Filomat 38:33 (2024), 11747–11772
https://doi.org/10.2298/FIL2433747D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Exploring gradient soliton structures and metric deformations on Riemannian manifolds with parallel vector fields

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Abstract. In this present paper, we delve into an investigation of the distinctive characteristics and properties exhibited by various Ricci soliton structures that manifest within Riemannian manifolds equipped with parallel vector fields. Expanding upon these discoveries, we offer systematic classifications related to the conformal and semi-conformal modifications applied to a Riemannian metric denoted as g. To enhance the clarity and practical applicability of our research, we provide illustrative examples that serve to validate and exemplify the theoretical findings and relationships presented throughout the paper. These examples serve as tangible instances that concretely exemplify the presence and behavior of manifolds endowed with the aforementioned soliton structures.

1. Introduction

Ricci solitons on Riemannian manifolds have become an intriguing and actively studied topic in recent years, largely inspired by the groundbreaking work of Richard S. Hamilton and Grigori Perelman on the Ricci flow, which is a fundamental tool in the field of differential geometry and geometric analysis. Hamilton and Perelman's contributions, documented in their respective papers [14] and [16], have provided a deep understanding of the geometry and topology of Riemannian manifolds through the study of Ricci solitons. Since the seminal works of Hamilton and Perelman, numerous researchers have been captivated by the study and exploration of Ricci solitons on Riemannian manifolds. This area of research has garnered significant attention and has led to a wealth of interesting results and developments. One particular focus within the study of Ricci solitons is on Riemannian manifolds that admit certain special vector fields. These vector fields play a crucial role in understanding the geometry and dynamics of the manifold. Some notable contributions and references in this direction include [3], [17], and [19]. These works delve into the properties and implications of vector fields on manifolds with Ricci solitons and provide valuable insights into the underlying structures. Overall, the study of Ricci solitons on Riemannian manifolds has evolved into a rich and multifaceted research area, offering a deeper understanding of geometric properties, differential

2020 *Mathematics Subject Classification*. Primary 53C18, 53C21; Secondary 53C25.
Keywords. Parallel vector field, gradient Ricci soliton, gradient m-QE, gradient ρ-Einstein soliton, gradient η-Ricci soliton, conformal metric, semi-conformal metric.
Received: 26 March 2024; Accepted: 31 July 2024
Communicated by Mića Stanković
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equations, and their connections to various fields of mathematics and physics. Researchers continue to explore and expand upon this topic, making it a fascinating and continually evolving area of study.

In this paper, the focus is on the study of various types of gradient Ricci soliton structures on a Riemannian manifold denoted as (M^n, g), with a particular emphasis on cases where the manifold admits parallel vector fields. The research presented in this paper explores several intriguing relationships and results concerning these structures. Let us break down the key findings and contributions presented in the paper. In the initial section of the paper, we establish fundamental connections between the parallel vector field denoted as P and the potential function ψ for Riemannian metrics that are gradient Ricci solitons. These relationships are stated as Theorem 3.1 and Corollary 3.5, providing insights into the interplay between vector fields and the potential function associated with gradient Ricci soliton metrics. In section 2, the paper explores Riemannian metrics that belong to the category of gradient m-quasi-Einstein metrics. It is proven in Theorem 3.8 that either the gradient of the potential function ψ aligns with the parallel vector field P, or a specific constant parameter λ is equal to zero. Additionally, Corollary 3.10 introduces further implications, including cases where the manifold exhibits constant scalar curvature or satisfies a particular equation involving $P(\psi)$. In section 3, we proceed to investigate gradient ρ-Einstein soliton metrics, showing that in such cases, the Riemannian manifold M^n must have constant scalar curvature (Theorem 3.16). Section 5 is dedicated to investigating gradient η-Ricci soliton metrics. Theorems and corollaries, specifically Theorem 3.18 to Corollary 3.23, provide crucial insights into the intricate interplay between parallel vector fields and another vector field, denoted as V, associated with a 1-form η. These findings shed light on the manifold's geometric properties in this context. To illustrate these results, we provide a series of concrete examples (Example 4.1 to Example 4.3) that showcase the practical applications of their theoretical discoveries. In the closing sections of the paper, the paper explores the behavior of manifolds subject to conformal and semi-conformal deformations of the Riemannian metric. We provide interesting classifications and insights into how these deformations impact the properties of the underlying manifolds. In summary, this research paper contributes to the understanding of various types of gradient soliton structures on Riemannian manifolds, shedding light on their relationships with parallel vector fields, potential functions, and other geometric properties. The presented theorems, corollaries, and examples offer valuable tools for further exploration of these topics in differential geometry and related fields.

2. Preliminaries

A Ricci soliton is a concept in the realm of Riemannian geometry, specifically defined on an n-dimensional Riemannian manifold denoted as (M^n, g). This definition, elucidated in works such as [5] and [12], comprises a triple denoted as (g, Z, λ), where Z represents a vector field on the manifold, and λ is a real constant. These components are subject to the equation

$$\frac{1}{2}\mathcal{L}_Z g + Ric = \lambda g. \tag{1}$$

Here, $\mathcal{L}_Z$ denotes the Lie derivative operator along the vector field Z, and Ric represents the Ricci curvature tensor. The nature of a Ricci soliton can be categorized as shrinking, expanding, or steady, depending on whether λ is positive, negative, or zero, respectively.

A special case arises when the vector field Z is a gradient of a smooth function ψ. In this scenario, the Ricci soliton is termed a gradient Ricci soliton, as discussed in references like [17], [19], and [20]. In this context, the equation (1) can be expressed as

$$Ric + H^{\psi} = \lambda g,$$

where H^{ψ} represents the Hessian operator of the function ψ. It is worth noting that on any compact manifold, every Ricci soliton can be considered a gradient Ricci soliton, as demonstrated in [16]. Additionally, it's important to recognize that any Einstein metric inherently provides a trivial gradient Ricci soliton. This concept plays a significant role in understanding the geometric properties and curvature of Riemannian manifolds, offering valuable insights into their structural characteristics.

Various extensions and generalizations of gradient Ricci solitons have been explored within the field of Riemannian geometry. Among these generalizations, a notable metric is defined as follows

$$Ric + H^{\psi} - \frac{1}{m} d\psi \otimes d\psi = \lambda g, \tag{2}$$

where $0 < m \leq \infty$ is an integer, and this metric is referred to as a gradient m-quasi-Einstein metric, succinctly termed a gradient m-QE metric (with m indicating the degree of generalization). In the case where $m = \infty$, this metric reduces to a standard gradient Ricci soliton, as discussed in [6]. These gradient m-QE metrics offer a broader framework for understanding geometric structures on Riemannian manifolds, encompassing both the traditional gradient Ricci solitons and their extended variations. They provide valuable tools for investigating the curvature properties and behaviors of these manifolds under different conditions.

In the papers by Catino and Mazzieri, namely [7] and [8], they introduced the concept of a gradient ρ-Einstein soliton. This concept pertains to a Riemannian metric g defined on an n-dimensional Riemannian manifold (M^n, g), with $n \geq 3$. A metric is considered a gradient ρ-Einstein soliton if there exist three crucial components: a smooth function $\psi \in C^{\infty}(M^n)$, a real constant $\lambda \in \mathbb{R}$, and a non-zero real constant ρ, such that the following equation holds

$$Ric + H^{\psi} = (\lambda + \rho S)g, \tag{3}$$

where S denotes the scalar curvature of the metric g. This soliton concept can further be classified based on specific values of ρ. For instance, when ρ takes on values of $\frac{1}{2}$, $\frac{1}{n}$, and $\frac{1}{2n-1}$, the gradient ρ-Einstein soliton is respectively referred to as a gradient Einstein soliton, a gradient traceless Ricci soliton, and a gradient Schouten soliton. For more detailed information, you can refer to [13] and [21].

An η-Ricci soliton, introduced by Cho and Kimura in [9], is defined on a Riemannian manifold (M^n, g) and is denoted by (g, Z, λ, μ). This soliton is characterized by the equation

$$Ric + \frac{1}{2}\mathcal{L}_Z g - \mu \eta \otimes \eta = \lambda g, \tag{4}$$

where η represents a 1-form, and λ and μ are real constants. If the vector field Z is the gradient of a smooth function ψ, then this soliton is referred to as a gradient η-Ricci soliton, as discussed in [1]. The equation (4) can be rewritten as

$$Ric + H^{\psi} - \mu \eta \otimes \eta = \lambda g.$$

In cases where λ, ρ, and μ are smooth functions, the equations (1), (2), (3), and (4) are collectively referred to as almost Ricci soliton, almost gradient m-QE metric, almost gradient ρ-Einstein soliton, and almost gradient η-Ricci soliton, respectively. These terms are discussed in more detail in [2], [11], and [18].

Consider a Riemannian manifold M^n equipped with the Levi-Civita connection ∇ corresponding to the Riemannian metric g. A vector field P on M^n is categorized as a parallel vector field if it satisfies the condition

$$\nabla P = 0.$$

This condition signifies that the derivative (covariant derivative) of the vector field P is zero, indicating that P does not change as one moves along the manifold in any direction. Moreover, let us assume that the norm of the vector field P is constant, and for simplicity, we will set it equal to 1. Under this assumption, the following lemma holds true.

Lemma 2.1. *Let (M^n, g) be a Riemannian manifold. If P is a parallel vector field on M^n, then for all vector fields X, Y, Z on M^n we have*

$$\begin{aligned} R(X,Y)P &= 0, \\ g(R(X,Y)Z, P) &= 0, \\ Ric(X,P) &= 0, \\ Q(P) &= 0, \end{aligned}$$

where R, Ric and Q are the Riemannian curvature tensor, Ricci curvature tensor and the Ricci operator, respectively.

Theorem 2.2. *Let (M^n, g) be a Riemannian manifold. If P is a unit parallel vector field on M^n, then either the scalar curvature S is constant or the gradient of the function S is g-orthogonal with the vector P.*

Proof. Using the well known equation [15]:

$$dS(X) = 2div(Ric)(X).$$

Taking into account that

$$\begin{aligned} div(Ric)(X) &= \sum_{i=1}^{n} (\nabla_{e_i} Ric)(X, e_i) \\ &= g(\nabla_{e_i} Q(X), e_i) - g(Q(e_i), \nabla_{e_i} X), \end{aligned}$$

we get

$$dS(P) = g(grad(S), P) = 0,$$

which means either $grad(S) = 0$ or $grad(S) \perp P$. □

3. Main results

In this section, we provide characterizations of various gradient Ricci soliton structures on a Riemannian manifold that accommodates parallel vector fields. Our investigation yields several intriguing results and insights into these geometric configurations.

