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Chen's inequality for hemi-slant warped products in locally metallic Riemannian manifolds

Lamia Saeed Alqahtani[a,*], **Ebtehal M. AL-Husainy**[a,b]

[a]*Department of Mathematics, Faculty of Science, King Abdulaziz University, 21589 Jeddah, Saudi Arabia*
[b]*Mathematics Department, Faculty of Science, Taibah University, 344 AL-Madinah AL-Munawara, Saudi Arabia*

Abstract. Cristina-Elena and Adara-Monica have previously begun studying warped product hemi-slant submanifolds in metallic Riemannian manifolds. In this study, we extend their work by examining warped product hemi-slant submanifolds of two specific forms: $M = M_\theta \times_f M_\perp$ and $M = M_\perp \times_f M_\theta$, where $M_\perp$ and M_θ are anti-invariant and proper slant submanifolds of a locally metallic Riemannian manifold. We provide examples of warped product hemi-slant submanifolds and establish a sharp inequality for the squared norm of the second fundamental form of these submanifolds in metallic Riemannian manifolds, considering the equality case.

1. Introduction

Warped product manifolds are a type of Riemannian manifolds that are constructed by taking the product of manifolds with a warping metric tensor on one of its factors. This warping function can create a variety of interesting geometries, including submanifolds with non-constant curvature.

The study of warped product submanifolds has been an active area of research in recent years. Researchers have examined these submanifolds in various settings, including almost Hermitian, locally product, and almost contact metric structures on manifolds, such as Kaehler manifolds, locally product Riemannian manifolds, Sasakian, Kenmotsu and metallic Riemannian manifolds.

Warped products were first studied by J. F. Nash in the 1950s [11] and have since been used to construct a variety of interesting geometric objects. Recently, B. Sahin in [12] studied warped product submanifolds of Kaehler manifolds with a slant factor. He showed that the geometry of these submanifolds is completely determined by the geometry of the slant submanifolds and the warping function. S. Uddin et al.[13] constructed examples of warped product bi-slant submanifolds in complex Euclidean spaces. L. S. Alqahtani et al. [1] showed that there is no proper warped product bi-slant submanifold other than pseudo-slant warped product in cosymplectic manifolds.

On the other hand, the metallic structure is a further the generalization of the Golden structure. It was introduced in [7] by C. E. Hretcanu and M. Crasmareanu. A metallic structure is characterized by the existence of two vector fields, P and Q that satisfy certain conditions. These conditions are more general than

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* Corresponding author: Lamia Saeed Alqahtani
Email addresses: Lalqahtani@kau.edu.sa (Lamia Saeed Alqahtani), ealhusainy@stu.kau.edu.sa (Ebtehal M. AL-Husainy)

those for a Golden structure. Different types of submanifolds in metallic and Golden Riemannian manifolds were studied in [4, 8]. The authors obtained ntegrability conditions for the distributions involved in the definition of such submanifolds.

Recently, the study conducted by C. E. Hretcanu and A. M. Blaga [9] is considered one of the most significant contributions to the study of warped product submanifolds. They have studied slant, bi-slant, semi-slant, and hemi-slant submanifolds in locally metallic Riemannian manifolds. Their results have shed light on the geometry of these submanifolds and opened up new possibilities for research in this area.

This paper continues the research on warped products of two specific forms of hemi-slant submanifolds in locally metallic Riemannian manifolds: $M = M_\theta \times_f M_\perp$ and $M = M_\perp \times_f M_\theta$, where $M_\perp$ and M_θ are anti-invariant and proper slant submanifolds of a metallic Riemannian manifold. In the first part of the paper, we study $M = M_\theta \times_f M_\perp$. In the second part, we study $M = M_\perp \times_f M_\theta$. In both sections, we make significant contributions to the field of differential geometry by constructing new examples of warped product hemi-slant submanifolds in metallic Riemannian manifolds, deriving useful lemmas that build up to the proofs of our main theorems, and proving more general theorems on the squared norm of the second fundamental form in terms of the components of the gradient of the warping function. We also examine the equality case, which corresponds to when the submanifold is minimal.

2. Preliminaries

Let $\tilde{M}$ be a smooth manifold of dimension m. A metallic structure on $\tilde{M}$ is a (1,1) tensor field defined by the following equation

$$J^2 = pJ + qI, \tag{1}$$

where $p, q \in N$, and I is the identity operator on the set of all vector fields on $\tilde{M}$, denoted by$\Gamma(T\tilde{M})$, [7]. A metallic Riemannian manifold is a Riemannian manifold $(\tilde{M}, g)$ where the Riemannian metric g is compatible with the metallic structure J. This means that the following equation holds for all vector fields X and Y on $\tilde{M}$

$$g(JX, Y) = g(X, JY). \tag{2}$$

The equation

$$g(JX, JY) = pg(JX, Y) + qg(X, Y), \tag{3}$$

can be derived from equations (1) and (2), for all vector fields $X, Y \in \Gamma(T\tilde{M})$, [7]. A locally metallic Riemannian manifold is a metallic Riemannian manifold $(\tilde{M}, g, J)$ in which the metallic structure J is preserved by parallel transport along geodesics with respect to the Levi-Civita connection $\tilde{\nabla}$ on $\tilde{M}$. This means that $\tilde{\nabla} J = 0$, so J is constant along geodesics [10]. If f is a smooth real valued function on a Riemannian manifold $(\tilde{M}, g)$, then the gradient of f, denoted by $\vec{\nabla} f$, can be calculated using the equation:

$$g(\vec{\nabla} f, X) = X(f), \tag{4}$$

for any X tangents to $\tilde{M}$. Accordingly, we have the following:

$$\| \vec{\nabla} f \|^2 = \sum_{i=1}^{m} (e_i(f))^2, \tag{5}$$

where $e_1, \cdots, e_m$ is a set of mutually orthogonal vectors with unit norm, which forms a local frame field on $\tilde{M}$ of dimension m [3].

A hemi-slant submanifold is an immersed submanifold M of $\tilde{M}$ that has two orthogonal distributions, $\mathfrak{D}^1$ and $\mathfrak{D}^2$, characterized by the following properties:

1. $TM = \mathfrak{D}^1 \oplus \mathfrak{D}^2$.

2. The distribution $\mathfrak{D}^1$ is anti-invariant, meaning $J(\mathfrak{D}^1) \subseteq \Gamma(T^{\perp}M)$, where $\Gamma(T^{\perp}M)$ denoted to the normal bundle of M.
3. The distribution $\mathfrak{D}^2$ is slant, meaning the angle θ between JX and $J\mathfrak{D}^2$ is constant for all $X \in \mathfrak{D}^2$. The angle θ is called a slant angle.

Moreover, if the two distributions $\mathfrak{D}^1$ and $\mathfrak{D}^2$ are not both trivial, and the angle between J and any vector in $\mathfrak{D}^2$ is strictly less than $\frac{\pi}{2}$, then the submanifold is called a proper hemi-slant submanifold [8].

