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Initial and terminal value problem for fractional differential equations of variable order

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Abstract. An initial and terminal value problem for fractional differential equations of variable order is introduced. The existence and uniqueness properties are analyzed based on the fixed point theorems of Schaefer and Banach. The results obtained are supported by approximate numerical examples.

1. Introduction

The idea of fractional-order integration and differentiation goes back to sixteenth century, but it wasn't until the 19th century that Augustin-Louis Cauchy and Liouville made significant advances, so that, the theory of fractional derivatives and integrals was formalized [16, 21, 24]. Since then, fractional calculus has been used in a variety of fields. It is used in engineering to simulate intricate systems incorporating electrical circuits, control theory, and viscoelasticity. It is essential to the description of processes in physics like diffusion, wave propagation, and fractional quantum mechanics. Fractional calculus also provides useful tools for deciphering non-Markovian processes and irregular data patterns in biology, finance, and signal processing. Fractional calculus is an essential component of contemporary mathematics and applied sciences due to its versatility [1–3, 19, 20, 33, 37]. In the recent years, there have seen a huge increase in the number of research publications that examine various qualitative aspects of differential equations while also involving various fractional operators, see the papers [4–6, 23].

Variable-order differentiation and integration are a logical progression from their counterpart in constant order. In this situation, the order can continuously change depending on dependent or independent variables of differentiation or integration. The mentioned extension of the order is more flexible than the conventional fractional order and is a natural progression [8, 29, 31, 32]. These notions have been

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successfully used to represent complicated real-world problems in a variety of fields, including biology, mechanics, control theory, and transport systems. This is due to the capability of developing evolutionary governing equations. Due to this widespread area of applications, the scientific community has been actively researching variable order fractional applications to the modeling of engineering and physical systems, see for instance [28, 30].

Fractional differential equations of variable order are a flexible expansion of the traditional fractional calculus, wherein the degree of differentiation or integration is variable in relation to the independent variable or, other parameters. The notion of variable-order fractional derivatives has its origins in the early 20th century and has since garnered considerable interest in contemporary times owing to its wide-ranging applicability across several fields [17, 34].

The utilization of variable-order fractional differential equations encompasses a wide range of applications. The equations utilized in the field of physics are employed to elucidate the characteristics of materials that exhibit dynamic features, such as porous media or viscoelastic materials. In the field of biology, computational models are employed to simulate and analyze many biological phenomena, such as the distribution of drugs through tissues or the activity of neurons. Variable-order fractional differential equations are employed by economists and finance experts for the purpose of modeling intricate market dynamics and asset pricing. Furthermore, control engineers employ these methodologies to analyze and regulate systems exhibiting diverse dynamics, thereby enhancing the precision and efficiency of control procedures. In general, variable-order fractional differential equations provide a robust foundation for improving modeling and analysis in various fields, rendering them a subject of ongoing research and practical implementation in modern scientific and technological progress [35, 36].

Recent research in this area has been particularly performed by many researchers who focused on the study of the existence, uniqueness, and stability of solutions to many different problems of fractional differential equations of variable-order under different conditions [10, 15, 22] . The measure of noncompactness technique, the upper-lower solutions method, and the fixed point theory are the foundations upon which all of the above-mentioned results are proved. Further, the stability of the proposed problems in the sense of Ulam-Hyers or Ulam-Hyers-Rassias was under observation [9, 11, 12, 14]. It is important to note that the investigation relies heavily on the concept of piece-wise constant function which plays a crucial role. For this purpose, the interval of existence $[0, L]$ has been divided into subintervals via the partition $P := \{I_1 = [0, L_1], I_2 = (L_1, L_2], I_3 = (L_2, L_3], \ldots, I_n = (L_{n-1}, L]\}$, where n is a given natural number. Further, the piecewise constant function $\chi(\zeta) : [0, L] \to (1, 2]$ with respect to P, is defined as

$$\chi(\zeta) = \sum_{k=1}^{n} \chi_k I_k(\zeta),\ \zeta \in [0, L],$$

where $1 < \chi_k \leq 2, k = 1, 2, \ldots, n$ are constants. Here $L_0 = 0$ and $L_n = L$, that is, $I_k = 1$ for $\zeta \in [L_{k-1}, L_k]$, and $I_k = 0$ elsewhere. The majority of the aforementioned results are obtained using this approach, which first divides the existence interval into subintervals and then defines the differential and integral operators with respect to those subintervals. Using this technique, researchers were able to convert the fractional problems of variable-order into their equivalent conventional fractional problems of constant order.

Agarwal *et al.* [7] studied the following constant fractional order problem

$$\begin{cases} \mathbb{D}_{0^+}^{\chi} \mu(\zeta) = \eta(\zeta, \mu(\zeta)),\ \ \zeta \in [0, \infty),\ \ \chi \in]1, 2], \\ \mu(0) = 0,\ \ \mu \textit{ bounded on } [0, \infty), \end{cases}$$

where $\mathbb{D}_{0^+}^{\chi}$ stand for the Riemann-Liouville fractional derivative of order χ, respectively, η is a given continuous function.