3.1. The Gradient Ricci soliton

Theorem 3.1. *Suppose (M^n, g) is a Riemannian manifold, and let P be a unit parallel vector field on M^n. If (M^n, g, ψ, λ) is a gradient Ricci soliton manifold, then the gradient of the function $P(\psi)$ is parallel to P.*

Proof. If (M^n, g, ψ, λ) is a gradient Ricci soliton manifold, then we have

$$Ric(X, Y) + H^{\psi}(X, Y) = \lambda g(X, Y).$$

From Lemma 2.1, for any vector field X on M^n we obtain

$$\begin{aligned} Ric(X, P) &= 0 \\ &= \lambda g(X, P) - g(\nabla_X grad(\psi), P) \\ &= \lambda g(X, P) - X(g(grad(\psi), P)) \\ &= g(X, \lambda P - grad(P(\psi))). \end{aligned}$$

Hence, we deduce $\lambda P = grad(P(\psi))$. Thus, the proof is completed. □

Moreover, we can state the following.

Corollary 3.2. *Under the hypotheses of Theorem 3.1 we find that*

$$\lambda = P(P(\psi)).$$

Proof. One can easily see that

$$\lambda = g(\lambda P, P) = g(grad(P(\psi)), P) = P(P(\psi)).$$

□

Proposition 3.3. *Consider a Riemannian manifold (M^n, g) with a unit parallel vector field P. If g is an Einstein metric, then $\lambda = 0$, and M^n is a Ricci-flat manifold.*

Proof. If g is an Einstein metric, it naturally induces a trivial gradient Ricci soliton with a constant function ψ. By applying Corollary 3.2, we conclude that $\lambda = 0$, which further implies $Ric = 0$. □

Theorem 3.4. *If (M^n, g, ψ, λ) is a gradient Ricci soliton manifold, then for any vector fields X and Y defined on M^n, we have the following relations*

1.
$$\nabla_X grad(\psi) = \lambda X - Q(X), \tag{5}$$

2.
$$R(X,Y)grad(\psi) = (\nabla_Y Q)X - (\nabla_X Q)Y, \tag{6}$$

3.
$$Ric(Y, grad(\psi)) = \frac{1}{2}Y(S),$$

where $(\nabla_X Q)Y = \nabla_X Q(Y) - Q(\nabla_X Y)$.

Proof. Let (M^n, g, ψ, λ) be a gradient Ricci soliton manifold, i.e.,

$$Ric + H^{\psi} = \lambda g.$$

1. For all vector fields X, Y on M^n, we have
$$\begin{aligned}\lambda g(X,Y) &= Ric(X,Y) + H^{\psi}(X,Y)\\ &= g(Q(X),Y) + g(\nabla_X grad(\psi), Y)\\ &= g(Q(X) + \nabla_X grad(\psi), Y),\end{aligned}$$
from which we deduce the equation (5).

2. From the equation (5), we obtain
$$\nabla_Y \nabla_X grad(\psi) = \lambda \nabla_Y X - \nabla_Y Q(X). \tag{7}$$
Interchanging X and Y in (7), we get
$$\nabla_X \nabla_Y grad(\psi) = \lambda \nabla_X Y - \nabla_X Q(Y). \tag{8}$$
Substituting the equations (7) and (8) into $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, we obtain (6).

3. Let $x \in M^n$ and $(e_i)_{i=1}^n$ be an orthonormal frame on M^n such that $(\nabla_{e_i} e_j)_x = 0$. By using the definition of Ricci curvature tensor and from (6) we get
$$\begin{aligned}Ric(Y, grad(\psi)) &= \sum_{i=1}^{n} g(R(e_i, Y)grad(\psi), e_i)\\ &= \sum_{i=1}^{n} \Big[g(\nabla_Y Q(e_i), e_i) - g(Q(\nabla_Y e_i), e_i) - g((\nabla_{e_i} Q)Y, e_i)\Big]\\ &= Y(S) - \sum_{i=1}^{n} \Big[2g(Q(e_i), \nabla_Y e_i) + g((\nabla_{e_i} Q)Y, e_i)\Big]\\ &= Y(S) - div(Ric)(Y),\end{aligned}$$
where $div(Ric)(Y) = \sum_{i=1}^{n} (\nabla_{e_i} Ric)(Y, e_i)$, by applying the well-known formula ($2div(Ric)(Y) = Y(S)$) the result immediately follows.

□

Corollary 3.5. *Suppose (M^n, g) is a Riemannian manifold with a unit parallel vector field P. If g is a gradient Ricci soliton metric, then*

$$(\nabla_P Q)(X) = 0$$

for any vector field X on M^n.

Proof. The proof follows immediately from (6) and by virtue of $R(X,P)grad(\psi) = 0$. □

3.2. The Gradient m-QE

Theorem 3.6. *If (M^n, g) is a Riemannian manifold with a unit parallel vector field P, and (M^n, g, ψ, λ) is a gradient m-QE manifold, then we can state the following*

$$\lambda = P(P(\psi)) - \frac{1}{m}P(\psi)P(\psi).$$

Proof. If (M^n, g, ψ, λ) forms a gradient *m-QE* manifold, then we can state the following:

$$Ric(X,Y) + H^{\psi}(X,Y) - \frac{1}{m}d\psi(X)d\psi(Y) = \lambda g(X,Y).$$

By utilizing Lemma 2.1, we can derive the following result for any vector field X defined on M^n:

$$\begin{aligned} Ric(X,P) &= 0 \\ &= \lambda g(X,P) - g(\nabla_X grad(\psi), P) + \frac{1}{m}d\psi(X)d\psi(P) \\ &= \lambda g(X,P) - X(g(grad(\psi), P)) + \frac{1}{m}P(\psi)X(\psi) \\ &= g(X, \lambda P - grad(P(\psi)) + \frac{1}{m}P(\psi)grad(\psi)). \end{aligned}$$

Thus, we deduce

$$\lambda P = grad(P(\psi)) - \frac{1}{m}P(\psi)grad(\psi).$$

Notice that $\lambda = g(\lambda P, P)$ from which we get

$$\lambda = g(grad(P(\psi)), P) - \frac{1}{m}P(\psi)g(grad(\psi), P).$$

Hence, Theorem 3.6 follows. □

Theorem 3.7. *If (M^n, g) is a Riemannian manifold and (M^n, g, ψ, λ) is a gradient m-QE manifold, then for all vector fields X and Y defined on M^n, we have the followings*

1.

$$\nabla_X grad(\psi) = \lambda X - Q(X) + \frac{1}{m}g(grad(\psi), X)grad(\psi), \tag{9}$$

2.

$$\begin{aligned} R(X,Y)grad(\psi) &= (\nabla_Y Q)X - (\nabla_X Q)Y + \frac{1}{m}Y(\psi)[\lambda X - Q(X)] \\ &\quad -\frac{1}{m}X(\psi)[\lambda Y - Q(Y)], \end{aligned} \tag{10}$$

3.

$$(\frac{m-1}{m})Ric(Y, grad(\psi)) = \frac{\lambda(n-1)-S}{m}Y(\psi) + \frac{1}{2}Y(S), \tag{11}$$

where $(\nabla_X Q)Y = \nabla_X Q(Y) - Q(\nabla_X Y)$.

Proof. In the context of a gradient m-QE manifold (M^n, g, ψ, λ), we can deduce the following from the equation (2):

$$Ric + H^{\psi} - \frac{1}{m}d\psi \otimes d\psi = \lambda g.$$

1. For all vector fields X, Y on M^n, we have

$$\begin{aligned} \lambda g(X,Y) &= Ric(X,Y) + H^{\psi}(X,Y) - \frac{1}{m}d\psi(X)d\psi(Y) \\ &= g(Q(X),Y) + g(\nabla_X grad(\psi), Y) - \frac{1}{m}g(grad(\psi),X)g(grad(\psi),Y) \\ &= g(Q(X) + \nabla_X grad(\psi) - \frac{1}{m}g(grad(\psi),X)grad(\psi), Y), \end{aligned}$$

from which we obtain the equation (9).

2. From the equation (9), we obtain

$$\begin{aligned} \nabla_Y\nabla_X grad(\psi) &= \lambda\nabla_Y X - \nabla_Y Q(X) + \frac{1}{m}Y(X(\psi))grad(\psi) \\ &+ \frac{1}{m}g(X, grad(\psi))[\lambda Y - Q(Y) + \frac{1}{m}Y(\psi)grad(\psi)]. \end{aligned} \tag{12}$$

In (12), by exchanging X and Y, we derive

$$\begin{aligned} \nabla_X\nabla_Y grad(\psi) &= \lambda\nabla_X Y - \nabla_X Q(Y) + \frac{1}{m}X(Y(\psi))grad(\psi) \\ &+ \frac{1}{m}g(Y, grad(\psi))[\lambda X - Q(X) + \frac{1}{m}X(\psi)grad(\psi)]. \end{aligned} \tag{13}$$

The equation (10) is obtained by substituting (12) and (13) into the expression $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$.

3. Suppose $x \in M^n$ and let $(e_i)_{i=1}^n$ is an orthonormal frame on M^n such that $(\nabla_{e_i}e_j)_x = 0$, by using (10), we see

$$\begin{aligned} Ric(Y, grad(\psi)) &= \sum_{i=1}^{n} g(R(e_i, Y)grad(\psi), e_i) \\ &= \sum_{i=1}^{n} \Big[g(\nabla_Y Q(e_i), e_i) - g(Q(\nabla_Y e_i), e_i) - g((\nabla_{e_i}Q)Y, e_i) \\ &\quad + \frac{1}{m}[Y(\psi)g(\lambda e_i - Q(e_i), e_i) - e_i(\psi)g(\lambda Y - Q(Y), e_i)]\Big] \\ &= Y(S) - div(Ric)(Y) + \frac{1}{m}Y(\psi)(\lambda n - S) - \frac{1}{m}g(\lambda Y - Q(Y), grad(\psi)), \\ &= Y(S) - div(Ric)(Y) + \frac{1}{m}Y(\psi)(\lambda(n-1) - S) + \frac{1}{m}Ric(Y, grad(\psi)). \end{aligned}$$

Taking into account $\frac{1}{2}Y(S) = div(Ric)(Y)$ and $\sum_{i=1}^n e_i(\psi)e_i = grad(\psi)$, the result immediately follows.