Suppose that $\tilde{\nabla}$ and ∇ are the Levi-Civita connections on $(\tilde{M}, g)$ and M, respectively. Then the Gauss and Weingarten formulas are represented by the following equations

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{6}$$
$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{7}$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$. The second fundamental form h and the shape operator A_V are two tensors that describe the curvature of a submanifold. These two tensors are related by the following equation [6]:

$$g(h(X, Y), V) = g(A_V X, Y). \tag{8}$$

Let TX and FX be the tangential and normal components of JX, respectively, for any vector field X in the tangent bundle $\Gamma(TM)$. Similarly, let tV and nV be the tangential and normal components of JV, respectively, for any vector field V in the normal bundle $\Gamma(T^{\perp}M)$. Then, we have the following equations:

$$JX = TX + FX, \tag{9}$$
$$JV = tV + nV. \tag{10}$$

These equations lead to

$$g(TX, Y) = g(X, TY), \tag{11}$$
$$g(nU, V) = g(U, nV), \tag{12}$$
$$g(FX, V) = g(X, tV). \tag{13}$$

Then, the maps T and n are symmetric with respect to the Riemannian metric g [4].

If M is a slant submanifold with a slant angle θ in a metallic Riemannian manifold $(\tilde{M}, g, J)$, then

$$g(TX, TY) = \cos^2\theta[pg(X, TY) + qg(X, Y)], \tag{14}$$
$$g(FX, FY) = \sin^2\theta[pg(X, TY) + qg(X, Y)], \tag{15}$$

which imply that

$$T^2 = \cos^2\theta[pT + qI] \quad \text{and} \quad \nabla(T^2) = p\cos^2\theta(\nabla T), \tag{16}$$

where $X, Y \in \Gamma(TM)$ and I is the identity on $\Gamma(TM)$ [4].

Given two Riemannian manifolds, (M_1, g_1) and (M_2, g_2), their warped product with a warping function f is a Riemannian manifold $(M_1 \times_f M_2, g)$. The Riemannian metric g is defined as $g = g_1 + f^2 g_2$, where f is a positive smooth function on M_1. The dimension of the warped product manifold is the sum of $dim(M_1) = n_1$ and $dim(M_2) = n_2$, i.e., $n = n_1 + n_2$, [3]. In a warped product manifold $M_1 \times_f M_2$, M_1 and M_2 are totally geodesic and totally umbilical submanifolds of M, respectively [3]. A warped product hemi-slant submanifold is a warped product submanifold $M = M_1 \times_f M_2$ such that one of the components is an anti-invariant submanifold and the other is a slant submanifold with a slant angle $\theta \in [0, \frac{\pi}{2}]$ [9]. On a warped product manifold $M = M_1 \times_f M_2$, it is known that for any $X, Y \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$, the following hold:

$$\nabla_X Y \in \Gamma(TM_1), \tag{17}$$
$$\nabla_X Z = \nabla_Z X = X(\ln f)Z, \tag{18}$$
$$\nabla_Z W = \acute{\nabla}_Z W - g(Z, W)\vec{\nabla} \ln f, \tag{19}$$

where ∇ is the Levi-Civita connection on M [2, 5].

3. Warped product hemi-slant submanifold of the form $M = M_\theta \times_f M_\perp$

In this section, we study a special type of submanifold in a locally metallic Riemannian manifold $\tilde{M}$, called a warped product hemi-slant submanifold of the form $M = M_\theta \times_f M_\perp$, where $M_\perp$ is an anti-invariant submanifold and M_θ is a proper slant submanifold.

Lemma 3.1. *A slant submanifold M_θ with a slant angle θ, isometrically immersed in a metallic Riemannian manifold $\tilde{M}$, satisfies the following equations for any $X \in \Gamma(TM_\theta)$,*

$$tFX = \sin^2\theta(pTX + qX), \tag{20}$$
$$fFX = pFX - FTX. \tag{21}$$

Proof. Let $X \in \Gamma(TM)$, then from (9), we have

$$J^2X = JTX + JFX. \tag{22}$$

By using (1), (9), and (10) in (22), we can show that

$$pJX + qX = T^2X + FTX + tFX + fFX. \tag{23}$$

Applying (9) and (16) in (23), we get

$$\sin^2\theta(pTX + qX) + pFX = FTX + tFX + fFX. \tag{24}$$

By comparing both sides of equation (24), we can arrive at the proof of the lemma. □

Lemma 3.2. *Let $M = M_\theta \times_f M_\perp$ be a warped product hemi-slant submanifold of a locally metallic Riemannian manifold $\tilde{M}$. Then, for any $X, Y \in \Gamma(TM_\perp)$ and $Z, W \in \Gamma(TM_\theta)$, the following equations hold*

$$g(h(X, Z), JY) = 0, \tag{25}$$
$$g(h(X, Y), FZ) = (TZ \ln f)\, g(X, Y), \tag{26}$$
$$g(h(X, Z), FW) = 0, \tag{27}$$
$$g(h(Z, W), JX) = 0. \tag{28}$$

Proof. For any $X, Y \in \Gamma(TM_\perp)$ and $Z, W \in \Gamma(TM_\theta)$ in a warped product hemi-slant submanifold $M = M_\theta \times_f M_\perp$ of a locally metallic Riemannian manifold $\tilde{M}$, then we can observe that

$$g(h(X, Z), JY) = g(\tilde{\nabla}_X Z, JY). \tag{29}$$

Using (3) and (18) in (29), we get

$$g(h(X, Z), JY) = \frac{1}{p} g(J\tilde{\nabla}_X Z, JY) - \frac{q}{p} Z(\ln f) g(X, Y). \tag{30}$$

Applying equation (9) in (30), we have

$$g(h(X,Z),JY) = \frac{1}{p}g(\tilde{\nabla}_X(TZ+FZ),JY) - \frac{q}{p}Z(\ln f)g(X,Y),$$
$$= \frac{1}{p}g(\tilde{\nabla}_X TZ, FY) + \frac{1}{p}g(\tilde{\nabla}_X tFZ, Y) + \frac{1}{p}g(\tilde{\nabla}_X fFZ, Y) - \frac{q}{p}Z(\ln f)g(X,Y). \tag{31}$$

By using equations (20) and (21) in (31), we get

$$g(h(X,Z),JY) = \frac{1}{p}g(h(X,TZ),FY) + \sin^2\theta(TZ\ln f)g(X,Y) + \frac{q}{p}\sin^2\theta(Z\ln f)g(X,Y) \tag{32}$$
$$- g(h(X,Y),FZ) + \frac{1}{p}g(h(X,Y),FTZ) - \frac{q}{p}(Z\ln f)g(X,Y).$$

By substituting TZ for Z in the above equation and applying equation (16), we obtain that

$$g(h(X,TZ),JY) = \frac{q\cos^2\theta}{p\sin^2\theta}g(h(X,Z),FY) + p\cos^2\theta(TZ\ln f)g(X,Y) + q\cos^2\theta(Z\ln f)g(X,Y) \tag{33}$$
$$- \frac{q\cos^2\theta}{p\sin^2\theta}(TZ\ln f)g(X,Y) - g(h(X,Y),FTZ) + \frac{q\cos^2\theta}{p\sin^2\theta}g(h(X,Y),FZ).$$

When we insert equation (33) into equation (32), it follows that

$$g(h(X,Z),JY) = (TZ\ln f)g(X,Y) - g(h(X,Y),FZ). \tag{34}$$

From a different perspective, we have

$$g(h(X,Z),JY) = g(\tilde{\nabla}_Z X, JY), \tag{35}$$
$$g(h(X,Z),JY) = g(\tilde{\nabla}_Z FX, Y), \tag{36}$$
$$g(h(X,Z),JY) = -g(h(Y,Z),FX). \tag{37}$$

When we replace Y with X in equation (34), we get

$$g(h(Y,Z),JX) = g(h(X,Z),JY). \tag{38}$$

Based on equations (37), we can conclude that

$$g(h(X,Z),JY) = 0.$$

Employing equations (25) and (34) as supporting evidence, we can provide a compelling proof of equation (26).

