Filomat 38:33 (2024), 11773–11784
https://doi.org/10.2298/FIL2433773S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Riemann solitons on perfect fluid spacetimes

Avijit Sarkar[a], Urmila Biswas[a]

[a]*Department of Mathematics, University of Kalyani, Kalyani 741235, West Bengal, India*

Abstract. In this article, we characterize Riemann solitons on perfect fluid spacetimes. Some relationship between perfect fluid spacetimes and Riemann solitons with certain soliton vector fields are established. We investigate Ricci-symmetric perfect fluid spacetimes whose metrics are Riemann solitons. Pseudo-projectively flat perfect fluid spacetimes with the metrics as Riemann solitons have been studied. It is also proved that a gradient Riemann soliton on a perfect fluid spacetime is Einstein. An example of a gradient Riemann soliton on perfect fluid spacetime has been constructed.

1. Introduction

The physically emanting waves that maintain their shapes after collision with another waves, are called solitons. Mathematically, soliton waves are the self-similar solutions of wave equations. The notion of solitons has been abstracted by Hamilton [19] as Ricci solitons which are self similar solutions of some flows represented by pseudo-parabolic PDEs. Nowadays, the study of solitons has become an important topic. in the research area of Differential Geometry. Ricci flow, a type of geometric flows, was invented by Hamilton [19] and successfully applied by Perelman [29] to solve Poincare conjecture. A Ricci flow is given by the equation

$$\frac{\partial}{\partial t}\mathfrak{g} = -2\mathcal{S},$$

satisfying $\mathfrak{g}_0 = \mathfrak{g}(0)$ where $\mathfrak{g}$ is the time dependent metric tensor and $\mathcal{S}$ is the Ricci tensor of type $(0, 2)$. After the introduction of the Ricci flow by Hamilton, many geometers have been interested to study geometric solitons. As a result, Udriste [36] introduced a special type of geometric flow which is named as Riemann flow and its equation is presented by

$$\frac{\partial}{\partial t}\mathcal{G}(t) = -2\mathcal{R}(\mathfrak{g}(t)), \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 53C50; Secondary 53C35, 53E20, 53E40.
Keywords. Lorentzian manifold, Riemann soliton, Perfect fluid spacetime, Gradient Riemann soliton, Ricci-symmetric perfect fluid spacetime, Pseudo-projectively flat perfect fluid spacetime.
Received: 20 February 2024; Accepted: 31 July 2024
Communicated by Mića Stanković
Avijit Sarkar is supported by DST-FIST, Sanction No. SR/FST/MS-I/2019/42(C) dated 30. 08. 2022.
Urmila Biswas is financially supported by CSIR-UGC NET fellowship, India, Ref. ID- 201610057626.
Email addresses: avjaj@yahoo.co.in (Avijit Sarkar), biswasurmila50@gmail.com (Urmila Biswas)

for some Riemann curvature tensor $\mathcal{R}$ of type $(0,4)$ generated by the metric $\mathfrak{g}$ and $\mathcal{G} = \frac{1}{2}\mathfrak{g} \odot \mathfrak{g}$, where $\odot$ indicates the Kulkarni-Nomizu product [3] defined by

$$\begin{aligned}(\mathcal{P}_1 \odot \mathcal{P}_2)(G_1, H_1, F_1, W_1) =& \mathcal{P}_1(G_1, W_1)\mathcal{P}_2(H_1, F_1) + \mathcal{P}_1(H_1, F_1)\mathcal{P}_2(G_1, W_1)\\ &- \mathcal{P}_1(G_1, F_1)\mathcal{P}_2(H_1, W_1) - \mathcal{P}_1(H_1, W_1)\mathcal{P}_2(G_1, F_1),\end{aligned} \tag{2}$$

for any vector fields G_1, H_1, F_1 and W_1 on the manifold $\mathcal{N}^n$, $\mathcal{P}_1$ and $\mathcal{P}_2$ are two $(0,2)$ tensor fields. A self similar solution of the Riemann flow is known as Riemann soliton which is related with the theory of Relativistic spacetimes [2, 17] and Riemann soliton exhibits symmetry that agrees with symmetry of spacetime geometry. A Riemann soliton on a manifold $\mathcal{N}^n$ is specially a particular solution of the above Riemann flow equation and it is given by [36]

$$\mathfrak{g} \odot \pounds_X \mathfrak{g} + \lambda \mathfrak{g} \odot \mathfrak{g} + 2\mathcal{R} = 0, \tag{3}$$

where $\pounds_X$ and λ are the Lie-derivative operator along the vector field X and a real constant respectively. The vector field X is called the soliton vector field of the Riemann soliton. The soliton is known as shrinkig, steady and expanding according as the value of λ is negative, zero and positive respectively. A Riemann soliton is denoted by $(\mathfrak{g}, X, \lambda)$. If the soliton function λ is a smooth function then the Riemann solitons are known as almost Riemann solitons. For a Riemann soliton [7], the equations (2) and (3) provide the following

$$(\pounds_X \mathfrak{g})(G_1, H_1) + \frac{2}{n-2}\mathcal{S}(G_1, H_1) + \frac{2[(n-1)\lambda + \mathrm{div}X]}{n-2}\mathfrak{g}(G_1, H_1) = 0, \tag{4}$$

where $\mathrm{div}X = -\frac{\mathfrak{r} + \mathfrak{n}(\mathfrak{n} - g)\lambda}{2(n-1)}$, $\mathcal{S}$ denotes the Ricci tensor of type $(0,2)$ and $\mathfrak{r}$ indicates the scalar curvature of the manifold.

Therefore, by the above expression, the Riemann soliton is presented by the following equation

$$(\pounds_X \mathfrak{g})(G_1, H_1) + \frac{2}{n-2}\mathcal{S}(G_1, H_1) = \Big(\frac{\mathfrak{r}}{(n-1)(n-2)} - \lambda\Big)\mathfrak{g}(G_1, H_1), \tag{5}$$

for any vector fields G_1, H_1, F_1 and W_1 on the manifold $\mathcal{N}^n$.