In this paper, we introduce a novel approach to replace the use of the piecewise constant function and existence interval splitting. The creation of a new operator that is more adaptable and does not need any additional phases is the keystone of our strategy. We apply the new technique on the following initial and terminal value problem (ITVP) of variable order

$$\begin{cases} \mathbb{D}_{0^+}^{\chi(\zeta)} \mu(\zeta) = \eta(\zeta, \mu(\zeta)),\ \ \zeta \in A := [0, L], \\ \mu(0) = 0,\ \ \mu(L) = 0, \end{cases} \tag{1}$$

where $0 < L < +\infty$, $1 < \chi(\zeta) < 2$, $\eta : A \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\mathbb{D}_{0^+}^{\chi(\zeta)}$ is the Riemann-Liouville fractional derivative of variable-order $\chi(\zeta)$.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.
Note that the set $\mathbb{E} = C(A, \mathbb{R})$ is a Banach space of continuous functions μ from A into $\mathbb{R}$, such that, $\mu(0) = \mu(L) = 0$ with a norm defined as

$$\|\mu\| = \sup_{\zeta \in A} |\mu(\zeta)|.$$

Definition 2.1. *([25, 26, 30]) Let $\chi : A \to (1, 2)$ be a continuous function. The left Riemann-Liouville fractional integral of variable order $\chi(\zeta)$ for function $\mu(\zeta)$ is defined by*

$$\mathbb{I}_{0^+}^{\chi(\zeta)} \mu(\zeta) = \int_0^\zeta \frac{(\zeta - \varpi)^{\chi(\varpi)-1}}{\Gamma(\chi(\varpi))} \mu(\varpi) d\varpi, \ \zeta > 0, \tag{2}$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.2. *([25, 26, 30]) Let $\chi : A \to (1, 2)$ be a continuous function. The left Riemann-Liouville fractional derivative of variable order $\chi(\zeta)$ for function $\mu(\zeta)$ is defined by*

$$\mathbb{D}_{0^+}^{\chi(\zeta)} \mu(\zeta) = \Big(\frac{d}{dt}\Big)^2 \mathbb{I}_{0^+}^{2-\chi(\zeta)} \mu(\zeta) = \Big(\frac{d}{dt}\Big)^2 \int_0^\zeta \frac{(\zeta - \varpi)^{1-\chi(\varpi)}}{\Gamma(2 - \chi(\varpi))} \mu(\varpi) d\varpi, \ \ \zeta > 0. \tag{3}$$

Remark 2.3. *([13]) For general functions $\chi(\zeta)$, $v(\zeta)$, we notice that the semi group property does not hold, i. e:*

$$\mathbb{I}_{a^+}^{\chi(\zeta)} \mathbb{I}_{a^+}^{v(\zeta)} \mu(\zeta) \neq \mathbb{I}_{a^+}^{\chi(\zeta)+v(\zeta)} \mu(\zeta).$$

Lemma 2.4. *([37]) Let $\chi : A \to (1, 2)$ be a continuous function. Then for $y \in C_\sigma(A, \mathbb{R}) = \{y(\zeta) \in C(A, \mathbb{R}),\ \zeta^\sigma y(\zeta) \in C(A, \mathbb{R}),\ (0 < \sigma < 1)\}$, the variable order fractional integral $\mathbb{I}_{0^+}^{\chi(\zeta)} y(\zeta)$ exists for $\zeta \in A$.*

Lemma 2.5. *([37]) Let $\chi \in C(A, (1, 2))$ be a continuous function. Then $\mathbb{I}_{0^+}^{\chi(\zeta)} y(\zeta) \in C(A, \mathbb{R})$ for $y \in C(A, \mathbb{R})$.*

Theorem 2.6. *([27]) Suppose $\mathfrak{I}$ is a Banach space. If $\wp : \mathfrak{I} \to \mathfrak{I}$ is a completely continuous operator and $\ominus = \{\mu \in \mathfrak{I} : \mu = \lambda \wp \mu, 0 < \lambda < 1\}$ is bounded, then $\wp$ has a fixed point in $\mathfrak{I}$.*

Theorem 2.7. *([18]) Let $\mathfrak{I}$ be a Banach space and $\wp : \mathfrak{I} \to \mathfrak{I}$ be a mapping such that, $\wp^n$ is a contraction, for some $n \in N$. Then $\wp$ has a unique fixed point in $\mathfrak{I}$.*

3. Existence criteria

We start by introducing the following assumptions.

(AS1) There exist constants $0 < \sigma < 1$, $p > 0$, such that,
$\zeta^\sigma \mid \eta(\zeta, \mu(\zeta)) - \eta(\zeta, y(\zeta)) \mid \leq p \mid \mu(\zeta) - y(\zeta)|, \quad \forall\ \mu, y \in R, \zeta \in A.$

(AS2) $\chi : A \to (1, \chi^*]$ is a continuous function, such that, $1 < \chi^* < 2$.

Remark 3.1. 1. *The function* $\Gamma(2\text{-}\chi(\zeta))$ *is continuous as a composition of two continuous function. We set* $M_\Gamma = \max_{\zeta\in A} \mid \frac{1}{\Gamma(2-\chi(\zeta))} \mid$.