With the aim of validating equation (27), we have

$$g(h(X,Z),FW) = \frac{1}{p}g(J\tilde{\nabla}_X Z, JW). \tag{39}$$

From (9) and (20), we get

$$g(h(X,Z),FW) = \frac{1}{p}g(\tilde{\nabla}_X TZ, FW) + \frac{1}{p}g(\tilde{\nabla}_X fFZ, W). \tag{40}$$

Employing equation (21) in (40), we obtain the following result

$$g(h(X,Z),FW) = \frac{1}{p}g(h(X,TZ),FW) + g(\tilde{\nabla}_X FZ, W) - \frac{1}{p}g(\tilde{\nabla}_X FTZ, W). \tag{41}$$

By interchanging Z with TZ in equation (41) and applying equation (16), we reveal the following

$$g(h(X,TZ),FW) = \frac{q\cos^2\theta}{p\sin^2\theta}g(h(X,Z),FW) + g(\tilde{\nabla}_X FTZ, W) - \frac{q\cos^2\theta}{p\sin^2\theta}g(\tilde{\nabla}_X FZ, W). \tag{42}$$

Referring to equations (42) and (41), we can see that

$$g(h(X,Z),FW) = -g(h(X,W),FZ). \tag{43}$$

Conversely, we have

$$g(h(X,Z),FW) = g(\tilde{\nabla}_Z X, JW), \tag{44}$$
$$g(h(X,Z),FW) = g(\tilde{\nabla}_Z FX, W), \tag{45}$$
$$g(h(X,Z),FW) = -g(h(Z,W),FX). \tag{46}$$

Substituting W for Z in (43), we obtain

$$g(h(X,W),FZ) = g(h(X,Z),FW). \tag{47}$$

According to the above equation and equation (43), we can derive that

$$g(h(X,Z),FW) = 0.$$

Now, let's replicate the successful techniques we employed to establish equation (28), that means

$$\begin{aligned} g(h(Z,W),JX) &= \frac{1}{p}g(J\tilde{\nabla}_Z W, JX), \\ &= \frac{1}{p}g(\tilde{\nabla}_Z FTW, X) + \frac{1}{p}g(\tilde{\nabla}_Z fFW, X). \end{aligned} \tag{48}$$

By applying equation (21), and subsequently utilize equation (27), we arrive at the following outcome

$$g(h(Z,W),JX) = 0.$$

□

Example 3.3. *We consider a warped product hemi-slant submanifold immersion of a subset M of a manifold $\tilde{M}$ into $\tilde{M}$. This immersion i is defined by the following equation:*

$$\begin{aligned} i(u,\theta,\phi) = (&u\cos\theta, u\sin\theta, u\cos\phi, u\sin\phi, \sqrt{\frac{2}{q}}\sigma u, \sqrt{\frac{1}{q}}\tilde{\sigma}u\cos\theta, \sqrt{\frac{1}{q}}\tilde{\sigma}u\sin\theta, \sqrt{\frac{1}{q}}\tilde{\sigma}u\cos\phi \\ &, \sqrt{\frac{1}{q}}\tilde{\sigma}u\sin\phi, \sqrt{\frac{1}{q}}\sigma u\cos\theta, \sqrt{\frac{1}{q}}\sigma u\sin\theta, u\cos\theta, u\sin\theta), \end{aligned} \tag{49}$$

where $u > 0$, θ and ϕ are in the open interval $(0, \frac{\pi}{2})$, σ is a metallic number, $\tilde{\sigma} := p - \sigma$, also p and q are positive *integers. The tangent bundle of M can be easily shown to be spanned by the set of vectors* $\{Z_1, Z_2, Z_3\}$, *where*

$$Z_1 = \cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2} + \cos\phi \frac{\partial}{\partial x_3} + \sin\phi \frac{\partial}{\partial x_4} + \sqrt{\frac{2}{q}}\sigma \frac{\partial}{\partial x_5} + \sqrt{\frac{1}{q}}\tilde{\sigma}\cos\theta \frac{\partial}{\partial x_6} + \sqrt{\frac{1}{q}}\tilde{\sigma}\sin\theta \frac{\partial}{\partial x_7} + \sqrt{\frac{1}{q}}\tilde{\sigma}\cos\phi \frac{\partial}{\partial x_8} + \sqrt{\frac{1}{q}}\tilde{\sigma}\sin\phi \frac{\partial}{\partial x_9} + \sqrt{\frac{1}{q}}\sigma\cos\theta \frac{\partial}{\partial x_{10}} + \sqrt{\frac{1}{q}}\sigma\sin\theta \frac{\partial}{\partial x_{11}} + \cos\theta \frac{\partial}{\partial x_{12}} + \sin\theta \frac{\partial}{\partial x_{13}}, \quad (50)$$

$$Z_2 = -u\sin\theta \frac{\partial}{\partial x_1} + u\cos\theta \frac{\partial}{\partial x_2} - \sqrt{\frac{1}{q}}\tilde{\sigma}u\sin\theta \frac{\partial}{\partial x_6} + \sqrt{\frac{1}{q}}\tilde{\sigma}u\cos\theta \frac{\partial}{\partial x_7} - \sqrt{\frac{1}{q}}\sigma u\sin\theta \frac{\partial}{\partial x_{10}} + \sqrt{\frac{1}{q}}\sigma u\cos\theta \frac{\partial}{\partial x_{11}} - u\sin\theta \frac{\partial}{\partial x_{12}} + u\cos\theta \frac{\partial}{\partial x_{13}}, \quad (51)$$

$$Z_3 = -u\sin\phi \frac{\partial}{\partial x_3} + u\cos\phi \frac{\partial}{\partial x_4} - \sqrt{\frac{1}{q}}\tilde{\sigma}u\sin\phi \frac{\partial}{\partial x_8} + \sqrt{\frac{1}{q}}\tilde{\sigma}u\cos\phi \frac{\partial}{\partial x_9}. \quad (52)$$

When we apply the metallic structure J *of* $\tilde{M}$, *which is a linear transformation that flips the signs of some of the coordinates, to the coordinates of a point on* $\tilde{M}$, *we obtain the following new coordinates*

$$J(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}) = (\sigma x_1, \sigma x_2, \tilde{\sigma} x_3, \tilde{\sigma} x_4, \tilde{\sigma} x_5, \sigma x_6, \sigma x_7, \sigma x_8, \sigma x_9, \tilde{\sigma} x_{10}, \tilde{\sigma} x_{11}, \tilde{\sigma} x_{12}, \tilde{\sigma} x_{13}). \quad (53)$$

