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Symmetry analysis of ornaments in Serbian medieval frescoes art

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Abstract. The ornaments from the Serbian medieval frescoes belong to the religious decorative art with the restricted set of possible motifs, frequently related to a cross, and thus the basic motifs can fit in a very limited number of symmetry groups. The main criterion for the quality of such ornamental art could be the richness and variety of patterns obtained from a very small number of symmetry groups, proving the creativity of their authors - their ability to create variety with a very restricted number of initial symmetry groups. Here we analyze these ornaments and their symmetry group and give an introductions to further work on automated symmetry group recognition and pattern reconstruction.

1. Introduction

Ornament (lat. *Ornamentum* – decoration) is the oldest and basic decorative element in visual arts, closely connected to the concepts of repetition and symmetry. Artists and artisans of different epochs, cultures and civilizations used the repetition and combination of motifs for the creation and construction of different decorative patterns on bone, textile, ceramics, paintings, or jewelry... During the complete historical development of humanity, there existed unbreakable connections between geometry and art, where the visual presentation often served as the basis for geometrical consideration. This especially applies to the ornamental art, which is called by Hermann Weyl [35] ”the oldest aspect of higher mathematics given implicitly”. Relatively independent development of geometry and painting resulted in formation of two different languages, which are using completely different terms for describing symmetrical forms. The moment when painting and sculpture were differentiated from decorative arts, the period of a relative stagnation of ornamental art has started, which acquired a rather subordinate role and remained on the margins of the dominant aesthetics.

The meaning of the word ”symmetry”, which originates from the Greek science and philosophy, is the ”common measure” ($\sigma\upsilon\mu$ = ”common”, $\mu\epsilon\tau\rho\upsilon$ = ”measure”), directly pointing to one of the most important problems of Greek geometry: the question of proportionality (commensurability) of two line segments. Two line segments are commensurable if their common measure exists: a line segment (unit)

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contained in the first line segment p times, and in the other q times, where p and q are natural numbers. The proportion between these two segments is expressed by the fraction (rational number) $\frac{p}{q}$. For centuries, Greek philosophy, aesthetics, mathematics, physics, acoustics, and musical theory have been based on the assumption that every two line segments are commensurable (denied by the discovery of irrational numbers, e.g., $\sqrt{2}$, and that the edge and the diagonal of a square are incommensurable line segments), and on the theory of proportions.

Examples of symmetry in nature are present everywhere. We can say that "nature loves patterns". The reason for that is the universal principle of the economy in nature. One of its manifestations is the principle of modularity: the possibility to create diverse and variable structures, originating from some (finite and restricted) set of basic elements, by their recombination (see [12], [25]).

1.1. Symmetry and Ornaments in Art

In fine arts, the term "symmetry" preserved for centuries the meaning it had in Ancient Greek aesthetics: in its wider sense it indicated harmony, accord, regularity, while in the more narrow sense it was identified with mirror symmetry (reflection). Very often, even now-days, people presume symmetrical object as mirror symmetrical one.The connections between symmetry and aesthetics have strengthened and become deeper during the history, where the important place in the study of their mutual connection occupies the works of ornamental art. The oldest examples of ornamental art can already be found in the time of Paleolithic and Neolithic, when geometrical shapes, zigzag lines, circular and spiral forms were mostly used, drawn with relative precision, repeated in harmonic and rhythmic manner (for more see [12], [14]). During the development of the ornamental art, the spectrum of motifs used in design with naturalistic and stylized representations of plants and animals, was extended. The representative examples of such symmetry are different kinds of two-headed animals: the two-headed Mayan snake, the double-headed eagle, or similar heraldic motifs. Such an irrational symmetry can be interpreted as a human need for symmetry (in the visual and symbolic sense).

Gestalt psychology suggests that, at an unconscious level, the perception of visual stimuli involves generalizations, simplifications, and perceptual grouping by the viewer . This is particularly evident in the visual perception of ornaments, which "invite" the observer to discern the underlying rules of their construction. Research shows that viewers can recognize symmetrical objects in less than 1/20 of a second, especially those with vertical mirror symmetry(see [33]). These processes shape the aesthetic impact that ornamental art has on the observer.

The classification of ornaments according to their symmetries can help us to find answers to many questions: when and where a certain kind of symmetry appears in the ornamental art; which forms prevail; how to classify colored ornaments; how, when, where, and why man created ornaments at all... That kind of classification can also be used as the indicator of connections between different cultures. Except by a direct, physical transfer, ornaments can be transferred by making sketches, as well as by memorizing patterns, and then reproduced by remembering the symmetry of the original motif. In that process, many details are changed, or lost. The masters of ornamental art have varied patterns and changed the shapes of original motifs, but preserved the symmetry. Therefore, in many cases it is possible to follow the changes of some motifs in time and reconstruct their "path" and variations, from the initial version to the regarded state. The typical examples are "Swastika" or "Pawn feather" motifs, which represent some kind of ornamental archetypes.

In this paper we analize ornaments from The Catalog of the exhibition *Memory Update – Ornaments of Serbian Medieval Frescoes*, and book by Z. Janc, *Ornaments in the Serbian and Macedonian frescoes from the XII to the middle of the XV century* ([23], [11]). Z. Janc analyzed ornaments from history of art point of view, trough typology of ornaments (the ornaments composed of the zigzag lines and tapes; the ornaments composed of the undulated lines and tapes; the ornaments composed of circles; the egg shaped stick; the ornaments developed from the Kufic alphabet; palmettes; various kinds of leaves;fan-shaped ornaments; birds; branchlets and vine leaflets and the mixed group of ornaments - see [11]). The mathematical approach to analyzing the symmetry of ornaments is different.

1.2. Mathematical approach to ornaments

Dynamic progress of the mathematical theory of symmetry during XX centery caused that the first more significant incitement for the study of ornamental art came from mathematicians. In the appendix of his monograph about infinite groups, Andreas Speiser [28] proposed to use ornaments from Ancient world (like Egyptian ornaments) as the best possible illustration of symmetry groups. The approach to the classification and analysis of ornaments based on symmetries was enriched by the contributions of different authors (E.A. Müller, A.O. Shepard, N.V.Belov, D. Washburn, D. Crowe, B. Grünbaum...). In their works, the descriptive language was replaced by geometric-crystallographic terminology. The approach to the ornamental art from the point of view of the theory of symmetry offers the possibilities for the more profound study of the complete historical development of ornamental art, regularities and laws on which constructions of ornaments are based, as well as an efficient method for the classification, comparative analysis and reconstruction of ornaments.

The mathematicians usually working with the patterns from different sources and try to show the representative examples of the particular symmetry groups. Working with from patterns from some particular locations (e.g., Alhambra patterns) the main criterion of the richness of the material was the number of different symmetry groups that can be found in this particular location. The typical example of this was discussion between B. Grünbaum and several Spanish mathematicians (C. Ruiz, R. Perez-Gomez)about the number of different symmetry groups present in Alhambra. In his famous paper "The Emperor's New Clothes: Full Regalia, Gstring, or Nothing" Grünbaum showed that one can find only 13 from 17 symmetry groups of plane ornaments in Alhambra [6]. This is the same number of patterns recognized by E.A. Müller [24], the student of A. Speiser. The simmilar hold for Moorish ornaments, analyzed by B. Grünbaum [7].

In this study we work with the restricted material from Serbian medieval ornaments (in sense of symmetry groups). One of the reasons for restriction in reachness is the symbolical meaning of the ornaments: the ornaments from the Serbian medieval frescoes belong to the religious ornamental art with the very restricted set of possible motifs, mostly related to a cross, so such basic motifs can fit in a very limited number of symmetry groups, based on the rectangular, rhombic, or square grid. As the main criterion for the quality of such ornamental art could be the richness and variety of patterns obtained from a very small number of symmetry groups, proving the creativity of their authors – their ability to create variety with the very restricted number of initial symmetry groups. The other important property of these ornaments was that they were mostly copied from textile patterns – details of religious clothes.

