is denoted by $\mathrm{Coz}\,(L)$. For every $\alpha, \beta \in \mathcal{R}(L)$, we frequently use the following properties:

(1) $\mathrm{coz}(\alpha\beta) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta)$,

(2) $\mathrm{coz}(\alpha + \beta) \leq \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) = \mathrm{coz}(\alpha^2 + \beta^2)$,

(3) $\alpha \in \mathcal{R}(L)$ is invertible if and only if $\mathrm{coz}(\alpha) = \top$,

(4) $\mathrm{coz}(\alpha) = \bot$ if and only if $\alpha = 0$.

From (1) and (4), it follows that $\mathcal{R}(L)$ has no nonzero nilpotent element. Consequently, a prime ideal $P \in \mathcal{R}(L)$ is minimal prime if and only if for every $\varphi \in P$, there exists $\psi \notin P$ such that $\varphi\psi = 0$.

For any x and y in a frame L, we say that x is **completely below** y in L and write $x \lll y$ if there exists a trail $\{x_i\}_{i \in [0,1] \cap \mathbb{Q}} \subseteq L$ such that $x_0 = x$, $x_1 = y$, and for every$p, q \in [0, 1] \cap \mathbb{Q}$ with $p < q$, $x_p \prec x_q$. A frame L is called **completely regular** if for every $a \in L$, we have $a = \bigvee_{b \lll a} b$. An ideal I of L is called completely regular if for any $a \in I$, there exists $b \in I$ such that $a \lll b$. The frame βL is the frame of all completely regular ideals of L, and βL is the Stone-Čech compactification of a completely regular frame L. The map

$$r_L\Big(x \mapsto \Big\{a \in L \colon a \lll x\Big\}\Big) : L \to \beta L$$

is the right adjoint of the join map

$$\bigvee \Big(I \mapsto \bigvee I\Big) : \beta L \to L.$$

We recall from [13, Definition 4.10] that an ideal I of $\mathcal{R}(L)$ is called a **z-ideal** if, for any $\alpha \in \mathcal{R}(L)$ and $\beta \in I$, $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ implies $\alpha \in I$ and it is called **d-ideal** (it is discussed in this paper under the title **$\mathbf{z^0}$-ideal**) if, for any $\alpha \in \mathcal{R}(L)$ and $\beta \in I$, $\mathrm{coz}(\alpha) \leq \Big(\mathrm{coz}(\beta)\Big)^{**}$ implies $\alpha \in I$. Also, we can see equivalence for it in [1, Proposition 4.1]; for example, an ideal I of $\mathcal{R}(L)$ is a z^0-ideal if, for any $(\alpha, \beta) \in I \times \mathcal{R}(L)$, $\Big(\mathrm{coz}(\alpha)\Big)^{*} = \Big(\mathrm{coz}(\beta)\Big)^{*}$ implies $\beta \in I$. Also, we remember from [13] that for each $I \in \beta L$, the ideal M^I of $\mathcal{R}(L)$ is defined by $M^I := \Big\{\alpha \in \mathcal{R}(L) \colon r_L\Big(\mathrm{coz}(\alpha)\Big) \subseteq I\Big\}$, which is a z-ideal, and the ideal O^I of $\mathcal{R}(L)$ is defined by $O^I := \Big\{\alpha \in \mathcal{R}(L) \colon r_L\Big(\mathrm{coz}(\alpha)\Big) \lll I\Big\}$, which is a z^0-ideal.

2.3. *Sublocale*

For a locale L, a subset $S \subseteq L$ is a **sublocale** if and only if

$$M \subseteq L \Rightarrow \bigwedge M \in S \quad \text{and} \quad (x \in L, s \in S) \Rightarrow x \to s \in S.$$

The subset S is a frame in the order of L and inherits its Heyting structure. The smallest sublocale of L is $\mathsf{O} = \{\top\}$ and is called the void sublocale, and the largest sublocale of L is L. The open and the closed sublocales corresponding to each $a \in L$ are, respectively, the sublocales

$$\mathfrak{o}_L(a) = \{a \to x \,|\, x \in L\} = \{x \,|\, x = a \to x\} \quad \text{and} \quad \mathfrak{c}_L(a) = \uparrow a = \{x \in L \,|\, x \geq a\}.$$

Some of their properties, which we shall freely use, are as follows:

(1) $\mathfrak{o}_L(\bot) = \mathfrak{c}_L(\top) = \mathsf{O}$ and $\mathfrak{o}_L(\top) = \mathfrak{c}_L(\bot) = L$.

(2) $\mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(b)$ if and only if $a \vee b = \top$ and $\mathfrak{o}_L(a) \subseteq \mathfrak{c}_L(b)$ if and only if $a \wedge b = \bot$.

(3) $\mathfrak{o}_L(a) \cap \mathfrak{o}_L(b) = \mathfrak{o}_L(a \wedge b)$ and $\mathfrak{c}_L(a) \vee \mathfrak{c}_L(b) = \mathfrak{c}_L(a \wedge b)$.

(4) $\bigvee_i \mathfrak{o}_L(a_i) = \mathfrak{o}_L\big(\bigvee_i a_i\big)$ and $\bigcap_i \mathfrak{c}_L(a_i) = \mathfrak{c}_L\big(\bigvee_i a_i\big)$.

(5) $\mathrm{int}_L\big(\mathfrak{c}_L(a)\big) = \mathfrak{o}_L(a^*)$.

(6) $\mathrm{cl}_L\big(\mathfrak{o}_L(a)\big) = \mathfrak{c}_L(a^*)$.

3. On cozero complemented frames

In this section, we examine the r-ideals in the cozero complemented frames. We show that in these frames, every prime r-ideal of $\mathcal{R}(L)$ is a z^0-ideal, and every prime z^0-ideal in $\mathcal{R}(L)$ is a minimal prime ideal of $\mathcal{R}(L)$. Also, based on the r-ideal concept, we state and prove other equivalents for cozero complemented frames.

We recall from [21] that a space X is called a **cozero complemented space** if, for each cozero set B of X, there exists a cozero set D in X such that $B \cap D = \varnothing$ and $B \cup D$ is dense in X. These spaces were first studied in [21, 24], and they were also studied under the name of m-space in [6].

The cozero complemented frame was introduced and reviewed in [15]. A frame L has been defined in [15] to be **cozero complemented** if for every $c \in \mathrm{Coz}(L)$, there is $d \in \mathrm{Coz}(L)$ such that $c \wedge d = \bot$ and $c \vee d$ is dense. In [15], it was shown that a frame L is cozero complemented if and only if for each $\alpha \in \mathcal{R}(L)$, there is an element β in $\mathcal{R}(L) \setminus \mathrm{Zdv}\big(\mathcal{R}(L)\big)$ such that $\alpha\beta = \alpha^2$ if and only if for every $\alpha \in \mathcal{R}(L)$, there is $\beta \in \mathcal{R}(L)$ such that $\mathrm{coz}(\alpha)^{**} = \mathrm{coz}(\beta)^*$ (see [15, Corollary 3.2]).

Throughout this paper, for every $\alpha \in \mathcal{R}(L)$, we define

$$h(\alpha) := \Big\{P \in \mathrm{Min}\big(\mathcal{R}(L)\big) \colon \alpha \in P\Big\}$$

Then, we use the following lemma many times in proving propositions.

Lemma 3.1. *Let $\alpha \in \mathcal{R}(L)$ be given. Then, the following statements are equivalent:*

(1) $\mathrm{Ann}(\alpha) = (\mathbf{0})$.

(2) $\mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big)\Big) = \mathsf{O}$.

(3) $\big(\mathrm{coz}(\alpha)\big)^* = \bot$.

(4) $h(\alpha) = \emptyset$.

Proof. (1) ⇒ (2). We argue by contradiction. Let us assume that $\mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big)\Big) \neq \mathsf{O}$. Then, there exists an element $b \neq \top$ in $\mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big)\Big) = \mathfrak{o}_L\big(\mathrm{coz}(\alpha)^*\big)$. Hence, by [18, Proposition 3.4], there exists an element $\mathbf{0} \neq \delta$ in $\mathcal{R}^*(L)$ such that

$$\mathfrak{c}_L\Big(\big(\mathrm{coz}(\alpha)\big)^*\Big) \subseteq \mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\delta)\big)\Big) \subseteq \mathfrak{c}_L\big(\mathrm{coz}(\delta)\big).$$

Therefore, we have

$$\begin{aligned} L &= \mathfrak{c}_L\big(\mathrm{coz}(\alpha)^*\big) \vee \mathfrak{o}_L\big(\mathrm{coz}(\alpha)^*\big) \subseteq \mathfrak{c}_L\big(\mathrm{coz}(\delta)\big) \vee \mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big)\Big) \subseteq \mathfrak{c}_L\big(\mathrm{coz}(\delta)\big) \vee \mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \\ &= \mathfrak{c}_L\big(\mathrm{coz}(\delta) \wedge \mathrm{coz}(\alpha)\big) = \mathfrak{c}_L\big(\mathrm{coz}(\delta\alpha)\big), \end{aligned}$$

which implies that $\delta\alpha = \mathbf{0}$, and this is a contradiction.

(2) ⇒ (3). It is evident.

(3) ⇒ (4). We suppose, by way of contradiction, that $h(\alpha) \neq \emptyset$. Then there exists an element P in $\mathrm{Min}\big(\mathcal{R}(L)\big)$ such that $\alpha \in P$, which implies from [19, Corollary 1.2] that there is an element β in $\mathcal{R}(L) \setminus P$ such that $\alpha\beta = \mathbf{0}$, and we obtain

$$\mathrm{coz}(\beta) \leq \Big(\mathrm{coz}(\beta)\Big)^{**} = \Big(\mathrm{coz}(\alpha)\Big)^{**} \wedge \Big(\mathrm{coz}(\beta)\Big)^{**} = \Big(\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta)\Big)^{**} = \bot,$$

and this is a contradiction.

(4) ⇒ (1). Let us assume that $\mathrm{Ann}(\alpha) \neq (\mathbf{0})$. We are seeking a contradiction. Then there exists an element $\mathbf{0} \neq \beta$ in $\mathcal{R}(L)$ such that $\alpha\beta = \mathbf{0}$, which implies from $h(\alpha) = \emptyset$ that $\beta \in \bigcap \mathrm{Min}\big(\mathcal{R}(L)\big) = (\mathbf{0})$, which is a contradiction. □

In the following proposition, we examine the relationship between r-ideals and z-ideals in cozero complemented frames, and we give equivalent definitions for these frames.

Proposition 3.2. *The following statements are equivalent for a completely regular frame L:*

(1) Every r-ideal of $\mathcal{R}(L)$ is a z-ideal.

(2) Every prime r-ideal of $\mathcal{R}(L)$ is a z-ideal.