It follows that we have

$$JZ_1 = \sigma\cos\theta \frac{\partial}{\partial x_1} + \sigma\sin\theta \frac{\partial}{\partial x_2} + \tilde{\sigma}\cos\phi \frac{\partial}{\partial x_3} + \tilde{\sigma}\sin\phi \frac{\partial}{\partial x_4} + \sqrt{\frac{2}{q}}\sigma\tilde{\sigma} \frac{\partial}{\partial x_5} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}\cos\theta \frac{\partial}{\partial x_6} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}\sin\theta \frac{\partial}{\partial x_7} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}\cos\phi \frac{\partial}{\partial x_8} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}\sin\phi \frac{\partial}{\partial x_9} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}\cos\theta \frac{\partial}{\partial x_{10}} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}\sin\theta \frac{\partial}{\partial x_{11}} + \tilde{\sigma}\cos\theta \frac{\partial}{\partial x_{12}} + \tilde{\sigma}\sin\theta \frac{\partial}{\partial x_{13}}, \quad (54)$$

$$JZ_2 = -u\sigma\sin\theta \frac{\partial}{\partial x_1} + u\sigma\cos\theta \frac{\partial}{\partial x_2} - \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}u\sin\theta \frac{\partial}{\partial x_6} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}u\cos\theta \frac{\partial}{\partial x_7} - \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}u\sin\theta \frac{\partial}{\partial x_{10}} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}u\cos\theta \frac{\partial}{\partial x_{11}} - u\tilde{\sigma}\sin\theta \frac{\partial}{\partial x_{12}} + u\tilde{\sigma}\cos\theta \frac{\partial}{\partial x_{13}}, \quad (55)$$

$$JZ_3 = -u\tilde{\sigma}\sin\phi \frac{\partial}{\partial x_3} + u\tilde{\sigma}\cos\phi \frac{\partial}{\partial x_4} - \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}u\sin\phi \frac{\partial}{\partial x_8} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}u\cos\phi \frac{\partial}{\partial x_9}. \quad (56)$$

Let's define two distributions on the manifold $\tilde{M}$:

- $\mathfrak{D}^{\perp}$ *is spanned by the vectors* Z_2, *and* Z_3. *It is the anti-invariant distribution under the metallic structure* J.
- $\mathfrak{D}^{\theta^*}$ *is spanned by the vectors* Z_1. *It is the slant distribution with slant angle* θ^*, *which means that* $\theta^* = \cos^{-1} \frac{2\sqrt{q}\sigma}{\sqrt{(\sigma^2+2\tilde{\sigma}^2+5q)(3q+3\sigma^2+2\tilde{\sigma}^2)}}$, *under the metallic structure* J.

The metric tensor that defines the distance between points on the manifold M is given by the following equation:

$$g := \Big(3 + \frac{3\sigma^2}{q} + \frac{2\tilde{\sigma}^2}{q}\Big) d^2u + u^2\Big(2 + \frac{\tilde{\sigma}^2}{q} + \frac{\sigma^2}{q}\Big) d^2\theta + \Big(1 + \frac{\tilde{\sigma}^2}{q}\Big) d^2\phi.$$

This metric tensor defines a type of submanifold called a warped product hemi-slant submanifold, which is embedded in a metallic Riemannian manifold. The manifold M is the product of two manifolds, $M = M_\theta \times_f M_\perp$*, and the warping function f stretches the manifold* $M_\perp$ *by a factor of u.*

Now, we need to define a specific frame field for a warped product hemi-slant submanifold of dimension n of a locally metallic Riemannian manifold $\tilde{M}$ with m-dimensional. Let $\dim(M_\perp) = t_1$ and $\dim(M_\theta) = t_2$, where $n = t_1 + t_2$. Also, let the tangent bundles of the anti-invariant $M_\perp$ and the slant M_θ are $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$, respectively. Let $\{e_1, e_2, ..., e_{t_1}\}$ be the orthonormal frames of $\mathfrak{D}^\perp$ and

$$\{e_{t_1+1}, \cdots, e_{t_1+w}, e_{t_1+w+1} = \frac{\sec\theta}{\sqrt{q}} Te_{t_1+1}, e_{t_1+w+2} = \frac{\sec\theta}{\sqrt{q}} Te_{t_1+2}, \cdots, e_{t_1+t_2} = e_{t_1+2w} = e_n = \frac{\sec\theta}{\sqrt{q}} Te_{t_1+w}\} \tag{57}$$

be the orthonormal frame of $\mathfrak{D}^\theta$. Also, the orthonormal frame field of the normal sub bundle of $JD^\perp$ and $F\mathfrak{D}^\theta$ are respectively,

$$\{e_{n+1} = \tilde{e_1} = Je_1, e_{n+2} = \tilde{e_2} = Je_2, ..., e_{n+t_1} = \tilde{e_{t_1}} = Je_{t_1}\}, \tag{58}$$

$$\begin{aligned} \{e_{n+t_1+1} = \tilde{e}_{t_1+1} = \frac{\csc\theta}{\sqrt{q}} Fe_{t_1+1}, e_{n+t_1+2} = \tilde{e}_{t_1+2} = \frac{\csc\theta}{\sqrt{q}} Fe_{t_1+2}, ..., e_{n+t_1+w} = \tilde{e}_{t_1+w} = \frac{\csc\theta}{\sqrt{q}} Fe_{t_1+w}, \\ e_{n+t_1+w+1} = \tilde{e}_{t_1+w+1} = \frac{\sec\theta\csc\theta}{\sqrt{qp^2\cos^2\theta + q^2}} FTe_{t_1+1}, ..., e_{n+t_1+2w} = \tilde{e}_{t_1+t_2} = \frac{\sec\theta\csc\theta}{\sqrt{qp^2\cos^2\theta + q^2}} FTe_{t_1+w}\}. \end{aligned} \tag{59}$$

Theorem 3.4. *Let* $M = M_\theta \times_f M_\perp$ *be a warped product hemi-slant submanifold in a locally metallic Riemannian manifold* $\tilde{M}$*, where* $M_\perp$ *is an anti-invariant submanifold and* M_θ *is a slant submanifold of* $\tilde{M}$*. Then,*

(i) the squared norm of the second fundamental form h of M is at least

$$\|h\|^2 \geq \frac{t_1 \cos^4\theta \csc^2\theta}{p^2\cos^2\theta + q} \|\nabla^\theta \ln f\|^2, \tag{60}$$

where $t_1 = \dim M_\perp$ *and* $\nabla^\theta \ln f$ *is the gradient of* $\ln f$ *along* M_θ*.*

(ii) If the squared norm of the second fundamental form h of M in (60) is equal to it is lower bound, then both M_θ *and* $M_\perp$ *are totally geodesic and totally umbilical submanifolds of* $\tilde{M}$*, respectively.*

Proof. For a frame field $\{e_1, \dots, e_n\}$ of a submanifold M of dimension n, the squared norm of the second fundamental form h can be expressed as