Many of the analyzed ornaments are not geometrically precise and look more as the sketches of geometrical patterns, than as their exact realizations, inviting a viewer to recognize their symmetry and regularity and disregard details. As it was mentioned by Zagorka Janc [11], "all ornaments in our frescoes are painted very freely, almost imprecisely, usually with strong illusionistic tendencies". According to Vladimir Dvorniković [5], "after the first lost battles against the Turks, in the painting is a clearly visible new, rude, but strong natural stroke", and that "after the Turkish occupation, popular fresco-painters, stone-cutters and engravers appear, which transformed that old church art into a semi-popular art and craft..." The lack of precision can be explained as the intent of the fresco-painter to give a free interpretation of the patterns copied from textile (where the patterns are more precise and regular), and use them as a decorative part of frescoes, or parts of the space surrounding them. Such ornaments can be treated as approximately symmetric. Through our visual perception, we recognize them as symmetrical. In general, we perceive every approximately symmetrical object as symmetrical, by observing the whole (object) and avoiding the details [33].

1.3. Symbolic meaning of ornaments

The original meaning of ornaments was primarily symbolical, and then decorative, so it is possible to talk about the language of ornaments. This thesis confirm the names of certain ornaments preserved in the ethnic ornamental art. Some of names of the details from the Pirot kilims and the ornaments from this exhibition are known: "Upright man in the home of Good", or "Holy Trinity". In order to understand the symbolical meanings of the complex ornaments, we need to know the basics of the visual-symbolic language of their elementary building blocks – rosettes, which represent the ornamental archetypes ([25], [15]). Even the elementary symmetry structures and their corresponding geometrical figures, as the square, the circle, the cross, have a symbolic meaning. The most usual symbols in Serbian fresco-painting are:

The point: the primeval element, the beginning, and kernel; symbol of the number. It is the symbol of the beginning (grain of seed) and of the end (grain of dust); it represents the smallest substance (atom, nucleus). The point is in fact imaginary: it occupies no space.

The vertical line: the sign of life, health, activity, certainty, effective stability, manliness. It is the symbol of the spirit directed upward, of grandeur and loftiness, and of man running erect; it is the sign of the right and might.

The horizontal line: the polar opposite of the vertical, and a symbol of the earth, the passive, woman, death and rest; the material and the earthbound.

The cross: one of the oldest and most universal signs, uniting the polar contrasts of the vertical and horizontal, of God and the world, of the spiritual and the material, of life and death, of man and woman. It indicates the four points of the compass and the point of an intersection. After the Crucifixion it became a holy symbol in Christianity and was used in many variations (including the motif the double cross from the monastery Veluće). Except in the standard forms of the cross, the double cross can be found very often in fresco-paintings. It symbolizes the state independence represented by the scepter of the ruler, in the form of the double cross, with two horizontal lines symbolizing layman and church power. It can also be interpreted as the Crucifixion cross, which has two horizontal bars.

The circle: together with the square and triangle, one of the primeval signs, the motion with no beginning or an end, the sign of infinity, eternity, perfection, and God, Sun, Cosmos, Earth and Universe. The spiral: a very popular symbol in the ornamental art, dynamic symbol indicating that all life develops from a one point, symbol of the motion, of the rising Sun and growth. The double spiral: symbol of death and life, god and evil, a cosmic symbol...

The square: symbol of massiveness, limited area, standing firm on the ground. Symbolizes the four seasons, the four points of the compass, the four elements, the four rivers of Paradise...

The triangle: Symbol of the Trinity and, with a point in the middle, a sign of the all-seeing eye of God. A triangle standing with its base firmly planted on the ground and its point striving upwards has a womanly character, as opposed to a triangle balanced on its point with broad ”shoulders” above, which has a more manly character...

Hexagram: two triangles passing through each other create a symmetrical star. The hexagram is the magic sign of the preservation and protection against destruction, a very old Jewish sign, an emblem of the cosmos, the divine Creator, and His work...

For their descriptions and explanations see [27]. Deeper meaning in psychology are discussed in [22].

The visual form of the ornaments was harmonized with their (symbolic) meaning. Information about ”visual forces” (static and dynamic impression that some ornament produces, the existence of the ”left” and ”right” form, etc.) are contained in the symmetry group of an ornament, so the choice of a certain symmetry group and its frequency of occurrence depends from its visual/symbolical meaning. A typical example is the swastika symbol, which has two forms, ”left” and ”right”, each with different meaning: one of them is the prehistoric symbol of Sun and life, and the other the symbol associated to one of the darkest moments of the history: the time of fascism. Except from its symbolical meaning, the frequency of an occurrence of a certain symmetry group can be predicted, according to the criterion of the maximal visual and constructional simplicity.

2. Symmetry analyses of Ornaments

We are interested in the symmetric figures and plane patterns, remaining unchanged by the action of nontrivial transformations – symmetries. We distinguish three kinds of symmetric figures in a plane:

1. the figures which have preserved a single invariant (unchanged) point of the figure (the center) under the action of symmetries – the *rosettes* (circular patterns);

2. the figures without the invariant points, with an invariant direction, the translation axis – the *friezes* (linear patterns);
3. the patterns without invariant points, and with two invariant directions – the *plane ornaments*.

Instead of a descriptive classification of the patterns (their division into the zoomorphic, plant, floral, geometrical, etc.), we apply the geometrical classification: to each ornament we will assign the corresponding symmetry group (and its symbol) and divide the rosettes, friezes, and ornaments into a (finite) number of classes, using their symmetry groups as the classification criterion. There are two infinite classes of the rosettes: cyclic and dihedral. Every frieze belongs to one of the seven types, and every ornament belongs to one of the 17 types of plane ornaments. For more about discrete summetry groups of ornaments, their generatores and subgroups see for example [13].

2.1. Rosetts

In the ornamental art, the rosettes are decorative elements of a polygonal or circular shape. The symmetry transformations of the rosettes can be only transformations preserving the invariant point – the center of the rosette. Hence, such transformations could only be rotations (by an arbitrary angle, an integer part of a full circle), or mirror reflections with regards to the line - the axis containing the center of the rosette. A rotation by an angle which is the n^{th} part of the full circle is called the rotation of order n. A special case of this rotation is a half-turn, rotation by 180^o – central symmetry, the composition of two reflections with mutually perpendicular axes.

Rosettes can be divided into two infinite classes: the class of cyclic symmetry groups C_n, which contain only rotations, and class of dihedral symmetry groups D_n containing rotations and reflections. Since every rotation can be ″left″ or ″right″, the rosettes with a cyclic symmetry group can appear in two (enantiomorphic) forms: the ″left″ and the ″right″, introducing a possibility to suggest the motion, so cyclic groups produce the visual impression of dynamics and rotation (turn). On the other hand, thanks to the mirror reflections, dihedral groups have only one possible form. Typical geometrical figures with the dihedral symmetry group D_n are regular polygons. Usually, they are placed in such a way that one edge, the basis, coincides with the horizontal line, and the other with the vertical. Because of that, dihedral symmetry groups possess a specific balance between the dynamic visual component, caused by the rotations, and the static component produced by the reflections. It is interesting to notice that Leonardo was the first who recognized that exactly two kinds of rosettes exist: these with a cyclic and a dihedral symmetry group. The rosettes with a symmetry group D_1, based on a mirror reflection, prevail in the ornamental

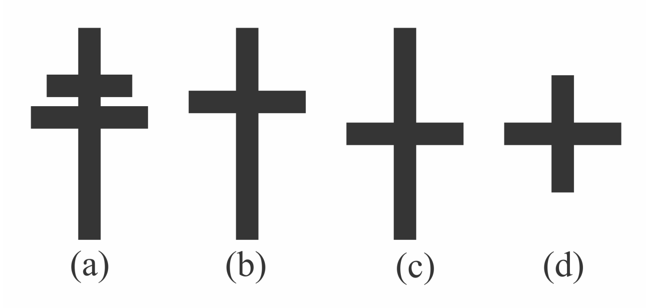


Figure 1: Different form of cross

art. The reflection axis is almost always vertical. Let us notice the difference between ″mathematical″ and ″visual″ objects: the ″mathematical″ objects live in the idealized mathematical space, where all directions are equivalent and where the plane is infinite in all directions. ″Visual″ objects live in the ″real″ space, the plane of drawing, where the vertical and horizontal line dominate over all the other directions, and where the plane is a finite drawing, only potentially infinite in the sense that can be extended in some direction (directions).