(3) For every $\alpha \in \mathcal{R}(L)$, there exists an element β in $\mathcal{R}(L)$ such that

$$\mathfrak{c}_L\Big(\mathrm{coz}(\alpha)\Big) \vee \mathfrak{c}_L\Big(\mathrm{coz}(\beta)\Big) = L \text{ and } \mathrm{int}_L\Big(\mathfrak{c}_L\Big(\mathrm{coz}(\alpha)\Big) \wedge \mathfrak{c}_L\Big(\mathrm{coz}(\beta)\Big)\Big) = \mathsf{O}.$$

(4) For every $\alpha \in \mathcal{R}(L)$, there exists an element β in $\mathcal{R}(L)$ such that

$$\mathrm{cl}_L\Big(\mathrm{int}_L\Big(\mathfrak{c}_L(\mathrm{coz}(\alpha))\Big)\Big) = \mathrm{cl}_L\Big(\mathfrak{o}_L(\mathrm{coz}(\beta))\Big).$$

(5) The frame L is a cozero complemented frame.

(6) For every $\alpha \in \mathcal{R}(L)$, there exists an element β in $\mathcal{R}(L)$ such that

$$\mathrm{cl}_L\Big(\mathfrak{o}_L\Big(\mathrm{coz}(\alpha)\Big) \vee \mathfrak{o}_L\Big(\mathrm{coz}(\beta)\Big)\Big) = L \text{ and } \mathfrak{o}_L\Big(\mathrm{coz}(\alpha)\Big) \wedge \mathfrak{o}_L\Big(\mathrm{coz}(\beta)\Big) = \mathsf{O}.$$

(7) For each $\alpha \in \mathcal{R}(L)$, $(\alpha)_r = (\alpha^2)_r$.

Proof. (1) ⇒ (2). It is evident.

(2) ⇒ (3). If $\alpha \in r\big(\mathcal{R}(L)\big)$, then it is enough to consider $\beta = 0$. Thus, let $\alpha \in \mathcal{R}(L) \setminus r\big(\mathcal{R}(L)\big)$ be given. Then, by [26, Theorem 2.20], if $P \in \mathrm{Min}((\alpha)_r)$, then it is an r-ideal of $\mathcal{R}(L)$, which implies from our hypothesis that it is a z-ideal of $\mathcal{R}(L)$. Hence, by [28, Corollary 7.2.2], $(\alpha)_r$ is a z-ideal, which implies that $\alpha^{\frac{1}{3}} \in (\alpha)_r$. In consequence, there exists an element γ in $r\big(\mathcal{R}(L)\big)$ such that $\gamma\alpha^{\frac{1}{3}} \in (\alpha)$, and we deduce that there exists an element δ in $\mathcal{R}(L)$ such that $\gamma\alpha^{\frac{1}{3}} = \alpha\delta$. We set $\beta := \gamma - \alpha^{\frac{2}{3}}\delta$. Now it is trivial that

$$\mathfrak{c}_L\Big(\mathrm{coz}(\alpha)\Big) \vee \mathfrak{c}_L\Big(\mathrm{coz}(\beta)\Big) = \mathfrak{c}_L\Big(\mathrm{coz}(\alpha\beta)\Big) = \mathfrak{c}_L(\mathbf{0}) = L.$$

Let $a \in \mathfrak{c}_L\Big(\mathrm{coz}(\alpha)\Big) \wedge \mathfrak{c}_L\Big(\mathrm{coz}(\gamma)\Big)$ be given. Then

$$\mathrm{coz}(\beta) = \mathrm{coz}(\gamma - \alpha^{\frac{2}{3}}\delta) \leq \Big(\mathrm{coz}(\gamma) \vee \mathrm{coz}(\alpha)\Big) \wedge \Big(\mathrm{coz}(\gamma) \vee \mathrm{coz}(\delta)\Big) \leq \Big(\mathrm{coz}(\gamma) \vee \mathrm{coz}(\alpha)\Big) \leq a,$$

which implies that $a \in \mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \wedge \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)$. Now, suppose that $a \in \mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \wedge \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)$. Then

$$\mathrm{coz}(\gamma) = \mathrm{coz}(\beta + \alpha^{\frac{2}{3}}\delta) \leq \big(\mathrm{coz}(\beta) \vee \mathrm{coz}(\alpha)\big) \wedge \big(\mathrm{coz}(\beta) \vee \mathrm{coz}(\delta)\big) \leq \big(\mathrm{coz}(\beta) \vee \mathrm{coz}(\alpha)\big) \leq a,$$

which implies that $a \in \mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \wedge \mathfrak{c}_L\big(\mathrm{coz}(\gamma)\big)$. Therefore,

$$\mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \wedge \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)\Big) = \mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \wedge \mathfrak{c}_L\big(\mathrm{coz}(\gamma)\big)\Big) \leq \mathrm{int}_L\mathfrak{c}_L\big(\mathrm{coz}(\gamma)\big) = \mathsf{O}.$$

(3) ⇒ (4). Let $\alpha \in \mathcal{R}(L)$ be given. Then, by our hypothesis, there exists an element β in $\mathcal{R}(L)$ such that $\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \vee \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big) = L$ and $\mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \wedge \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)\Big) = \mathsf{O}$, which implies that $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \bot$ and $\big(\mathrm{coz}(\alpha)\big)^* \wedge \big(\mathrm{coz}(\beta)\big)^* = \bot$. We deduce that $\big(\mathrm{coz}(\alpha)\big)^{**} = \big(\mathrm{coz}(\beta)\big)^*$. Therefore, $\mathrm{cl}_L\Big(\mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big)\Big)\Big) = \mathrm{cl}_L\Big(\mathfrak{o}_L\big(\mathrm{coz}(\beta)\big)\Big)$.

(4) ⇒ (5). Let $\alpha \in \mathcal{R}(L)$ be given. Then, by our hypothesis, there exists an element β in $\mathcal{R}(L)$ such that $\mathrm{cl}_L\Big(\mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big)\Big)\Big) = \mathrm{cl}_L\Big(\mathfrak{o}_L\big(\mathrm{coz}(\beta)\big)\Big)$, which implies that $\big(\mathrm{coz}(\alpha)\big)^{**} = \big(\mathrm{coz}(\beta)\big)^*$. Therefore, L is a cozero complemented frame.

(5) ⇒ (6). Let $\alpha \in \mathcal{R}(L)$ be given. Then, by our hypothesis, there exists an element β in $\mathcal{R}(L)$ such that $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \bot$ and $\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta)$ is a dense element of L, which implies that

$$\mathrm{cl}_L\Big(\mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big) \vee \mathfrak{o}_L\big(\mathrm{coz}(\beta)\big)\Big) = \mathfrak{c}_L\Big(\big(\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta)\big)^*\Big) = L$$

and

$$\mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big) \wedge \mathfrak{o}_L\big(\mathrm{coz}(\beta)\big) = \mathfrak{o}_L\big(\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta)\big) = \mathsf{O}.$$

(6) ⇒ (5) and (6) ⇒ (7). Let $\alpha \in \mathcal{R}(L)$ be given. Then, by our hypothesis, there exists an element β in $\mathcal{R}(L)$ such that

$$\mathfrak{c}_L\Big(\big(\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta)\big)^*\Big) = \mathrm{cl}_L\Big(\mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big) \vee \mathfrak{o}_L\big(\mathrm{coz}(\beta)\big)\Big) = L$$

and

$$\mathfrak{o}_L\big(\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta)\big) = \mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big) \wedge \mathfrak{o}_L\big(\mathrm{coz}(\beta)\big) = \mathsf{O},$$

which implies that $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \bot$ and $\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta)$ is a dense element of L. Therefore, L is a cozero complemented frame. Thus, by [15, Proposition 1.1], there is a nonzero-divisor γ in $\mathcal{R}(L)$ such that $\alpha\gamma = \alpha^2$. It is evident that $(\alpha^2)_r \subseteq (\alpha)_r$. Now, suppose that $\mu \in (\alpha)_r$. Then there exists an element τ in $r(\mathcal{R}(L))$ such that $\mu\tau \in (\alpha)$, which implies that there exists an element δ in $\mathcal{R}(L)$ such that $\mu\tau = \delta\alpha$. We conclude that $\mu\tau\gamma = \delta\alpha\gamma = \delta\alpha^2 \in (\alpha^2)$, and so $\mu \in (\alpha^2)_r$. Therefore, $(\alpha^2)_r = (\alpha)_r$.

(7) ⇒ (5) and (7) ⇒ (1). Let $\alpha \in \mathcal{R}(L)$ be given. Then, by our hypothesis, there exists an element β in $r\big(\mathcal{R}(L)\big)$ such that $\alpha^2 = \beta\alpha$. Therefore, L is a cozero complemented frame.

Now, suppose that I is an r-ideal of $\mathcal{R}(L)$. Let $\alpha, \gamma \in I \times \mathcal{R}(L)$ with $\mathrm{coz}(\alpha) = \mathrm{coz}(\gamma)$ be given. Then, there exists an element β in $\mathcal{R}(L)$ such that $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \bot$ and $\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta)$ is a dense element of L, which implies from Lemma 3.1 that $\alpha\beta = 0$ and $\delta := \alpha^2 + \beta^2 \in \mathrm{r}\big(\mathcal{R}(L)\big)$. We deduce that $\gamma\beta = 0$, and so from $\gamma\delta = \gamma\alpha^2 \in I$, we conclude that $\gamma \in I$. Therefore, I is a z-ideal of $\mathcal{R}(L)$. □

In the following corollary, according to Proposition 3.2, for an arbitrary ideal I in $\mathcal{R}(L)$, where L is a cozero complemented frame, we express the relationship between the smallest r-ideal containing I and the smallest z-ideal containing I.

Corollary 3.3. *The following statements are equivalent for a completely regular frame L:*

(1) A frame L is a cozero complemented frame.

(2) Every r-ideal of $\mathcal{R}(L)$ is semiprime.

(3) For each ideal I of $\mathcal{R}(L)$, $I_z \subseteq I_r$.

Proof. (1) ⇒ (2). Let I be an r-ideal of $\mathcal{R}(L)$. Then, by Proposition 3.2, I is a z-ideal of $\mathcal{R}(L)$, which implies that it is a semiprime ideal of $\mathcal{R}(L)$.

(2) ⇒ (1). By our hypothesis, $(\alpha^2)_r$ is semiprime for each $\alpha \in \mathcal{R}(L)$, which implies that $\alpha \in (\alpha^2)_r$. Therefore, for each $\alpha \in \mathcal{R}(L)$, $(\alpha)_r = (\alpha^2)_r$ and by using Proposition 3.2, L is a cozero complemented frame.

(2) ⇒ (3). Let I be an ideal of $\mathcal{R}(L)$. Then, by Proposition 3.2, every prime r-ideal of $\mathcal{R}(L)$ is a z-ideal, which implies that $I_z \subseteq I_r$.

(3) ⇒ (2). If I is an r-ideal of $\mathcal{R}(L)$, then, by our hypothesis, $I_z \subseteq I_r = I$, which implies that $I = I_z$, that is, I is a z-ideal. We deduce that I is a semiprime ideal of $\mathcal{R}(L)$. □

To state the next proposition, we first prove the following lemma and show that a frame L is a cozero completed frame if and only if every prime z^0-ideal in $\mathcal{R}(L)$ is a minimal prime ideal.