If the soliton vector field is gradient of some smooth function ω, called potential function, that is $X = D\omega$, then the Riemann soliton is known as gradient Riemann soliton and its equation is given by

$$\mathrm{Hess}(\omega)(G_1, H_1) + \frac{1}{n-2}\mathcal{S}(G_1, H_1) = \frac{1}{2}\Big(\frac{\mathfrak{r}}{(n-1)(n-2)} - \lambda\Big)\mathfrak{g}(G_1, H_1), \tag{6}$$

where $\mathrm{Hess}(\omega)$ indicates the Hessian operator of the smooth function ω.

A vector field X is called torse-forming [4] if

$$\nabla_{G_1} X = \alpha G_1 + \mathcal{A}(G_1)X, \tag{7}$$

for some smooth function α and a non-vanishing 1-form $\mathcal{A}$, where $\mathcal{A}(G_1) = \mathfrak{g}(G_1, \zeta)$, ζ is a unit time-like vector field, that is $\mathfrak{g}(\zeta, \zeta) = -1$.

A vector field X is named as concircular [8] if it satisfies the following

$$\nabla_{G_1} X = bG_1, \tag{8}$$

for any smooth function b and any vector field G_1 on the manifold $\mathcal{N}^n$ and a vector field X is Killing if $\pounds_X \mathfrak{g} = 0$. For a Killing soliton vector field, the soliton metric becomes Einstein. Moreover, the manifold is of constant sectional curvature.

Riemann solitons have been studied by many researchers. Such as, in [4], Blaga investigated some interesting results of almost Riemann solitons with the solitons vector fields as gradient or torse-forming. In [7], Biswas et al. proved a necessary and sufficient condition of a 3-dimensional Riemannian manifold to be a Riemann soliton. They also proved some fascinating results of Riemann solitons in the framework of almost co-Kähler manifolds. For details of Riemann solitons, the readers can follow ([6], [12], [21]) etc.

A perfect fluid spacetime in general relativity theory, is a Lorentzian manifold $(\mathcal{N}^n, \mathfrak{g})$ of dimension $n(n \geq 4)$, which satisfies certain condition of Ricci tensor. Perfect fluid spacetime deals with Physics, Astrophysics, Nuclear Physics and many other topics. Various kind of spacetimes have been studied by many researchers in different perspectives. Such as, De et al. proved that a H-flat perfect fluid spacetime is Einstein and many other attractive results of perfect fluid spacetimes have been studied in [13]. For instances, we refer ([14], [15], [23], [26], [28]). Some latest development on perfect fluid spacetimes and solitons can be found in ([1],[5], [22], [32], [33], [34] and [35]).

Recently, De et al. [14] studied the characteristics of Gradient solitons in perfect fluid spacetimes. They proved that Lorentzian manifold admitting Ricci soliton is a perfect fluid spacetime under certain restrictions of soliton vector field. They have also studied some concentrated results of Gradient solitons. Again, in [31] Siddiqi studied Ricci ρ-solitons in the frame of dust fluid and viscous fluid spacetimes and many other authors ([10],[11],[15]) have investigated several types of solitons on perfect fluid spaceimes. Motivated from the above studies, the following questions have arised in our mind:
Question 1. What are the characteristics of Riemann solitons and gradient Riemann solitons in the frame of perfect fluid spacetimes?
Question 2. Does there exist an example of a Riemann soliton in perfect fluid spacetime?
For an affirmative answer of the above two question, we are interested to study Riemann solitons in the frame of perfect fluid spacetimes and an example of a gradient Riemann soliton in perfect fluid spacetime has been provided.

The present article is structured as follows: After introduction in Section 1, a time-oriented, connected, four-dimensional Lorentzian manifold has been modeled using spacetime in Section 2. We give some relationship between perfect fluid spacetimes and Riemann solitons with certain soliton vector fields in Section 3. Section 4 contains some study of Ricci-symmetric perfect fluid spacetimes and Riemann solitons. Next, we investigate, pseudo-projectively flat perfect fluid spacetimes with the metric as Riemann solitons in Section 5. In Section 6, we study gradient Riemann solitons and perfect fluid spacetimes. Lastly, in Section 7, we provide an example of a gradient Riemann soliton in perfect fluid spacetime.

2. Perfect fluid spacetimes

Let $\mathcal{N}^n$ be a Lorentzian manifold of dimension n $(n \geq 4)$ with the metric $\mathfrak{g}$ of signature $(+, +, ..., -)$ and this metric admits a globally time-oriented vector field. This metric $\mathfrak{g}$ is called Lorentzian metric. **Definition 2.1.** *A Lorentzian manifold $\mathcal{N}^n$ is called the perfect fluid spacetime [38], if the non-zero Ricci tensor $\mathcal{S}$ satisfies*

$$\mathcal{S}(G_1, H_1) = \beta_1 \mathfrak{g}(G_1, H_1) + \beta_2 \mathcal{A}(G_1)\mathcal{A}(H_1), \tag{9}$$

where $\mathcal{A}(G_1) = \mathfrak{g}(G_1, \zeta)$ and β_1, β_2 are real scalars but not simultaneously zero. ζ is a unit time-like vector field under the metric $\mathfrak{g}$, that is $\mathfrak{g}(\zeta, \zeta) = -1$.