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=n+1}^{m} \sum_{i,j=1}^{n} g(h(e_i, e_j), e_r)^2. \tag{61}$$

By using the frame fields given in (57), (58), and (59) in (61), we see that

$$\begin{aligned} \|h\|^2 = &\sum_{r=1}^{t_1} \sum_{i,j=1}^{t_1} g(h(e_i, e_j), \tilde{e_r})^2 + \sum_{r=t_1+1}^{t_1+t_2} \sum_{i,j=1}^{t_1} g(h(e_i, e_j), \tilde{e_r})^2 + 2\sum_{r=1}^{t_1} \sum_{i=1}^{t_1} \sum_{j=t_1+1}^{t_1+t_2} g(h(e_i, e_j), \tilde{e_r})^2 \\ &+ 2\sum_{r=t_1+1}^{t_1+t_2} \sum_{i=1}^{t_1} \sum_{j=t_1+1}^{t_1+t_2} g(h(e_i, e_j), \tilde{e_r})^2 + \sum_{r=1}^{t_1} \sum_{i,j=t_1+1}^{t_1+t_2} g(h(e_i, e_j), \tilde{e_r})^2 + \sum_{r=t_1+1}^{t_1+t_2} \sum_{i,j=t_1+1}^{t_1+t_2} g(h(e_i, e_j), \tilde{e_r})^2. \end{aligned} \tag{62}$$

Substituting equations (25) and (27) in (62), we get that

$$||h||^2 \geq \sum_{r=t_1+1}^{t_1+t_2} \sum_{i,j=1}^{t_1} g(h(e_i, e_j), \tilde{e_r})^2. \tag{63}$$

With the aid of (59), we can rewrite the last inequality (63) as

$$||h||^2 \geq \frac{\csc^2\theta}{q} \sum_{r=t_1+1}^{t_1+w} \sum_{i,j=1}^{t_1} g(h(e_i, e_j), Fe_r)^2 + \frac{\sec^2\theta \csc^2\theta}{q(p^2\cos^2\theta + q)} \sum_{r=t_1+1}^{t_1+w} \sum_{i,j=1}^{t_1} g(h(e_i, e_j), FTe_r)^2. \tag{64}$$

Using equation (26) in (64), we find

$$||h||^2 \geq \frac{t_1 \csc^2\theta}{q} \sum_{r=t_1+1}^{t_1+w} (Te_r \ln f)^2 + \frac{t_1 \csc^2\theta}{(p^2\cos^2\theta + q)} \sum_{r=t_1+w+1}^{t_1+t_2} (Te_r \ln f)^2. \tag{65}$$

Applying equations (57) and (16), we obtain that

$$||h||^2 \geq \frac{t_1 \csc^2\theta}{\sec^2\theta} \sum_{r=t_1+w+1}^{t_1+t_2} (e_r \ln f)^2 + \frac{t_1 p^2 \cos^2\theta \csc^2\theta}{\sec^2\theta(p^2\cos^2\theta + q)} \sum_{r=t_1+w+1}^{t_1+t_2} (e_r \ln f)^2 + \frac{t_1 q \cos^2\theta \csc^2\theta}{(p^2\cos^2\theta + q)} \sum_{r=t_1+1}^{t_1+w} (e_r \ln f)^2.$$

Clearly, by taking $\frac{t_1 \cos^2\theta \csc^2\theta}{\sec^2\theta(p^2\cos^2\theta + q)}$ as a common factor, we get

$$||h||^2 \geq \frac{t_1 \cos^2\theta \csc^2\theta}{\sec^2\theta(p^2\cos^2\theta + q)} \left(\sum_{r=t_1+w+1}^{t_1+t_2} (e_r \ln f)^2 + \sum_{r=t_1+1}^{t_1+w} (e_r \ln f)^2 \right). \tag{66}$$

Finally,

$$||h||^2 \geq \frac{t_1 \cos^4\theta \csc^2\theta}{(p^2\cos^2\theta + q)} ||\nabla^\theta \ln f||^2.$$

For the equality case, from the leaving fifth and sixth terms of (62), we find

$$h(D^\theta, D^\theta) = 0. \tag{67}$$

Similarly, from the leaving first term of (62), we find that

$$h(\mathfrak{D}^\perp, \mathfrak{D}^\perp) \subseteq F\mathfrak{D}^\theta. \tag{68}$$

On the other hand, the leaving fourth term in (62), we get

$$h(\mathfrak{D}^\perp, \mathfrak{D}^\theta) \subseteq JD^\perp. \tag{69}$$

Since M_θ is totally geodesic in M and from (67), one can find that M_θ is totally geodesic in $\tilde{M}$. Similarly, as $M_\perp$ being totally umbilical in M, then (68) and (69) imply that $M_\perp$ is totally umbilical in $\tilde{M}$, which ends the proof. □

4. Warped product hemi-slant submanifold of the form $M = M_\perp \times_f M_\theta$

Next, we will study another type of warped product hemi-slant submanifold. This submanifold can be expressed as $M = M_\perp \times_f M_\theta$ in a locally metallic Riemannian manifold $\tilde{M}$, where $M_\perp$ and M_θ are anti-invariant and slant submanifolds of $\tilde{M}$.

Lemma 4.1. *Let* $M = M_\perp \times_f M_\theta$ *be a warped product hemi-slant submanifold in a locally metallic Riemannian manifold* $\tilde{M}$*, then we have the following equations:*

$$g(h(X,Z),JY) = 0, \tag{70}$$
$$g(h(X,Y),FZ) = 0, \tag{71}$$
$$g(h(X,Z),FW) = 0, \tag{72}$$
$$g(h(Z,W),JX) = -(X\ln f)\,g(Z,TW), \tag{73}$$

for all $X, Y \in \Gamma(TM_\perp)$ *and* $Z, W \in \Gamma(TM_\theta)$.

Proof. To prove the equations in the lemma (4.1), let X, Y be vector fields tangent to $M_\perp$ and Z, W be vector fields tangent to M_θ in a locally metallic Riemannian manifold $\tilde{M}$, where $M = M_\perp \times_f M_\theta$ is a warped product hemi-slant submanifold. We can then define $g(h(X,Z),JY) = g(\tilde{\nabla}_X Z, JY)$, and so

$$g(h(X,Z),JY) = \frac{1}{p} g(J\tilde{\nabla}_X Z, JY). \tag{74}$$

By using equations (2) and (9), we have

$$g(h(X,Z),JY) = \frac{1}{p} g(\tilde{\nabla}_X(TZ), FY) + \frac{1}{p} g(\tilde{\nabla}_X(tFZ), Y) + \frac{1}{p} g(\tilde{\nabla}_X(fFZ), Y). \tag{75}$$

From equations (20) and (21), we get

$$g(h(X,Z),JY) = \frac{1}{p}(g(h(X,TZ),FY) + \frac{1}{p} g(h(X,Y),FTZ)) - g(h(X,Y),FZ). \tag{76}$$

By substituting TZ for Z in equation (76) and applying equation (16), we have

$$g(h(X,TZ),FY) = \frac{q\cos^2\theta}{p\sin^2\theta}(g(h(X,Z),FY) + \frac{q\cos^2\theta}{p\sin^2\theta} g(h(X,Y),FZ)) - g(h(X,Y),FTZ). \tag{77}$$

By applying equations (77) in (76), we obtain

$$g(h(X,Z),JY) = -g(h(X,Y),FZ). \tag{78}$$

However, we have

$$g(h(X,Z),JY) = g(\tilde{\nabla}_Z X, JY), \tag{79}$$
$$g(h(X,Z),JY) = g(\tilde{\nabla}_Z JX, Y), \tag{80}$$
$$g(h(X,Z),JY) = -g(h(Y,Z),FX). \tag{81}$$

By exchanging X for Y in equation (78) and utilizing equation (81), we arrive at

$$g(h(X,Z),JY) = 0.$$

Furthermore, employing equation (70) in conjunction with equation (78) serves as a means to validate equation (71).