Lemma 3.4. *The following statements are true, for every $\alpha, \beta \in \mathcal{R}(L)$:*

(1) $h(\alpha) \cap h(\beta) = \emptyset$ if and only if $h(\beta) \subseteq h\big(\mathrm{Ann}(\alpha)\big)$.

(2) $h\big(\mathrm{Ann}(\alpha)\big) \subseteq h(\beta)$ if and only if $\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \vee \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big) = L$.

(3) $h(\beta) \subseteq h\big(\mathrm{Ann}(\alpha)\big)$ if and only if $\mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \wedge \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)\Big) = \mathsf{O}$.

Proof. (1). *Necessity.* Let $P \in h(\beta)$ be given. Then, by our hypothesis, $\alpha \notin P$. Hence, we have

$$\delta \in \mathrm{Ann}(\alpha) \Rightarrow \delta\alpha = 0 \in P \Rightarrow \delta \in P.$$

Therefore, $h(\beta) \subseteq h\big(\mathrm{Ann}(\alpha)\big)$.

Sufficiency. We proceed by contradiction. Assume that $h(\alpha) \cap h(\beta) \neq \emptyset$. Then there exists an element P in $h(\alpha) \cap h(\beta)$, which implies from our hypothesis that $P \in h\big(\mathrm{Ann}(\alpha)\big)$. By [19, Theorem 2.3], $h(\alpha)$ and $h\big(\mathrm{Ann}(\alpha)\big)$ are disjoint open and closed sets, but this is a contradiction to the fact that $P \in h(\alpha) \cap h\big(\mathrm{Ann}(\alpha)\big)$.

(2). *Necessity.* Since $\mathrm{Ann}(\alpha)$ is a z-ideal of $\mathcal{R}(L)$, we infer from our hypothesis that

$$\beta \in \bigcap h(\beta) \subseteq \bigcap h\big(\mathrm{Ann}(\alpha)\big) = \mathrm{Ann}(\alpha)$$

which implies that

$$\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \vee \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big) = \mathfrak{c}_L\big(\mathrm{coz}(\alpha\beta)\big) = \mathfrak{c}_L(0) = L.$$

Sufficiency. Let $P \in h\big(\mathrm{Ann}(\alpha)\big)$ be given. Then, by [19, Theorem 2.3], $P \in \mathrm{Min}\big(\mathcal{R}(L)\big) \setminus h(\alpha)$, which implies that $\alpha \notin P$. Since, by our hypothesis, $\alpha\beta = 0 \in P$, we conclude that $P \in h(\beta)$. Thus, $h\big(\mathrm{Ann}(\alpha)\big) \subseteq h(\beta)$.

(3). We always have $h(\beta) \subseteq h\big(\mathrm{Ann}(\alpha)\big)$ if and only if $h(\alpha) \cap h(\beta) = \emptyset$ if and only if $h(\alpha^2 + \beta^2) = \emptyset$ if and only if, by Lemma 3.1,

$$\mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \wedge \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)\Big) = \mathrm{int}_L \mathfrak{c}_L\big(\mathrm{coz}(\alpha^2 + \beta^2)\big) = \mathsf{O}.$$

□

We recall from [5] that for every a in a ring R, $P_a = \bigcap h(a)$. Also, an ideal I in a commutative ring R is said to be a z^0-ideal if I consists of zero-divisors and for each $a \in I$, the intersection of all minimal prime ideals containing a is contained in I (for any $a \in I$ implies that $P_a \subseteq I$). In a ring $\mathcal{R}(L)$, by [4, Proposition 1.5], we have

$$P_\alpha = \big\{\beta \in \mathcal{R}(L)\colon \operatorname{Ann}(\alpha) \subseteq \operatorname{Ann}(\beta)\big\},$$

and by [13, Lemma 4.1], we have

$$P_\alpha = \Big\{\beta \in \mathcal{R}(L)\colon \big(\operatorname{coz}(\alpha)\big)^* \leq \big(\operatorname{coz}(\beta)\big)^*\Big\}$$

for every $\alpha \in \mathcal{R}(L)$. Also, we recall from [1] that $P_\alpha \cap P_\beta = P_{\alpha\beta}$ and $P_\alpha + P_\beta \subseteq P_{\alpha^2+\beta^2}$ for every $\alpha, \beta \in \mathcal{R}(L)$.

Proposition 3.5. *The following statements are equivalent for a completely regular frame L:*

(1) The frame L is a cozero complemented frame.

(2) Every prime z^0-ideal in $\mathcal{R}(L)$ is a minimal prime ideal.

(3) For every $\alpha \in \mathcal{R}(L)$, there exists an element β in $\mathcal{R}(L)$ such that

$$\mathfrak{c}_L\big(\operatorname{coz}(\alpha)\big) \vee \mathfrak{c}_L\big(\operatorname{coz}(\beta)\big) = L \textit{ and } \operatorname{int}_L\Big(\mathfrak{c}_L\big(\operatorname{coz}(\alpha)\big) \wedge \mathfrak{c}_L\big(\operatorname{coz}(\beta)\big)\Big) = \mathsf{O}.$$

Proof. (1) ⇒ (2). Let a prime z^0-ideal P be given. Suppose that prime ideal Q in $\mathcal{R}(L)$ such that $Q \subseteq P$ with $Q \neq P$. Then, there exists an element $\alpha \in P \setminus Q$, which implies by our hypothesis that there is an element β in $\mathcal{R}(L)$ such that $\mathfrak{c}_L\big(\operatorname{coz}(\alpha\beta)\big) = \mathfrak{c}_L\big(\operatorname{coz}(\alpha)\big) \vee \mathfrak{c}_L\big(\operatorname{coz}(\beta)\big) = L$ and $\operatorname{int}_L\big(\mathfrak{c}_L\big(\operatorname{coz}(\alpha)\big) \wedge \mathfrak{c}_L\big(\operatorname{coz}(\beta)\big) = \mathsf{O}$. Since $0 = \alpha\beta \in Q \subseteq P$, we deduce that $\beta \in Q \subseteq P$. Hence $\alpha^2 + \beta^2 \in P$. On the other hand, we have

$$\operatorname{int}_L\Big(\mathfrak{c}_L\big(\operatorname{coz}(\alpha^2+\beta^2)\big)\Big) = \operatorname{int}_L\Big(\mathfrak{c}_L\big(\operatorname{coz}(\alpha)\big) \wedge \mathfrak{c}_L\big(\operatorname{coz}(\beta)\big)\Big) = \mathsf{O} = \operatorname{int}_L\big(\mathfrak{c}_L(\mathbf{1})\big).$$

Since P is a z^0-ideal, it follows that $1 \in P$, and this is a contradiction. Therefore, P is a minimal prime ideal.

(2) ⇒ (3). Let $\alpha \in \mathcal{R}(L)$ be given. Then, by [6, Proposition 1.5], there exists an element β in $\mathcal{R}(L)$ such that $\operatorname{Ann}(\alpha) = P_\beta$. It is evident that $h(\beta) = h(P_\beta) = h\big(\operatorname{Ann}(\alpha)\big)$, which implies from Lemma 3.4 that

$$\mathfrak{c}_L\big(\operatorname{coz}(\alpha)\big) \vee \mathfrak{c}_L\big(\operatorname{coz}(\beta)\big) = L \text{ and } \operatorname{int}_L\Big(\mathfrak{c}_L\big(\operatorname{coz}(\alpha)\big) \wedge \mathfrak{c}_L\big(\operatorname{coz}(\beta)\big)\Big) = \mathsf{O}.$$

(3) ⇒ (1). By Proposition 3.2, it is evident. □

In the last result of this section, we derive another equivalent for cozero complemented frames based on the notion of r-ideal, which shows that there exists a prime r-ideal that is not z^0-ideal.

Corollary 3.6. *A frame L is a cozero complemented frame if and only if every prime r-ideal of $\mathcal{R}(L)$ is a z^0-ideal.*

Proof. Necessity. Let I be a prime r-ideal of $\mathcal{R}(L)$ with $\big(\operatorname{coz}(\alpha)\big)^* = \big(\operatorname{coz}(\beta)\big)^*$ for $(\alpha, \beta) \in I \times \mathcal{R}(L)$. According to our assumption and Proposition 3.2, there exists $\delta \in \mathcal{R}(L)$ such that

$$\mathfrak{c}_L\big(\operatorname{coz}(\beta)\big) \vee \mathfrak{c}_L\big(\operatorname{coz}(\delta)\big) = L \quad \text{and} \quad \operatorname{int}_L\Big(\mathfrak{c}_L\big(\operatorname{coz}(\beta)\big)\Big) \wedge \operatorname{int}_L\Big(\mathfrak{c}_L\big(\operatorname{coz}(\delta)\big)\Big) = \mathsf{O}.$$

Thus, $\beta\delta = \mathbf{0}$ and $(\beta^2 + \delta^2) \in \mathrm{r}\big(\mathcal{R}(L)\big)$. Since $\operatorname{int}_L\Big(\mathfrak{c}_L\big(\operatorname{coz}(\beta)\big)\Big) = \operatorname{int}_L\Big(\mathfrak{c}_L\big(\operatorname{coz}(\alpha)\big)\Big)$, it is obtained that $(\alpha^2 + \delta^2) \in \mathrm{r}\big(\mathcal{R}(L)\big)$. Since I is a prime ideal of $\mathcal{R}(L)$ and $\beta\delta \in I$, it is obtained that $\beta \in I$ or $\delta \in I$. If $\delta \in I$, then $(\alpha^2 + \delta^2) \in I$, which contradicts with I being an r-ideal. Therefore, $\beta \in I$.

Sufficiency. It is clear by Proposition 3.2.
□

Remark 3.7. The converse of parts (1) and (6) of [8, Lemma 2.2] is not necessarily true. For this, since $\mathrm{coz}(\alpha)\vee\left(\mathrm{coz}(\alpha)\right)^{*}=\top$ for every $\alpha\in\mathcal{R}(L)$ if and only if $\mathfrak{c}_L\left(\mathrm{coz}(\alpha)\right)\subseteq\mathfrak{o}_L\left(\left(\mathrm{coz}(\alpha)\right)^{*}\right)=\mathrm{int}_L\left(\mathfrak{c}_L\left(\mathrm{coz}(\alpha)\right)\right)$, therefore $\mathfrak{c}_L\left(\mathrm{coz}(\alpha)\right)$ is open for every $\alpha\in\mathcal{R}(L)$ if and only if L is a P-frame (see [9, Defnition 8.4.6]).