2. *By the continuity of the function* $\chi(\zeta)$*, we let*
$L^{1-\chi(\zeta)} \leq 1$ *if* $1 \leq L < \infty$, $L^{1-\chi(\zeta)} \leq L^{1-\chi^*}$ *if* $0 \leq L \leq 1$.
We conclude that $L^{1-\chi(\zeta)} \leq \max(1, L^{1-\chi^*}) = L^*$.

We will need the following lemma about the solution of the ITVP (1).

Lemma 3.2. *The ITVP (1) is equivalent to the integral equation*

$$\int_0^\zeta \frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ d\varpi = \int_0^\zeta (\zeta-\varpi)\eta(\varpi,\mu(\varpi))\ d\varpi$$

$$+\frac{\zeta}{L}\int_0^L \frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ d\varpi - \frac{\zeta}{L}\int_0^L (L-\varpi)\eta(\varpi,\mu(\varpi))\ d\varpi, \tag{4}$$

such that, $\mu(0) = \mu(L) = 0$ *holds.*

Proof. By the definition of fractional derivative of variable order given in (3), the ITVP (1) can be written in the form:

$$\frac{d^2}{d\zeta^2}\int_0^\zeta \frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ d\varpi = \eta(\zeta,\mu(\zeta)).$$

Then,

$$\frac{d}{d\zeta}\int_0^\zeta \frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ d\varpi = \int_0^\zeta \eta(\varpi,\mu(\varpi))\ d\varpi + c_1.$$

Thus,

$$\int_0^\zeta \frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ d\varpi = \int_0^\zeta (\zeta-\varpi)\eta(\varpi,\mu(\varpi))\ d\varpi + c_1\zeta + c_2. \tag{5}$$

Evaluating equation (5) at $\zeta = 0$ and $\zeta = L$ gives us $c_2 = 0$ and

$$c_1 = \frac{1}{L}\Big[\int_0^L \frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ d\varpi - \int_0^L (L-\varpi)\eta(\varpi,\mu(\varpi))\ d\varpi\Big].$$

Then,

$$\int_0^\zeta \frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ d\varpi = \int_0^\zeta (\zeta-\varpi)\eta(\varpi,\mu(\varpi))\ d\varpi + \frac{\zeta}{L}\int_0^L \frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ d\varpi$$

$$-\frac{\zeta}{L}\int_0^L (L-\varpi)\eta(\varpi,\mu(\varpi))\ d\varpi.$$

Conversely, by taking the derivative of both sides of the equation (4), we have

$$\frac{d}{d\zeta}\Big(\int_0^\zeta \frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)d\varpi\Big) = \int_0^\zeta \eta(\varpi,\mu(\varpi))\ d\varpi$$

$$+\frac{1}{L}\Big(\int_0^L \frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ d\varpi - \int_0^L (L-\varpi)\eta(\varpi,\mu(\varpi))\ d\varpi\Big).$$

Taking the derivative again, we get

$$\frac{d^2}{d\zeta^2}\Big(\int_0^\zeta \frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)d\varpi\Big) = \eta(\zeta,\mu(\zeta)),$$

which gives the ITVP (1). □

The first result is based on Theorem 2.6.

Theorem 3.3. *Assume that conditions the (AS1) and (AS2) hold. Then the ITVP (1) has at least one solution on $\mathbb{E}$.*

Proof. We construct the following operator

$$C : \mathbb{E} \to \mathbb{E},$$

as follows,

$$C\mu(\zeta) = \mu(\zeta) - \int_0^\zeta \frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ d\varpi + \int_0^\zeta (\zeta-\varpi)\eta(\varpi,\mu(\varpi))\ d\varpi$$
$$+\frac{\zeta}{L}\int_0^L \frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ d\varpi - \frac{\zeta}{L}\int_0^L (L-\varpi)\eta(\varpi,\mu(\varpi))\ d\varpi.$$

Set

$$E_r = \{\mu \in \mathbb{E}, \|\mu\| < r, r > 0\}.$$

Clearly, E_r is non empty, closed and convex subset of $\mathbb{E}$.
Now, we will prove that the operator C satisfies the hypothesis of Theorem 2.6.
Step 1: C is continuous.
We presume that the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to μ in $\mathbb{E}$. Then, we have