For the perfect fluid spacetime the energy momentum tensor $\mathcal{T}$ is given by [24]

$$\mathcal{T}(G_1, H_1) = \mathfrak{p}\mathfrak{g}(G_1, H_1) + (\mathfrak{p} + \mu)\mathcal{A}(G_1)\mathcal{A}(H_1), \tag{10}$$

where $\mathfrak{p}$ and μ denote isotropic pressure and energy density respectively, $\mathcal{A}$ is a non-vanishing 1-form defined by $\mathcal{A}(G_1) = \mathfrak{g}(G_1, \zeta)$. The Einstein field equations except cosmological constant are presented by

$$\mathcal{S}(G_1, H_1) = \frac{\mathfrak{r}}{2}\mathfrak{g}(G_1, H_1) + \kappa\mathcal{T}(G_1, H_1), \tag{11}$$

where $\mathfrak{r}$ denotes scalar curvature and κ indicates gravitaional constant which is greater than zero. Combining the equations (9), (10) and (11), we have the following [13]

$$\beta_1 = \frac{\kappa(\mathfrak{p}-\mu)}{2-n} \quad \text{and} \quad \beta_2 = \kappa(\mathfrak{p}+\mu). \tag{12}$$

According to ([13], [14]), if the equation of state is $\mathfrak{p}+\mu = 0$, then the spacetime represents a dark matter era and if $\mathfrak{p} = 0$, the spacetime represents a dust matter era. A perfect fluid spacetime is called isentropic [20], if in the equation of state $\mathfrak{p} = \mathfrak{p}(\mu)$ holds.

The pseudo-projective curvature tensor ([25], [30]) P^* is defined by

$$\begin{aligned} P^*(G_1,H_1)F_1 =& a_0\mathcal{R}(G_1,H_1)F_1 + a_1\Big(\mathcal{S}(H_1,F_1)G_1 - \mathcal{S}(G_1,F_1)H_1\Big) - \frac{\mathfrak{r}}{n}\Big(\frac{a_0}{n-1} + a_1\Big) \\ &\Big(\mathfrak{g}(H_1,F_1)G_1 - \mathfrak{g}(G_1,F_1)H_1\Big), \end{aligned} \tag{13}$$

for any vector fields G_1, H_1 and F_1 on the Lorentzian manifold N^n, $\mathcal{R}$ being the Riemann curvature tensor of type $(1,3)$ and a_0, a_1 are real numbers. A perfect fluid spacetime is known as pseudo-projectively flat if $P^*(G_1,H_1)F_1 = 0$.

3. Relationship between Perfect fluid spacetimes and Riemann solitons with certain soliton vector fields

Theorem 3.1. *A Lorentzian manifold of dimension n $(n \geq 4)$ admitting a Riemann soliton whose soliton vector field is unit time-like, is a perfect fluid spacetime with closed non-vanishing 1-form if and only if its soliton vector field is torse-forming.*

Proof. Let the metric $\mathfrak{g}$ of a Lorentzian manifold of dimension n $(n \geq 4)$ be a Riemann soliton whose soliton vector field is a unit time-like vector field ζ, then the equation (5) gives

$$\mathfrak{g}(\nabla_{G_1}\zeta, H_1) + \mathfrak{g}(\nabla_{H_1}\zeta, G_1) + \frac{2}{n-2}\mathcal{S}(G_1,H_1) = \Big(\frac{\mathfrak{r}}{(n-1)(n-2)} - \lambda\Big)\mathfrak{g}(G_1,H_1), \tag{14}$$

for any vector fields G_1, H_1 on the Lorentzian manifold $\mathcal{N}^n$. For the metric of a perfect fluid spacetime, the equation (14) leads to

$$\mathfrak{g}(\nabla_{G_1}\zeta, H_1) + \mathfrak{g}(\nabla_{H_1}\zeta, G_1) + \frac{2\beta_1}{n-2}\mathfrak{g}(G_1,H_1) + \frac{2\beta_2}{n-2}\mathcal{A}(G_1)\mathcal{A}(H_1) = \Big(\frac{\mathfrak{r}}{(n-1)(n-2)} - \lambda\Big)\mathfrak{g}(G_1,H_1). \tag{15}$$

Again, if the non-vanishing 1-form $\mathcal{A}$ associates with the perfect fluid spacetime is closed, then

$$\mathfrak{g}(\nabla_{G_1}\zeta, H_1) = \mathfrak{g}(\nabla_{H_1}\zeta, G_1). \tag{16}$$

Using the equation (16) in the equation (15), we have

$$\nabla_{G_1}\zeta = \frac{1}{2}\Big(\frac{\mathfrak{r}}{(n-1)(n-2)} - \frac{2\beta_1}{n-2} - \lambda\Big)G_1 - \frac{\beta_2}{n-2}\mathcal{A}(G_1)\zeta, \tag{17}$$

for any vector field H_1 on the manifold $\mathcal{N}^n$. Therefore, the unit time-like soliton vector field is torse-forming. Conversely, in a Lorentzian manifold, let us suppose that the soliton vector field of the Riemann soliton be a unit time-like torse-forming vector field ζ, then the equations (5) and (7) infer that

$$\mathcal{S}(G_1,H_1) = \Big(\frac{\mathfrak{r}}{2(n-1)} - \frac{(n-2)\lambda}{2} - (n-2)\alpha\Big)\mathfrak{g}(G_1,H_1) - (n-2)\mathcal{A}(G_1)\mathcal{A}(H_1). \tag{18}$$

Therefore, the Lorentzian manifold is a perfect fluid spacetime and for a unit time-like torse-forming vector field ζ, the non-vanishing 1-form $\mathcal{A}$ associated with ζ is closed.
Hence the theorem. □

Theorem 3.2. *If the metric of a perfect fluid spacetime is a Riemann soliton with concircular soliton vector field , then the spacetime represents a dark matter era. Also, the value of the smooth function b is* $-\frac{1}{2}\Big(\lambda + \frac{\mathfrak{r}}{n(n-1)}\Big)$.