Now suppose that L is a cozero complemented frame and it is not a P-frame. Therefore, there exists a nonzero element $\alpha\in\mathcal{R}(L)\smallsetminus\mathrm{r}\left(\mathcal{R}(L)\right)$ such that $\mathfrak{c}_L\left(\mathrm{coz}(\alpha)\right)$ is not open. Since L is a cozero complemented frame by Proposition 3.2, $\left(\mathrm{coz}(\alpha)\right)_r$ is a z-ideal. Indeed $\left(\mathrm{coz}(\alpha)\right)$ is not a semiprime ideal because $\mathfrak{c}_L\left(\mathrm{coz}(\alpha)\right)$ is not open.

4. Some of the connections between almost *P*-frames and *r*-ideals

We recall from [9, Defnition 8.4.6] that a frame L is called a P-frame if $a\vee a^{*}=\top$ for every $a\in\mathrm{Coz}(L)$. A frame L is said to be an almost P-frame if $a=a^{**}$ for all $a\in\mathrm{Coz}(L)$. Almost P-frames first appeared in [9] and were also studied in [13, 20]. Dube [13] showed that a frame L is an almost P-frame if and only if $\mathcal{R}(L)=\mathrm{Zdv}(\mathcal{R}(L))\cup\mathrm{Inv}\left(\mathcal{R}(L)\right)$, where $\mathrm{Zdv}\left(\mathcal{R}(L)\right)$ denotes the set of all zero-divisor elements of $\mathcal{R}(L)$ and $\mathrm{Inv}\left(\mathcal{R}(L)\right)$ denotes the set of all invertible elements of $\mathcal{R}(L)$.

It has already been shown that frame L is an almost P-frame if and only if every z-ideal is a z^0-ideal (see [13, Proposition 4.13]). In the following proposition, we state another proof based on the concept of r-ideals. We also express and prove other equivalents for these frames in the following proposition.

Proposition 4.1. *The following statements are equivalent for a completely regular frame L:*

(1) A frame L is an almost P-frame.

(2) Every proper ideal in $\mathcal{R}(L)$ is an r-ideal.

(3) Every z-ideal in $\mathcal{R}(L)$ is a z^0-ideal.

(4) Every z-ideal in $\mathcal{R}(L)$ is an r-ideal.

(5) For each ideal I of $\mathcal{R}(L)$, $I_r\subseteq I_z$.

(6) Every prime z-ideal of $\mathcal{R}(L)$ is an r-ideal.

(7) Every maximal ideal of $\mathcal{R}(L)$ is an r-ideal.

Proof. (1) ⇒ (2). Let I be an ideal in $\mathcal{R}(L)$ and let $(\alpha,\tau)\in\mathcal{R}(L)\times\mathrm{r}(\mathcal{R}(L))$ with $\alpha\tau\in I$ be given. Then, by our hypothesis, τ is an invertible element in $\mathcal{R}(L)$, which implies that $\alpha\in I$. Therefore, I is an r-ideal.

(2) ⇒ (3). First, we show that for every $\alpha\in\mathcal{R}(L)$, if $\left(\mathrm{coz}(\alpha)\right)^{*}=\bot$, then $\mathrm{coz}(\alpha)=\top$. Then we show that for every $\alpha\in\mathcal{R}(L)$, $\left(\mathrm{coz}(\alpha)\right)^{**}=\mathrm{coz}(\alpha)$. Let $\alpha\in\mathcal{R}(L)$ with $\left(\mathrm{coz}(\alpha)\right)^{*}=\bot$ be given. Then

$$\mathrm{int}_L\left(\mathfrak{c}_L\left(\mathrm{coz}(\alpha)\right)\right)=\mathfrak{o}_L\left(\left(\mathrm{coz}(\alpha)\right)^{*}\right)=\mathfrak{o}_L(\bot)=\mathsf{O},$$

which implies that $\alpha\in r\left(\mathcal{R}(L)\right)$. If $\mathrm{coz}(\alpha)\neq\top$, then (α) is a proper ideal, which implies from our hypothesis that (α) is an r-ideal in $\mathcal{R}(L)$. Since $(\alpha)\cap r\left(\mathcal{R}(L)\right)\neq\emptyset$, we conclude that $(\alpha)=(\alpha)_r=\mathcal{R}(L)$, which is a contradiction. Therefore, $\mathrm{coz}(\alpha)=\top$.

Let $\alpha\in\mathcal{R}(L)$ be given. Then, we have

$$\begin{aligned}x\ll\left(\mathrm{coz}(\alpha)\right)^{**}&\Rightarrow\text{There exists }\beta\in\mathcal{R}(L)\left(\mathrm{coz}(\beta)\wedge x=\bot\text{ and }\mathrm{coz}(\beta)\vee\left(\mathrm{coz}(\alpha)\right)^{**}=\top\right)\\&\Rightarrow\text{There exists }\beta\in\mathcal{R}(L)\,(\mathrm{coz}(\beta)\wedge x=\bot\text{ and}\\&\qquad\left(\mathrm{coz}(\beta)\vee\mathrm{coz}(\alpha)\right)^{*}=\left(\mathrm{coz}(\beta)\vee\left(\mathrm{coz}(\alpha)\right)^{**}\right)^{*}=\bot)\\&\Rightarrow\text{There exists }\beta\in\mathcal{R}(L)\,(\mathrm{coz}(\beta)\wedge x=\bot\text{ and }\mathrm{coz}(\beta)\vee\mathrm{coz}(\alpha)=\top)\\&\Rightarrow x\prec\mathrm{coz}(\alpha).\end{aligned}$$

Hence, $\left(\mathrm{coz}(\alpha)\right)^{**} = \bigvee_{x \ll \left(\mathrm{coz}(\alpha)\right)^{**}} x \leq \bigvee_{x \prec \mathrm{coz}(\alpha)} x = \mathrm{coz}(\alpha) \leq \left(\mathrm{coz}(\alpha)\right)^{**}$, which implies that $\mathrm{coz}(\alpha) = \left(\mathrm{coz}(\alpha)\right)^{**}$.

Let I be a z-ideal and let $(\alpha, \beta) \in I \times \mathcal{R}(L)$ with $\left(\mathrm{coz}(\alpha)\right)^{*} = \left(\mathrm{coz}(\beta)\right)^{*}$ be given. Then $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$, which implies that $\beta \in I$. Hence, I is a z^0-ideal.

(3) ⇒ (4). It is evident.

(4) ⇒ (5). Using our hypothesis, I_z is an r-ideal containing I, and so $I_r \subseteq I_z$.

(5) ⇒ (6). If P is a prime z-ideal, then $P_r \subseteq P_z = P$, which implies that P is an r-ideal.

(6) ⇒ (7). It is evident.

(7) ⇒ (1). Suppose that L is not an almost P-frame. Then, there is an element α in $\mathrm{r}(\mathcal{R}(L)) \setminus \mathrm{Inv}\left(\mathcal{R}(L)\right)$, which implies that there is a maximal ideal M of $\mathcal{R}(L)$ such that $(\alpha) \subseteq M$. Now, by our hypothesis, M is an r-ideal. This is a contradiction, since $\alpha \in M$ is a nonzero-divisor element. Therefore, L is an almost P-frame. □

Example 4.2. Suppose that L is not an almost P-frame. Then, there exists an element α in $\mathrm{r}\left(\mathcal{R}(L)\right)$ such that it is a noninvertible element in $\mathcal{R}(L)$. Consequently, there is a maximal ideal M in $\mathcal{R}(L)$ such that $(\alpha) \subseteq M$. So, M is a prime z-ideal, which is not an r-ideal.

According to Proposition 3.2, if a frame L is not a cozero complemented frame, then there is a prime r-ideal such that is not a z-ideal, or if the frame L is not a P-frame but is an almost P-frame, then by [3, Theorem 4.1], there is a prime ideal Q such that it is not a z-ideal. On the other hand, by Proposition 4.1, Q is an r-ideal. Then Q is a prime r-ideal such that it is not a z-ideal. Also, according to Proposition 4.1, if a frame L is not an almost P-frame, then there is a prime z-ideal such that it is not an r-ideal.

For an arbitrary ideal I in the ring $\mathcal{R}(L)$, we see the relation between I_r and I_z in Corollary 3.3 and Proposition 4.1. In the next corollary, we show that, in P-frames, every r-ideal is a z-ideal and vice versa, that is, $I_r = I_z$.

Corollary 4.3. *The following statements are equivalent for a completely regular frame L:*

(1) *The frame L is a P-frame.*

(2) *The frame L is a cozero complemented frame and almost P-frame.*

(3) *For every ideal I of $\mathcal{R}(L)$, it is a z-ideal of $\mathcal{R}(L)$ if and only if it is an r-ideal of $\mathcal{R}(L)$.*

(4) *For each ideal I of $\mathcal{R}(L)$, $I_z = I_r$.*

Proof. (1) ⇒ (2). From [12, Proposition 3.9] and Proposition 3.2, L is a cozero complemented frame. Also, from [13, Proposition 3.3], L is an almost P-frame.

(2) ⇒ (3). Let I be an ideal of $\mathcal{R}(L)$. Then, by Proposition 4.1, I is an r-ideal of $\mathcal{R}(L)$, which implies form Proposition 3.2 that I is a z-ideal of $\mathcal{R}(L)$. Hence, for every ideal I of $\mathcal{R}(L)$, it is an r-ideal of $\mathcal{R}(L)$ and also, it is a z-ideal of $\mathcal{R}(L)$.

(3) ⇒ (4). It is evident.

(4) ⇒ (2) and (4) ⇒ (1). By Propositions 3.3 and 4.1, L is a cozero complemented frame and an almost P-frame. Let I be a proper ideal of $\mathcal{R}(L)$. Then, by Proposition 4.1, I is an r-ideal of $\mathcal{R}(L)$, which implies form Proposition 3.2 that I is a z-ideal of $\mathcal{R}(L)$. Therefore, by [12, Proposition 3.9], L is a P-frame. □

It was shown in [26] that the intersection of any family of r-ideals is an r-ideal, but their product and sum are not necessarily an r-ideal. In the following lemma and proposition, we will investigate what happens if the product or sum of a prime ideal in another ideal becomes an r-ideal. For frames that are almost P-frames, we give another equivalent.

Lemma 4.4. *Let R be a reduce commutative ring and let $(I, P) \in \mathrm{Id}(R) \times \mathrm{Spec}(R)$. Then, the following statements are true:*

(1) *If IP is an r-ideal, then I or P is an r-ideal.*

(2) *If IP is an r-ideal and $I \nsubseteq P$, then P is an r-ideal.*

(3) *If $I \cap P$ is an r-ideal, then I or P is an r-ideal.*

(4) *Let I and P be prime ideals that are not in a chain. If $I \cap P$ is an r-ideal, then I and P are r-ideals.*

Proof. (1). It is evident that if $P \cap r(R) = \emptyset$, then P is an r-ideal. Now, suppose that $r \in P \cap r(R)$. Then for every $i \in I$, $ir \in IP$, which implies that $i \in IP$, and we get that $IP = I$ is an r-ideal.