A typical symbol illustrating this difference is the sign of a cross. It expresses the fundamental geometrical property: the perpendicularity. However, a pair of perpendicular lines can be placed in any position, even in a slanting position with regard to the fundamental directions – vertical and horizontal. On the other hand, a cross, as a visual archetype, is almost always vertical. Depending from its construction, this symbol appears in three different symmetrical forms: D_1 (Fig.1a,b), D_2 (Fig.1c), and D_4 (Fig.1d).

Mirror symmetry and proportions of the human body have inspired many artworks. In art and architecture, we can find many examples of the drawings and architectural constructions, where the human body resembles some geometrical shape. Typical examples are the ways how humans are buried in the Neolithic time, with the body forming a shape of a triangle, or "Vitruvian man" by Leonardo. In the Serbian fresco-painting, we can recognize characteristic examples of floor plans, where the shapes of the figures of saints perfectly fit in the geometry of the floor plan.

Knowing that rosettes are geometrical figures with an invariant point, the center of the rosette, the only questions we need to pose in order to make a distinction between symmetry groups C_n and D_n are: what is the order of rotations, and are there reflections. If the order of rotations is n, the answer to the other question makes the difference between the groups C_n and D_n.

Figure 2: Table 1.

The superposition of the identical rosettes (Monastery Petkovica, Fig.2a; Monastery Petkovica, Fig.2b) can also result in a symmetrization: by composing two rosettes with the symmetry D_4, we obtain an octagram with the symmetry D_8. We discus superposition and symmetrization in next section.

Different examples of the rosettes can be found in the catalogue of this exhibition. The cyclic symmetry groups of rosettes are not used so often: C_2 (Holy Mother of Ljeviš, Fig.2c; rosette in the middle) and C_4 (Monastery Staro Nagoričino, lower rosette, Fig.2f). Dihedral rosettes are much more frequent: D_1 (Holy Mother of Ljeviš, Fig.2i); D_2 (Monastery Holy Mother of Slimnica, Fig.2g; Monastery Sopoćani, Fig.2j); D_4 (Church of St. Panteleimon, Nerezi, Fig.2k); D_8 (Church of St. Panteleimon, Nerezi, Fig.2h). Depending on taking details in account or not, the examples Fig.2d (Monastery Sopoćani) and Fig.2e (Monastery Krušedol) can give different results: the rosette Fig.2d can be recognized as D_2, but details does not follow this symmetry precise enough, so if they are taken in account the result will be assymetrical rosette C_1. In the similar way, the rosette Fig.2e aproximatelly satisfies the symmetry D_2, but if all details are taken in account, its symmetry will be C_2. There are some representative examples of the groups D_{12} (see Fig.3) taken from the catalogue by Zagorka Janc ([11], Table XXXIX, 242 and 243).

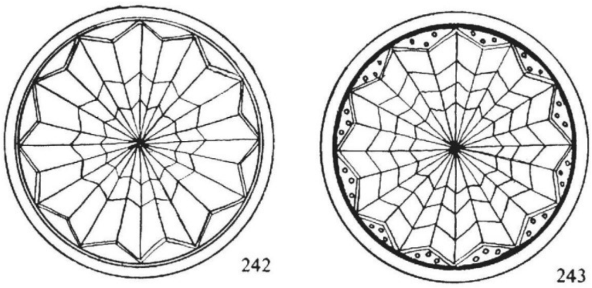


Figure 3: Table XXXIX, 242 and 243 from [11].

2.2. *Friezes*

The next classes of the plane patterns are the linear patterns - *friezes*: figures with an invariant direction – the axis of translation. In the friezes, beside translations, reflections and glide reflections, only rotations of order 2 – half-turns appear as symmetries, preserving invariant the axis of a frieze. Exactly seven combinations of the mentioned symmetries exist. This means that every frieze belongs to one of the seven possible schematic patterns, i.e., symmetry groups.

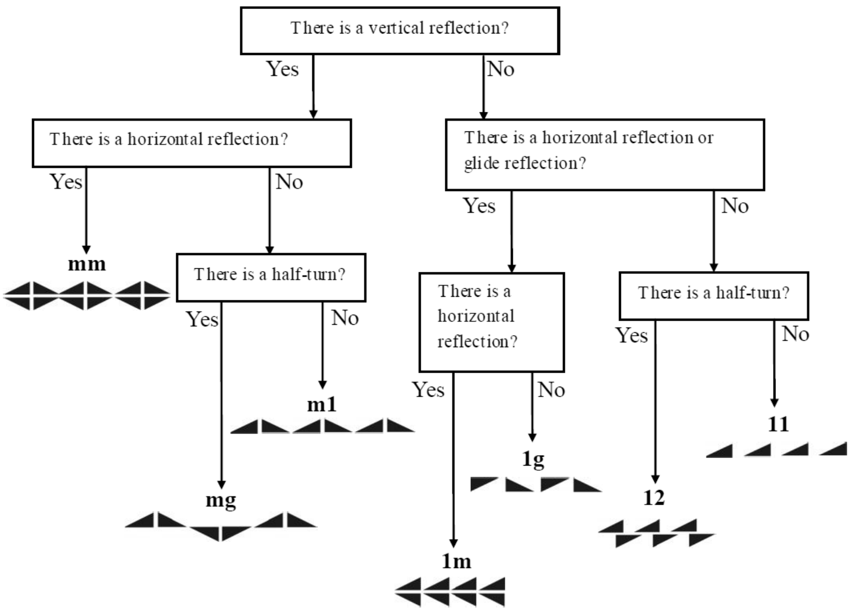


Figure 4: Flow charts for the recognition of friezes.

Every symmetry group of friezes can be denoted by the symbol, which consists of the elementary symbols: **1** – trivial rotation of order 1, **g** – glide reflection, **2** – rotation of order 2 (half-turn), and **m** – mirror reflection. As the result we obtain the notation for seven symmetry groups of friezes: **11**, **1g**, **12**, **m1**, **1m**, **mg**, and **mm**. In this notation, the first symbol (**1** or **m**) represents symmetry (trivial rotation of the order 1 or mirror reflection **m**), perpendicular to the translation axis, and the second the element of symmetry parallel to the translation axis (rotation of the order 1 or 2, reflection, or glide reflection).

Friezes are usually horizontal. If the axis is polar (none of the symmetry transformations from the symmetry group of a frieze changes the direction of the axis), it is possible to suggest an oriented motion. If some symmetry group has no indirect (sense reversing) transformations, the corresponding patterns appear in two forms: the "left" and the "right", so there is the enantiomorphism. In the case of friezes, this means that only friezes **11** and **12** have the "left" and the "right" form, since they have no indirect transformations: mirror reflections, or glide reflections. By using vertical oriented friezes, it is possible to suggest an upward motion. In the case of the "real" and the "mathematical" friezes, we have the same problem as in the case of rosettes: the "mathematical" friezes are infinite symmetrical linear patterns. The "real" friezes are finite, usually placed along an axis coinciding with one of the fundamental natural directions - vertical or horizontal line, which have different visual-symbolical meaning. Oriented friezes with the slanted axis are very rare. For the recognition of friezes we can use the flow charts (Fig.4 taken from [34]).

Every "real" frieze can be extended to infinity in two directions: by continuing the frieze along its axis, and by making parallel copies of the original frieze. The extension in the other direction as the result always gives some plane ornament. In a visual sense, friezes are linear patterns that represent homogeneous visual entities, with repetitions occurring in only one direction (translation) and without any 'visual forces' that tend to extend them in other directions, typically perpendicular to the axis. In many illustrations, plane ornaments are represented only by their linear parts, which, according to their visual logic, need to be extended in both directions. In contrast, some friezes are 'well-behaved', self-contained forms that can logically be extended in only one direction. Therefore, we can conclude that a linear structure with translation in only one direction does not necessarily represent a frieze. Such examples can be found in the monograph by Zagorka Janc [11], where many of the ornaments are shown by their cut linear parts - friezes.