Proof. Suppose that the soliton vector field X of a Riemann soliton is a concircular vector field, then by the virtue of (8), the equation (5) is transferred to

$$2b\mathfrak{g}(G_1, H_1) + \frac{2}{n-2}\mathcal{S}(G_1, H_1) = \Big(\frac{\mathfrak{r}}{(n-1)(n-2)} - \lambda\Big)\mathfrak{g}(G_1, H_1), \tag{19}$$

for any vector fields G_1, H_1 on the Lorentzian manifold $\mathcal{N}^n$. Taking contraction of the foregoing equation, we have

$$2b + \frac{2}{n(n-2)}\mathfrak{r} = \Big(\frac{\mathfrak{r}}{(n-1)(n-2)} - \lambda\Big). \tag{20}$$

Setting $G_1 = H_1 = \zeta$ in the equation (19), we get

$$2b - \frac{2}{n-2}\mathcal{S}(\zeta, \zeta) = \Big(\frac{\mathfrak{r}}{(n-1)(n-2)} - \lambda\Big). \tag{21}$$

For a perfect fluid spacetime, the equation (9) gives $\mathcal{S}(\zeta, \zeta) = \beta_2 - \beta_1$. Then from (21), we obtain

$$2b + \frac{2}{n-2}(\beta_1 - \beta_2) = \Big(\frac{\mathfrak{r}}{(n-1)(n-2)} - \lambda\Big). \tag{22}$$

Comparing (20) and (22), we infer that

$$\mathfrak{r} = n(\beta_1 - \beta_2). \tag{23}$$

Again, in a perfect fluid spacetime, contraction of (9) provides $\mathfrak{r} = n\beta_1 - \beta_2$. Hence from (23), we find

$$\beta_2 = 0. \tag{24}$$

Therefore, from the equation (12) we get $\mathfrak{p} + \mu = 0$, that is the spacetime represents a dark matter era.

Since $\beta_2 = 0$, from the equation (23), we infer

$$\beta_1 = \frac{\mathfrak{r}}{n}. \tag{25}$$

Use of (24) and (25) in (22) gives

$$b = -\frac{1}{2}\Big(\lambda + \frac{\mathfrak{r}}{n(n-1)}\Big). \tag{26}$$

This completes the proof. □

4. Ricci-symmetric perfect fluid spacetimes and Riemann solitons

In this section, we study Ricci-symmetric perfect fluid spacetime. In this connection, it is to be mentioned that the notion of symmetric spaces was introduced by E. Cartan [9]. Symmetric spaces are important in view of their physical relevance as they are used to interpret gauge theoretic importance of gravity [16, 27]. Ricci-symmetric spaces are weaker versions of symmetric spaces. This is why such spaces are important in view of their physical relevance. For the notion of Ricci-symmetric spaces, we refer [37].

Theorem 4.1. *A n-dimensional Ricci-symmetric perfect fluid spacetime is isentropic. Moreover, the scalar curvature* $\mathfrak{r} = 2\kappa\mu$.

Proof. For a n-dimensional perfect fluid spacetime, differentiating (9) covariantly with respect to any vector field F_1, we get

$$(\nabla_{F_1}\mathcal{S})(G_1,H_1) =(F_1\beta_1)\mathfrak{g}(G_1,H_1) + (F_1\beta_2)\mathcal{A}(G_1)\mathcal{A}(H_1) + \beta_2\Big(\mathfrak{g}(\nabla_{F_1}\zeta,G_1)\mathcal{A}(H_1) + \mathfrak{g}(\nabla_{F_1}\zeta,H_1)\mathcal{A}(G_1)\Big), \quad (27)$$

for any vector fields G_1, H_1 and F_1 on the manifold $\mathcal{N}^n$. A perfect fluid spacetime is called Ricci-symmetric [11] if $(\nabla_{F_1}\mathcal{S})(G_1,H_1) = 0$. For a Ricci-symmetric perfect fluid spacetime, setting $G_1 = H_1 = \zeta$ in the equation (27), we have

$$F_1\beta_1 = F_1\beta_2, \quad (28)$$

where $\mathfrak{g}(\zeta,\zeta) = 0$ and $\mathfrak{g}(\nabla_{F_1}\zeta,\zeta) = 0$ are used. Putting the value of β_1 and β_2 from (12) in the above equation, we have

$$d\mathfrak{p} = \frac{(3-n)}{(n-1)}d\mu, \quad (29)$$

for any vector field F_1. By integrating (29), we infer the following

$$\mathfrak{p} = \frac{(3-n)}{(n-1)}\mu, \quad (30)$$

where the integration constant is treatd as 0. Therefore, the perfect fluid spacetime is isentropic.
Again, the equations (12) and (30) imply that

$$\beta_1 = \beta_2 = \frac{2\kappa}{(n-1)}\mu. \quad (31)$$

Using (31), the equation (9) transforms into

$$\mathcal{S}(G_1,H_1) = \frac{2\kappa}{(n-1)}\Big(\mathfrak{g}(G_1,H_1) + \mathcal{A}(G_1)\mathcal{A}(H_1)\Big)\mu. \quad (32)$$

Contracting the equation (32), we obtain

$$\mathfrak{r} = 2\kappa\mu. \quad (33)$$

This proves the theorem. □

The above theorem leads the following:

Corollary 4.2. *A Riemann soliton $(\mathfrak{g},\zeta,\lambda)$ on a Ricci-symmetric perfect fluid spacetime is shrinking, steady and expanding if $\mathfrak{r} <, =$ and > 0 respectively. Moreover, the integral curve of the time-like vector field is geodesic.*

Proof. The equation of the Riemann soliton $(\mathfrak{g},\zeta,\lambda)$ is given by

$$\mathfrak{g}(\nabla_{G_1}\zeta,H_1) + \mathfrak{g}(\nabla_{H_1}\zeta,G_1) + \frac{2}{n-2}\mathcal{S}(G_1,H_1) = \Big(\frac{\mathfrak{r}}{(n-1)(n-2)} - \lambda\Big)\mathfrak{g}(G_1,H_1). \quad (34)$$

Choosing $G_1 = H_1 = \zeta$ in (34), we get

$$\mathcal{S}(\zeta,\zeta) = \frac{(n-2)\lambda}{2} - \frac{\mathfrak{r}}{2(n-1)}, \quad (35)$$

where $\mathfrak{g}(\nabla_\zeta\zeta,\zeta) = 0$ is used. Again for a Ricci-symmetric perfect fluid spacetime, putting $G_1 = H_1 = \zeta$ in (32), gives that $\mathcal{S}(\zeta,\zeta) = 0$. Then the equation (35) becomes

$$\lambda = \frac{\mathfrak{r}}{(n-1)(n-2)}. \quad (36)$$

Therefore, the Riemann soliton is shrinking, steady and expanding if $\mathfrak{r} <, =$ and > 0 respectively.
Again, putting $G_1 = \zeta$ in (34), using (36) and $\mathfrak{g}(\nabla_{H_1}\zeta, \zeta) = 0$, we have

$$\mathfrak{g}(\nabla_\zeta \zeta, H_1) + \frac{2}{n-2}\mathcal{S}(H_1, \zeta) = 0. \tag{37}$$