(2). Let $(a, b) \in r(R) \times R$ with $ab \in P$ be given. By our hypothesis, there exists an element i in $I \setminus P$, such that $iab \in IP$, which implies that $ib \in IP \subseteq P$. We obtain $b \in P$. Therefore, P is an r-ideal.

(3). If $I \subseteq P$, that is $I \cap P = I$, then, by our hypothesis, I is an r-ideal. Now, suppose that $I \nsubseteq P$. Then, there exists an element i in $I \setminus P$. Let $(a, b) \in r(R) \times R$ with $ab \in P$ be given. Then, $iab \in I \cap P$, which implies that $ib \in I \cap P \subseteq P$, and we obtain $b \in P$. Therefore, P is an r-ideal.

(4). The proof is similar to the proof of part (3). □

Proposition 4.5. *The following statements are equivalent for a completely regular frame L:*

(1) The frame L is an almost P-frame.

(2) For every $(I, P) \in \mathrm{Id}\big(\mathcal{R}(L)\big) \times \mathrm{Spec}\big(\mathcal{R}(L)\big)$, if $I \cap P$ is an r-ideal in $\mathcal{R}(L)$, then I and P are r-ideals.

(3) For every $(I, P) \in \mathrm{Id}\big(\mathcal{R}(L)\big) \times \mathrm{Spec}\big(\mathcal{R}(L)\big)$, if IP is an r-ideal in $\mathcal{R}(L)$, then I and P are r-ideals.

Proof. By proposition 4.1, (1) ⇒ (2) and (1) ⇒ (3) are evident.

(2) ⇒ (1). Suppose that L is not an almost P-frame. Then, there is an element α in $\mathrm{r}(\mathcal{R}(L)) \setminus \mathrm{Inv}\big(\mathcal{R}(L)\big)$, which implies that there is a maximal ideal M of $\mathcal{R}(L)$ such that $(\alpha) \subseteq M$ and $M_r = \mathcal{R}(L)$. Let Q be a minimal prime ideal of $\mathcal{R}(L)$ such that $Q \subseteq M$. Then, by [26, Remark 2.3], $Q \cap M = Q$ is an r-ideal, which implies from our hypothesis that M is an r-ideal, and this is a contradiction.

(3) ⇒ (1). Suppose that L is not an almost P-frame. Then, there exists an element α in $\mathrm{r}\big(\mathcal{R}(L)\big) \setminus \mathrm{Inv}\big(\mathcal{R}(L)\big)$, which implies that there exists an element I in $\Sigma\beta L$ such that $(\alpha) \subseteq M^I$. It is evident that $O^I = O^I \cap M^I = O^I M^I$ is a z^0-ideal in $\mathcal{R}(L)$, which implies from [26, Theorem 2.19] that $O^I M^I$ is an r-ideal in $\mathcal{R}(L)$. Then, by our hypothesis, M^I is an r-ideal in $\mathcal{R}(L)$, and this is a contradiction to the fact that $\alpha \in M^I \cap \mathrm{r}\big(\mathcal{R}(L)\big)$. Therefore, L is an almost P-frame. □

A **weakly almost *P*-space** is a topological space X such that for every two zerosets Z and F with $\mathrm{int}\, Z \subseteq \mathrm{int}\, F$, there exists a zeroset E in X with empty interior such that $Z \subseteq F \cup E$. This space was studied for the first time in [6]. Every almost P-space is a weakly almost P-space. More generally, any space in which every closed set (boundary of any zeroset) is contained in a zeroset with empty interior (for example, a metric space), is a weakly almost P-space. In 2015, the concept of weak almost P-frame and some of its features were studied and investigated [16]. It was shown that if βL is a weak almost P-frame, so is L (see [16, Corollary 2.10]), and conversely, if L is a continuous Lindeläof frame, so is βL (see [16, Proposition 2.12]). We recall from [16, Definition 2.1] that a completely regular frame L is a **weak almost *P*-frame** if a and b are cozero elements of L with $a^* \leq b^*$, then there is a dense cozero element c such that $b \wedge c \leq a$. Every almost P-frame and every cozero complemented frame is a weakly almost P-frame (see [16, Examples 2.2 and 2.3]).

In the following proposition, we express and prove a definition equivalent to weakly almost P-frames based on closed sublocales.

Proposition 4.6. *A frame L is a weakly almost P-frame if and only if for every $\alpha, \beta \in \mathcal{R}(L)$ with $\mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big)\Big) \subseteq \mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)\Big)$, there exists an element γ in $\mathrm{r}\big(\mathcal{R}(L)\big)$ such that $\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \subseteq \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big) \vee \mathfrak{c}_L\big(\mathrm{coz}(\gamma)\big)$.*

Proof. *Necessity.* Suppose L is a weakly almost P-frame. Let $\alpha, \beta \in \mathcal{R}(L)$ with $\mathrm{int}_L\big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big)\big) \subseteq \mathrm{int}_L\big(\mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)\big)$ be given. Then $\big(\mathrm{coz}(\alpha)\big)^* \le \big(\mathrm{coz}(\beta)\big)^*$, which implies from our hypothesis that there exists an element γ in $\mathcal{R}(L)$ with $\big(\mathrm{coz}(\gamma)\big)^* = \bot$ such that $\mathrm{coz}(\gamma) \wedge \mathrm{coz}(\beta) \le \mathrm{coz}(\alpha)$. We deduce that $\gamma \in \mathrm{r}\big(\mathcal{R}(L)\big)$ and

$$\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \subseteq \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big) \vee \mathfrak{c}_L\big(\mathrm{coz}(\gamma)\big).$$

Sufficiency. Let $\alpha, \beta \in \mathcal{R}(L)$ with $\big(\mathrm{coz}(\alpha)\big)^* \le \big(\mathrm{coz}(\beta)\big)^*$ be given. Then,

$$\mathrm{int}_L\big(\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big)\big) = \mathfrak{o}_L\big(\big(\mathrm{coz}(\alpha)\big)^*\big) \subseteq \mathfrak{o}_L\big(\big(\mathrm{coz}(\beta)\big)^*\big) = \mathrm{int}_L\big(\mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)\big),$$

which implies from our hypothesis that there exists an element γ in $\mathrm{r}\big(\mathcal{R}(L)\big)$ such that $\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \subseteq \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big) \vee \mathfrak{c}_L\big(\mathrm{coz}(\gamma)\big)$. We deduce that $\big(\mathrm{coz}(\gamma)\big)^* = \bot$ and $\mathrm{coz}(\gamma) \wedge \mathrm{coz}(\beta) \le \mathrm{coz}(\alpha)$. Therefore, L is a weakly almost P-frame. □

Below we give an example of the connection between the r-ideal and the classical ideals of the ring $\mathcal{R}(L)$ in weakly almost P-frames.

Example 4.7. By [16, Proposition 3.1], if L is not a weakly almost P-frame, then there exists a prime z-ideal P in $\mathcal{R}(L)$ with $P \cap \mathrm{r}\big(\mathcal{R}(L)\big) = \emptyset$, which is not a z^0-ideal. On the other hand, by [26, Remark 2.3(f)], P is an r-ideal. So if L is not a weakly almost P-frame, there exists an r-ideal that is a z-ideal but not a z^0-ideal.

Examples 4.8. By the definition of an r-ideal, every element of a proper r-ideal is a zero-divisor element. Below are some examples that show that the above statement is not always true in the ring $\mathcal{R}(L)$.

- For each $(a, r) \in \mathrm{Zdv}(R), \times \mathrm{r}(R)$ in any reduced ring R, we have $(a)_r = (ra)_r$ (see [8, Remark 2.4]). Now, we assume that $(\alpha, \beta) \in \mathrm{r}\big(\mathcal{R}(L)\big) \times \mathrm{Zdv}\big(\mathcal{R}(L)\big)$ such that $\mathrm{coz}(\alpha) \nleq \mathrm{coz}(\beta)$. Therefore, every element of $(\alpha\beta)$ is a zero-divisor element, but $(\alpha\beta)$ is not an r-ideal. Since if $(\alpha\beta)$ is an r-ideal, then $(\alpha)_r = (\alpha\beta)_r = (\alpha\beta)$ implies $(\alpha) \subseteq (\alpha\beta)$, which is a contradiction.

- Suppose $\alpha \notin \mathrm{r}\big(\mathcal{R}(L)\big)$ and $\beta \in \mathrm{r}\big(\mathcal{R}(L)\big)$ such that $\big(\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta)\big) = \top$. Therefore, every element $I = (\alpha\beta)$ is a zero-divisor element, but I is not an r-ideal. For this, suppose I is an r-ideal. Then $I_r = I$ implies that $\alpha\beta \in I$. Since $\beta \in \mathrm{r}\big(\mathcal{R}(L)\big)$ implies that $\alpha \in I$. Therefore, there is $\delta \in \mathcal{R}(L)$ such that $\alpha = \alpha\beta\delta$, which implies $\mathrm{coz}(\alpha) \le \mathrm{coz}(\beta)$. So it is followed

  $$\mathfrak{c}_L\big(\mathrm{coz}(\beta)\big) = \mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \wedge \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big) = \mathfrak{c}_L\big(\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta)\big) = \mathfrak{c}_L(\top) = \mathsf{O}.$$

  Therefore, $(\mathrm{coz}(\beta) = \top$, which is a contradiction.

- Suppose that $(\alpha, \beta) \in \mathrm{r}\big(\mathcal{R}(L)\big) \times \mathcal{R}(L)$ are noninvertible such that $\mathfrak{c}_L\big(\mathrm{coz}(\beta)\big) \subseteq \mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big)$ and $\big(\mathrm{coz}(\beta)\big)^{**} = \mathrm{coz}(\beta)$. We consider $J := \big\{\gamma \in \mathcal{R}(L) \colon \mathrm{coz}(\gamma) \le \mathrm{coz}(\alpha\beta)\big\}$. Therefore, J is a z-ideal of $\mathcal{R}(L)$ consisting entirely of zero-divisors which it is not r-ideal. It is clear that J is a z-ideal and $\alpha\beta \in J$. Now suppose by contradiction that $\gamma \in J \cap \mathrm{r}\big(\mathcal{R}(L)\big)$. So, by Proposition 3.1, we have

  $$\bot = \big(\mathrm{coz}(\gamma)\big)^* \ge \big(\mathrm{coz}(\alpha\beta)\big)^* \ge \big(\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta)\big)^* \ge \big(\mathrm{coz}(\alpha)\big)^* \vee \big(\mathrm{coz}(\beta)\big)^*,$$

  which implies that $\mathrm{coz}(\beta) = \big(\mathrm{coz}(\beta)\big)^{**} = \top$, which contradicts our assumption. Now suppose by contradiction that J is an r-ideal. Since $\alpha \in \mathrm{r}\big(\mathcal{R}(L)\big)$ implies that $\beta \in J$, therefore, $\mathrm{coz}(\beta) \le \mathrm{coz}(\alpha\beta) \le \mathrm{coz}(\alpha)$. On the other hand, according to the assumption, we have $\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) = \top$, which is obtained $\mathrm{coz}(\beta) = \top$, a contradiction.