Figure 5: Table 2.

In fresco painting of Medieval Serbia it is possible to find all seven types of friezes. They are represented as the decorative borders of priest's garments, and as decorations of pillars, arcs, doors, windows... Important classes of friezes are those, derived by a simple multiplication of a rosette. Although these rosettes can be of types D_3, C_4, D_4 and so on, the corresponding friezes cannot have rotations of order greater than 2.

For example, a translational repetition of rosettes of type C_4 results in frieze pattern **12**, while a translational repetition of rosettes of type D_4 results in frieze pattern **mm**, and so on. For example, despite that the frieze Fig.5a (Monastery Dečani) contains rosettes D_4, symmetry of the frieze is **mm** (rotations of order 4 cannot be the elements of symmetry groups of friezes). The other possibility is to consider this frieze as a part of the ornament with the symmetry group **p4m**, but in the visual sense this frieze represents a homogenous visual entity, without needing the introduction of new translations perpendicular to the frieze axis.

Here are some examples of different symmetry groups of friezes from Serbian Medieval Frescoes art: **m1** (Church if St. George, Kurbinovo, Fig.5b); **mm** (Monastery Dečani, Fig.5a; Monastery Sopoćani, Fig.5c), usually created by repeating a motif with the symmetry D_4; **mg** (Monastery Kalenić, Fig.5f and Fig.5d), which is the composition of different friezes multiplied by the translation perpendicular to the frieze axis; **1g** (Monastery Bela Crkva Karanska, Fig.5e), the symmetry group corresponding to a large number of friezes constructed by using zigzag or meander lines; **12** (Monastery Sopoćani, Fig.5g).

2.3. Ornaments

The most important classes of the plane schemes are plane ornaments – infinite plane patterns without invariant points or lines, which have two translations in different directions. Those two translations make so-called translational subgroup of symmetry groups of ornaments or wallpaper groups. There are exactly 17 different symmetry groups of ornaments, meaning that all plane ornaments can be constructed (in the sense of symmetry) in exactly 17 different ways, and divided (in the sense of symmetry classification) in 17 classes (Fig.8).

From pencils of parallel lines in the plane we obtain plane lattices, that can be used as the basis for the construction of all ornaments. Depending from the mutual position of such two pencils (the angle between lines belonging to different directions), and from distances between adjacent lines, we obtain parallelogramic, rhombic, rectangular, square and hexagonal grid (Fig.6). Three pencils of parallel lines generate a triangular lattice, from which we can extract a hexagonal grid. Notice that by joining the mid-points of adjacent edges of a parallelogramic grid, or by constructing its diagonals, we obtain a rhombic lattice. If a grid consists from the regular polygons, as the result we have the regular grids (i.e., tessellations – complete edge-to-edge tiling of a plane, without gaps or overlaps): square, triangular and hexagonal regular tessellation.

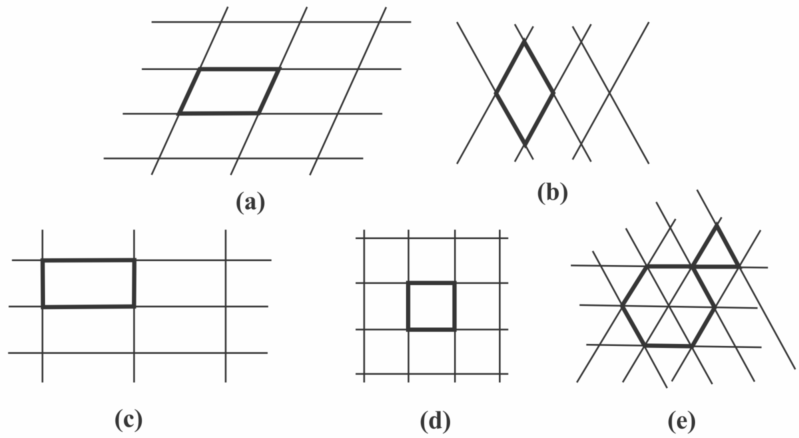


Figure 6: Plane grids.

Symmetry groups of plane ornaments can contain all kinds of symmetries. There is a special restriction (called crystallographic restrictions) for rotations: permitted order of rotations in plane ornaments is 1, 2, 3, 4, and 6. This restriction means that if in an ornament, as local symmetry, appears rotation of another order, e.g. 5, it cannot be the global symmetry of the ornament. A simple proof can is given by Stevens in

[29]. For example, despite that in the ornament Holy Mother of Ljeviš, Fig.71, as local symmetry appears rotation of the order 8, the symmetry of this ornament is **p4m**, because the order of rotations in symmetry group of ornaments can be only 1, 2, 3, 4, or 6.