The equation (32) provides that $\mathcal{S}(H_1, \zeta) = 0$, then the equation (37) infer that

$$\nabla_\zeta \zeta = 0, \tag{38}$$

for any vector field H_1 on $\mathcal{N}^n$. That is the integral curve of the time-like vector field is geodesic. □

5. Pseudo-projectively flat perfect fluid spacetimes and Riemann solitons

Theorem 5.1. *A pseudo-projectively flat perfect fluid spacetime is Einstein. Also, the spacetime represents a dark matter era, provided $a_0 + (n-1)a_1 \neq 0$.*

Proof. For a pseudo-projectively flat perfect fluid spacetime, from (13) we get

$$a_0\mathcal{R}(G_1, H_1)F_1 + a_1\Big(\mathcal{S}(H_1, F_1)G_1 - \mathcal{S}(G_1, F_1)H_1\Big) - \frac{\mathfrak{r}}{n}\Big(\frac{a_0}{n-1} + a_1\Big)\Big(\mathfrak{g}(H_1, F_1)G_1 - \mathfrak{g}(G_1, F_1)H_1\Big) = 0. \tag{39}$$

Taking contraction of the equation (39) with respect to G_1, we infer

$$\mathcal{S}(H_1, F_1) = \frac{\mathfrak{r}}{n}\mathfrak{g}(H_1, F_1), \tag{40}$$

provided $a_0 + (n-1)a_1 \neq 0$. Comparing the equation (40) with (9), we obtain

$$\beta_2 = 0. \tag{41}$$

Combining (12) and (41), we conclude that

$$\kappa(\mathfrak{p} + \mu) = 0, \tag{42}$$

κ is gravitational constant, so $\kappa > 0$. Then $\mathfrak{p} + \mu = 0$, that means that the spacetime represents a dark matter era. □

Corollary 5.2. *A Riemann soliton $(\mathfrak{g}, \zeta, \lambda)$ on a pseudo-projectively flat perfect fluid spacetime is shrinking, steady and expanding according as $\mathfrak{r} >, =,$ or < 0 respectively, provided $a_0 + (n-1)a_1 \neq 0$.*

Proof. For a pseudo-projectively flat perfect fluid spacetime, the Ricci tensor (40) is of the following form

$$\mathcal{S}(H_1, F_1) = \frac{\mathfrak{r}}{n}\mathfrak{g}(H_1, F_1). \tag{43}$$

Combining (35) and (43), we have

$$\lambda = -\frac{\mathfrak{r}}{n(n-1)}. \tag{44}$$

Therefore, a pseudo-projectively flat perfect fluid spacetime is shrinking, steady and expanding according as $\mathfrak{r} >, =,$ or < 0 respectively, provided $a_0 + (n-1)a_1 \neq 0$. □

Theorem 5.3. *Let $(\mathfrak{g}, X, \lambda)$ be a Riemann soliton on a pseudo-projectively flat perfect fluid spacetime and the soliton vector field X is Killing. Then the isotropic pressure $\mathfrak{p}$ is given by $\frac{(n-1)(n-2)\lambda}{2\kappa}$, provided $a_0 + (n-1)a_1 \neq 0$.*

Proof. Let $(\mathfrak{g}, X, \lambda)$ be a Riemann soliton and the soliton vector field X is Killing. Then from (5), the Ricci tensor is of the following form

$$\mathcal{S}(G_1, H_1) = \Big(\frac{\mathfrak{r}}{2(n-1)} - \frac{(n-2)\lambda}{2}\Big)\mathfrak{g}(G_1, H_1). \tag{45}$$

For a pseudo-projectively flat perfect fluid spacetime, comparing (9) and (43), we infer the following

$$\beta_1 = \frac{\mathfrak{r}}{n} \quad \text{and} \quad \beta_2 = 0. \tag{46}$$

Combining (40) and (45), we have

$$\Big(\frac{\mathfrak{r}}{n} + \frac{(n-2)\lambda}{2} - \frac{\mathfrak{r}}{2(n-1)}\Big)\mathfrak{g}(G_1, H_1) = 0. \tag{47}$$

Now, for any vector fields G_1, H_1, we deduce that

$$\mathfrak{r} = -n(n-1)\lambda. \tag{48}$$

Using the equation (48) in the equation (46), we find that

$$\beta_1 = -(n-1)\lambda \quad \text{and} \quad \beta_2 = 0. \tag{49}$$

Since $\beta_2 = 0$, from (12), we conclude that $\mathfrak{p} + \mu = 0$, where κ being gravitational constant > 0. Therefore, also from (12) we have

$$\beta_1 = \frac{2\kappa\mathfrak{p}}{2-n}. \tag{50}$$

Comparing (49) and (50), we obtain

$$\mathfrak{p} = \frac{(n-1)(n-2)\lambda}{2\kappa}. \tag{51}$$

Hence the theorem.
□

Corollary 5.4. *If the metric of a pseudo-projectively flat perfect fluid spacetime admits a steady Riemann soliton $(\mathfrak{g}, X, \lambda)$ with the Killing soliton vector field, then the spacetime represents a dust matter era, provided $a_0 + (n-1)a_1 \neq 0$.*

Proof. From the above theorem, since $\lambda = 0$, this implies that $\mathfrak{p} = 0$. That means the spacetime represents a dust matter era. □

Corollary 5.5. *If the metric of a pseudo-projectively flat perfect fluid spacetime admits Riemann soliton with the Killing soliton vector field, then the soliton is shrinking and expanding according as $\mathfrak{p} < 0$ and $\mathfrak{p} > 0$ respectively, provided $a_0 + (n-1)a_1 \neq 0$.*