5. The concept of z_r-ideal and s_r-ideal in the ring $\mathcal{R}(L)$

The concept of z_r-ideal and s_r-ideals in the ring $C(X)$ was studied for the first time in [8]. They investigated the properties of these ideals in the ring $C(X)$ and stated some of their properties in any reduced ring.

In this section, we determine the concept of z_r-ideals and s_r-ideals in the ring $\mathcal{R}(L)$ according to the concept of r-ideals and examine their characteristics and relationships with each other. We also indicate the frames L for which z_r-ideals coincide with some other types of ideals.

Definition 5.1. An ideal I of $\mathcal{R}(L)$ is said to be a $\mathbf{z_r}$**-ideal** if it is an r-ideal which is also a z-ideal.

Remark 5.2. Let L be a completely regular frame. Then we have:

(1) By [26, Theorem 2.19(a)], every z^0-ideal in a ring R is an r-ideal which implies that every z^0-ideal of $\mathcal{R}(L)$ is a z_r-ideal of $\mathcal{R}(L)$. Also, by [26, Remark 2.3(f)], every minimal prime ideal is an r-ideal in $\mathcal{R}(L)$. Hence, every minimal prime ideal in $\mathcal{R}(L)$ is a z_r-ideal of $\mathcal{R}(L)$.

(2) It is well known that the intersection of any family of z-ideals is a z-ideal. Also, by [26, Remark 2.3] the intersection of any family of r-ideals is an r-ideal. Hence, the intersection of any family of z_r-ideals of $\mathcal{R}(L)$ is a z_r-ideal of $\mathcal{R}(L)$.

(3) It is well known that if I and J are z-ideals of $\mathcal{R}(L)$, then $IJ = I \cap J$. Hence, the product of two z_r-ideals in $\mathcal{R}(L)$ is a z_r-ideal.

By Remark 5.2, the smallest z_r-ideal containing a given ideal I exists and we denote it by I_{z_r}. In fact I_{z_r} is the intersection of all z_r-ideals containing I.

Proposition 5.3. *For each ideal I of $\mathcal{R}(L)$, the following statements are true.*

(1) $I_{z_r} = \big((I_r)_z\big)_r = (I_z)_r = \big((I_z)_r\big)_z$.

(2) $I_{z_r} = \Big\{\alpha \in \mathcal{R}(L) \colon \operatorname{coz}(\tau\alpha) \leq \operatorname{coz}(\beta)$ *for some* $(\beta,\tau) \in I \times \mathrm{r}\big(\mathcal{R}(L)\big)\Big\}$.

Proof. (1). Let I be a proper ideal of $\mathcal{R}(L)$. If $I \cap \mathrm{r}\big(\mathcal{R}(L) \neq \emptyset$, then $I_{z_r} = \big((I_r)_z\big)_r = (I_z)_r = \mathcal{R}(L)$. Now, we can choose $I \cap \mathrm{r}\big(\mathcal{R}(L) = \emptyset$. Let $(\alpha,\beta) \in (I_z)_r \times \mathcal{R}(L)$ with $\operatorname{coz}(\alpha) = \operatorname{coz}(\beta)$ be given. Then there exists an element τ in $\mathrm{r}\big(\mathcal{R}(L)\big)$ such that $\tau\alpha \in I_z$, and from $\operatorname{coz}(\tau\alpha) = \operatorname{coz}(\tau\beta)$, we conclude that $\tau\beta \in I_z \subseteq (I_z)_r$, which implies that $\beta \in (I_z)_r$. Thus $(I_z)_r$ is a z_r-ideal. Now suppose J is a z_r-ideal contains I. Take $\alpha \in (I_z)_r$, then $\tau\alpha \in I_z$ for some $\tau \in \mathrm{r}\big(\mathcal{R}(L)\big)$. But $I_z \subseteq J$, so $\tau\alpha \in J$. Since J is an r-ideal, then $\alpha \in J$. Therefore, $(I_z)_r = I_{z_r}$.

Since $I \subseteq I_r$, we infer that $(I_z)_r \subseteq \big((I_r)_z\big)_r$. On the other hand, if $\alpha \in (I_r)_z$, then there exists an element β in I_r such that $\operatorname{coz}(\alpha) = \operatorname{coz}(\beta)$, which implies that for some $\gamma \in \mathrm{r}\big(\mathcal{R}(L), \gamma\beta \in I \subseteq I_z$ and $\operatorname{coz}(\gamma\beta) = \operatorname{coz}(\gamma\alpha)$, and we deduce that $\alpha \in (I_z)_r$. Thus we have $(I_r)_z \subseteq (I_z)_r$, which implies that $\big((I_r)_z\big)_r \subseteq (I_z)_r$. Therefore, $\big((I_r)_z\big)_r = (I_z)_r$. The rest is trivial.

(2). We set

$$T := \Big\{\alpha \in \mathcal{R}(L) \colon \operatorname{coz}(\tau\alpha) \leq \operatorname{coz}(\beta) \text{ for some } (\beta,\tau) \in I \times \mathrm{r}\big(\mathcal{R}(L)\big)\Big\}.$$

If $\alpha \in (I_z)_r$, then there exists an element τ in $\mathrm{r}\big(\mathcal{R}(L)\big)$ such that $\tau\alpha \in I_z$, which implies that there exists an element β in I such that $\operatorname{coz}(\tau\alpha) = \operatorname{coz}(\beta)$, and we deduce that $\alpha \in T$. Hence, $(I_z)_r \subseteq T$. On the other hand, if $\alpha \in T$, then there exists an element (β,τ) in $I \times \mathrm{r}\big(\mathcal{R}(L)\big)$ such that $\operatorname{coz}(\tau\alpha) \leq \operatorname{coz}(\beta)$, which implies that $\tau\alpha \in I_z$, and so $\alpha \in (I_z)_r$. Hence, $(I_z)_r = T$. □

In the following remark, we intend to provide a basic z_r-ideal with respect to the basic z-ideal and use it to express and prove an algebraic equivalent for the concept of z_r-ideal.

Remark 5.4. It is well known that $M_\alpha := \big\{\beta \in \mathcal{R}(L)\colon \operatorname{coz}(\beta) \le \operatorname{coz}(\alpha)\big\}$ is a basic z-ideal of $\mathcal{R}(L)$ for every $\alpha \in \mathcal{R}(L)$. Then, by Proposition 5.3,

$$\begin{aligned}(M_\alpha)_{z_r} = (M_\alpha)_r &= \Big\{\beta \in \mathcal{R}(L)\colon \gamma\beta \in M_\alpha \text{ for some } \gamma \in \mathrm{r}\big(\mathcal{R}(L)\big)\Big\}\\ &= \Big\{\beta \in \mathcal{R}(L)\colon \operatorname{coz}(\gamma\beta) \le \operatorname{coz}(\alpha) \text{ for some } \gamma \in \mathrm{r}\big(\mathcal{R}(L)\big)\Big\}.\end{aligned}$$

Suppose that $\beta \in (M_\alpha)_r$, then there is an element δ in $\mathrm{r}\big(\mathcal{R}(L)\big)$ such that

$$\operatorname{coz}(\delta) \wedge \operatorname{coz}(\beta) = \operatorname{coz}(\delta\beta) \le \operatorname{coz}(\alpha),$$

which implies from $\delta \in \mathrm{r}\big(\mathcal{R}(L)\big)$ that

$$\operatorname{coz}(\beta)^{**} = \operatorname{coz}(\delta)^{**} \wedge \operatorname{coz}(\beta)^{**} = \big(\operatorname{coz}(\delta) \wedge \operatorname{coz}(\beta)\big)^{**} \le \big(\operatorname{coz}(\alpha)\big)^{**},$$

and we deduce from [13, Lemma 4.1] that $\operatorname{Ann}(\alpha) \subseteq \operatorname{Ann}(\beta)$. Therefore, $\beta \in P_\alpha$. Hence, $M_\alpha \subseteq (M_\alpha)_r \subseteq P_\alpha$ for each $\alpha \in \mathcal{R}(L)$.

Lemma 5.5. *An ideal I in the ring $\mathcal{R}(L)$ is a z_r-ideal if and only if $(M_\alpha)_r \subseteq I$ for each $\alpha \in I$.*

Proof. Necessity. Suppose $\alpha \in I$. By remark 5.4, if $\beta \in (M_\alpha)_r$, there is $\gamma \in \mathrm{r}\big(\mathcal{R}(L)\big)$ such that $\operatorname{coz}(\gamma\beta) \le \operatorname{coz}(\alpha)$. Since I is a z_r-ideal, we infer that $\beta \in I$. Hence, $(M_\alpha)_r \subseteq I$.

Sufficiency. Suppose $(\alpha, \beta) \in \mathcal{R}(L)\times\mathrm{r}\big(\mathcal{R}(L)\big)$ such that $\alpha\beta \in I$. Since $\operatorname{coz}(\alpha\beta) \le \operatorname{coz}(\alpha\beta)$ by remark 5.4 implies that $\alpha \in (M_{\alpha\beta})_r$. It follows from the assumption that $\alpha \in I$ and I is an r-ideal. Now suppose $\operatorname{coz}(\beta) \le \operatorname{coz}(\alpha)$ and $\alpha \in I$. Since $\top \in \mathrm{r}\big(\mathcal{R}(L)\big)$, by remark 5.4 and our assumption implies that $\beta \in I$. Therefore I is a z_r-ideal. □

Lemma 5.6. *If I is an ideal of $\mathcal{R}(L)$ and $\beta \in \sum_{\alpha\in I}(M_\alpha)_r$. Then, there is $\alpha \in I$ such that $\beta \in (M_\alpha)_r$.*

Proof. Suppose $\beta \in \sum_{\alpha\in I}(M_\alpha)_r$. Therefore, there are $\alpha_1, \cdots, \alpha_n \in I$ such that $\beta \in \sum_{i=1}^{n}(M_{\alpha_i})_r$. For every $1 \le i \le n$, there exists an element $\beta_i \in (M_{\alpha_i})_r$ such that $\beta = \beta_1 + \cdots + \beta_n$. By remark 5.4, there is $\gamma_i \in \mathrm{r}\big(\mathcal{R}(L)\big)$ such that $\operatorname{coz}(\gamma_i\beta_i) \le \operatorname{coz}(\alpha_i)$. If we put $\gamma := \gamma_1\gamma_2\cdots\gamma_n \in \mathrm{r}\big(\mathcal{R}(L)\big)$, then $\operatorname{coz}(\gamma\beta_i) \le \operatorname{coz}(\gamma_i\beta_i) \le \operatorname{coz}(\alpha_i)$ for every i. Therefore,

$$\operatorname{coz}(\gamma\beta) = \operatorname{coz}\big(\gamma(\beta_1 + \cdots + \beta_n)\big) \le \bigvee_{i=1}^{n} \operatorname{coz}(\gamma\beta_i) \le \bigvee_{i=1}^{n} \operatorname{coz}(\alpha_i) = \operatorname{coz}(\alpha_1^2 + \cdots + \alpha_n^2).$$

Since $\gamma \in \mathrm{r}\big(\mathcal{R}(L)\big)$, we conclude from remark 5.4 that $\beta \in (M_{\alpha_1^2+\cdots+\alpha_n^2})_r$ and $\alpha_1^2 + \cdots + \alpha_n^2 \in I$. □

Corollary 5.7. *An ideal I of $\mathcal{R}(L)$ is a z_r-ideal if and only if $I = \sum_{\alpha\in I}(M_\alpha)_r$.*

Proof. It is evident by using Lemmas 5.5 and 5.6. □

Now, in the next proposition, we present other equivalents for the concept of z_r-ideals based on cozero elements.