□

Theorem 3.8. *Suppose (M^n, g) is a Riemannian manifold with a unit parallel vector field P. If g is a gradient m-QE metric, then one of the following conditions holds: either $\lambda = 0$, or the gradient of the potential function ψ is parallel to the vector field P.*

Proof. Using (10) and by virtue of $g(R(X,Y)grad(\psi), P) = 0$, we find

$$\begin{aligned} 0 &= g(R(X,Y)grad(\psi), P) \\ &= g(\nabla_Y Q(X) - Q(\nabla_Y X), P) - g(\nabla_X Q(Y) - Q(\nabla_X Y), P) \\ &\quad + \frac{1}{m} Y(\psi) g(\lambda X - Q(X), P) - \frac{1}{m} X(\psi) g(\lambda Y - Q(Y), P) \\ &= g(\nabla_Y Q(X), P) - g(\nabla_X Q(Y), P) + \frac{\lambda}{m} Y(\psi) g(X,P) - \frac{\lambda}{m} X(\psi) g(Y,P) \\ &= Y(g(Q(X), P)) - g(Q(X), \nabla_Y P) - X(g(Q(Y), P)) \\ &\quad + g(Q(Y), \nabla_X P) + \frac{\lambda}{m}[Y(\psi) g(X,P) - g(Y,P) X(\psi)] \\ &= \frac{\lambda}{m}[Y(\psi) g(X,P) - g(Y,P) X(\psi)], \end{aligned}$$

which means either $\lambda = 0$ or $[Y(\psi)g(X,P) - g(Y,P)X(\psi)] = 0$.

The second condition gives

$$\begin{aligned} [Y(\psi)g(X,P) - g(Y,P)X(\psi)] = 0 &\Rightarrow Y(\psi)g(X,P) = g(Y,P)X(\psi) \\ &\Rightarrow g(Y, g(X,P)grad(\psi)) = g(Y, X(\psi)P) \\ &\Rightarrow grad(\psi) = P(\psi)P. \end{aligned}$$

Therefore, the gradient of the potential function ψ is collinear with the vector P. □

Lemma 3.9. *If (M^n, g) is a Riemannian manifold with a unit parallel vector field P, and g is a gradient m-QE metric, then*

$$0 = P(\psi)[\lambda(n-1) - S].$$

Proof. The formula (11), Lemma 2.1 and Theorem 2.2 all lead directly to the proof. □

Corollary 3.10. *Suppose (M^n, g) is a Riemannian manifold with a unit parallel vector field P. If g is a gradient m-QE metric, then we can state one of the following two cases*

1. *If $\lambda = 0$, then either the gradient of the potential function ψ and the vector field P are orthogonal or $S = 0$.*
2. *If $\lambda \neq 0$, then either the function ψ is constant or (M^n, g) has a constant scalar curvature $S = \lambda(n-1)$.*

Proof. Applying the result from Theorem 3.8 to the gradient *m-QE* metric g, we conclude that either the gradient of the potential function ψ is collinear with P or $\lambda = 0$. When we use this conclusion in conjunction with Lemma 3.9, we obtain the followings

1. If $\lambda = 0$ then $P(\psi) = 0$ or $S = 0$. Here comes the first case.
2. Otherwise, $grad(\psi) = P(\psi)P$ with $P(\psi) = 0$ or $S = \lambda(n-1)$ we find the second case.

□

An important observation follows from the Corollary above.

Remark 3.11. *If $\lambda \neq 0$ and the function ψ is non-constant, then we have*

$$\lambda = \Delta(\psi) - \frac{1}{m}||grad(\psi)||^2.$$

3.3. Gradient ρ-Einstein soliton

Theorem 3.12. *Suppose (M^n, g) is a Riemannian manifold with a unit parallel vector field P. If (M^n, g, ψ, λ) is a ρ-Einstein soliton manifold, then the gradient of the function $P(\psi)$ is collinear with P.*

Proof. If (M^n, g, ψ, λ) is a ρ-Einstein soliton manifold, then

$$Ric(X, Y) + H^{\psi}(X, Y) = (\rho S + \lambda)g(X, Y).$$

For any vector field X on M^n, from Lemma 2.1, we can derive the following

$$\begin{aligned} Ric(X, P) &= 0 \\ &= (\rho S + \lambda)g(X, P) - g(\nabla_X grad(\psi), P) \\ &= (\rho S + \lambda)g(X, P) - X(g(grad(\psi), P)) \\ &= g(X, (\rho S + \lambda)P - grad(P(\psi))). \end{aligned}$$

So, we arrive to $(\rho S + \lambda)P = grad(P(\psi))$. The proof is so finished. □

Moreover, we can state the following.

Corollary 3.13. *Under the hypotheses of Theorem 3.12, we find that*

$$(\rho S + \lambda) = P(P(\psi)).$$

Proof. One can easily see that

$$(\rho S + \lambda) = g((\rho S + \lambda)P, P) = g(grad(P(\psi)), P) = P(P(\psi)).$$

□

Remark 3.14. *Under the hypotheses of Theorem 3.12, if ψ is a constant function, then we have*

$$S = \lambda = 0.$$

Proof. If ψ is a constant function, then from Corollary 3.13 we have $(\rho S + \lambda) = 0$.
On the other hand, from (3), we have

$$S = (\rho S + \lambda)n.$$

Combine the two results we find $S = 0$. □

Theorem 3.15. *If (M^n, g) is a Riemannian manifold and (M^n, g, ψ, λ) is a gradient ρ-Einstein soliton manifold, then for all vector fields X and Y defined on M^n, we have the following relationships*

1.
$$\nabla_X grad(\psi) = (\rho S + \lambda)X - Q(X), \tag{14}$$

2.
$$R(X, Y)grad(\psi) = (\nabla_Y Q)X - (\nabla_X Q)Y + \rho[X(S)Y - Y(S)X]. \tag{15}$$

3.
$$Ric(Y, grad(\psi)) = Y(S)[\rho(1 - n) + \frac{1}{2}], \tag{16}$$

where $(\nabla_X Q)Y = \nabla_X Q(Y) - Q(\nabla_X Y)$.

Proof. Suppose we have a Riemannian manifold (M^n, g, ψ, λ) that forms a gradient ρ-Einstein soliton, i.e.,

$$Ric + H^{\psi} = (\rho S + \lambda)g.$$

1. We have the following equality for all vector fields X and Y defined on the Riemannian manifold (M^n, g) with the gradient ρ-Einstein soliton structure

$$\begin{aligned}(\rho S + \lambda)g(X, Y) &= Ric(X, Y) + H^{\psi}(X, Y)\\ &= g(Q(X), Y) + g(\nabla_X grad(\psi), Y)\\ &= g(Q(X) + \nabla_X grad(\psi), Y).\end{aligned}$$

From this equation, we can derive (14).

2. Starting from the equation (14), we can derive the following

$$\nabla_Y \nabla_X grad(\psi) = (\rho S + \lambda)\nabla_Y X - \nabla_Y Q(X) + \rho Y(S)X. \tag{17}$$

Interchanging the roles of X and Y in the equation (17), we obtain

$$\nabla_X \nabla_Y grad(\psi) = (\rho S + \lambda)\nabla_X Y - \nabla_X Q(Y) + \rho Y(S)X. \tag{18}$$

Now, by substituting the equations (17) and (18) into the expression for the Riemann tensor $R(X, Y)$: $R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, we obtain (15).

3. Consider a point x in the Riemannian manifold M^n, and let $(e_i)_{i=1}^n$ be an orthonormal frame at x such that $(\nabla_{e_i} e_j)_x = 0$. Utilizing the equation (15), we can give the following expression

$$\begin{aligned}Ric(Y, grad(\psi)) &= \sum_{i=1}^{n} g(R(e_i, Y)grad(\psi), e_i)\\ &= \frac{1}{2}Y(S) + \sum_{i=1}^{n}\big[\rho e_i(S)g(Y, e_i) - \rho Y(S)g(e_i, e_i)\big]\\ &= \frac{1}{2}Y(S) + Y(S)(\rho(1-n)),\end{aligned}$$

which gives the result.

□

Theorem 3.16. *Suppose we have a Riemannian manifold (M^n, g) and a unit parallel vector field P defined on it. If the metric g is a gradient ρ-Einstein soliton metric, then it follows that the scalar curvature S is constant.*

Proof. Using (15) and by virtue of $g(R(X, Y)grad(\psi), P) = 0$, we find

$$\begin{aligned}0 &= g(R(X, Y)grad(\psi), P)\\ &= g(\nabla_Y Q(X) - Q(\nabla_Y X), P) - g(\nabla_X Q(Y) - Q(\nabla_X Y), P) + \rho[X(S)g(Y, P) - Y(S)g(X, P)]\\ &= \rho[Y(S)g(X, P) - X(S)g(Y, P)],\end{aligned}$$

which gives us $[Y(S)g(X, P) - X(S)g(Y, P)] = 0$. This last equation can be expressed as

$$[Y(S)g(X, P) - X(S)g(Y, P)] = 0 = g(g(X, P)grad(S) - X(S)P, Y)$$

for all vector fields X, Y on M^n. As a consequence $g(X, P)grad(S) - X(S)P = 0$ for any vector field X on M^n. By taking $X = P$ we obtain

$$grad(S) = P(S)P.$$

Applying Theorem 2.2, the result follows. □

Corollary 3.17. *Consider a Riemannian manifold* (M^n, g) *and a unit parallel vector field P defined on it. If the metric g is a gradient* ρ*-Einstein metric, then*

$$Q(grad(\psi)) = 0,$$

where ψ *is a potential function.*

Proof. The proof follows immediately from (16) and Theorem 3.16 .
□

3.4. Gradient η*-Ricci soliton*

Theorem 3.18. *Given a Riemannian manifold* (M^n, g) *with a unit parallel vector field P, if* $(M^n, g, \psi, \lambda, \mu)$ *forms a gradient* η*-Ricci soliton manifold, we can state that*

$$\lambda = P(P(\psi)) - \mu g(P, V)^2,$$

where V is the corresponding vector field to the 1*-form* η*.*

Proof. In the case of a manifold $(M^n, g, \psi, \lambda, \mu)$ being a gradient η-Ricci soliton, then

$$Ric(X, Y) + H^{\psi}(X, Y) - \mu\eta(X)\eta(Y) = \lambda g(X, Y).$$

Utilizing Lemma (2.1), it follows that for any vector field X defined on M^n, we can deduce the following result

$$\begin{aligned} Ric(X, P) &= 0 \\ &= \lambda g(X, P) - g(\nabla_X grad(\psi), P) + \mu\eta(X)\eta(P) \\ &= \lambda g(X, P) - X(g(grad(\psi), P)) + \mu g(P, V)\eta(X) \\ &= g(X, \lambda P - grad(P(\psi)) + \mu g(P, V)V). \end{aligned}$$