Proof. The equation (51) gives this corollary. □

6. Perfect fluid spacetimes and gradient Riemann solitons

Theorem 6.1. *If the metric of a perfect fluid spacetime is a gradient Riemann soliton and the scalar curvature is invariant under the non-solenoidal time-like vector field, then the spacetime represents an Einstein spacetime, provided $\mathfrak{p} \neq \dfrac{(n-3)\mu}{(1-n)}$. Moreover, the spacetime becomes a dark matter era.*

Proof. For a gradient Riemann soliton (6), we have the following

$$\nabla_{G_1} D\omega + \frac{1}{n-2} Q G_1 = \frac{1}{2}\Big(\frac{\mathfrak{r}}{(n-1)(n-2)} - \lambda\Big) G_1, \tag{52}$$

for any vector field G_1 and a potential function ω. Utilizing the above equation and by computation, we have

$$\mathcal{R}(G_1, H_1) D\omega = \frac{1}{n-2}\Big((\nabla_{H_1} Q) G_1 - (\nabla_{G_1} Q) H_1\Big) + \frac{1}{2(n-1)(n-2)}\Big((G_1 \mathfrak{r}) H_1 - (H_1 \mathfrak{r}) G_1\Big). \tag{53}$$

Contraction of (53) with respect to G_1 gives that

$$\mathcal{S}(H_1, D\omega) = 0. \tag{54}$$

Covariant derivative of (54) deduce that

$$(\nabla_{F_1} \mathcal{S})(H_1, D\omega) = 0. \tag{55}$$

The equation (55) provides

$$(\nabla_{F_1} \mathcal{S})(H_1, D\omega) = (\nabla_{H_1} \mathcal{S})(F_1, D\omega). \tag{56}$$

Again, for a perfect fluid spacetime (9), we obtain

$$\begin{aligned}(\nabla_{G_1} Q) H_1 - (\nabla_{H_1} Q) G_1 = &(G_1 \beta_1) H_1 - (H_1 \beta_1) G_1 + \Big((G_1 \beta_2) \mathcal{A}(H_1) - (H_1 \beta_2) \mathcal{A}(G_1)\Big)\zeta \\ &+ \beta_2\Big[\Big((\nabla_{G_1} \mathcal{A}) H_1 - (\nabla_{H_1} \mathcal{A}) G_1\Big)\zeta + \mathcal{A}(H_1) \nabla_{G_1} \zeta - \mathcal{A}(G_1) \nabla_{H_1} \zeta\Big].\end{aligned} \tag{57}$$

Taking contraction of (57) for some orthonormal basis $\{e_i\}_{i=1}^n$ with respect to the vector field H_1, we find

$$G_1 \mathfrak{r} = 2(n-1) G_1 \beta_1 - 2\Big((G_1 \beta_2) + \mathcal{A}(G_1) \zeta \beta_2\Big) - 2\beta_2\Big[\sum_{i=1}^n \mathfrak{g}(\nabla_{e_i} \zeta, G_1) \mathfrak{g}(e_i, \zeta) + \mathcal{A}(G_1) \sum_{i=1}^n \mathfrak{g}(\nabla_{e_i} \zeta, e_i)\Big]. \tag{58}$$

Putting $G_1 = \zeta$ in (58), we get

$$\zeta \mathfrak{r} = 2(n-1) \zeta \beta_1 + 2\beta_2 \mathrm{div} \zeta. \tag{59}$$

For a perfect fluid spacetime (9), we also have

$$\begin{aligned}&(\nabla_{F_1} \mathcal{S})(H_1, D\omega) - (\nabla_{H_1} \mathcal{S})(F_1, D\omega) = [(F_1 \beta_1) H_1 \omega - (H_1 \beta_1) F_1 \omega] + [(F_1 \beta_2) \mathcal{A}(H_1) - (H_1 \beta_2) \mathcal{A}(F_1)]\zeta\omega \\ &+ \beta_2\Big[\Big((\nabla_{F_1} \mathcal{A}) H_1 - (\nabla_{H_1} \mathcal{A}) F_1\Big)\zeta\omega + \mathcal{A}(H_1) F_1(\zeta\omega) - \mathcal{A}(F_1) H_1(\zeta\omega)\Big].\end{aligned} \tag{60}$$

Using the equations (9), (54) and setting $H_1 = \zeta$, we infer

$$(\beta_1 - \beta_2)\zeta\omega = 0. \tag{61}$$

If $\beta_1 \neq \beta_2$, that imply $\mathfrak{p} \neq \frac{(n-3)\mu}{(1-n)}$, then the potential function ω is invariant under a time-like vector field ζ, that is $\zeta\omega = 0$, comparing (56) and (60), we acquire that

$$(F_1 \beta_1) H_1 \omega - (H_1 \beta_1) F_1 \omega = 0. \tag{62}$$

Choosing $F_1 = \zeta$, the equation (62) becomes

$$(\zeta \beta_1)(H_1 \omega) = 0. \tag{63}$$

Now, two cases may arise:
Case I : If $H_1\omega = 0$, it implies that ω=constant. Therefore, the manifold is Einstein.
Case II : If $\zeta\beta_1 = 0$, it follows from (59) that

$$\zeta\mathfrak{r} = 2\beta_2 \mathrm{div}\zeta. \tag{64}$$

If the scalar curvature $\mathfrak{r}$ is invariant under a time-like vector field ζ, that is $\zeta\mathfrak{r} = 0$, then by (64) we find that

$$\beta_2 \mathrm{div}\zeta = 0. \tag{65}$$

Here, for a non-solenoidal time-like vector field ζ, we obtain $\beta_2 = 0$. It shows from (9) that the spacetime is Einstein.
By the above two cases, we conclude that the spacetime is Einstein.