$$| C\mu_n(\zeta) - C\mu(\zeta) |$$

$$\begin{aligned}
\leq\ & | \mu_n(\zeta) - \mu(\zeta) | + \int_0^\zeta \frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} | \mu(\varpi) - \mu_n(\varpi) | d\varpi \\
+\ & \int_0^\zeta (\zeta-\varpi)\varpi^{-\sigma}\varpi^{\sigma} | \eta(\varpi,\mu_n(\varpi)) - \eta(\varpi,\mu(\varpi)) | d\varpi \\
+\ & \int_0^L \frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} | \mu_n(\varpi) - \mu(\varpi) | d\varpi \\
+\ & \int_0^L (L-\varpi)\varpi^{-\sigma}\varpi^{\sigma} | \eta(\varpi,\mu(\varpi)) - \eta(\varpi,\mu_n(\varpi)) | d\varpi \\
\leq\ & \| \mu_n - \mu \| + M_\Gamma \| \mu_n - \mu \| \int_0^\zeta (\zeta-\varpi)^{1-\chi(\varpi)} d\varpi \\
+\ & p \| \mu_n - \mu \| \int_0^\zeta (\zeta-\varpi)\varpi^{-\sigma} d\varpi + M_\Gamma \| \mu_n - \mu \| \int_0^L (L-\varpi)^{1-\chi(\varpi)} d\varpi \\
+\ & p \| \mu_n - \mu \| \int_0^L (L-\varpi)\varpi^{-\sigma} d\varpi \\
\leq\ & \| \mu_n - \mu \| + M_\Gamma L^* \| \mu_n - \mu \| \int_0^\zeta \Big(\frac{\zeta-\varpi}{L}\Big)^{1-\chi^*} d\varpi + p \| \mu_n - \mu \| \frac{\zeta^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
+\ & M_\Gamma L^* \| \mu_n - \mu \| \int_0^L \Big(\frac{L-\varpi}{L}\Big)^{1-\chi^*} d\varpi + p \| \mu_n - \mu \| \frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
\leq\ & \| \mu_n - \mu \| + \frac{M_\Gamma L^*}{L^{1-\chi^*}} \frac{(\zeta)^{2-\chi^*}}{(2-\chi^*)} \| \mu_n - \mu \| + 2p \| \mu_n - \mu \| \frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
+\ & \frac{M_\Gamma L^*}{L^{1-\chi^*}} \frac{(L)^{2-\chi^*}}{2-\chi^*} \| \mu_n - \mu \| \\
\leq\ & \| \mu_n - \mu \| + \frac{M_\Gamma L L^*}{2-\chi^*} \| \mu_n - \mu \| + 2p \| \mu_n - \mu \| \frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
+\ & \frac{M_\Gamma L L^*}{2-\chi^*} \| \mu_n - \mu \|
\end{aligned}$$

$$
\begin{aligned}
&\leq \ \| \mu_n - \mu \| + \frac{2M_\Gamma L L^*}{2-\chi^*} \| \mu_n - \mu \| + 2p \| \mu_n - \mu \| \frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&\leq \ \Big(1 + \frac{2M_\Gamma L L^*}{2-\chi^*} + 2p \frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}\Big) \| \mu_n - \mu \|,
\end{aligned}
$$

which implies that,

$$\| C\mu_n(\zeta) - C\mu(\zeta) \| \to 0, as \ n \to \infty.$$

The above relation shows that the operator C is continuous on $\mathbb{E}$.
Step 2: C maps bounded sets into bounded sets in $\mathbb{E}$.
Let $\eta^* = \sup_{\zeta \in A} \mid \eta(\zeta, 0) \mid$. Then, for $\mu \in E_r$, we have