Thus, we deduce

$$\lambda P = grad(P(\psi)) - \mu g(P, V)V.$$

By applying $\lambda = g(\lambda P, P)$, we find $\lambda = P(P(\psi)) - \mu g(P, V)^2$. Hence, the proof is completed. □

Corollary 3.19. *Suppose we have a steady gradient* η*-Ricci soliton on a Riemannian manifold* (M^n, g)*, and let P denotes a unit parallel vector field on* M^n*. If both P and the gradient of the potential function* ψ *are orthogonal, then, for* $\mu \neq 0$*, it follows that the vector field P is orthogonal to the corresponding vector field associated with* η*.*

Proof. If we consider a steady gradient η-Ricci soliton on the Riemannian manifold (M^n, g) and assume that the vector field P and the gradient of the potential function ψ are orthogonal, denoted as $P(\psi) = 0$, then, according to Theorem 3.18, we can derive the following

$$\mu g(P, V)^2 = 0.$$

Moreover, it is important to note that Corollary 3.19 remains valid even when ψ is constant. □

Remark 3.20. *Under the hypotheses of Theorem 3.18, if* ψ *is a constant function then* λ *and* μ *have opposite signs.*

Theorem 3.21. *In the context of a gradient* η*-Ricci soliton manifold* $(M^n, g, \psi, \lambda, \mu)$*, then for all vector fields X, Y on* M^n *we can express the statement as follows*

1.

$$\nabla_X grad(\psi) = \lambda X - Q(X) + \mu\eta(X)V, \tag{19}$$

2.

$$\begin{aligned} R(X,Y)grad(\psi) &= (\nabla_Y Q)X - (\nabla_X Q)Y + \mu[\eta(Y)\nabla_X V - \eta(X)\nabla_Y V] \\ &\quad +\mu[g(Y,\nabla_X V) - g(X,\nabla_Y V)]V, \end{aligned} \tag{20}$$

3.

$$\begin{aligned} Ric(Y,grad(\psi)) &= \frac{1}{2}Y(S) + \mu[\eta(Y)div(V) \\ &\quad -2g(\nabla_Y V,V) + g(Y,\nabla_V V)]. \end{aligned} \tag{21}$$

In here, V is the corresponding vector field of the 1-form η.

Proof. Let $(M^n, g, \psi, \lambda, \mu)$ be a gradient η-Ricci soliton manifold, as described by the equation

$$Ric + H^{\psi} - \mu\eta \otimes \eta = \lambda g,$$

where η is 1-form satisfies $\eta(X) = g(V,X)$ and V is the corresponding vector field to η.

1. For all vector fields X, Y on M^n, we have

$$\begin{aligned} \lambda g(X,Y) &= Ric(X,Y) + H^{\psi}(X,Y) - \mu\eta(X)\eta(Y) \\ &= g(Q(X),Y) + g(\nabla_X grad(\psi),Y) - g(\mu\eta(X)V,Y) \\ &= g(Q(X) + \nabla_X grad(\psi) - \mu\eta(X)V, Y). \end{aligned}$$

   Thus, we deduce the equation (19).

2. From the equation (19), we can establish the equation

$$\nabla_Y\nabla_X grad(\psi) = \lambda\nabla_Y X - \nabla_Y Q(X) + \mu\eta(X)\nabla_Y V + \mu Y(\eta(X))V. \tag{22}$$

   Interchanging X and Y in (22), we get

$$\nabla_X\nabla_Y grad(\psi) = \lambda\nabla_X Y - \nabla_X Q(Y) + \mu\eta(Y)\nabla_X V + \mu X(\eta(Y))V. \tag{23}$$

   Substituting the equations (22) and (23) into $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, we obtain (20).

3. Finally, considering an orthonormal frame $(e_i)_{i=1}^n$ on M^n with $(\nabla_{e_i}e_j)_x = 0$ at a point $x \in M^n$, from (20), we can derive the equation

$$\begin{aligned} Ric(Y,grad(\psi)) &= \sum_{i=1}^{n} g(R(e_i,Y)grad(\psi),e_i) \\ &= \sum_{i=1}^{n}\Big[g((\nabla_Y Q)(e_i),e_i) - g((\nabla_{e_i}Q)Y,e_i) \\ &\quad +\mu[\eta(Y)g(\nabla_{e_i}V,e_i) - \eta(e_i)g(\nabla_Y V,e_i)] \\ &\quad +\mu[g(Y,\nabla_{e_i}V)g(V,e_i) - g(e_i,\nabla_Y V)g(V,e_i)]\Big] \\ &= Y(S) - div(Ric)(Y) + \mu[\eta(Y)div(V) - g(\nabla_Y V,V) \\ &\quad +g(Y,\nabla_V V) - g(V,\nabla_Y V)\Big] \\ &= \frac{1}{2}Y(S) + \mu[\eta(Y)div(V) - 2g(\nabla_Y V,V) + g(Y,\nabla_V V)\Big]. \end{aligned}$$

□

Theorem 3.22. *Consider a Riemannian manifold (M^n, g) with a unit parallel vector field P defined on it. If g possesses a gradient η-Ricci soliton metric, then we can conclude that either $\mu = 0$ or the following relation holds*

$$\nabla_P fV = grad(f^2),$$

where $f = g(P, V)$ and V represents the corresponding vector field to 1-form η.

Proof. Using (20) and the condition $g(R(X, Y)\nabla\psi, P) = 0$, we can derive the following

$$\begin{aligned} 0 &= g(R(X, Y)grad(\psi), P) \\ &= g(\nabla_Y Q(X) - Q(\nabla_Y X), P) - g(\nabla_X Q(Y) - Q(\nabla_X Y), P) \\ &\quad +\mu[\eta(Y)g(\nabla_X V, P) - \eta(X)g(\nabla_Y V, P)] \\ &\quad +\mu[g(Y, \nabla_X V)g(V, P) - g(X, \nabla_Y V)g(V, P)] \\ &= \mu[g(\nabla_X V, P)g(V, Y) - \eta(X)g(\nabla_Y V, P) + g(Y, \nabla_X V)g(V, P) - g(X, \nabla_Y V)g(V, P)]. \end{aligned}$$

Replacing $Y = P$ in the above equation, we obtain

$$\begin{aligned} 0 &= \mu[g(\nabla_X V, P)g(V, P) - \eta(X)g(\nabla_P V, P) \\ &\quad +g(P, \nabla_X V)g(V, P) - g(X, \nabla_P V)g(V, P)] \\ &= \mu[g(-g(V, P)\nabla_P V - g(\nabla_P V, P)V + grad(g(V, P)^2), X)] \end{aligned}$$

for any vector field X on M^n. By taking $f = g(V, P)$ we obtain

$$\mu[-f\nabla_P V - P(f)V + grad(f^2)] = 0.$$

Therefore, we can conclude that either $\mu = 0$ or $\nabla_P fV = \text{grad}(f^2)$. □

Corollary 3.23. *Suppose we have a Riemannian manifold (M^n, g) with a unit parallel vector field P defined on it. If g is a gradient η-Ricci manifold, then we can conclude that either $\mu = 0$ or the following relation holds*

$$div(fV) = P(|V|^2),$$

where $f = g(P, V)$ and V represents the corresponding vector field to 1-form η.

Proof. The proof follows immediately from (21) by applying $Ric(P, grad(\psi)) = 0$. □

4. Examples

To demonstrate the findings derived in the preceding sections, we will now present a set of relevant examples.

4.1. *Euclidean space $\mathbb{R}^n$*

Let $(\mathbb{R}^n, <, >)$ be an Euclidean space equipped with the standard metric

$$g = dx_1^2 + dx_2^2 + ... + dx_n^2.$$

A unit parallel vector field P is expressed as

$$P = \frac{1}{|a|}(a_1, a_2, ..., a_n), \qquad |a| = \sqrt{\sum_{i=1}^{n} a_i^2},$$

where a is constant.

Consider a function $\psi(x) = \alpha\|x\|^2 = \alpha(x_1^2 + x_2^2 + ... + x_n^2)$, where $\alpha \in \mathbb{R}$, then we have

$$H_{ij}^{\psi} = \frac{\partial^2\psi}{\partial x_i \partial x_j} = 2\alpha g_{ij}.$$

Since the Ricci curvature Ric and the scalar curvature S are zero, we can write

$$Ric + H^{\psi} = 2\alpha g.$$

Therefore $(\psi, 2\alpha)$ defines a gradient Ricci soliton on $(\mathbb{R}^n, <,>)$.

As we have already demonstrated in Corollary 3.2, we have $P(P(\psi)) = 2\alpha$.

Let $f(x) = \alpha - \ln(x_1 + x_2 + ... + x_n + \beta)$ be a smooth function on $\mathbb{R}^n$, where α and β are constants. By a direct calculation we obtain

$$H_{ij}^{f} = \frac{1}{(\beta + \sum_{i=1}^{n} x_i)^2} = df(\partial_i)df(\partial_j) \qquad 1 \leq i, j \leq n,$$

from which we have $H_{ij}^f - df(\partial_i)df(\partial_j) = 0$. Then $(\mathbb{R}^n, g)$ is a steady gradient m-QE manifold with $\lambda = 0$ and $m = 1$. It is clear that $P(\psi) \neq 0$ but $S = 0$.
This example supports the results obtained in Theorem 3.8 and Corollary 3.10.