Moreover, if $\beta_2 = 0$, then (12) gives $\mathfrak{p} + \mu = 0$, that is the spacetime represents a dark matter era.
This proves the theorem.
□

7. Example

We consider a Lorentzian metric $\mathfrak{g}$ on $\mathbb{R}^4$ [18] by

$$ds^2 = \mathfrak{g}_{ij} du^i du^j = (e^{2u^4})(du^1)^2 + (u^4)^2(du^2)^2 + (e^{2u^4})(du^3)^2 - (du^4)^2,$$

where $\frac{1}{2} < u^4 < 1$ and u^1, u^2, u^3, u^4 are the standard coordinates of $\mathbb{R}^4$.
For some orthogonal basis $\{e_i\}_{i=1}^4$ on the manifold $(\mathbb{R}^4, \mathfrak{g})$, the non-vanishing components of metric tensor are

$$\mathfrak{g}_{11} = e^{2u^4}, \quad \mathfrak{g}_{22} = (u^4)^2, \quad \mathfrak{g}_{33} = e^{2u^4}, \quad \mathfrak{g}_{44} = -1, \tag{66}$$

and the contravariant components of the above metric tensors are

$$\mathfrak{g}^{11} = \frac{1}{e^{2u^4}}, \quad \mathfrak{g}^{22} = \frac{1}{(u^4)^2}, \quad \mathfrak{g}^{33} = \frac{1}{e^{2u^4}}, \quad \mathfrak{g}^{44} = -1. \tag{67}$$

By using (66) and (67), the non-vanishing components of Christoffel symbols, the curvature tensor and Ricci tensor are determined as

$$\Gamma^4_{11} = \Gamma^4_{33} = e^{2u^4}, \quad \Gamma^1_{14} = \Gamma^3_{34} = 1, \quad \Gamma^4_{22} = u^4, \quad \Gamma^2_{24} = \frac{1}{u^4},$$

$$\mathcal{R}_{1331} = -e^{4u^4}, \quad \mathcal{R}_{1221} = \mathcal{R}_{2332} = -u^4 e^{2u^4}, \quad \mathcal{R}_{1441} = \mathcal{R}_{3443} = e^{2u^4},$$

$$\mathcal{S}_{11} = \mathcal{S}_{33} = -e^{2u^4}\left(\frac{1}{u^4} + 2\right), \quad \mathcal{S}_{22} = -2u^4, \quad \mathcal{S}_{44} = 2,$$

and the symmetric properties give other non-vanishing components.
If we consider $\beta_1 = -2$, $\beta_2 = -1$ and the 1-form $\mathcal{A}$ as follows

$$\mathcal{A}_i = \begin{cases} \sqrt{\dfrac{e^{2u^4}}{u^4}}, & \text{for} \quad i = 1,3 \\ \sqrt{2u^4(1-u^4)}, & \text{for} \quad i = 2 \\ 0, & \text{for} \quad i = 4, \end{cases}$$

where $\mathcal{A}_i = \mathfrak{g}(e_i, \zeta)$. Then

$$\mathcal{S}_{11} = \beta_1 \mathfrak{g}_{11} + \beta_2 \mathcal{A}_1 \mathcal{A}_1 = -e^{2u^4}(\frac{1}{u^4} + 2),$$

$$\mathcal{S}_{22} = \beta_1 \mathfrak{g}_{22} + \beta_2 \mathcal{A}_2 \mathcal{A}_2 = -2u^4,$$

$$\mathcal{S}_{33} = \beta_1 \mathfrak{g}_{33} + \beta_2 \mathcal{A}_3 \mathcal{A}_3 = -e^{2u^4}(\frac{1}{u^4} + 2),$$

$$\mathcal{S}_{44} = \beta_1 \mathfrak{g}_{44} + \beta_2 \mathcal{A}_4 \mathcal{A}_4 = 2.$$

Hence, the scalar curvature $\mathfrak{r} = -2e^{2u^4}(\frac{1}{u^4} + 2) - 2u^4 + 2$.
In a 4-dimensional Lorentzian manifold $(\mathbb{R}^4, \mathfrak{g})$, from (6) the equation of gradient Riemann soliton is given by

$$\mathfrak{g}(\nabla_{G_1} D\omega, H_1) + \frac{1}{2}\mathcal{S}(G_1, H_1) = \frac{1}{2}(\frac{\mathfrak{r}}{6} - \lambda)\mathfrak{g}(G_1, H_1), \tag{68}$$

for any vector fields G_1, H_1 on $(\mathbb{R}^4, \mathfrak{g})$. If the potential function ω is chosen by the following fashion

$$\nabla_{e_i} D\omega = \begin{cases} \frac{1}{2}(\frac{1}{u^4} + 2)(1 - \frac{1}{3}e^{2u^4})e_1, & \text{for} \quad i = 1 \\ \Big(-\frac{1}{6}e^{2u^4}(\frac{1}{u^4} + 2) + \frac{1}{u^4}\Big)e_2, & \text{for} \quad i = 2 \\ \frac{1}{2}(\frac{1}{u^4} + 2)(1 - \frac{1}{3}e^{2u^4})e_3, & \text{for} \quad i = 3 \\ \Big(-\frac{1}{6}e^{2u^4}(\frac{1}{u^4} + 2) + 1\Big)e_4, & \text{for} \quad i = 4 \end{cases}$$

and the soliton function λ is defined by $-\frac{1}{3}u^4 + \frac{1}{3}$. Then from (68), the Lorentzian metric $\mathfrak{g}$ is a gradient Riemann soliton. Moreover, the soliton is expanding, since $\frac{1}{2} < u^4 < 1$.
Therefore, the Lorentzian manifold $\mathfrak{g}$ is an expanding gradient Riemann soliton in the perfect fluid spacetime $(\mathbb{R}^4, \mathfrak{g})$.
Acknowledgement. The authors are very much thankful to the referee for his/her valuable suggestions towards the improvement of the paper.