$$\mid C\mu(\zeta) \mid$$

$$
\begin{aligned}
&\leq \ \mid \mu(\zeta) \mid + \int_0^\zeta \frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} \mid \mu(\varpi) \mid d\varpi + \int_0^\zeta (\zeta-\varpi) \mid \eta(\varpi, \mu(\varpi)) \mid d\varpi \\
&+ \ \frac{\zeta}{L} \int_0^L \frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} \mid \mu(\varpi) \mid d\varpi + \frac{\zeta}{L} \int_0^L (L-\varpi) \mid \eta(\varpi, \mu(\varpi)) \mid d\varpi \\
&\leq \ \mid \mu(\zeta) \mid + \int_0^\zeta \frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} \mid \mu(\varpi) \mid d\varpi + \int_0^\zeta (\zeta-\varpi) \mid \eta(\varpi, \mu(\varpi)) - \eta(\varpi, 0) \mid d\varpi \\
&+ \ \int_0^\zeta (\zeta-\varpi) \mid \eta(\varpi, 0) \mid d\varpi + \frac{\zeta}{L} \int_0^L \frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} \mid \mu(\varpi) \mid d\varpi \\
&+ \ \frac{\zeta}{L} \int_0^L (L-\varpi) \mid \eta(\varpi, \mu(\varpi)) - \eta(\varpi, 0) \mid d\varpi + \frac{\zeta}{L} \int_0^L (L-\varpi) \mid \eta(\varpi, 0) \mid d\varpi \\
&\leq \ \mid \mu(\zeta) \mid + M_\Gamma L^* \int_0^\zeta \Big(\frac{\zeta-\varpi}{L}\Big)^{1-\chi^*} \mid \mu(\varpi) \mid d\varpi \\
&+ \ \int_0^\zeta (\zeta-\varpi) \varpi^{-\sigma} \varpi^{\sigma} \mid \eta(\varpi, \mu(\varpi)) - \eta(\varpi, 0) \mid d\varpi \\
&+ \ \int_0^\zeta (\zeta-\varpi) \mid \eta(\varpi, 0) \mid d\varpi + M_\Gamma L^* \int_0^L \Big(\frac{L-\varpi}{L}\Big)^{1-\chi^*} \mid \mu(\varpi) \mid d\varpi \\
&+ \ \int_0^L \varpi^{-\sigma} \varpi^{\sigma} (L-\varpi) \mid \eta(\varpi, \mu(\varpi)) - \eta(\varpi, 0) \mid d\varpi + \int_0^L (L-\varpi) \mid \eta(\varpi, 0) \mid d\varpi \\
&\leq \ \| \mu \| + \frac{M_\Gamma L^*}{L^{1-\chi^*}} \frac{L^{2-\chi^*}}{2-\chi^*} \| \mu \| + \int_0^\zeta (\zeta-\varpi) \varpi^{-\sigma} p \mid \mu(\varpi) \mid d\varpi + \int_0^\zeta (\zeta-\varpi) \eta^* d\varpi \\
&+ \ \frac{M_\Gamma L^*}{L^{1-\chi^*}} \frac{L^{2-\chi^*}}{2-\chi^*} \| \mu \| + \int_0^L (L-\varpi) \varpi^{-\sigma} p \mid \mu(\varpi) \mid d\varpi + \int_0^L (L-\varpi) \eta^* d\varpi \\
&\leq \ \| \mu \| + \frac{2M_\Gamma L L^*}{2-\chi^*} \| \mu \| + p \| \mu \| \int_0^\zeta (\zeta-\varpi) \varpi^{-\sigma} d\varpi + \eta^* \frac{\zeta^2}{2} \\
&+ \ p \| \mu \| \int_0^L (L-\varpi) \varpi^{-\sigma} d\varpi + \eta^* \frac{L^2}{2} \\
&\leq \ \| \mu \| + \frac{2M_\Gamma L L^*}{2-\chi^*} \| \mu \| + p \| \mu \| \frac{\zeta^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} + \eta^* L^2 + p \| \mu \| \frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \; \| \mu \| + \frac{2M_\Gamma L L^*}{2-\chi^*} \| \mu \| + 2p \| \mu \| \frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} + \eta^* L^2 \\
&\leq \; \Big[1 + \frac{2M_\Gamma L L^*}{2-\chi^*} + 2p \frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}\Big] \| \mu \| + \eta^* L^2,
\end{aligned}
$$

which implies that,

$$
\| C\mu \| \leq \Big[1 + \frac{2M_\Gamma L L^*}{2-\chi^*} + 2p \frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}\Big] r + \eta^* L^2.
$$

Hence, $C(E_r)$ is uniformly bounded.
Step 3: C maps bounded sets into equicontinuous sets in $\mathbb{E}$.
Firstly, we can remark that the function $w_\chi(\varpi) = \left(\frac{\zeta_1-\varpi}{L}\right)^{1-\chi(\varpi)} - \left(\frac{\zeta_2-\varpi}{L}\right)^{1-\chi(\varpi)}$ is decreasing with respect to its exponent $1-\chi(\varpi)$, for $0 < \frac{\zeta_1-\varpi}{L} < \frac{\zeta_2-\varpi}{L} < 1$. Then, for $\zeta_1, \zeta_2 \in A$, $\zeta_1 < \zeta_2$ and $\mu \in E_r$, we have