4.2. *Example 2*

Let $(\mathbb{R}^4, g)$ be a 4-dimensional Riemannian manifold such that

$$g = e^{2\gamma(x)}(dx^2 + dy^2) + dz^2 + dw^2.$$

Denote by $\{e_1, e_2, e_3, e_4\}$ the orthonormal frame of $(\mathbb{R}^4, g)$, where

$$e_1 = e^{-\gamma(x)}\partial_x, \quad e_2 = e^{-\gamma(x)}\partial_y, \quad e_3 = \partial_z, \quad e_4 = \partial_w.$$

The Lie brackets between the orthonormal frame are zero expect for

$$[e_1, e_2] = -\gamma'(x)e^{-\gamma(x)}e_2.$$

The Levi-Civita connection is given by

$$\nabla_{e_i} e_j = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \gamma'(x)e^{-\gamma(x)}e_2 & -\gamma'(x)e^{-\gamma(x)}e_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The non-zero Riemannian curvature tensor is given by

$$R(e_1, e_2)e_1 = \gamma''(x)e^{-2\gamma(x)}e_2,$$
$$R(e_2, e_1)e_2 = \gamma''(x)e^{-2\gamma(x)}e_1.$$

Then we obtain the Ricci curvature tensor

$$Ric(e_1, e_1) = Ric(e_2, e_2) = -\gamma''(x)e^{-2\gamma(x)}$$

and the scalar curvature

$$S = -2\gamma''(x)e^{-2\gamma(x)}.$$

We take $\gamma'(x) \neq const.$ to avoid the case of null scalar curvature. Then, the parallel vector field on $(\mathbb{R}^4, g)$ is given by

$$\xi = (0, 0, a_3, a_4), \qquad where \quad a_i \in \mathbb{R}.$$

Moreover, the unit parallel vector field P is expressed as

$$P = \frac{a_3\partial_z + a_4\partial_w}{\sqrt{a_3^2 + a_4^2}}. \tag{24}$$

Let assume now that $\gamma''(x)e^{-2\gamma(x)} = e^{-2a}$ is constant (this relation involve $\gamma(x) = a - \ln(x + b)$ and $\gamma(x) = -\ln(\cos(e^{-a}x))$ as solutions), where a and b are real constants.

Consider the smooth function

$$\psi(x, y, z, w) = -\frac{e^{-2a}}{2}z^2 - \frac{e^{-2a}}{2}w^2.$$

The Hessian of the function ψ satisfies

$$\begin{aligned} H^\psi(e_1, e_1) &= H^\psi(e_2, e_2) = 0, \\ H^\psi(e_3, e_3) &= H^\psi(e_4, e_4) = -e^{-2a}, \\ H^\psi(e_i, e_j) &= 0 \qquad \forall i \neq j. \end{aligned}$$

Therefore, we have

$$Ric(e_i, e_j) + H^\psi(e_i, e_j) = -e^{-2a}g(e_i, e_j), \tag{25}$$

which from we deduce that $(\mathbb{R}^4, g)$ is an expanding gradient Ricci soliton manifold ($\lambda = -e^{-2a}$).
After an easy computation, we get

$$P(\psi) = \frac{-e^{-2a}}{\sqrt{a_3^2 + a_4^2}}(a_3 z + a_4 w).$$

It is clear that P is collinear with the gradient of $P(\psi)$ and $\lambda = P(P(\psi))$, thus Theorem 3.1 and Corollary 3.2 are verified.

The equation (25) can be reformulated as follows

$$\begin{aligned} Ric(e_i, e_j) + H^\psi(e_i, e_j) &= -e^{-2a}g(e_i, e_j) \\ &= [\frac{1}{2}(-2e^{-2a}) + 0]g(e_i, e_j) \\ &= [\rho S + \lambda]g(e_i, e_j). \end{aligned}$$

Hence, we conclude that $(\mathbb{R}^4, g)$ is a gradient ρ-Einstein soliton manifold, with $\rho = \frac{1}{2}$ and $\lambda = 0$. We can easily verify that $(\rho S + \lambda)P = grad(P(\psi))$ and $(\rho S + \lambda) = P(P(\psi))$, that is, Theorem 3.12 and Corollary 3.13 are verified.

Consider now the function

$$h(x, y, z, w) = \ln(z + w) - \frac{e^{-2a}}{2}z^2 - \frac{e^{-2a}}{2}w^2.$$

The Hessian of the function h satisfies

$$\begin{array}{rcl} H^h(e_3,e_3) & = & H^h(e_4,e_4) = \dfrac{-1}{(z+w)^2} - e^{-2a}, \\ H^h(e_3,e_4) & = & \dfrac{-1}{(z+w)^2}, \\ H^h(e_1,e_i) & = & H^h(e_2,e_i) = 0, \qquad 1 \leq i \leq 4. \end{array}$$

Let $\eta = \frac{1}{z+w}(dz+dw)$. η verifies the followings

$$\eta(e_1) = \eta(e_2) = 0, \qquad \eta(e_3) = \eta(e_4) = \frac{1}{z+w}.$$

Therefore, we can write

$$Ric(e_i,e_j) + H^h(e_i,e_j) + \eta \otimes \eta(e_i,e_j) = -e^{-2a} g(e_i,e_j).$$

From the last equation we state that $(g, h, -e^{-2a}, -1)$ is a gradient η-Ricci soliton on $\mathbb{R}^4$ with ($\lambda = -e^{-2a}$ and $\mu = -1$).

By taking $f = g(P,V)$, where P is the unit parallel vector field given in (24) and $V = \frac{1}{z+w}(\partial z + \partial w)$ is the associated vector field to 1-form η, we find that

$$f^2 = g(P,V)^2 = \frac{(a_3+a_4)^2}{(a_3^2+a_4^2)(z+w)^2}. \tag{26}$$

On other hand we have

$$P(P(h)) = \frac{-(a_3+a_4)^2}{(a_3^2+a_4^2)(z+w)^2} - e^{-2a}. \tag{27}$$

From (26) and (27) we obtain

$$P(P(h)) - \mu g(P,V)^2 = -e^{-2a} = \lambda.$$

Hence Theorem 3.18 is verified.

As we have already demonstrated in Theorem 3.22 and Corollary 3.23 , we have

$$\nabla_P fV = \frac{-2(a_3+a_4)^2}{(a_3^2+a_4^2)(z+w)^3}(\partial z + \partial w) = grad(f^2)$$

and

$$div(fV) = \frac{-4(a_3+a_4)}{(a_3^2+a_4^2)^{\frac{1}{2}}(z+w)^3} = P(g(V,V)).$$

4.3. *The product of Hamilton's cigar soliton N^2 with the real space $\mathbb{R}^n$*

In [14], Hamilton presents the first example of a complete steady soliton N^2 equipped with the metric

$$g = ds_N^2 = \frac{dx^2+dy^2}{1+x^2+y^2}$$

and the potential function

$$f = -\ln(1+x^2+y^2).$$

This space is asymptotic to cylinder at infinity. The three-dimensional $N^2 \times \mathbb{R}$ plays an important role of collapsed complete gradient steady Ricci solitons, in fact it is the only known example in three-dimensional (see [4]).

Let us take now the product $(N^2 \times \mathbb{R}^n, \overline{g})$ equipped with the metric

$$\begin{aligned} \overline{g} &= g + g_0 \\ &= \frac{dx^2 + dy^2}{1 + x^2 + y^2} + (dx_1^2 + dx_2^2 + ... + dx_n^2), \end{aligned}$$

where g_0 is the standard metric on $\mathbb{R}^n$. The parallel vector field on $(N^2 \times \mathbb{R}^n, \overline{g})$ is given by

$$\xi = (0, 0, a_1, ..., a_n), \qquad where \quad a_i \in \mathbb{R}.$$

Therefore, the unit parallel vector field P is

$$P = \frac{1}{|a|}(0, 0, a_1, a_2, ..., a_n), \quad where \quad |a| = \sqrt{\sum_{i=1}^{n} a_i^2}.$$

The Ricci curvature is given by

$$\begin{aligned} \overline{Ric}((X_1, X_2), (Y_1, Y_2)) &= Ric_{N^2}(X_1, Y_1) + Ric_{\mathbb{R}^n}(X_2, Y_2) \\ &= Ric_{N^2}(X_1, Y_1), \end{aligned}$$

where (X_1, X_2) and (Y_1, Y_2) are vector fields on $N^2 \times \mathbb{R}^n$. It is well known that

$$Ric_{N^2}(\partial_x, \partial_x) = Ric_{N^2}(\partial_y, \partial_y) = \frac{2}{(1 + x^2 + y^2)^2} \quad and \quad Ric_{N^2}(\partial_x, \partial_y) = 0.$$

Therefore the non-zero Ricci curvature on $N^2 \times \mathbb{R}^n$ is defined by

$$\overline{Ric}((\partial_x, X), (\partial_x, Y)) = \overline{Ric}((\partial_y, X), (\partial_y, Y)) = \frac{2}{(1 + x^2 + y^2)^2} \tag{28}$$

for all vector fields X, Y on $\mathbb{R}^n$.

Consider the smooth function f on $(N^2 \times \mathbb{R}^n, \overline{g})$ defined by

$$f(x, y, x_1, ..x_n) = -\ln(1 + x^2 + y^2) \qquad \forall (x, y, x_1, .., x_n) \in N^2 \times \mathbb{R}^n.$$

A direct calculation shows that the non-zero Hessian operator of f verifies

$$\overline{H}^f((\partial_x, X), (\partial_x, Y)) = \overline{H}^f((\partial_y, X), (\partial_y, Y)) = -\frac{2}{(1 + x^2 + y^2)^2}. \tag{29}$$

Finally, from (28) and (29), we get

$$\overline{Ric} + \overline{H}^f = 0.$$

Hence $(N^2 \times \mathbb{R}^n, \overline{g})$ is a gradient steady Ricci soliton.

One can easily see that $P(f) = 0$ and $\lambda = 0 = P(P(f))$, also

$$(\overline{\nabla}_P Q)(X, Y) = 0.$$

This example supports the results obtained in Corollary 3.2 and Corollary 3.5.

5. Conformal metric

In this section, we delve into the investigation of the behavior exhibited by conformal metrics on Riemannian manifolds that admit parallel vector fields. Building upon our earlier findings, we explore the characteristics of these conformal metrics in this specific context.