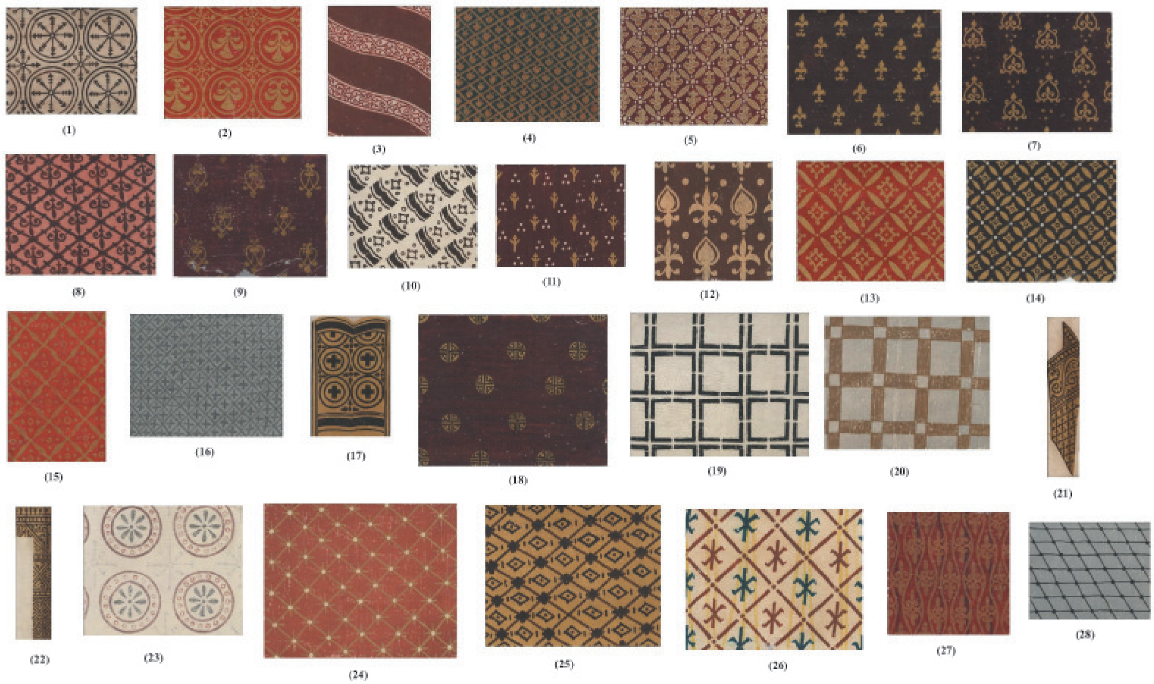
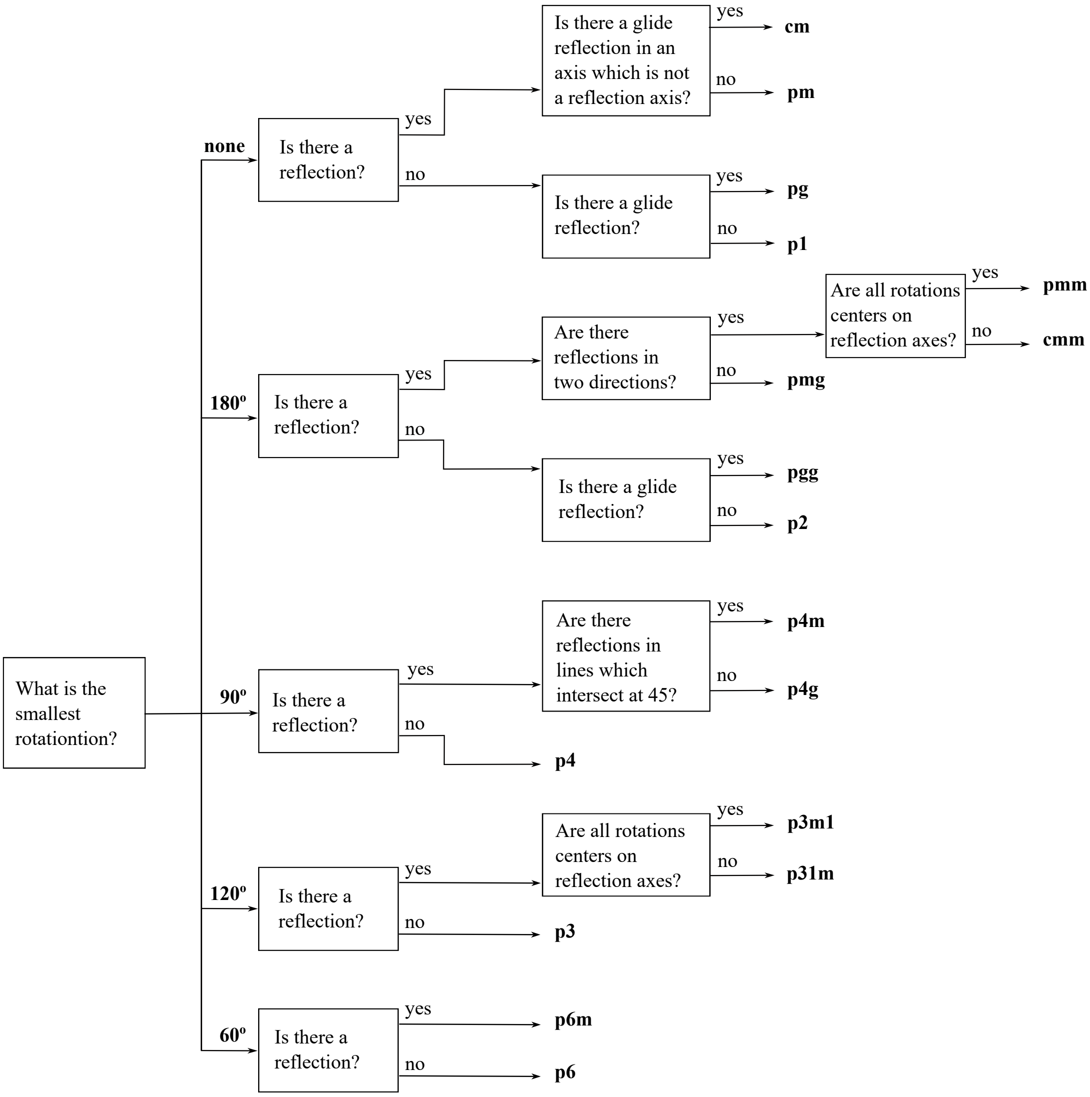


Figure 7: Table 3.

Each of the 17 plane symmetry groups of ornaments we have denoted by the corresponding symbol. In the concise crystallographic symbols of the symmetry groups of ornaments, the first coordinate is the translational part of the symmetry group: parallelogramic lattice **p**, or rhombic lattice **c**; the second symbol represents a reflection **m** perpendicular to the first translation axis or rotation, and the third symbol represents reflection **m** or glide reflection **g**. Such notation is chosen to completely describe each symmetry group in the simplest possible way. The flow charts for the recognition of plane symmetry groups are much more complicated than the flow charts for friezes (Fig.8).

In the catalogue, we can find many examples of symmetry groups of ornaments. The most frequent are: **pm** (Holy Mother of Ljeviš, Fig.7.1; Monastery Kalenić, Fig.7.2), mostly obtained by the multiplication of friezes; **cm** (Monastery Gračanica: Fig.7.3; Fig.7.4, Fig.7.5; Fig.7.6; Monastery Ljubostinja, Fig.7.7; Monastery Resava (Manasija), Fig.7.8; Monastery Novo Hopovo, Fig.7.9; Monastery Sopoćani, Fig.7.10; Monastery Studenica, Church of Holy Mother Evergetides, Fig.7.11); **p4m** (Holy Mother of Ljeviš: Fig.7.12; Fig.7.13; Monastery Dečani, Fig.7.14; Monastery Gračanica:Fig.7.15; Fig.7.16; Monastery Sopoćani, Fig.7.17; Monastery Novo Hopovo: Fig.7.18; Fig.7.19; Church of St. Panteleimon, Nerezi: Fig.7.20; Fig.7.21; Monastery Gradac, Fig.7.22; Monastery Novo Hopovo, Fig.7.23); **cmm** (Holy Mother of Ljeviš, Fig.7.24; Monastery Studenica, Church of Holy Mother Evergetides, Fig.7.25; Lesnovo, Fig.7.26; Monastery Novo Hopovo: Fig.7.27; Fig.7.28; Church of Saint Petar and Paul, Ras, Fig.9.1); **p4g** (Church of St. Sofia, Ohrid, Fig.9.2); Church of St. Panteleimon, Nerezi, Fig.9.3); **p4** (Monastery Zaum (Ohrid Lakeshore), Church of Holy Mother, Fig.9.4); **pmg** (Monastery Gračanica, Fig.9.5). Some of them are parts of the superposition (compound) ornaments, i.e., superposition of symmetry groups, or subgroups in the antisymmetry groups. However, we have not succeeded to find the examples of groups **p6**, **pgg**, and **p31m**.



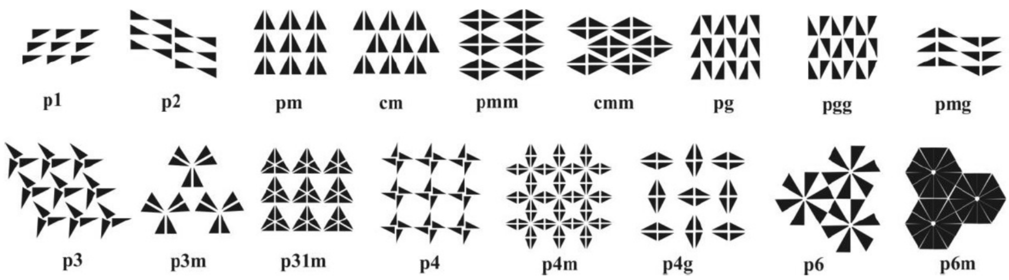


Figure 8: Flow charts for the recognition of ornaments [34].