$$
| C\mu(\zeta_2) - C\mu(\zeta_1) |
$$

$$
\begin{aligned}
&\leq \; | \mu(\zeta_2) - \mu(\zeta_1) | + \Big| \int_0^{\zeta_2} \frac{(\zeta_2-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} \mu(\varpi) d\varpi - \int_0^{\zeta_1} \frac{(\zeta_1-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} \mu(\varpi) d\varpi \Big| \\
&+ \; \Big| \int_0^{\zeta_2} (\zeta_2-\varpi)\eta(\varpi, \mu(\varpi)) d\varpi - \int_0^{\zeta_1} (\zeta_1-\varpi)\eta(\varpi, \mu(\varpi)) d\varpi \Big| \\
&+ \; \Big| \frac{\zeta_2}{L} \int_0^{L} \frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} \mu(\varpi) d\varpi - \frac{\zeta_1}{L} \int_0^{L} \frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} \mu(\varpi) d\varpi \Big| \\
&+ \; \Big| \frac{\zeta_2}{L} \int_0^{L} (L-\varpi)\eta(\varpi, \mu(\varpi)) d\varpi - \frac{\zeta_1}{L} \int_0^{L} (L-\varpi)\eta(\varpi, \mu(\varpi)) d\varpi \Big| \\
&\leq \; | \mu(\zeta_2) - \mu(\zeta_1) | + \int_0^{\zeta_1} \Big| \frac{1}{\Gamma(2-\chi(\varpi))} \Big\| (\zeta_2-\varpi)^{1-\chi(\varpi)} - (\zeta_1-\varpi)^{1-\chi(\varpi)} \Big\| \mu(\varpi) \Big| d\varpi \\
&+ \; \Big| \int_{\zeta_1}^{\zeta_2} \frac{(\zeta_2-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} \mu(\varpi) d\varpi \Big| + \Big| \int_0^{\zeta_1} (\zeta_2-\varpi)\eta(\varpi, \mu(\varpi)) - (\zeta_1-\varpi)\eta(\varpi, \mu(\varpi)) d\varpi \Big| \\
&+ \; \Big| \int_{\zeta_1}^{\zeta_2} (\zeta_2-\varpi)\eta(\varpi, \mu(\varpi)) d\varpi \Big| + \Big| \frac{1}{L} \int_0^{L} \frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} \mu(\varpi) d\varpi \big[\zeta_2 - \zeta_1\big] \Big| \\
&+ \; \Big| \frac{1}{L} \int_0^{L} (L-\varpi)\eta(\varpi, \mu(\varpi)) d\varpi \big[\zeta_2 - \zeta_1\big] \Big| \\
&\leq \; | \mu(\zeta_2) - \mu(\zeta_1) | + M_\Gamma \| \mu \| \int_0^{\zeta_1} \Big[(\zeta_1-\varpi)^{1-\chi(\varpi)} - (\zeta_2-\varpi)^{1-\chi(\varpi)}\Big] d\varpi \\
&+ \; \int_{\zeta_1}^{\zeta_2} \frac{(\zeta_2-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} | \mu(\varpi) | d\varpi + \int_0^{\zeta_1} \Big[(\zeta_2-\varpi) - (\zeta_1-\varpi)\Big] | \eta(\varpi, \mu(\varpi)) | d\varpi \\
&+ \; \int_{\zeta_1}^{\zeta_2} (\zeta_2-\varpi) | \eta(\varpi, \mu(\varpi)) | d\varpi + \frac{\zeta_2-\zeta_1}{L} \int_0^{L} \frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} | \mu(\varpi) | d\varpi \\
&+ \; \frac{\zeta_2-\zeta_1}{L} \int_0^{L} (L-\varpi) | \eta(\varpi, \mu(\varpi)) | d\varpi
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & |\ \mu(\zeta_2)-\mu(\zeta_1)\ |+M_\Gamma \parallel \mu \parallel \int_0^{\zeta_1} L^{1-\chi(\varpi)}\Big[\Big(\frac{\zeta_1-\varpi}{L}\Big)^{1-\chi(\varpi)}-\Big(\frac{\zeta_2-\varpi}{L}\Big)^{1-\chi(\varpi)}\Big]d\varpi\\
+\ & M_\Gamma \parallel \mu \parallel L^* \int_{\zeta_1}^{\zeta_2}\Big(\frac{\zeta_2-\varpi}{L}\Big)^{1-\chi^*}d\varpi\\
+\ & \int_0^{\zeta_1}\Big[(\zeta_2-\varpi)-(\zeta_1-\varpi)\Big]\ |\ \eta(\varpi,\mu(\varpi))-\eta(\varpi,0)+\eta(\varpi,0)\ |\ d\varpi\\
+\ & \int_{\zeta_1}^{\zeta_2}(\zeta_2-\varpi)\ |\ \eta(\varpi,\mu(\varpi))-\eta(\varpi,0)+\eta(\varpi,0)\ |\ d\varpi\\
+\ & \frac{M_\Gamma}{L}(\zeta_2-\zeta_1) \parallel \mu \parallel L^* \int_0^{L}\Big(\frac{L-\varpi}{L}\Big)^{1-\chi^*}d\varpi\\
+\ & \frac{\zeta_2-\zeta_1}{L}\int_0^{L}(L-\varpi)\ |\ \eta(\varpi,\mu(\varpi))-\eta(\varpi,0)+\eta(\varpi,0)\ |\ d\varpi\\