Let us consider an n-dimensional Riemannian manifold denoted as (M^n, g), furnished with the Riemannian metric g. Furthermore, let ∇ represent the Levi-Civita connection that corresponds to this metric g. The metric $\widetilde{g}$ on M^n is referred to as a conformal metric of g if there exists a positive function $\gamma \in C^\infty(M^n)$ such that it conforms to the following relationship:

$$\widetilde{g} = e^{2\gamma} g,$$

i.e.,

$$\widetilde{g}(X, Y) = e^{2\gamma} g(X, Y)$$

for all vector fields X, Y on M^n. Here, γ assumes the role of a dilation function in this context. The Levi-Civita connection $\widetilde{\nabla}$ corresponding to $\widetilde{g}$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + X(\gamma)Y + Y(\gamma)X - g(X, Y)grad(\gamma). \tag{30}$$

Throughout this section we consider $\nabla grad(\gamma) = 0$. This condition implies that the norm of the vector field $grad(\gamma)$ is constant, let us suppose it equal to 1. In this situation we have

$$\begin{aligned}\widetilde{R}(X, Y)Z = R(X, Y)Z &+ Y(\gamma)Z(\gamma)X - X(\gamma)Z(\gamma)Y \\ &+ [X(\gamma)g(Y, Z) - Y(\gamma)g(X, Z)]grad(\gamma) \\ &+ [g(X, Z)Y - g(Y, Z)X],\end{aligned} \tag{31}$$

where R (resp., $\widetilde{R}$) denotes the Riemannian curvature tensor corresponding to the Levi-Civita connection ∇ (resp., $\widetilde{\nabla}$).

Proposition 5.1. *Let (M^n, g) be a Riemannian manifold and let $\widetilde{g} = e^{2\gamma} g$ be the conformal metric of g. Then, for any vector field X on M^n we have*

$$\widetilde{Q}(X) = e^{-2\gamma}[Q(X) + (n-2)[X(\gamma)grad(\gamma) - X]], \tag{32}$$

where Q and $\widetilde{Q}$ denote the Ricci operators associated to g and $\widetilde{g}$, respectively.

Proof. Let $(e_i)_{i=1,..,n}$ be an orthonormal frame associated to g, then the family $\{\widetilde{e_i}\}_{i=1,..,n}$ such that $\widetilde{e_i} = e^{-\gamma} e_i$ is an orthonormal frame associated to $\widetilde{g}$. By using the definition of the Ricci operator, we obtain

$$\widetilde{Q}(X) = \sum_{i=1}^{n} \widetilde{R}(X, \widetilde{e_i})\widetilde{e_i} = e^{-2\gamma} \sum_{i=1}^{n} \widetilde{R}(X, e_i)e_i.$$

Substituting the last equation into the equation (31), we get

$$\begin{aligned}\widetilde{Q}(X) &= \sum_{i=1}^{n} e^{-2\gamma}\widetilde{R}(X, e_i)e_i \\ &= e^{-2\gamma} \sum_{i=1}^{n} \{R(X, e_i)e_i + e_i(\gamma)e_i(\gamma)X - X(\gamma)e_i(\gamma)e_i \\ &\quad + [X(\gamma)g(e_i, e_i) - e_i(\gamma)g(X, e_i)]grad\gamma \\ &\quad + [g(X, e_i)e_i - g(e_i, e_i)X]\},\end{aligned}$$

and by applying the following equalities

$$\begin{aligned} Q(X) &= \sum_{i=1}^{n} R(X, e_i)e_i, \\ grad\gamma &= \sum_{i=1}^{n} e_i(\gamma)e_i, \\ X(\gamma) &= g(X, grad\gamma), \end{aligned}$$

we obtain the formula (32). □

From the previous Proposition it follows the following.

Proposition 5.2. *Let (M^n, g) be a Riemannian manifold and let $\widetilde{g} = e^{2\gamma}g$ be the conformal metric of g. Then for all vector fields X, Y on M^n we have*

$$\widetilde{Ric}(X, Y) = Ric(X, Y) + (n-2)[X(\gamma)Y(\gamma) - g(X, Y)], \tag{33}$$

where Ric (resp., $\widetilde{Ric}$) is the Ricci curvature corresponding to g (resp., $\widetilde{g}$).

Proof. To prove the equation (33), we use

$$\begin{aligned} \widetilde{Ric}(X, Y) &= \widetilde{g}(\widetilde{Q}(X), Y) \\ &= e^{2\gamma}g(\widetilde{Q}(X), Y) \\ &= g(Q(X), Y) + (n-2)[X(\gamma)Y(\gamma) - g(X, Y)] \\ &= Ric(X, Y) + (n-2)[X(\gamma)Y(\gamma) - g(X, Y)]. \end{aligned}$$

Thus the proof of (33) is completed. □

Proposition 5.3. *Let (M^n, g) be a Riemannian manifold and let $\widetilde{g} = e^{2\gamma}g$ be the conformal metric of g. Then for any vector field X on M^n we have*

$$\widetilde{S} = e^{-2\gamma}[S - (n-1)(n-2)],$$

where S and $\widetilde{S}$ denote the scalar curvatures associated to g and $\widetilde{g}$, respectively.

Proof. The proof comes immediately from Proposition 5.2. □

Proposition 5.4. *Let (M^n, g) be a Riemannian manifold and let $\widetilde{g} = e^{2\gamma}g$ be the conformal metric of g. Then for any smooth function ψ on M^n we have*

$$\widetilde{H}^{\psi}(X, Y) = H^{\psi}(X, Y) - X(\gamma)Y(\psi) - Y(\gamma)X(\psi) + g(grad\gamma, grad\psi)g(X, Y), \tag{34}$$

where H^{ψ} (resp., $\widetilde{H}^{\psi}$) denotes the Hessian of the function ψ with respct to g (resp., $\widetilde{g}$).

Proof. We have

$$\begin{aligned} \widetilde{H}^{\psi}(X, Y) &= \widetilde{g}(\widetilde{\nabla}_X \widetilde{grad}(\psi), Y) \\ &= e^{2\gamma}g(\widetilde{\nabla}_X e^{-2\gamma} grad(\psi), Y) \\ &= g(\widetilde{\nabla}_X grad(\psi) - 2X(\gamma)grad(\psi), Y) \end{aligned}$$

and by using the equation (30) in the above relation we obtain the result. □

From the previous Proposition we deduce the following corollary.

Corollary 5.5. *Let (M^n, g) be a Riemannian manifold. If $\widetilde{g} = e^{2\gamma}g$ is a conformal metric of g, we have*

$$\widetilde{H}^{\gamma}(X,Y) = g(X,Y) - 2X(\gamma)Y(\gamma),$$

where $\widetilde{H}^{\gamma}$ is the Hessian of γ with respect to $\widetilde{g}$.

Theorem 5.6. *Let (M^n, g) be a flat Riemannian manifold and let $\widetilde{g} = e^{2\gamma}g$ be the conformal metric of g such that $n > 4$. Then $\widetilde{g}$ is an almost expanding gradient m-QE metric.*

Proof. From Proposition 5.2 and Corollary 5.5, we have

$$\begin{aligned}\widetilde{Ric}(X,Y) + \widetilde{H}^{\gamma}(X,Y) &= Ric(X,Y) + (n-2)[X(\gamma)Y(\gamma) - g(X,Y)] \\ &\quad + g(X,Y) - 2X(\gamma)Y(\gamma) \\ &= (n-4)X(\gamma)Y(\gamma) + (3-n)g(X,Y) \\ &= (n-4)d\gamma \otimes d\gamma(X,Y) + (3-n)e^{-2\gamma}\widetilde{g}(X,Y).\end{aligned}$$

By taking $m = \frac{1}{n-4}$ and $f = (3-n)e^{-2\gamma}$, we get

$$\widetilde{Ric}(X,Y) + \widetilde{H}^{\gamma}(X,Y) - \frac{1}{m}d\gamma \otimes d\gamma(X,Y) = f\widetilde{g}(X,Y),$$

where $f \in C^{\infty}(M^n)$ is a negative function, which means $\widetilde{g}$ is an almost expanding gradient m-QE metric.
□

Remark 5.7. *If (M^n, g) is a 4-dimensional flat manifold, then $\widetilde{g}$ is an almost expanding gradient Ricci soliton and we have*

$$\widetilde{Ric}(X,Y) + \widetilde{H}^{\gamma}(X,Y) = -e^{-2\gamma}\widetilde{g}(X,Y).$$

Corollary 5.8. *Let (M^n, g) be a flat Riemannian manifold. If $\widetilde{g} = e^{2\gamma}g$ is the conformal metric of g, then $\widetilde{g}$ is an almost expanding gradient η-Ricci soliton.*

Proof. From Proposition 5.2, we have

$$\begin{aligned}\widetilde{Ric}(X,Y) &= Ric(X,Y) + (n-2)[X(\gamma)Y(\gamma) - g(X,Y)] \\ &= (n-2)[X(\gamma)Y(\gamma) - g(X,Y)] \\ &= (n-2)d\gamma \otimes d\gamma(X,Y) - (n-2)g(X,Y).\end{aligned}$$

For any function ψ on M^n such that $\widetilde{\nabla} grad(\psi) = 0$, we have

$$\widetilde{Ric}(X,Y) + \widetilde{H}^{\psi}(X,Y) - (n-2)d\gamma \otimes d\gamma(X,Y) = -(n-2)g(X,Y).$$

By taking $\mu = (n-2)$, $f = -(n-2)e^{-2\gamma}$ and $\eta = d\gamma$, we get

$$\widetilde{Ric}(X,Y) + \widetilde{H}^{\psi}(X,Y) - \mu\eta \otimes \eta(X,Y) = f\widetilde{g}(X,Y).$$

Moreover, $f < 0$ for $n > 2$, then $\widetilde{g}$ is an almost expanding gradient η-Ricci soliton metric. □

Remark 5.9. *Under the hypotheses of Corollary 5.8, $\widetilde{g}$ can be seen as an almost quasi-Einstein metric*

$$\widetilde{Ric} = -(n-2)e^{-2\gamma}\widetilde{g} + (n-2)d\gamma \otimes d\gamma.$$

Theorem 5.10. *Let (M^n, g) be a Riemannian manifold and let $\widetilde{g} = e^{2\gamma}g$ be the conformal metric of g. If g is a steady gradient Ricci soliton metric such that the gradient of the potential function ψ is orthoganal to the gradient of γ, then we have*

$$\widetilde{Ric} + \widetilde{H}^{\psi} = (n-2)d\gamma \otimes d\gamma - (d\psi \otimes d\gamma + d\gamma \otimes d\psi) - (n-2)e^{-2\gamma}\widetilde{g}.$$

We call $\widetilde{g}$ a gradient mixed quasi-Einstein metric (see [3]).