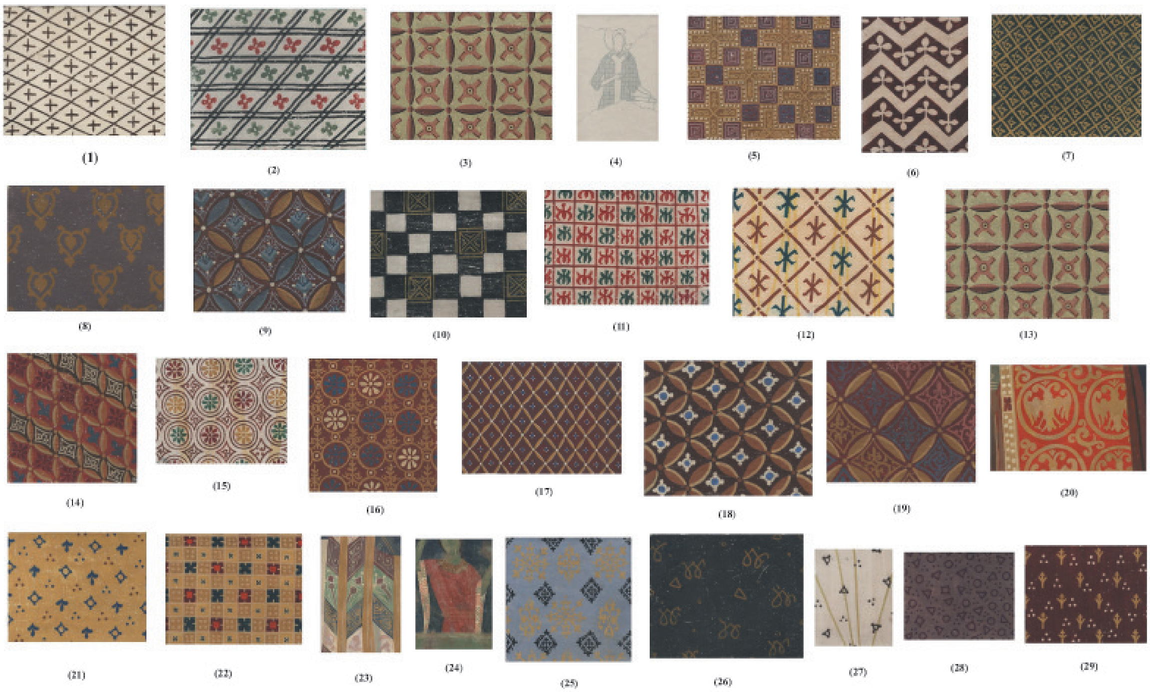


Figure 9: Table 4.

3. Colored Symmetry of Ornaments

3.1. *Symmetrization and desymmetrization*

The problem of recognition of ornaments is very closely related to the question: how to construct symmetrical patterns? There are two approaches to solve this problem: we can start from a local symmetry, and obtain global symmetry, or directly use global symmetry and then, if necessary, reduce it to some lower level (degree) of symmetry (e.g., by using subgroups of the given symmetry group). The first approach can be called *symmetrization*, and the other *desymmetrization*.

In the first case we start from some asymmetric figure, surrounded by its symmetrically arranged copies and continue this algorithm. This is the simplest way to construct ornaments: the symmetrization. In order to construct ornaments, we can start from the corresponding fundamental region of some symmetry group - the smallest asymmetric part of the plane that can be multiplied by symmetries from some group, resulting in a complete plane tiling. In order to avoid overlapping of asymmetric figures forming some pattern, we can choose an asymmetric figure, completely belonging to the fundamental region, and then multiply it by symmetries from the symmetry group. To obtain complex structures, we usually use some intermediate structures derived from the basic ones. Instead to start from an asymmetrical basic motif (figure) and multiply it by using transformations from some symmetry group, we can start from some already derived non-elementary symmetrical structure, rosette or frieze, and multiply it by some symmetry transformations (usually, translations) in order to obtain symmetry groups of ornaments. The hierarchy: rosettes → friezes → ornaments corresponds to the series of symmetry group extensions.

The other possibility is to start from the global symmetry. For example, we are not going to construct a square grid by drawing it square-by-square. Instead we will draw a set of parallel, mutually perpendicular horizontal and vertical equidistant lines to create the square grid. By placing some asymmetrical motif in every of the squares (respecting regularity), we obtain the subgroup of the preceding symmetry group of the square grid. We initially started with the symmetry group of the square grid, denoted by **p4m**, and eventually ended up with a symmetry group containing only translations in two directions, denoted by **p1** (see Fig.9.7, Holy Mother of Ljeviš). If we use a symmetrical figure, some rosette, e.g., a double-headed eagle with the symmetry group D_1 instead of an asymmetrical figure, as the result we obtain the ornament **pm**, *etc*. This method, frequently used in ornamental art, appears in many frescoes featuring a rhombic lattice with the symmetry group **cmm**. In this lattice, the rhombuses are filled with rosettes exhibiting D_1 symmetry, resulting in the symmetry group **cm** (see Fig.9.8, Gračanica)

In the ornament Fig.9.5 (Monastery Zaum, Ohrid Lakeshore, Church of Holy Mother), the dominating geometrical structure (based on crosses) is the symmetry group **p4m**. The decorations in the form of the letter *S* are placed in the centers of the rotations of order 2 (half-turns). The visual dynamics is introduced by (non-consequently used) decorations covering the sides of the crosses, disturbing reflections, and suggesting the rotations of order 4. Considering this ornament together with decorations, its symmetry reduces to **p4**.

3.2. *Antisymmetry and Colored symmetry*

Similar desymmetrization can be achieved by (regular) black-white colorings or colorings with more than two colors. These kind of desymmetrizations, and the corresponding symmetry groups, are called antisymmetry and colored symmetry groups.

One of the favorite methods of desymmetrizations in ornamental art is the use of black-white coloring, i.e., the antisymmetry. The oldest antisymmetric patterns can be found already in the Neolithic ornamental art, in two-colored ceramics. Through the history, antisymmetry served as a very useful tool for presenting dualism, internal dynamics, alternating, space component (by suggesting the relations ”in front-behind”, ”over-under”), and a specific kind of balance: the unity of opposites (complements), making a whole.

In the Neolithic ornamental art and ornamental art of the Pirot kilims (Fig.10), we can recognize a higher level of the antisymetry organization: perfect two-colored plane tessellations, where the figure (a black part) is congruent to the ground (a white part), covering together plane without gaps or overlaps. Those kind of black-white or colored ornaments, with zoomorphic motifs, where the bordering regions are of different colors, are created by M.C. Escher. Such geometrical ornaments can be found in the Neolithic ornamental art. In that sense, the patterns from kilims can be considered as the ornamental treasury, preserving some more ancient knowledge, dating even from the prehistory. Some details from Pirot kilims, constructed in the algorithmic way (e.g., the ornament ”Soveljka” = ”Shuttle”, Fig.10c), can be considered as the examples of iterative black-white systems, in the form of a rosette with a fractal alternating structure of the figure and ground.

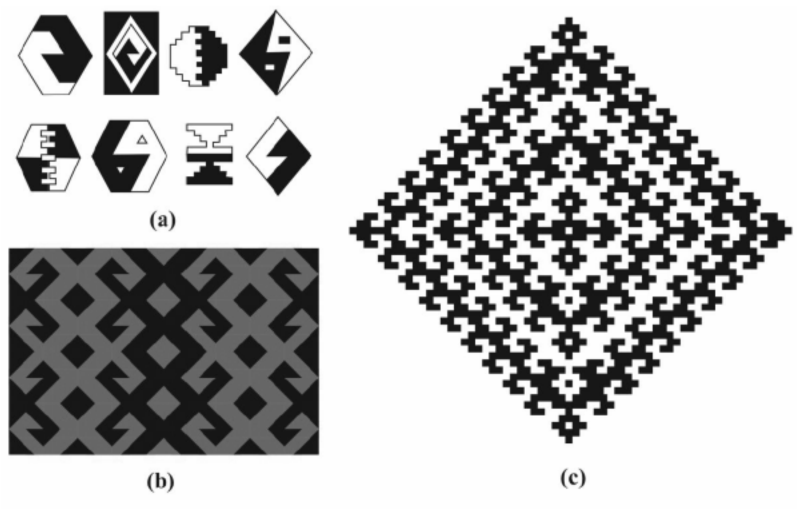


Figure 10: (a) ”Yin-Yang” type symbols; (b) modular reconstruction of a Pirot kilim; (c) pattern ”Soveljka”.