Proposition 5.8. *The following statements are equivalent for an ideal I of $\mathcal{R}(L)$.*

1. *The ideal I is a z_r-ideal.*
2. *If $(\alpha, \beta, \tau) \in I \times \mathcal{R}(L) \times \mathrm{r}\big(\mathcal{R}(L)\big)$ with $\operatorname{coz}(\tau\alpha) = \operatorname{coz}(\tau\beta)$, then $\beta \in I$.*

(3) If $(\alpha, \beta, \tau) \in I \times \mathcal{R}(L) \times \mathrm{r}\big(\mathcal{R}(L)\big)$ *with* $\mathrm{coz}(\tau\beta) \leqslant \mathrm{coz}(\alpha)$*, then* $\beta \in I$*.*

Proof. (1) ⇒ (2). Let $(\alpha, \beta, \tau) \in I \times \mathcal{R}(L) \times \mathrm{r}\big(\mathcal{R}(L)\big)$ with $\mathrm{coz}(\tau\alpha) = \mathrm{coz}(\tau\beta)$ be given. Since I is a z-ideal and $\tau\alpha \in I$, we infer that $\tau\beta \in I$, which implies that $\beta \in I$, because I is an r-ideal.

(2) ⇒ (3). Let $(\alpha, \beta, \tau) \in I \times \mathcal{R}(L) \times \mathrm{r}\big(\mathcal{R}(L)\big)$ with $\mathrm{coz}(\tau\beta) \leq \mathrm{coz}(\alpha)$. Then $\mathrm{coz}(\tau\beta) = \mathrm{coz}(\tau\beta\alpha)$, which implies from $(\alpha\beta, \beta, \tau) \in I \times \mathcal{R}(L) \times \mathrm{r}\big(\mathcal{R}(L)\big)$ that $\beta \in I$.

(3) ⇒ (1). If we put $\tau = \top$ in (3), we deduce that I is a z-ideal. Let $(\alpha, \tau) \in \mathcal{R}(L) \times \mathrm{r}\big(\mathcal{R}(L)\big)$ with $\tau\alpha \in I$ be given. From $\mathrm{coz}(\tau\alpha) \leqslant \mathrm{coz}(\tau\alpha)$, we infer from part (3) that $\alpha \in I$. Hence, I is a z_r-ideal. □

Proposition 5.9. *Let* I *be an ideal of* $\mathcal{R}(L)$ *with* $I \cap \mathrm{r}\big(\mathcal{R}(L) = \emptyset$*. If* I *is a* z_r*-ideal, then* P *is a* z_r*-ideal for every* $P \in \mathrm{Min}(I)$*. The converse is also true if* I *is a semiprime ideal.*

Proof. The first part is evident by [26, Theorem 2.20] and [25, corollary after Theorem 1.1]. Now, let I be a semiprime ideal of $\mathcal{R}(L)$ such that P is a z_r-ideal for every $P \in \mathrm{Min}(I)$. Since any intersection of z_r-ideals is a z_r-ideal of $\mathcal{R}(L)$, we conclude that I is a z_r-ideal of $\mathcal{R}(L)$ and we are through. □

We recall from [9] that if the open quotient of every dense cozero element is a C^*-quotient, the frame L is called **quasi F-frame**. In [14], the properties of quasi-F-frame were investigated and equivalents for these frames were proved, which we use to prove the following theorem. In the following theorem, we show that the sum of z_r-ideals in $\mathcal{R}(L)$ behaves similar to the sum of z^0-ideals in $\mathcal{R}(L)$.

Theorem 5.10. *The sum of every two* z_r*-ideals in* $\mathcal{R}(L)$ *is a* z_r*-ideal or all of* $\mathcal{R}(L)$ *if and only if* L *is a quasi-F-frame.*

Proof. Necessity. Let $\alpha, \beta \in \mathcal{R}(L)$ with $\big(\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta)\big)^* = \bot$ be given. If $\alpha \in \mathrm{r}\big(\mathcal{R}(L)\big)$ or $\beta \in \mathrm{r}\big(\mathcal{R}(L)\big)$, then, by Lemma 3.1, $\big(\mathrm{coz}(\alpha)\big)^{**} \vee \big(\mathrm{coz}(\beta)\big)^{**} = \top$. Now, suppose that α and β are zero-divisors in $\mathcal{R}(L)$. Then, by [4, Remark 1.1], P_α and P_β are z^0-ideal, which implies from remark 5.2 that they are z_r-ideal. Thus, according to the assumption, $P_\alpha + P_\beta$ is a z_r-ideal or all of $\mathcal{R}(L)$. Since $\alpha^2 + \beta^2 \in \mathrm{r}\big(\mathcal{R}(L)\big)$ and $\alpha^2 + \beta^2 \in P_\alpha + P_\beta$, so $P_\alpha + P_\beta = \mathcal{R}(L)$. Hence, there exists $\delta \in P_\alpha$ and $\gamma \in P_\beta$ such that $\delta + \gamma = 1$. So we have

$$\top = \mathrm{coz}(1) = \mathrm{coz}(\delta + \gamma) \leq \mathrm{coz}(\delta) \vee \mathrm{coz}(\gamma).$$

On the other hand, by [1, proposition 4.2],

$$\big(\mathrm{coz}(\alpha)\big)^* \leq \big(\mathrm{coz}(\delta)\big)^* \text{ and } \big(\mathrm{coz}(\beta)\big)^* \leq \big(\mathrm{coz}(\gamma)\big)^*,$$

which implies that

$$\top = \mathrm{coz}(\delta) \vee \mathrm{coz}(\gamma) \leq \big(\mathrm{coz}(\delta)\big)^{**} \vee \big(\mathrm{coz}(\gamma)\big)^{**} \leq \big(\mathrm{coz}(\alpha)\big)^{**} \vee \big(\mathrm{coz}(\beta)\big)^{**}.$$

Therefore, by [14, proposition 3.1], L is a quasi-F-frame.

Sufficiency. Let L be a quasi-F-frame and I, J be two z_r-ideals of $\mathcal{R}(L)$ and $I + J \neq \mathcal{R}(L)$. Since, by [17, Proposition 5.1], the sum of two z-ideals of $\mathcal{R}(L)$ is always a z-ideal of $\mathcal{R}(L)$, it suffices to show that $I + J$ is an r-ideal of $\mathcal{R}(L)$. Let $T \in \mathrm{Min}(I + J)$ be given. Since T is a prime ideal and $I \subseteq T$, we infer that there exists an element P in $\mathrm{Min}(I)$ such that $P \subseteq T$. Thus, by [26, Theorem 2.20], P is an r-ideal of $\mathcal{R}(L)$, and by [28, Corollary 7.2.2], P is a z_r-ideal of $\mathcal{R}(L)$. Similarly, there exists an element Q in $\mathrm{Min}(J)$ with $Q \subseteq T$ such that Q is a z_r-ideal of $\mathcal{R}(L)$. If P and Q are in a chain, say $P \subseteq Q$, we have $I + J \subseteq P + Q = Q \subseteq T$, which implies from $T \in \mathrm{Min}(I + J)$ that $T = Q$ is a z_r-ideal of $\mathcal{R}(L)$. Now, we suppose that P and Q are not in a chain. Let I_P and I_Q are minimal prime ideals of $\mathcal{R}(L)$ such that $I_P \subseteq P$ and $I_Q \subseteq Q$. Then, by [2, Lemma 4.8], [17, Proposition 5.1], and [25, corollary after Theorem 1.1], $I_P + I_Q$ is a prime z-ideal of $\mathcal{R}(L)$, which implies from [11, Proposition 3.7] that $P + Q = I_P + I_Q$, and because T is a minimal prime over $I + J$, we conclude that T is equal to $P + Q$. Consequently, in both cases T is a z_r-ideal of $\mathcal{R}(L)$ and this means that T is a z_r-ideal of $\mathcal{R}(L)$ for every $T \in \mathrm{Min}(I + J)$. Since $I + J$ is a z-ideal of $\mathcal{R}(L)$, we conclude from proposition 5.9 that $I + J$ is a z_r-ideal of $\mathcal{R}(L)$. □

Corollary 5.11. *In every almost P-frame the sum of every two z_r-ideals in $\mathcal{R}(L)$ is a z_r-ideal or all of $\mathcal{R}(L)$.*

Proof. According to [14, Corollary 3.3] and Theorem 5.10, it is obvious. □

According to the Theorem 5.10, whenever L is a quasi-F-frame, then there is the largest z_r-ideal included in I for every ideal I of $\mathcal{R}(L)$, that with I^{z_r} it is displayed. Actually I^{z_r}, the sum of all z_r-ideals included in I.

Corollary 5.12. *If L is a quasi-F-frame and I is an ideal in $\mathcal{R}(L)$, then*

$$I^{z_r} = \sum_{(M_\alpha)_r \subseteq I} (M_\alpha)_r$$

Proof. Suppose that $J := \sum_{(M_\alpha)_r \subseteq I}(M_\alpha)_r$. Since L is a quasi-F-frame, we conclude from Theorem 5.10 that J is a z_r-ideal in $\mathcal{R}(L)$. On the other hand, if K is a z_r-ideal in $\mathcal{R}(L)$ included in I and $\beta \in K$, then, by Lemma 5.5, $(M_\beta)_r \subseteq K$. Since $K \subseteq I$ implies that $\beta \in J$. Therefore, $K \subseteq J$. □

Proposition 5.13. *For two ideals I and J in $\mathcal{R}(L)$, the following relations hold:*

(1) $((I \cap J)_z)_r = (I_z)_r \cap (J_z)_r = ((IJ)_z)_r = (I_z)_r(J_z)_r$.

(2) $(I_z)_r + (J_z)_r \subseteq ((I + J)_z)_r$.

Proof. According to the definition and properties r-ideals and z-ideals, relationships are established. □

As we observed every z^0-ideal in $\mathcal{R}(L)$ is a z_r-ideal. The following theorem, characterizes the frames L for which the converse also holds, i.e., every z_r-ideal of $\mathcal{R}(L)$ is a z^0-ideal.