Proof. The proof comes immediately from Proposition 5.2 and Proposition 5.4, and the fact that if $g(grad(\gamma), grad(\psi)) = 0$, then g is a steady gradient Ricci soliton metric (see Theorem 3.1). □

An Application: Recall the Example 4.3 where we proof that the product $(N^2 \times \mathbb{R}^n, \overline{g})$ is a gradient steady Ricci soliton. Consider the function

$$\gamma(x, y, x_1, .., x_n) = a_1 x_1 + .. + a_n x_n + b, \qquad (x, y, x_1, .., x_n) \in (N^2 \times \mathbb{R}^n),$$

where a_i, b are real constants. The gradient of the function γ is a parallel vector field on $(N^2 \times \mathbb{R}^n)$:

$$grad(\gamma) = (0, 0, a_1, ..., a_n) = \xi,$$

which satisfies $g(grad(\gamma), grad(f)) = 0$, where f is the potential function defined in Example 4.3. By virtue of Theorem 5.10, we deduce that $(N^2 \times \mathbb{R}^n, \widetilde{g})$ such that $\widetilde{g} = e^{2\gamma}\overline{g}$ is an expanding gradient mixed quasi-Einstein manifold.

Proposition 5.11. *Let (M^n, g) be a Riemannian manifold and $\widetilde{g} = e^{2\gamma} g$ be the conformal metric of g. If g is a gradient m-QE metric, then $\widetilde{g}$ is either an almost gradient η-Ricci soliton or a type of a gradient mixed quasi-Einstein metric.*

Proof. Let g be a gradient m-QE metric, then from Theorem 3.8 either the gradient of the potential function ψ is collinear with $grad(\gamma)$ or $\lambda = 0$.

1. If $grad(\psi) = g(grad(\gamma), grad(\psi))grad(\gamma)$ then, by using the two equations (33) and (34), we get
$$\begin{aligned} \widetilde{Ric}(X, Y) + \widetilde{H}^{\psi}(X, Y) &= Ric(X, Y) + H^{\psi}(X, Y) + (n-2)X(\gamma)Y(\gamma) \\ &\quad -(n-2)g(X, Y) - X(\gamma)Y(\psi) - Y(\gamma)X(\psi) + g(grad\gamma, grad\psi)g(X, Y) \\ &= \Big[\frac{1}{m}h^2 - 2h + (n-2)\Big]X(\gamma)Y(\gamma) + \Big[\lambda + h - (n-2)\Big]e^{-2\gamma}\widetilde{g}(X, Y), \end{aligned}$$
where $h = g(grad\gamma, grad\psi)$. Hence
$$\widetilde{Ric}(X, Y) + \widetilde{H}^{\psi}(X, Y) - \mu\eta \otimes \eta = f\widetilde{g},$$
such that
$$\begin{cases} \mu = \frac{1}{m} g(grad\gamma, grad\psi)^2 - 2g(grad\gamma, grad\psi) + (n-2), \\ f = [\lambda + g(grad\gamma, grad\psi) - (n-2)]e^{-2\gamma}, \\ \eta = d\gamma, \end{cases}$$
which means $\widetilde{g}$ is an almost gradient η-Ricci soliton.

2. If $\lambda = 0$ then $Ric(X, Y) + H^{\psi}(X, Y) = \frac{1}{m} d\psi \otimes d\psi(X, Y)$, applying this equation, we find
$$\begin{aligned} \widetilde{Ric}(X, Y) + \widetilde{H}^{\psi}(X, Y) &= Ric(X, Y) + H^{\psi}(X, Y) + (n-2)[X(\gamma)Y(\gamma) \\ &\quad -(n-2)g(X, Y) - X(\gamma)Y(\psi) - Y(\gamma)X(\psi) \\ &\quad +g(grad\gamma, grad\psi)g(X, Y) \\ &= \frac{1}{m} d\psi \otimes d\psi(X, Y) + (n-2)d\gamma \otimes d\gamma(X, Y) \\ &\quad -(d\gamma \otimes d\psi + d\psi \otimes d\gamma)(X, Y) \\ &\quad +[g(grad(\gamma), grad(\psi)) - (n-2)]g(X, Y). \end{aligned}$$
For $grad(\gamma) \perp grad(\psi)$, we obtain
$$\widetilde{Ric} + \widetilde{H}^{\psi} = \alpha d\psi \otimes d\psi + \beta d\gamma \otimes d\gamma + \delta(d\gamma \otimes d\psi + d\psi \otimes d\gamma) + f\widetilde{g},$$
where
$$\begin{cases} \alpha = \frac{1}{m}, \\ \beta = (n-2), \\ \delta = -1, \\ f = -(n-2)e^{-2\gamma}. \end{cases}$$

□

Proposition 5.12. *Let (M^n, g) be a Riemannian manifold and $\widetilde{g} = e^{2\gamma} g$ be the conformal metric of g. If g is an $d\gamma$-Ricci soliton metric such that the gradient of γ is orthogonal with the gradient of the potential function ψ, then $\widetilde{g}$ is a type of a gradient mixed quasi-Einstein metric.*

Proof. Let g be an $d\gamma$-Ricci soliton metric, i.e.,

$$Ric + H^{\psi} - \mu d\gamma \otimes d\gamma = \lambda g.$$

Suppose that $g(grad(\gamma), grad(\psi)) = 0$, then from Theorem 3.18 we deduce that $\lambda = -\mu$. By using the two formulas (33) and (34), we get

$$\widetilde{Ric} + \widetilde{H}^{\psi} = [(n-2) - \lambda] d\gamma \otimes d\gamma - (d\gamma \otimes d\psi + d\psi \otimes d\gamma) + [\lambda - (n-2)] e^{-2\gamma} \widetilde{g}.$$

This completed the proof. □

6. Semi-conformal deformation of a Riemannian metric

In this section, we offer classifications of gradient Ricci solitons, gradient m-QE metrics, and η-Ricci solitons within the context of Riemannian manifolds subjected to semi-conformal deformations of the Riemannian metric. For a deeper understanding of semi-conformal metric deformations, you can refer to [10], which provides valuable background information on this topic.

Let (M^n, g) be an n-dimensional Riemannian manifold equipped with a Riemannian metric g and let f be a strictly positive smooth function on M^n. A semi-conformal deformation of the Riemannian metric g on M^n noted G is defined ([10]) by

$$G(X, Y)_x = f(x) g(X, Y)_x + g(\xi, X)_x g(\xi, Y)_x,$$

for all $x \in M$, vector fields X, Y on M^n and a vector field ξ on M^n such that $g(\xi, \xi) = 1$ and $\xi(f) = 0$.

It is important to note that the metric G is a conformal metric to g on the distribution orthogonal to ξ. In the subsequent discussion, we assume that ξ is a parallel vector field with respect to the Levi-Civita connection ∇ on the Riemannian manifold (M^n, g), meaning that $\nabla\xi = 0$. In this scenario, the Levi-Civita connection $\overline{\nabla}$ corresponding to the metric (M^n, G) can be defined as follows.

Theorem 6.1. *[10] Let (M^n, g) be a Riemannian manifold. The Levi-Civita connection $\overline{\nabla}$ of (M^n, G) is given by*

$$\overline{\nabla}_X Y = \nabla_X Y + \frac{X(f)}{2f} Y + \frac{Y(f)}{2f} X - \frac{g(X, Y)}{2f} grad\, f - \Big(\frac{X(f) g(\xi, Y)}{2f(f+1)} + \frac{Y(f) g(\xi, X)}{2f(f+1)} \Big) \xi \tag{35}$$

for all vector fields X, Y on M^n.

Theorem 6.2. *[10] Let (M^n, g) be a Riemannian manifold. If Ric (resp., $\overline{Ric}$) denotes the Ricci curvature of (M^n, g) (resp., (M^n, G)), then we have*

$$\begin{aligned} \overline{Ric}(X, Y) &= Ric(X, Y) + AG(X, Y) + BH^f(X, Y) + CX(f)Y(f) \qquad (36) \\ &\quad - [A + \frac{|grad\, f|^2}{2f^2(f+1)}] g(\xi, X) g(\xi, Y), \end{aligned}$$

where

$$A = \Big(\frac{((4-n)f + 5 - n) |grad\, f|^2}{4f^3(f+1)} - \frac{\Delta(f)}{2f^2} \Big), \qquad B = \frac{(2-n)f + 3 - n}{2f(f+1)},$$

$$C = \frac{((3n-6)f^2 + (6n-16)f + 3n - 9)}{4f^2(f+1)^2}$$

and H^f denotes the Hessian of f with respect to g.

Proposition 6.3. *Let (M^n, g) be a Riemannian manifold. If H^f (resp., $\overline{H}^f$) denotes the Hessian of f with respect to g (resp., G), then we have*

$$\overline{H}^f(X,Y) = H^f(X,Y) - \frac{1}{f}X(f)Y(f) + \frac{\left|grad\, f\right|^2}{2f}g(X,Y)$$

for all vector fields X, Y on M^n.