Now, we apply the same techniques used to prove equation (72). We have

$$g(h(X,Z),FW) = g(\tilde{\nabla}_X Z, JW) - g(\tilde{\nabla}_X Z, TW). \tag{82}$$

From equations (3) and (18), we have

$$g(h(X,Z),FW) = \frac{1}{p}(g(J\tilde{\nabla}_X(Z), JW) - \frac{q}{p}(X\ln f)g(Z,W)) - (X\ln f)g(Z,TW).$$

Using equations (9) and (10), we get

$$g(h(X,Z),FW) = \frac{1}{p}((X\ln f)g(TZ,TW) + \frac{1}{p}g(\tilde{\nabla}_X(TZ),FW) + \frac{1}{p}g(\tilde{\nabla}_X(tFZ),W) + \frac{1}{p}g(\tilde{\nabla}_X(fFZ),W) - \frac{q}{p}(X\ln f)g(Z,W)) - (X\ln f)g(Z,TW). \tag{83}$$

By employing equations (14), (20) and (21), we obtain

$$g(h(X,Z),FW) = \frac{1}{p}g(h(X,TZ),FW) - g(h(X,W),FZ) + \frac{1}{p}g(h(X,W),FTZ). \tag{84}$$

By exchanging Z for TZ in equation (84) and utilizing equation (16), we arrive at

$$g(h(X,TZ),FW) = \frac{q\cos^2\theta}{p\sin^2\theta}g(h(X,Z),FW) - g(h(X,W),FTZ) + \frac{q\cos^2\theta}{p\sin^2\theta}g(h(X,W),FZ). \tag{85}$$

Using equations (85) in conjunction with (84), we derive

$$g(h(X,Z),FW) = -g(h(X,W),FZ). \tag{86}$$

On the other hand,

$$g(h(X,Z),FW) = g(\tilde{\nabla}_Z X, JW) - g(\tilde{\nabla}_Z X, TW), = -g(h(Z,W),FX) - (X\ln f)g(Z,TW). \tag{87}$$

Replacing Z with W in equation (87), we obtain that

$$g(h(X,W),FZ) = g(h(X,Z),FW).$$

Through the application of equation (86), we can rigorously prove equation (72). Utilizing equation (72) within equation (87), we obtain

$$g(h(Z,W),JX) = -(X\ln f)\,g(Z,TW),$$

and with that, the lemma is proven. □

Example 4.2. *We consider an immersion of a subset M of a manifold $\tilde{M}$ into itself, defined as a warped product hemi-slant submanifold. This immersion is characterized by the following equation*

$$\begin{aligned} i(u,v,\theta,\phi) = (&\sqrt{2}u\cos\theta\sin\phi,\ \sqrt{2}u\sin\theta\sin\phi,\ \sqrt{2}u\sin\theta\cos\phi,\ \sqrt{2}u\cos\theta\cos\phi \\ &,\ \sqrt{2}v\cos\theta\sin\phi,\ \sqrt{2}v\sin\theta\sin\phi,\ \sqrt{2}v\sin\theta\cos\phi,\ \sqrt{2}v\cos\theta\cos\phi, \\ &\sqrt{\frac{1}{q}\sigma v}\cos\theta,\ \sqrt{\frac{1}{q}\sigma v}\sin\theta,\ \sqrt{\frac{1}{q}\sigma v}\cos\phi,\ \sqrt{\frac{1}{q}\sigma v}\sin\phi,\ \sqrt{\frac{1}{q}\sigma u}\cos\theta, \\ &\sqrt{\frac{1}{q}\sigma u}\sin\theta,\ \sqrt{\frac{1}{q}\sigma u}\cos\phi,\ \sqrt{\frac{1}{q}\sigma u}\sin\phi), \end{aligned} \tag{88}$$

where $u,v>0$, θ and ϕ belong to the interval $(0,\frac{\pi}{2})$, σ is a metallic number, $\tilde{\sigma} := p-\sigma$, and p and q are positive integers. By applying basic linear algebra principles, we can readily establish that the set of tangent vectors $\{Z_1,Z_2,Z_3,Z_4\}$

constitutes a basis for the tangent bundle of M, where

$$Z_1 = \sqrt{2}\cos\theta\sin\phi\frac{\partial}{\partial x_1} + \sqrt{2}\sin\theta\sin\phi\frac{\partial}{\partial x_2} + \sqrt{2}\sin\theta\cos\phi\frac{\partial}{\partial x_3} + \sqrt{2}\cos\theta\cos\phi\frac{\partial}{\partial x_4} + \sqrt{\frac{1}{q}}\sigma\cos\theta\frac{\partial}{\partial x_{13}} + \sqrt{\frac{1}{q}}\sigma\sin\theta\frac{\partial}{\partial x_{14}} + \sqrt{\frac{1}{q}}\sigma\cos\phi\frac{\partial}{\partial x_{15}} + \sqrt{\frac{1}{q}}\sigma\sin\phi\frac{\partial}{\partial x_{16}}, \tag{89}$$

$$Z_2 = \sqrt{2}\cos\theta\sin\phi\frac{\partial}{\partial x_5} + \sqrt{2}\sin\theta\sin\phi\frac{\partial}{\partial x_6} + \sqrt{2}\sin\theta\cos\phi\frac{\partial}{\partial x_7} + \sqrt{2}\cos\theta\cos\phi\frac{\partial}{\partial x_8} + \sqrt{\frac{1}{q}}\sigma\cos\theta\frac{\partial}{\partial x_9} + \sqrt{\frac{1}{q}}\sigma\sin\theta\frac{\partial}{\partial x_{10}} + \sqrt{\frac{1}{q}}\sigma\cos\phi\frac{\partial}{\partial x_{11}} + \sqrt{\frac{1}{q}}\sigma\sin\phi\frac{\partial}{\partial x_{12}}, \tag{90}$$