\leq\ & |\ \mu(\zeta_2)-\mu(\zeta_1)\ |+M_\Gamma \parallel \mu \parallel L^* \int_0^{\zeta_1}\Big[\Big(\frac{\zeta_1-\varpi}{L}\Big)^{1-\chi^*}-\Big(\frac{\zeta_2-\varpi}{L}\Big)^{1-\chi^*}\Big]d\varpi\\
+\ & M_\Gamma \parallel \mu \parallel L^* \int_{\zeta_1}^{\zeta_2}\Big(\frac{\zeta_2-\varpi}{L}\Big)^{1-\chi^*}d\varpi\\
+\ & \int_0^{\zeta_1}\Big[(\zeta_2-\varpi)-(\zeta_1-\varpi)\Big]\varpi^{-\sigma}\varpi^{\sigma}\ |\ \eta(\varpi,\mu(\varpi))-\eta(\varpi,0)\ |\ d\varpi\\
+\ & \int_0^{\zeta_1}\Big[(\zeta_2-\varpi)-(\zeta_1-\varpi)\Big]\ |\ \eta(\varpi,0)\ |\ d\varpi\\
+\ & \int_{\zeta_1}^{\zeta_2}(\zeta_2-\varpi)\varpi^{-\sigma}\varpi^{\sigma}\ |\ \eta(\varpi,\mu(\varpi))-\eta(\varpi,0)\ |\ d\varpi+\int_{\zeta_1}^{\zeta_2}(\zeta_2-\varpi)\ |\ \eta(\varpi,0)\ |\ d\varpi\\
+\ & \frac{M_\Gamma L^*}{2-\chi^*} \parallel \mu \parallel (\zeta_2-\zeta_1)+\frac{\zeta_2-\zeta_1}{L}\int_0^{L}(L-\varpi)\varpi^{-\sigma}\varpi^{\sigma}\ |\ \eta(\varpi,\mu(\varpi))-\eta(\varpi,0)\ |\ d\varpi\\
+\ & \frac{\zeta_2-\zeta_1}{L}\int_0^{L}(L-\varpi)\ |\ \eta(\varpi,0)\ |\ d\varpi\\
\leq\ & |\ \mu(\zeta_2)-\mu(\zeta_1)\ |+\Big[\frac{M_\Gamma \parallel \mu \parallel L^*}{L^{1-\chi^*}(2-\chi^*)}\Big]\Big[(\zeta_1)^{2-\chi^*}-(\zeta_2)^{2-\chi^*}+2(\zeta_2-\zeta_1)^{2-\chi^*}\Big]\\
+\ & p \parallel \mu \parallel \int_0^{\zeta_1}\Big[(\zeta_2-\varpi)-(\zeta_1-\varpi)\Big]\varpi^{-\sigma}d\varpi\\
+\ & \eta^*\Big[\zeta_1\zeta_2-\zeta_1^2\Big]+p \parallel \mu \parallel \int_{\zeta_1}^{\zeta_2}(\zeta_2-\varpi)\varpi^{-\sigma}d\varpi\\
+\ & \eta^*\Big[\frac{\zeta_2^2}{2}+\frac{\zeta_1^2}{2}-\zeta_1\zeta_2\Big]+\Big[\frac{M_\Gamma L^*}{2-\chi^*} \parallel \mu \parallel \Big(\zeta_2-\zeta_1\Big)\Big]\\
+\ & \frac{\zeta_2-\zeta_1}{L}p \parallel \mu \parallel \int_0^{L}(L-\varpi)\varpi^{-\sigma}d\varpi+\eta^*\frac{\zeta_2-\zeta_1}{L}\frac{L^2}{2}\\
\leq\ & |\ \mu(\zeta_2)-\mu(\zeta_1)\ |+\Big[\frac{M_\Gamma \parallel \mu \parallel L^*}{L^{1-\chi^*}(2-\chi^*)}\Big]\Big[(\zeta_1)^{2-\chi^*}-(\zeta_2)^{2-\chi^*}+2(\zeta_2-\zeta_1)^{2-\chi^*}\Big]\\
+\ & p \parallel \mu \parallel \Big[(\zeta_2-\zeta_1)\frac{\zeta_1^{-\sigma+1}}{-\sigma+1}\Big]+\eta^*\Big(\frac{-\zeta_1^2}{2}+\frac{\zeta_2^2}{2}\Big)\\
+\ & p \parallel \mu \parallel \Big[\Big(-(\zeta_2-\zeta_1)\frac{\zeta_1^{-\sigma+1}}{-\sigma+1}\Big)+\Big(\frac{\zeta_2^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}-\frac{\zeta_1^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}\Big)\Big]\\
+\ & \Big[\frac{M_\Gamma L^*}{2-\chi^*} \parallel \mu \parallel \Big(\zeta_2-\zeta_1\Big)\Big]+\frac{\zeta_2-\zeta_1}{L}p \parallel \mu \parallel \frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}+\eta^*(\zeta_2-\zeta_1)\frac{L}{2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \ |\ \mu(\zeta_2)-\mu(\zeta_1)\ |+\Big[\frac{M_\Gamma \parallel \mu \parallel L^*}{L^{1-\chi^*}(2-\chi^*)}\Big]\Big[(\zeta_1)^{2-\chi^*}-(\zeta_2)^{2-\chi^*}+2(\zeta_2-\zeta_1)^{2-\chi^*}\Big]\\
&+\ \eta^*\Big(\frac{\zeta_2^2-\zeta_1^2}{2}\Big)+p\parallel \mu \parallel\Big(\frac{\zeta_2^{-\sigma+2}-\zeta_1^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}\Big)\\
&+\ \Big[\frac{M_\Gamma L^*}{2-\chi^*}\parallel \mu \parallel+\frac{p\parallel \mu \parallel}{L}\frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}+\eta^*\frac{L}{2}\Big]\Big(\zeta_2-\zeta_1\Big).
\end{aligned}
$$