Proof. The proof comes immediately from (35) and the fact that $\overline{H}^f(X,Y) = X(Y(f)) - (\overline{\nabla}_X Y)(f)$. □

Corollary 6.4. *Let (M^n, g) be a Riemannian manifold and let ϕ and σ be two smooth functions on M^n such that*

$$grad(\phi) = (1-B)grad\, f = \frac{2f^2 + nf - 3 + n}{2f(f+1)}grad\, f \tag{37}$$

and

$$grad(\sigma) = (-B)grad\, f = -\frac{(2-n)f + 3 - n}{2f(f+1)}grad\, f,$$

where B is given in Theorem 6.2. If H^f denotes the Hessian of f with respct to g, then $\overline{H}^\phi$ (resp., $\overline{H}^\sigma$) the Hessian of ϕ (resp., of σ) with respect to G satisfies the followings

$$\begin{aligned}\overline{H}^\phi(X,Y) &= (1-B)H^f(X,Y) + DX(f)Y(f)\\ &\quad + \frac{(1-B)\left|grad\, f\right|^2}{2f^2}G(X,Y) - \frac{(1-B)\left|grad\, f\right|^2}{2f^2}g(\xi,X)g(\xi,Y),\end{aligned} \tag{38}$$

$$\overline{H}^\sigma(X,Y) = (-B)H^f(X,Y) + EX(f)Y(f) - \frac{B\left|grad\, f\right|^2}{2f^2}G(X,Y) + \frac{B\left|grad\, f\right|^2}{2f^2}g(\xi,X)g(\xi,Y), \tag{39}$$

where

$$D = \frac{-2f^2(f+n) + (9-4n)f + 2(3-n)}{2f^2(f+1)^2},$$

and

$$E = \frac{2(2-n)f^2 + (11-4n)f + 2(3-n)}{2f^2(f+1)^2}.$$

Proof. We have

$$g(X, grad\, f) = X(f) = G(X, \overline{grad}\, f) = fg(X, \overline{grad}\, f)$$

for any vector field X on M^n. The yield of the previous equation is

$$\overline{grad}\, f = \frac{1}{f}grad\, f.$$

On other hand, by the same way we have

$$g(X, grad(\phi)) = X(\phi) = G(X, \overline{grad}(\phi)) = fg(X, \overline{grad}(\phi)).$$

The yield of the last equation is

$$\overline{grad}(\phi) = \frac{1}{f}grad(\phi).$$

Using the equation (37) we obtain

$$\overline{grad}(\phi) = \frac{1-B}{f} grad\, f = (1-B)\overline{grad}\, f. \quad (40)$$

Now, by applying $\overline{H}^{\phi}(X, Y) = G(\overline{\nabla}_X \overline{grad}(\phi), Y)$ and using both of the equation (40) and Proposition 6.3, we obtain the result.

The same steps are used to prove the equation (39). □

Remark 6.5. *Let ω be a 1-form associated to ξ. Then we have*

$$\omega(X) = g(X, \xi) = G(X, \overline{\xi}),$$

where $\overline{\xi} = \frac{1}{f+1}\xi$. In the following we use $\omega \otimes \omega(X, Y)$ to indicate $g(X, \xi)g(\xi, Y)$ or $G(X, \overline{\xi})G(Y, \overline{\xi})$.

Remark 6.6. *In the rest of the paper, we will call* an almost generalized gradient quasi-Einstein manifold *every quadruple (g, ψ, η, ω) satisfies*

$$Ric + H^{\psi} = \lambda g + \mu \eta \otimes \eta + \tau \omega \otimes \omega,$$

where λ, μ and τ are smooth functions (for more details on generalized quasi-Einstein manifold, see [3]).

Theorem 6.7. *Let (M^n, g) be a flat Riemannian manifold. If G is a semi-conformal deformation of metric g, then $(M^n, G, \sigma, df, \omega)$ is an almost generalized gradient quasi-Einstein manifold, moreover, if $\Delta(f) = 0$ then $(M^n, G, \sigma, df, \omega)$ is shrinking.*

Proof. From Theorem 6.2 and Corollary 6.4, we get

$$\begin{aligned} \overline{Ric}(X, Y) + \overline{H}^{\sigma}(X, Y) &= [\frac{|grad\, f|^2}{2f^3} - \frac{\Delta(f)}{2f^2}]G(X, Y) + [C + E]df \otimes df(X, Y) \\ &\quad -[\frac{(2f+1)|grad\, f|^2}{2f^2(f+1)} - \frac{\Delta(f)}{2f^2}]\omega \otimes \omega(X, Y), \end{aligned}$$

where ω is the 1-form corresponding to ξ.

If $\Delta(f) = 0$, then

$$\frac{|grad\, f|^2}{2f^3} - \frac{\Delta(f)}{2f^2} = \frac{|grad\, f|^2}{2f^3},$$

where $f \in C^{\infty}(M)$ is a strictly positive function, which means $(M^n, G, \sigma, df, \omega)$ is an almost shrinking generalized gradient quasi-Einstein manifold. □

Corollary 6.8. *Under the hypotheses of Theorem 6.7, if f satisfies the equation $\Delta(f) = \frac{|grad\, f|^2}{f}$, then (M^n, G, df, ω) is steady.*

Example 6.9. *Recall the Example 4.1 where $(\mathbb{R}^n, <,>)$ is an Euclidean space equipped with the standard metric*

$$g = dx_1^2 + dx_2^2 + ... + dx_n^2.$$

Let ξ be a unit vector field on $\mathbb{R}^n$, defined by

$$\xi = \frac{1}{|a|}(0, a_2, ..., a_n) \quad \text{where} \quad |a| = \sqrt{\sum_{i=2}^{n} a_i^2}.$$

Consider the function $f(x) = e^{x_1}$,

$$\nabla\xi = 0 \quad and \quad \xi(f) = 0.$$

Since $\Delta(f) = \frac{|grad\, f|^2}{f}$, *then from Corollary 6.8,* $(\mathbb{R}^n, G, \sigma, df, \omega)$ *is an almost generalized gradient quasi-Einstein manifold, where* $G = e^{x_1} g + \omega\otimes\omega$, $\omega = \frac{1}{|a|}\sum_{i=2}^n a_i dx^i$ *and* $\sigma(x) = \frac{1}{2}((n-3)x_1 + \ln(e^{x_1}+1))$.

Theorem 6.10. *Let* (M^n, g) *be a Riemannian manifold and let* $G = fg + \omega\otimes\omega$ *be a semi-conformal deformation of the metric* g *where* $f \in C^\infty(M^n)$ *is a strictly positive function and* ω *is the* 1*-form corresponding to* ξ*. If* g *is a gradient Ricci soliton metric with the potential function* f*, then we have*

$$\begin{aligned}\overline{Ric} + \overline{H}^{\phi} &= [\frac{\lambda}{f} + \frac{(f+1)}{2f^3}|grad\, f|^2 - \frac{\Delta(f)}{2f^2}]G(X,Y) + [C+D]df\otimes df(X,Y)\\ &-[\frac{(f^2+3f+1)}{2f^3(f+1)}|grad\, f|^2 - \frac{\Delta(f)}{2f^2} + \frac{\lambda}{f}]\omega\otimes\omega(X,Y).\end{aligned}$$

Hence, (M^n, G, df, ω) *is an almost generalized gradient quasi-Einstein manifold.*

Proof. The proof comes immediately from Theorem 6.2 and Corollary 6.4. □

Theorem 6.11. *Let* (M^n, g) *be a Riemannian manifold and let* $G = fg + \omega\otimes\omega$ *be a semi-conformal deformation of the metric* g *where* $f \in C^\infty(M^n)$ *is a strictly positive function and* ω *is the* 1*-form corresponding to* ξ*. If* g *is a gradient* m*-QE metric with the potential function* f*, then* (M^n, G, df, ω) *is an almost generalized gradient quasi-Einstein manifold.*

Proof. Let g be a gradient m-*QE* metric, then from Theorem 6.2 and Corollary 6.4, we obtain

$$\begin{aligned}\overline{Ric}(X,Y) + \overline{H}^{\phi}(X,Y) &= [\frac{\lambda}{f} + \frac{(f+1)}{2f^3}|grad\, f|^2 - \frac{\Delta(f)}{2f^2}]G(X,Y) + [C+D+\frac{1}{m}]df\otimes df(X,Y)\\ &-[\frac{(f^2+3f+1)}{2f^3(f+1)}|grad\, f|^2 - \frac{\Delta(f)}{2f^2} + \frac{\lambda}{f}]\omega\otimes\omega(X,Y).\end{aligned}$$

□

Proposition 6.12. *Let* (M^n, g) *be a Riemannian manifold and let* $G = fg + \omega\otimes\omega$ *be a semi-conformal deformation of the metric* g*, where* $f \in C^\infty(M^n)$ *is a strictly positive function and* ω *is the* 1*-form corresponding to* ξ*. If* g *is an* df*-Ricci soliton metric with the gradient of the potential function* f*, then* $(M^n, G, \phi, df, \omega)$ *is an almost generalized gradient quasi-Einstein manifold.*

Proof. Let g be an df-Ricci soliton metric, i.e.,

$$Ric + H^f - \mu df\otimes df = \lambda g.$$

By using the equations (36) and (38), we get

$$\begin{aligned}\overline{Ric} + \overline{H}^{\phi} &= [\frac{\lambda}{f} + \frac{(f+1)}{2f^3}\left|grad\, f\right|^2 - \frac{\Delta(f)}{2f^2}]G(X,Y) + [C+D+\mu]df\otimes df(X,Y)\\ &-[\frac{(f^2+3f+1)}{2f^3(f+1)}\left|grad\, f\right|^2 - \frac{\Delta(f)}{2f^2} + \frac{\lambda}{f}]\omega\otimes\omega(X,Y).\end{aligned}$$

This completes the proof. □

Theorem 6.13. *Let* (M^n, g) *be a Riemannian manifold and let* $G = fg + \omega\otimes\omega$ *be a semi-conformal deformation of the metric* g*, where* $f \in C^\infty(M^n)$ *is a strictly positive function and* ω *is the* 1*-form corresponding to* ξ*. If* g *is an* ω*-Ricci soliton metric with the gradient of the potential function* f*, then* $(M^n, G, \phi, df, \omega)$ *is an almost generalized gradient quasi-Einstein manifold.*

Proof. Let g be an ω-Ricci soliton metric, i.e.,

$$Ric + H^f - \mu\omega \otimes \omega = \lambda g.$$

By using the two equations (36) and (38), we get

$$\begin{aligned} \overline{Ric} + \overline{H}^{\phi} &= [\frac{\lambda}{f} + \frac{(f+1)}{2f^3}\left|grad\, f\right|^2 - \frac{\Delta(f)}{2f^2}]G(X,Y) + [C+D]df \otimes df(X,Y) \\ &\quad -[\frac{(f^2+3f+1)}{2f^3(f+1)}\left|grad\, f\right|^2 - \frac{\Delta(f)}{2f^2} - \mu + \frac{\lambda}{f}]\omega \otimes \omega(X,Y). \end{aligned}$$

□

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