$$Z_3 = -\sqrt{2}u\sin\theta\sin\phi\frac{\partial}{\partial x_1} + \sqrt{2}u\cos\theta\sin\phi\frac{\partial}{\partial x_2} + \sqrt{2}u\cos\theta\cos\phi\frac{\partial}{\partial x_3} - \sqrt{2}u\sin\theta\cos\phi\frac{\partial}{\partial x_4} - \sqrt{2}v\sin\theta\sin\phi\frac{\partial}{\partial x_5} + \sqrt{2}v\cos\theta\sin\phi\frac{\partial}{\partial x_6} + \sqrt{2}v\cos\theta\cos\phi\frac{\partial}{\partial x_7} - \sqrt{2}v\sin\theta\cos\phi\frac{\partial}{\partial x_8} - \sqrt{\frac{1}{q}}\sigma v\sin\theta\frac{\partial}{\partial x_9} + \sqrt{\frac{1}{q}}\sigma v\cos\theta\frac{\partial}{\partial x_{10}} - \sqrt{\frac{1}{q}}\sigma u\sin\theta\frac{\partial}{\partial x_{13}} + \sqrt{\frac{1}{q}}\sigma u\cos\theta\frac{\partial}{\partial x_{14}}, \tag{91}$$

$$Z_4 = \sqrt{2}u\cos\theta\cos\phi\frac{\partial}{\partial x_1} + \sqrt{2}u\sin\theta\cos\phi\frac{\partial}{\partial x_2} - \sqrt{2}u\sin\theta\sin\phi\frac{\partial}{\partial x_3} - \sqrt{2}u\cos\theta\sin\phi\frac{\partial}{\partial x_4} + \sqrt{2}v\cos\theta\cos\phi\frac{\partial}{\partial x_5} + \sqrt{2}v\sin\theta\cos\phi\frac{\partial}{\partial x_6} - \sqrt{2}v\sin\theta\sin\phi\frac{\partial}{\partial x_7} - \sqrt{2}v\cos\theta\sin\phi\frac{\partial}{\partial x_8} - \sqrt{\frac{1}{q}}\sigma v\sin\phi\frac{\partial}{\partial x_{11}} + \sqrt{\frac{1}{q}}\sigma v\cos\phi\frac{\partial}{\partial x_{12}} - \sqrt{\frac{1}{q}}\sigma u\sin\phi\frac{\partial}{\partial x_{15}} + \sqrt{\frac{1}{q}}\sigma u\cos\phi\frac{\partial}{\partial x_{16}}. \tag{92}$$

Applying the linear transformation J of $\tilde{M}$, which flips signs of certain coordinates, to a point on $\tilde{M}$ will yield new coordinates as follows:

$$J(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) = (\sigma x_1, \sigma x_2, \sigma x_3, \sigma x_4, \sigma x_5, \sigma x_6, \sigma x_7, \sigma x_8, \tilde{\sigma} x_9, \tilde{\sigma} x_{10}, \tilde{\sigma} x_{11}, \tilde{\sigma} x_{12}, \tilde{\sigma} x_{13}, \tilde{\sigma} x_{14}, \tilde{\sigma} x_{15}, \tilde{\sigma} x_{16}). \tag{93}$$

As a consequence, we obtain that

$$JZ_1 = \sqrt{2}\sigma\cos\theta\sin\phi\frac{\partial}{\partial x_1} + \sqrt{2}\sigma\sin\theta\sin\phi\frac{\partial}{\partial x_2} + \sqrt{2}\sigma\sin\theta\cos\phi\frac{\partial}{\partial x_3} + \sqrt{2}\sigma\cos\theta\cos\phi\frac{\partial}{\partial x_4} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}\cos\theta\frac{\partial}{\partial x_{13}} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}\sin\theta\frac{\partial}{\partial x_{14}} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}\cos\phi\frac{\partial}{\partial x_{15}} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}\sin\phi\frac{\partial}{\partial x_{16}}, \tag{94}$$

$$JZ_2 = \sqrt{2}\sigma\cos\theta\sin\phi\frac{\partial}{\partial x_5} + \sqrt{2}\sigma\sin\theta\sin\phi\frac{\partial}{\partial x_6} + \sqrt{2}\sigma\sin\theta\cos\phi\frac{\partial}{\partial x_7} + \sqrt{2}\sigma\cos\theta\cos\phi\frac{\partial}{\partial x_8} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}\cos\theta\frac{\partial}{\partial x_9} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}\sin\theta\frac{\partial}{\partial x_{10}} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}\cos\phi\frac{\partial}{\partial x_{11}} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma}\sin\phi\frac{\partial}{\partial x_{12}}, \tag{95}$$

$$\begin{aligned}JZ_3 = &- \sqrt{2}\sigma u \sin\theta \sin\phi \frac{\partial}{\partial x_1} + \sqrt{2}\sigma u \cos\theta \sin\phi \frac{\partial}{\partial x_2} + \sqrt{2}\sigma u \cos\theta \cos\phi \frac{\partial}{\partial x_3} \\ &- \sqrt{2}\sigma u \sin\theta \cos\phi \frac{\partial}{\partial x_4} - \sqrt{2}\sigma v \sin\theta \sin\phi \frac{\partial}{\partial x_5} + \sqrt{2}\sigma v \cos\theta \sin\phi \frac{\partial}{\partial x_6} \\ &+ \sqrt{2}\sigma v \cos\theta \cos\phi \frac{\partial}{\partial x_7} - \sqrt{2}\sigma v \sin\theta \cos\phi \frac{\partial}{\partial x_8} - \sqrt{\frac{1}{q}}\sigma\tilde{\sigma} v \sin\theta \frac{\partial}{\partial x_9} \\ &+ \sqrt{\frac{1}{q}}\sigma\tilde{\sigma} v \cos\theta \frac{\partial}{\partial x_{10}} - \sqrt{\frac{1}{q}}\sigma\tilde{\sigma} u \sin\theta \frac{\partial}{\partial x_{13}} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma} u \cos\theta \frac{\partial}{\partial x_{14}},\end{aligned} \tag{96}$$

$$\begin{aligned}JZ_4 = &\sqrt{2}\sigma u \cos\theta \cos\phi \frac{\partial}{\partial x_1} + \sqrt{2}\sigma u \sin\theta \cos\phi \frac{\partial}{\partial x_2} - \sqrt{2}\sigma u \sin\theta \sin\phi \frac{\partial}{\partial x_3} \\ &- \sqrt{2}\sigma u \cos\theta \sin\phi \frac{\partial}{\partial x_4} + \sqrt{2}\sigma v \cos\theta \cos\phi \frac{\partial}{\partial x_5} + \sqrt{2}\sigma v \sin\theta \cos\phi \frac{\partial}{\partial x_6} \\ &- \sqrt{2}\sigma v \sin\theta \sin\phi \frac{\partial}{\partial x_7} - \sqrt{2}\sigma v \cos\theta \sin\phi \frac{\partial}{\partial x_8} - \sqrt{\frac{1}{q}}\sigma\tilde{\sigma} v \sin\phi \frac{\partial}{\partial x_{11}} \\ &+ \sqrt{\frac{1}{q}}\sigma\tilde{\sigma} v \cos\phi \frac{\partial}{\partial x_{12}} - \sqrt{\frac{1}{q}}\sigma\tilde{\sigma} u \sin\phi \frac{\partial}{\partial x_{15}} + \sqrt{\frac{1}{q}}\sigma\tilde{\sigma} u \cos\phi \frac{\partial}{\partial x_{16}}.\end{aligned} \tag{97}$$

On the manifold $\tilde{M}$, we define two distributions:

- $\mathfrak{D}^{\perp}$ *spanned by vectors Z_1 and Z_2, is anti-invariant under the metallic structure J.*
- $\mathfrak{D}^{\theta^*}$ *spanned by vector Z_3 and Z_4, is a slant distribution with slant angle $\theta^* = \cos^{-1}\frac{\sigma}{\sqrt{(2+\frac{\sigma^2}{q})(q+2\sigma^2)}}$, under J.*

The metric tensor on M is given by

$$g := \left(2 + \frac{2\sigma^2}{q}\right)(d^2u + d^2v) + \left((u^2+v^2)(2+\frac{\sigma^2}{q})\right)(d^2\theta + d^2\phi) \tag{98}$$

This defines a warped product hemi-slant submanifold M embedded in a metallic Riemannian manifold $\tilde{M}$. M can be expressed as the product of two manifolds, $M = M_{\perp} \times_f M_{\theta}$, where the warping function f stretches $M_{\perp}$ by a factor of u and v.