In order to construct some (black-white) antisymmetric pattern, we start from an uncolored ornament and its symmetry group. Then we choose some its subgroup (a pattern placed two times in the initial pattern), and color elements of the each of these two sets by different colors (usually, black and white). Every antisymmetry group (containing together with symmetry transformations also color-changing symmetries), can be completely described by knowing the symmetry group of the original uncolored pattern and its subgroup of index 2 (consisting from symmetries that preserve the color - black or white). In other words, in some black-white ornament we ignore colors, recognize the symmetry group of the basic (uncolored) pattern, then select the figures of the same color (black or white), and recognize the symmetry group of such one-colored pattern. In the notation for antisymmetry groups, we write the symbols of the initial symmetry group and its subgroup of index 2, separated by the slash.

This approach can be extended to colored symmetry groups, where instead of only two colors, we use more colors. Same as before, first we consider the symmetry group of the uncolored pattern. After that, we recognize the following two subgroups: the subgroup consisting from figures of the same color, and the symmetry group of the completely colored pattern. The first symmetry group and these two subgroups, written in this order, completely describe the colored symmetry group. If the two subgroups coincide, we use only one symbol. The history of the two-color and more hihly colored patterns is well described in the book [8].

In Serbian medieval fresco-painting very interesting examples of colored ornaments can be found. Here are some examples from the catalog of the exhibition [23]. As an example of antisymmetry, fresco-painting from Marko's Monastery (Fig.9.9) can be used. If we restrict our consideration to the antisymmetry group, consisting from yellow and blue leafs, the result is the antisymmetry group of the ”checkerboard” pattern **p4m/p4m**.

The fresco-painting from Monastery Peć Patriarchate, Church of Holy Apostoli (Fig.9.10) represents an excellent way for distinguishing crosses placed in an ordinary checkerboard pattern **p4m/p4m**. Interesting paraphrase on ”checkerboard” theme is the pattern Fig.9.11 (Monastery Sopoćani), where the antisymmetry group is **pm/cm**, thanks to details placed in the squares of the checkerboard. This group is distinguished by the use of complementary colors: red and green. The similar variation of the same theme is the ornament Fig.9.12 (Monastery Studenica, Church of Holy Mother Evergetides), with the antisymmetry group **pm/cm**.

The fresco-painting from Church of St. Sofia, Ohrid (Fig.9.13) is the interesting combination of symmetry and antisymmetry, where the dominating antisymmetry ornament **p4m/p4** (in which the system of small circles is not consequent in the sense of antisymmetry) is combined with the centered ornament **p4**.

Colored symmetry cannot be reduced only to a color change: it can be also considered as the change of some geometrical property. If the change of figures (details) is treated as a form of colored symmetry, the ornament from Monastery Lesnovo (Fig.9.14) is the example of 3-colored symmetry group **p1/p1**, where the index of the subgroup is 3 (because three kind of figures are used). The initial geometrical pattern corresponding to this example is ”Pawn feather”, with the symmetry group **cm**.

The uncolored ornament Fig.9.15 (Monastery Lesnovo), where some details are neglected, is **p4m**. The colored symmetry group, obtained by its coloring, is **p4m/p1**.

The ornament Fig.9.16 (Church if St. George, Kurbinovo) is the example of so-called *proportional colored symmetry*, where the colors are used in some proportion (in this case 1:2). The proportional colored symmetry is used in many cases, when some colors are more intensive than the others. The color balance is achieved in this way, that more intensive colors are less used (in a given proportion).

Antisymetry and colored symmetry can be used to represent the 3D space in a plane. By using the relation ”above-below”, i.e., the ”black-white”, the plane becomes two-sided. This can be illustrated by some fresco-paintings from this exhibition: some of them are interlacing symmetrical patterns, the images of the corresponding 3D structures. Such ornaments originated from basketry, matting, and plaiting.

Interesting example of the form that can be considered as a 2D or 3D is frieze representing meanders in space (such as ”Greek key” meander from Sopoćani). Some of these ornaments remind us to the floor mosaics from Antioch, based on the 3D space illusionism. This especially applies to the 3D paraphrases of meander ornaments, based on so-called ”Greek key”. The oldest examples of the 2D version of this meander can be found in the Paleolithic art, among ornaments from Mezin (23 000 B.C.). Ornaments of that kind are particularly interesting, and make some kind of labyrinth structure. Many of such examples can be found among the fresco-paintings from Hilandar Monastery, but they were not included in Catalog [23].

Some artists of the XX century (e.g., A. Kitaoka) used the small color changes of certain details (”underlining of edges”), in order to create different visual effects (e.g., 3D illusions, or illusions of motion). Medieval artist produced the illusion of the 3D space by highlighting one set of the edges of the pattern (Fig.9.17 and Fig.9.18 from Monastery Gradac). If we disregard the light edges, this ornament becomes the representation of the 3D symmetry group of layers. It represents the plane ornament **p4m**, extended by a mirror reflection (in the plane to which it belongs), i.e., a two-sided plane ornament. Considered as a plane ornament consisting only from the light contours, making a ”Pawn feather” pattern (despite from its ”square” structure), this is the ornament **cm**.

The geometrical basis of ornament Fig.9.19 (Monastery Lesnovo), consisting from the light lines is a plane covered by congruent parts (monohedral tessellation), with the symmetry group **p4g**. However, in order to obtain 3D effect, these light lines are added to the basic ornament **p4**.

Except the antisymmetry and colored symmetry, providing the possibilities for a ”dimensional transition”, the other possibility for the desymmatrization is the superposition of ornaments. The result is the intersection of superimposed symmetry groups, i.e., the set of symmetries belonging to the both symmetry groups. Sometimes is very hard to determine which symmetry group is the ”main”, and which is the ”secondary”. The typical example is the textile ornament Fig.9.20 (Church of Saint Peter and Paul, Bijelo Polje), with the motif of double-headed eagle. If we neglect only central asymmetrical decorations, its symmetry group is **pm**, same as before. The basic motif is not a half of the double-headed eagle, but the whole double-headed eagle with its appertaining decorations. However, notice the asymmetrical details placed between four circles with double-headed eagles. They completely break the symmetry of this ornament and reduce it to the translational symmetry group **p1**. Certainly, in the visual sense, our eye and brain disregards details (”noise”) and perceives the whole (”gestalt”), so the symmetry we recognize is **pm**. It corresponds to the first case, where all decorative details are neglected.

A similar example is the complex ornament from the Monastery Kalenić (Fig.9.21), the superposition of three ornaments. The first ornament **cm** consists from trefoils, the second **p4m** from circles, and the third is the antisymmetry ornament made from the two-colored point systems. As the final result, the symmetry group **p1** consisting only from translations is obtained. The reflections which coincide with the axes of trefoils are eliminated by coloring of point triplets.

Many ornaments from this exhibition are composed of different ornaments, joined by the common visual dominant: square grid (Holy Mother of Ljeviš, Fig.9.22). Some of them are compositions of various friezes, representing a small ”encyclopedia of friezes” composed in the basic pattern with the symmetry group **pmg** (Monastery Kalenić, Fig.9.23). The similar ”encyclopedia” is represented in the fresco-painting (Monastery Peć Patriarchate, Church of Holy Mother Hodegetria, Fig.9.24).

Almost all of the examples from the Monastery Resava (Manasija) are the superposition of ornaments. For example, the ornament Fig.9.25 is the superposition of three ornaments: **p4m** consisting from black rosettes D_4, **cmm** consisting from the golden rosettes D_4, and **cmm** consisting from the golden rosettes D_2. Also, superposition of ornaments can be found among the ornaments from Monastery Sopoćani. By analyzing the material, we can conclude that the most of ornaments are complex (composite) ornaments – superposition of ornaments.

As one of the important achievements of ornamental art, we can emphasize interlaced ornaments. They often occur in the miniature painting, or illuminated manuscripts [31].

The proposed method for the recognition and classification of ornaments can be applied to the ornaments in order to get answers to the following questions: which ornaments (and their corresponding symmetry groups) are friezes, and which are only linear parts of the plane ornaments; which ornaments cannot be completely analyzed, if treated as two-dimensional; which ornaments are modular structures; which symmetry groups are in common for the certain cities/monasteries/regions/territories; can we follow their appearance chronologically, or connect them with particular masters/schools; can we follow the variations of ornaments based on the same symmetry group; can we establish relations with the ornamental art belonging to the classical ornamental tradition (Egyptian, Islamic, Byzantine).