References

[1] Alkhaldi, A. H., Siddiqi, M. D., Khan, M. A. and Alqatani, L. S., Generalized Robertson Walker spacetime admitting Ricci-Yamabe metric, Adv. Math. Phys., 2021, 2485804.
[2] Azami, S., Jafari, M., Riemann solitons on relativistic spacetimes, Gravit. Cosmol., 30(2024), 36-311.
[3] Besse, A., Einstein manifolds, Springer, Berlin 1987.
[4] Blaga, A. M., Remarks on almost Riemann solitons with gradient or torse-forming vector field, arXiv:2008.06413v4[math. DG].
[5] Blaga, A. M., Solitons and geometrical structures in a perfect fluid spacetime, Rocky Mt. J. Math., 50(2020), 41-53.

[6] Blaga, A. M. and Latcu, D. R., Remarks on Riemann and Ricci solitons in (α, β)-contact metric manifolds, arXiv:2009.02506v4[math. DG].
[7] Biswas, G. G., Chen, X. and De, U. C., Riemann solitons on almost co-Kähler manifolds, Filomat, 2022(36), 1403-1413.
[8] Brickell, F. and Yano, K., Concurrent vector fields and Minkowski structure, Kodai Math. Ser. Rep. 26(1974), 22-28.
[9] Carten, E., Les espaces a connexions conforme, Ann. Soc. Pol. Math. 2(1923), 131-221.
[10] De, A., Özgür, C. and De, U. C., On conformally flat almost pseudo-Ricci symmetric spacetimes, International Journal of Theoretical Physics, 2012(51), 2878-2887.
[11] De, A. and Majhi, P., Weakly Ricci symmetric spacetimes, International Journal of Geometric Methods in Modern Physics, 2018(15), 1850007.
[12] De, K. and De, U. C., A note on almost Riemann Solitons and gradient almost Riemann Solitons, Afrika Matematika, 33(2022), 1-10.
[13] De, U.C. and Hazra, D., Characterizations of H-flat curvature tensor on spacetimes and f (r, T)-gravity, Physica Scripta, 98(2022) 025217.
[14] De, U. C., Mantica, C. A. and Suh, Y. J., Perfect fluid spacetimes and Gradient solitons, Filomat 36(2022), 829-842.
[15] De, K., De, U. C., Syied, A. A., Turki, N. B. and Alsaeed, S., Perfect fluid spacetimes and gradient solitons, Journal of Nonlinear Mathematical Physics, (2022) 1-16.
[16] Derek, K. W., Symmetric spaces Cartan connections and gravity in three and four dimensions, SIGMA, 5(2009), 080, 18pp.
[17] Fischer, A. E., An introduction to conformal Ricci Flow, Class. Quantum Grav., 21 (2004), 171-218.
[18] Güler, S. and Demirbağ, S. A., A study of generalized quasi Einstein spacetimes with applications in general relativity, International Journal of Theoretical Physics, 55(2016),548-562.
[19] Hamilton, R. S., The Ricci flow on surfaces, In Mathematics and General Relativity; Contemp. Math; American Mathematical Society, 71(1988), 237-262.
[20] Hawking, S. W. and Ellis, G.F., The large scale structure of space-time, Cambridge university press, (1973).
[21] Kenmotsu, K., A class of almost contact Riemannian manifolds, Tohoku Mathematical Journal, Second Series, 24(1972), 93-103.
[22] Kumara, H. A., Ricci solitons and geometrical structures in a perfect fluid spacetime with torse forming vector field, Afrika Matematika, 30(2019), 725-736.
[23] Mallick, S., Super Quasi-Einstein Manifolds with Applications to General Relativity, Kyungpook Math. J. 58(2018), 361-375.
[24] Mallick, S., De, U.C. and Suh, Y.J., Spacetimes with different forms of energy–momentum tensor, Journal of Geometry and Physics, 2020(151), 103622.
[25] Mallick, S., Suh, Y. J. and De, U. C., A spacetime with pseudo-projective curvature tensor, Journal of Mathematical Physics, 2016(57), 062501.
[26] Mantica, C. A., Molinari, L. G. and De, U. C., A condition for a perfect-fluid space-time to be a generalized Robertson-Walker space-time, Journal of Mathematical Physics, 57(2016), 022508.
[27] Misner, C. W., Thorne, K. S., Wheeler, J. A., Gravitation, W. H., Freeman and Co., San Francisco, Calif., 1973.
[28] O'Neil, B., Semi-Riemannian geometry with applications to relativity, Academic press (1983).
[29] Perelman, G., The entropy formula for the Ricci flow and its geometric applications, arXiv preprint math/0211159 (2002).
[30] Shivamurthy, T. R., Prakasha, D. G. and Chavan, V., On pseudo-projective curvature tensor with respect to semi-symmetric non-metric connection in a Kenmotsu manifold, Global Journal of Pure and Applied Mathematics, 2017(13), 4087-4097.
[31] Siddiqi, M. D., "Ricci ρ-soliton and geometrical structures in a dust fluid and viscous fluid spacetime," Bulgarian journal of Physics, vol. 46(2019), 163-173.
[32] Siddiqi, M. D. and Siddiqi, S. A., Conformal Ricci solitons and geometrical structure in a perfect fluid spacetime, Int. J. Geom. Methods in Mod. Phys., 2020, 17, 2050083.
[33] Siddiqi, M. D., De, U. C. and Deshmukh, S., Estimation of almost Ricci Yamabe solitons on static spacetimes, Filomat, 36(2022), 397-407.
[34] Siddiqi, M. D., De, U. C., Relativistic perfect fluid spacetime and Ricci-Yamabe soliton, Letters in Math. Phys., 112(2022), 1-12.
[35] Siddiqi, A. N. and Siddiqui, M. D., Almost Ricci-Bourguignon solitons and geometrical structures in a relativistic perfect fluid spacetime, Balkan J. Geom. Appl., 26(2021), 126-138.
[36] Udriste, C., Riemann flow and Riemann wave via bialternate product Riemannian metric, arXiv preprint arXiv:1112.4279 (2011).
[37] Yano, K and Kon, M., Structures on manifolds, Series in Pure Mathematics, Vol 3, World Scientific Publ, 1985.
[38] Zellovich, Y. B., The equation of state of ultrahigh densities and its relativistic limitations, Sov. Phy. J. Exp. Theor. Phys., 14(1962), 1143-1114.