Theorem 4.3. *Let $M = M_{\perp} \times_f M_{\theta}$ be a warped product hemi-slant submanifold in a locally metallic Riemannian manifold $\tilde{M}$, where $M_{\perp}$ is an anti-invariant submanifold and M_{θ} is a slant submanifold of $\tilde{M}$. Then,*

(i) The squared norm of the second fundamental form h of M is bounded below by the expression

$$||h||^2 \geq (2q + p^2\cos^2\theta) w \cos^2\theta ||\nabla^{\perp} \ln f||^2, \tag{99}$$

where $\dim M_{\theta} = t_2 = 2w$ and $\nabla^{\perp}\ln f$ is the gradient of $\ln f$ along $M_{\perp}$.

(ii) If equality holds, then both $M_{\perp}$ and M_{θ} become totally geodesic and totally umbilical submanifolds of $\tilde{M}$, respectively.

Proof. The squared norm of the second fundamental form h of M is given by the following equation

$$||h||^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j), h(e_i,e_j)) = \sum_{r=n+1}^{m}\sum_{i,j=1}^{n} g(h(e_i,e_j), e_r)^2. \tag{100}$$

Through the application of equation (100) and using the frame fields (57), (58), and (59), we see that

$$\begin{aligned}\|h\|^2 = &\sum_{r=1}^{t_1}\sum_{i,j=1}^{t_1} g(h(e_i,e_j),\tilde{e_r})^2 + \sum_{r=t_1+1}^{m-n}\sum_{i,j=1}^{t_1} g(h(e_i,e_j),\tilde{e_r})^2 + 2\sum_{r=1}^{t_1}\sum_{i=1}^{t_1}\sum_{j=t_1+1}^{t_1+t_2} g(h(e_i,e_j),\tilde{e_r})^2 \\ &+ 2\sum_{r=t_1+1}^{m-n}\sum_{i=1}^{t_1}\sum_{j=t_1+1}^{t_1+t_2} g(h(e_i,e_j),\tilde{e_r})^2 + \sum_{r=1}^{t_1}\sum_{i,j=t_1+1}^{t_1+t_2} g(h(e_i,e_j),\tilde{e_r})^2 + \sum_{r=t_1+1}^{m-n}\sum_{i,j=t_1+1}^{t_1+t_2} g(h(e_i,e_j),\tilde{e_r})^2. \qquad (101)\end{aligned}$$

Using equations (70) and (72) in (101) leads us to

$$\|h\|^2 \geq \sum_{r=1}^{t_1}\sum_{i,j=t_1+1}^{t_1+t_2} g(h(e_i,e_j),\tilde{e_r})^2. \qquad (102)$$

By equation (58), one can rewrite the last inequality as

$$\|h\|^2 \geq \sum_{r=1}^{t_1}\sum_{i,j=t_1+1}^{t_1+t_2} g(h(e_i,e_j),Fe_r)^2. \qquad (103)$$

By applying equation (73) and then using the frame field (57), we get

$$\begin{aligned}\|h\|^2 \geq &\sum_{r=1}^{t_1}\sum_{i,j=t_1+w+1}^{t_1+t_2} (e_r \ln f)^2 g(e_i,Te_j)^2 + \sum_{r=1}^{t_1}\sum_{i=t_1+1}^{t_1+w}\sum_{j=t_1+w+1}^{t_1+t_2} (e_r \ln f)^2 g(e_i,Te_j)^2 \\ &+ \sum_{r=1}^{t_1}\sum_{i=t_1+w+1}^{t_1+t_2}\sum_{j=t_1+1}^{t_1+w} (e_r \ln f)^2 g(e_i,Te_j)^2, \qquad (104)\end{aligned}$$

and so,

$$\begin{aligned}\|h\|^2 \geq &\frac{\sec^2\theta}{q}\sum_{r=1}^{t_1}\sum_{i=t_1+w+1}^{t_1+t_2}\sum_{j=t_1+1}^{t_1+w} (e_r \ln f)^2 g(e_i,T^2e_j)^2 + \frac{\sec^2\theta}{q}\sum_{r=1}^{t_1}\sum_{i,j=t_1+1}^{t_1+w} (e_r \ln f)^2 g(e_i,T^2e_j)^2 \\ &+ \frac{q}{\sec^2\theta}\sum_{r=1}^{t_1}\sum_{i,j=t_1+w+1}^{t_1+t_2} (e_r \ln f)^2 g(e_i,e_j)^2. \qquad (105)\end{aligned}$$

Using equation (16) in (105) allows us to get

$$\|h\|^2 \geq \frac{p^2\cos^2\theta}{q}\sum_{r=1}^{t_1}\sum_{i=t_1+w+1}^{t_1+t_2}\sum_{j=t_1+1}^{t_1+w} (e_r \ln f)^2 g(e_i,Te_j)^2 + \frac{wq^2\cos^2\theta}{q}\sum_{r=1}^{t_1}(e_r \ln f)^2 + \frac{qw}{\sec^2\theta}\sum_{r=1}^{t_1}(e_r \ln f)^2. \qquad (106)$$

By (57), we obtain that

$$\|h\|^2 \geq (2q + p^2\cos^2\theta) w \cos^2\theta \|\nabla^{\perp} \ln f\|^2.$$

In M, $M_{\perp}$ is totally geodesic and M_{θ} is totally umbilical. The vanishing of first, third, and sixth terms in equation (101) implies that $M_{\perp}$ and M_{θ} retain their geometric properties in $\tilde{M}$ under the condition that the squared norm of the second fundamental form h of M is equal, we have

$$h(\mathfrak{D}^{\perp},\mathfrak{D}^{\perp}) = 0, \qquad (107)$$

$$h(\mathfrak{D}^{\perp},\mathfrak{D}^{\theta}) \subseteq FD^{\theta}, \qquad (108)$$

$$h(D^{\theta},D^{\theta}) \subseteq JD^{\perp}. \qquad (109)$$

Specifically, equation (107) implies that $M_{\perp}$ is totally geodesic in $\tilde{M}$. Equations (108) and (109) together imply that M_{θ} is totally umbilical in $\tilde{M}$. □

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