Theorem 5.14. *A frame L is a weakly almost P-frame if and only if every z_r-ideal in $\mathcal{R}(L)$ is a z^0-ideal of $\mathcal{R}(L)$.*

Proof. Necessity. Let I be a z_r-ideal in $\mathcal{R}(L)$ and $P \in \mathrm{Min}(I)$. Then, by Proposition 5.9, P is a z_r-ideal, which implies from [16, Proposition 3.1] that P is a z^0-ideal of $\mathcal{R}(L)$. Since $I = \bigcap_{P\in \mathrm{Min}(I)} P$, we infer that I is a z^0-ideal of $\mathcal{R}(L)$.

Sufficiency. Let $\alpha, \beta \in \mathcal{R}(L)$ with $\left(\mathrm{coz}(\alpha)\right)^* \leq \left(\mathrm{coz}(\beta)\right)^*$ be given. According to our hypothesis, $(M_\alpha)_r$ is a z^0-ideal. From $\alpha \in (M_\alpha)_r$ and $\left(\mathrm{coz}(\alpha)\right)^* \leq \left(\mathrm{coz}(\beta)\right)^*$, we infer that $\beta \in (M_\alpha)_r$, which implies that there exists an element γ in $\mathrm{r}\big(\mathcal{R}(L)\big)$ such that

$$\mathrm{coz}(\beta) \wedge \mathrm{coz}(\gamma) = \mathrm{coz}(\beta\gamma) \leq \mathrm{coz}(\alpha).$$

Therefore, by Lemma 3.1 and definition, L is an weakly almost P-frame. □

Corollary 5.15. *If L is a weakly almost P-frame, then every z-ideals in the class of all r-ideals of $\mathcal{R}(L)$ is a z^0-ideal*

Proof. It is evident by Proposition 5.14. □

Corollary 5.16. *A frame L is an almost P-frame if and only if every z-ideal of $\mathcal{R}(L)$ is a z_r-ideal.*

Proof. By Proposition 4.1, it is evident. □

Corollary 5.17. *For an ideal I and a prime ideal Q in $\mathcal{R}(L)$, if $I \cap Q$ is a z_r-ideal, then one of them is a z_r-ideal.*

Proof. By [7, Proposition 2.8] and Proposition 4.4, it is evident. □

In the continuation of this section, by introducing the concept of s_r-ideal in the ring $\mathcal{R}(L)$, in the next remark and proposition, we express the connection of this ideal with z_r-ideals. We specify a frame where the s_r-ideals coincide with the z_r-ideals.

Definition 5.18. An ideal I of $\mathcal{R}(L)$ is said to be an $\mathbf{s_r}$**-ideal** if it is an r-ideal which is also a semiprime ideal.

Remark 5.19. It is clear that every z_r-ideal is an s_r-ideal. But every s_r-ideal is not necessarily a z_r-ideal. For this, if a frame L is not a cozero complemented frame, then, by Proposition 3.2, there exists a prime r-ideal Q that is not z-ideal. Therefore, Q is an s_r-ideal that is not a z_r-ideal.

Proposition 5.20. *A frame L is a cozero complemented frame if and only if every s_r-ideal in $\mathcal{R}(L)$ is a z_r-ideal.*

Proof. Necessity. By Proposition 3.2, it is evident that every s_r-ideal in $\mathcal{R}(L)$ is a z_r-ideal in $\mathcal{R}(L)$.

Sufficiency. Let P be a prime r-ideal in $\mathcal{R}(L)$. Then, by our hypothesis, P is a z-ideal. Hence, by Proposition 3.2, L is a cozero complemented frame. □

The intersection of any family of s_r-ideals is an s_r-ideal. Therefore, for every proper ideal I in the ring $\mathcal{R}(L)$ with $\mathrm{r}\big(\mathcal{R}(L)\big) \cap I = \emptyset$, there is the smallest s_r-ideal including I, which we represent by I_{s_r}.

Corollary 5.21. *For every ideal I of $\mathcal{R}(L)$, we have $I_{s_r} = \sqrt{I_r}$.*

Proof. By definition, we always have $I_r \subseteq I_{s_r}$. Since I_{s_r} is an s_r-ideal and according to [8, Lemma 4.1], implies that $\sqrt{I_r} \subseteq I_{s_r}$. On the other hand, since I_{s_r} is the smallest s_r-ideal including I, implies that $I_{s_r} \subseteq \sqrt{I_r}$. □

We recall from [4] that for a reduced ring R with the property A that for every ideal I with $\mathrm{r}(R) \cap I = \emptyset$ of R there is a smallest z^0-ideal including I. Therefore for every ideal I with $\mathrm{r}\big(\mathcal{R}(L)\big) \cap I = \emptyset$ of $\mathcal{R}(L)$, there is a smallest z^0-ideal including I which we denote by I_0 and $I_0 = \{\alpha \in \mathcal{R}(L) : \mathrm{Ann}(\beta) \subseteq \mathrm{Ann}(\alpha) \text{ for some } \beta \in I\}$.

Corollary 5.22. *For every proper ideal I of $\mathcal{R}(L)$ with $\mathrm{r}\big(\mathcal{R}(L)\big) \cap I = \emptyset$,*

$$I \subseteq I_r \subseteq I_{s_r} \subseteq I_{z_r} \subseteq I_0.$$

Proof. It is evident. □

We recall from [26] that the product of r-ideals is not necessarily an r-ideal, but by Remark 5.2, the product of z_r-ideals is a z_r-ideal. In the following proposition, we state the condition that if the product of two ideals becomes a z_r-ideal (or an s_r-ideal), then one of them is a z_r-ideal (or an s_r-ideal).

Proposition 5.23. *Suppose that I and J are two ideals in $\mathcal{R}(L)$ and $\mathrm{r}\big(\mathcal{R}(L)\big) \cap I \neq \emptyset$. Then, the following statements are true.*

(1) *If IJ is an s_r-ideal of $\mathcal{R}(L)$, then J is a s_r-ideal of $\mathcal{R}(L)$.*

(2) *If IJ is a z_r-ideal of $\mathcal{R}(L)$, then J is a z_r-ideal of $\mathcal{R}(L)$.*

Proof. (1). Suppose that $\gamma \in \mathrm{r}\big(\mathcal{R}(L)\big) \cap I$. If J is not a semiprime ideal of $\mathcal{R}(L)$, then there exists an element α in $\mathcal{R}(L)$ such that $\alpha \notin J$ and $\alpha^n \in J$ for some $n \in \mathbb{N}$, which implies that $\gamma^n\alpha^n \in IJ$, but IJ is a s_r ideal, hence $\alpha \in IJ \subseteq J$ and this is a contradiction. Accordingly, J is a semiprime ideal and it remains to show that J is a r-ideal of $\mathcal{R}(L)$. Let $(\alpha, \tau) \in \mathcal{R}(L) \times \mathrm{r}\big(\mathcal{R}(L)\big)$ with $\tau\alpha \in J$ be given. Then $\gamma\tau\alpha \in IJ$, which implies by our hypothesis that $\alpha \in IJ \subseteq J$.

(2). Suppose that $\gamma \in \mathrm{r}\big(\mathcal{R}(L)\big) \cap I$. Let $(\alpha, \beta) \in J \times \mathcal{R}(L)$ with $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ be given. Then $\mathrm{coz}(\gamma\alpha) = \mathrm{coz}(\gamma\beta)$, which implies from $\gamma\alpha \in IJ$ and Proposition 5.8 that $\beta \in IJ \subseteq J$, because IJ is a z_r-ideal of $\mathcal{R}(L)$. Therefore, J is a z-ideal of $\mathcal{R}(L)$. The proof of r-ideality of J is similar to the proof of the part (1). □

According to Theorem 5.10, in the next theorem, we show that the sum of s_r-ideals in $\mathcal{R}(L)$ behaves similar to the sum of z_r-ideals in $\mathcal{R}(L)$.

Theorem 5.24. *A frame L is a quasi-F-frame if and only if the sum of every two s_r-ideals in $\mathcal{R}(L)$ is a s_r-ideal or all of $\mathcal{R}(L)$.*

Proof. Necessity. Suppose I and J are two s_r-ideals of $\mathcal{R}(L)$ and $I + J \neq \mathcal{R}(L)$. Thus, by [30, Lemma 5.1], $I + J$ is a semiprime ideal of $\mathcal{R}(L)$. By a straightforward modification in the proof of Theorem 5.10, we obtain $I + J$ is an r-ideal of $\mathcal{R}(L)$. Therefore, $I + J$ is an s_r-ideal of $\mathcal{R}(L)$.

Sufficiency. Suppose I and J are two z_r-ideals of $\mathcal{R}(L)$. Then I and J are two s_r-ideals of $\mathcal{R}(L)$, and according to our hypothesis, $I + J$ is an s_r-ideal of $\mathcal{R}(L)$. On the other hand, by [17, Proposition 5.1], $I + J$ is a z-ideal, which implies that $I + J$ is a z_r-ideal of $\mathcal{R}(L)$. Therefore, by Theorem 5.10, L is a quasi-F-frame. □

We recall from [1] that for every ideal I with $\mathrm{r}\big(\mathcal{R}(L)\big) \cap I = \emptyset$ of $\mathcal{R}(L)$, if L is a quasi-F-frame, there is a largest z^0-ideal contained in I. We represent by I^0 which it is largest z^0-ideal contained in I and

$$I^0 = \{\, \alpha \in \mathcal{R}(L) : \mathrm{Ann}(\beta) \subseteq \mathrm{Ann}(\alpha) \text{ implies } \beta \in I \text{ for every } \beta \in \mathcal{R}(L)\}$$

Also, for ideal I with $\mathrm{r}\big(\mathcal{R}(L)\big) \cap I = \emptyset$ of $\mathcal{R}(L)$ and using Theorem 5.24, if our frame is a quasi-F-frame, then there exists the largest s_r-ideal contained in I, which we denote by I^{s_r}.

Corollary 5.25. *If L is a quasi-F-frame, then for every ideal I of $\mathcal{R}(L)$ with $\mathrm{r}\big(\mathcal{R}(L)\big) \cap I = \emptyset$, we have;*

$$I^0 \subseteq I^{z_r} \subseteq I^z \cap I^{s_r} \subseteq I^z + I^{s_r} \subseteq I.$$

Proof. According to definitions I^{z_r} and I^{s_r} and Remark 5.19, the proof is clear. □

In Corollaries 5.22 and 5.25, we saw a chain of ideals. At the end of this section, a systematic chain of well-known ideals and ideals introduced in this paper is presented in special frames.

Corollary 5.26. *If L is a quasi-F-frame and almost P-frame, then*

$$I^0 \subseteq I^z = I^{z_r} \subseteq I^{s_r} \subseteq I \subseteq I_r \subseteq I_{s_r} \subseteq I_z = I_{z_r} \subseteq I_0$$

for every ideal I of $\mathcal{R}(L)$ with $\mathrm{r}\big(\mathcal{R}(L)\big) \cap I = \emptyset$.

Proof. Using Proposition 4.1 and Theorems 5.10 and 5.24, as well as the characteristics of this class of ideals, the proof is obvious. □

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