From the foregoing, it is possible to get the impression that the proposed method for the analysis and classification of ornaments offers complete answers to all (or almost all) mentioned questions. However, we have to underline that the application of this method to the real ornamental material is relatively limited. This is especially true for the material from [23], where many patterns are superposition of ornaments. They are complex ornaments, containing several different ornaments simultaneously used on the different ”levels” of the composition (as the basis of construction, dominating ornament in the sense of visual perception, or the set of details that are ruining the exact symmetry). Unlike books dedicated to the symmetry of ornaments, which contain the selection of characteristic examples taken from different sources in order to illustrate forms of the symmetry of ornaments, the work with the real material is much more demanding. It shows that it is necessary to consider every ornament from several points of view, not only in the sense of symmetry (by describing all ”levels” of symmetry in the hierarchic order, beginning from the visually dominating component), but to use all other methods for the classification of ornaments, including the descriptive ones.

We have listed only some of the representative examples and possible answers to the above mentioned questions.Besides the number of the different symmetry groups that are present in some ornamental art, diversity of ornaments based on the same or similar themes serves as the criterion of its richness. Every symmetry group of ornaments offers an infinite number of possibilities for creating various designs. In the ornamental art of fresco-painting presented in this exhibition, most frequent are ornaments with maximal symmetry groups generated by reflections, which does not offer so much freedom. One of the reasons for this restriction is the religious function of presented ornaments, demanding the permanent use of symbols like a cross, trefoil, lily, double-headed eagle... Therefore, the square lattice **p4m** ihas been used very often, as the basis for paraphrasing different motifs on the theme of the cross – rosette D_4. Because of its constructional simplicity, the square grid is the most frequently used lattice in the entire ornamental art (Fig.11).

Figure 11: Ornaments in a square grid.

Also, they use the rhombic grid **cmm**, with the longer diagonal often placed vertically. The rhombic grid is usually derived from the rectangular grid by constructing diagonals, and combined with the mirror-symmetrical, or asymmetrical rosettes placed along the vertical, longer diagonal, suggesting the upward tendency, ascending (Fig.12).

We don’t know much about the visual sensibility of the medieval people, but abstract composition (Monastery Novo Hopovo, Fig.9.26) meets all the characteristics of the good modern design: freedom, simplicity and ”well-behaved form”. Visual perception of an abstract motif (as a visual entity on the level of perception, and not on the level of ”styles”, ”art epochs”, ”or ”spirit of time”) probably remained

Figure 12: Ornaments in a rhombic grid.

unchanged from the prehistory. Figurative art had contained a ”story” (contents): by looking a painted flower we are trying to recover (aesthetic, emotional) feelings that the author wanted to express while painting the image of the real, beautiful flower. By looking an abstract motif, we are trying to recover the feeling of ”universal harmony” (balance), that the author (maybe) felt when creating it.

From the point of view of the modern design, some fresco-paintings (such as Monastery Peć Patriarchate, Church of Holy Mother Hodegetria, Fig.9.28; Monastery Kalenić, Fig.9.29) represent a visual language, expressed with the minimum of visual tools, similar to the works of Paul Klee. The ornament from Monastery Novo Hopovo (Fig.7.19), based on the motif of the cross, produces the associations to the geometrical abstraction of the XX century. Many ornaments are based on the motif of the cross or its superposition of two ornaments **cm**, both using the motif of the cross, but in different form. These ornaments, so as many others, represent brilliant examples how to intelligently use the same motif of the cross, and avoid monotony and repetition.

4. Automated symmetry group recognition

Beside chart diagram and visual recognition of symmetry group, it is possible to use computer graphic and neural networks for automated symmetry group recognition. Early works in the field of symmetry identification began in the 1980s and 1990s, where researchers explored fundamental mathematical approaches to recognizing symmetry, such as Fourier and Hough transformations ([9, 21]). The focus initially was on lines and basic geometric figures. Subsequently, a theory of multiscale, curvature-based shape representation was proposed for images of varying quality and resolution [20], which has been successfully applied alongside methods for symmetry analysis. In 1999, the SIFT algorithm was introduced [18], becoming a standard in the detection of symmetric structures in images. A revolution in image analysis arrived with the advent of Convolutional Neural Networks (CNN), which enabled the automatic learning of complex features from large datasets. Special architectures for analyzing symmetry [1, 16, 36] and Deep Symmetric Networks (DSN) were developed during this period, specialized for recognizing symmetric transformations in images [4].

Recent advancements typically involve the application of some AI model. Graph Neural Networks (GNN) have become popular in recent years for recognizing complex symmetric structures, namely detecting symmetry not just in geometric shapes, but in complex structures such as repeating architectural elements (towers, windows, etc.) [2]. The introduction of attention mechanisms in deep learning has enabled models to focus on parts of the image where symmetry is likely to appear, significantly speeding up and improving detection accuracy [32].The introduction of Generative Adversarial Networks (GANs) has opened new possibilities for generating symmetrical images and training models to identify symmetries in complex and real-world scenes [37].

Numerous ornamental images, particularly those on historical buildings, artworks, or cultural artifacts, have suffered damage due to a range of factors, including weathering, air pollution, moisture, material degradation, biological factors, and even inadequate preservation efforts. These factors have led to significant deterioration or total loss of many ornamental details from the past, necessitating their digital reconstruction (image inpainting) or restoration. Given their generative capacity and ability to produce highly realistic results, especially when working with complex textures and detailed structures, which are critical in ornamentation and symmetrical designs – GANs and their advanced variants (Cycle–GAN, StyleGAN2 [17], BigGAN [3]) have proven to be highly effective for training models to learn symmetrical patterns from data and generate the corresponding missing elements. Their architecture facilitates adaptation to specific stylistic features, which is essential for accurately restoring symmetrical details in complex ornamental images.

The algorithms of symmetry recognition are included in AI models and are able to recognize symmetry group. We have used GPT 4.0 for symmetry group analyses and we were able to get correct underlying symmetry group for almost all examples. There were problems with examples with a lot of noises. Directions of our ongoing research is focused on training models to more effectively recognize and generate symmetrical patterns. By leveraging Conditional GANs (cGANs [10], [38]), we can incorporate symmetry labels to guide the generation of images that adhere to specific symmetry types. Additionally, models such as CycleGAN and StyleGAN could be utilized to restore damaged sections of ornamental images by learning from the remaining symmetrical structures. This would enable more precise digital reconstruction and restoration of historical ornamentation, as well as the creation of new designs based on learned symmetry patterns.

5. Conclusion

This is the very first study on Symmetry of Ornaments from Serbia from mathematical point of view (symmetry groups). Ornaments from the Serbian medieval frescoes respect relatively strong given rules coming from religious-symbolical function of ornaments. They are an impressive testimony that it is possible to create variability and richness in spite of that rules.

Compositions (superposition) are remarkable mixtures of different ornaments, perfectly balanced in the sense of the form and colors. Although it is based on the strict rules and most of them are copies from the textile, the Serbian medieval ornamental art characterizes a high degree of freedom: almost none of the ornaments are literal copies of the originals. In most of them we can recognize the lack of precision and imperfections, which are mainly the result of neglecting details, and not some kind of errors.

The ”variations on a given theme” remind us of the ”well-behaved forms” (term used by R. Arnheim to describe the characteristics of an abstract painting) and they are realized with a minimum of tools, as systems of points, or geometrical figures. Therefore, many of these ornaments look ”modern”: they are completely harmonized with the visual sensibility of a contemporary viewer, and represent an integral ”visual field” full of strong visual energy.

The results presented in this paper are comparable to studies from other cultures (see [7, 24]). Certain symmetry groups are generally rare in ornamental design, especially those derived from textiles or with religious themes. The ”variations on a given theme” contributed to the richness of symmetry groups found in ornaments. Combined with historical studies, this research on ornaments from Serbian medieval fresco art provides an excellent foundation for further work in this field.

6. Acknowledgement

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