Hence, $|\ C\mu(\zeta_2)-C\mu(\zeta_1)\ |\rightarrow 0$ as $\zeta_2\rightarrow\zeta_1$. It implies that $C(E_r)$ is equicontinuous. Consequently, the operator C is compact.

Step 4: The set $\ominus$ defined as

$$\ominus=\{\mu\in E:\mu=\lambda C\mu,\ \ 0<\lambda<1\},$$

is bounded.

Let $\mu\in\ominus$. Then, for any $\zeta\in A$, we have

$$\mu(\zeta)\ =\ \lambda C\mu(\zeta),\ \ 0<\lambda<1,$$

and

$$
\begin{aligned}
|\ \lambda C\mu(\zeta)\ | &= \lambda\Big|\mu(\zeta)-\int_0^\zeta\frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ \ d\varpi+\int_0^\zeta(\zeta-\varpi)\eta(\varpi,\mu(\varpi))d\varpi\\
&+\ \frac{\zeta}{L}\int_0^L\frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ \ d\varpi-\frac{\zeta}{L}\int_0^L(L-\varpi)\eta(\varpi,\mu(\varpi))d\varpi\Big|\\
&\leq \lambda\Big[\ |\ \mu(\zeta)\ |+\int_0^\zeta\frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\ |\ \mu(\varpi)\ |\ d\varpi+\int_0^\zeta(\zeta-\varpi)\ |\ \eta(\varpi,\mu(\varpi))\ |\ d\varpi\\
&+\ \frac{\zeta}{L}\int_0^L\frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\ |\ \mu(\varpi)\ |\ d\varpi+\frac{\zeta}{L}\int_0^L(L-\varpi)\ |\ \eta(\varpi,\mu(\varpi))\ |\ d\varpi\Big]\\
&\leq \Big[\ |\ \mu(\zeta)\ |+\int_0^\zeta\frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\ |\ \mu(\varpi)\ |\ d\varpi+\int_0^\zeta(\zeta-\varpi)\ |\ \eta(\varpi,\mu(\varpi))\ |\ d\varpi\\
&+\ \frac{\zeta}{L}\int_0^L\frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\ |\ \mu(\varpi)\ |\ d\varpi+\frac{\zeta}{L}\int_0^L(L-\varpi)\ |\ \eta(\varpi,\mu(\varpi))\ |\ d\varpi\Big]\\
&\leq \Big[1+\frac{2M_\Gamma LL^*}{2-\chi^*}+2p\frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}\Big]\parallel \mu \parallel+\eta^*L^2.
\end{aligned}
$$

Now, for every $\zeta\in A$, we have

$$\parallel \lambda C\mu \parallel\ \ \leq\ \ \Big[1+\frac{2M_\Gamma LL^*}{2-\chi^*}+2p\frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}\Big]\parallel \mu \parallel+\eta^*L^2<\infty.$$

This implies that the set $\ominus$ is bounded independently of $\lambda\in(0,1)$.

Then, all condition of Theorem 2.6 are satisfied and the ITVP (1) has at least one solution $\mu\in\mathbb{E}$.

□

4. Results on uniqueness

The next result is based on Theorem 2.7.

Theorem 4.1. *If the conditions (AS1) and (AS2) are satisfied, then the ITVP (1) has a unique solution on* $\mathbb{E}$.

Proof. We consider the same operator

$$C : \mathbb{E} \to \mathbb{E},$$

defined as follows:

$$\begin{aligned} C\mu(\zeta) &= \mu(\zeta) - \int_0^{\zeta} \frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ d\varpi + \int_0^{\zeta} (\zeta-\varpi)\eta(\varpi,\mu(\varpi))\ d\varpi \\ &+ \frac{\zeta}{L}\int_0^{L} \frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))}\mu(\varpi)\ d\varpi - \frac{\zeta}{L}\int_0^{L} (L-\varpi)\eta(\varpi,\mu(\varpi))\ d\varpi. \end{aligned}$$

For μ, $\mu^{*}\in \mathbb{E}$, we may write

$$| C\mu(\zeta) - C\mu^{*}(\zeta) |$$

$$\begin{aligned}
&\leq \ | \mu(\zeta) - \mu^{*}(\zeta) | + \int_0^{\zeta} \frac{(\zeta-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} \ | \mu^{*}(\varpi) - \mu(\varpi) | \, d\varpi \\
&+ \ \int_0^{\zeta} (\zeta-\varpi)\varpi^{-\sigma}\varpi^{\sigma} \ | \eta(\varpi,\mu(\varpi)) - \eta(\varpi,\mu^{*}(\varpi)) | \, d\varpi \\
&+ \ \int_0^{L} \frac{(L-\varpi)^{1-\chi(\varpi)}}{\Gamma(2-\chi(\varpi))} \ | \mu(\varpi) - \mu^{*}(\varpi) | \, d\varpi \\
&+ \ \int_0^{L} (L-\varpi)\varpi^{-\sigma}\varpi^{\sigma} \ | \eta(\varpi,\mu^{*}(\varpi)) - \eta(\varpi,\mu(\varpi)) | \, d\varpi \\
&\leq \ \| \mu - \mu^{*} \| + M_{\Gamma} \| \mu - \mu^{*} \| \int_0^{\zeta} (\zeta-\varpi)^{1-\chi(\varpi)} d\varpi \\
&+ \ p \| \mu - \mu^{*} \| \int_0^{\zeta} (\zeta-\varpi)\varpi^{-\sigma} d\varpi + M_{\Gamma} \| \mu - \mu^{*} \| \int_0^{L} (L-\varpi)^{1-\chi(\varpi)} d\varpi \\
&+ \ p \| \mu - \mu^{*} \| \int_0^{L} (L-\varpi)\varpi^{-\sigma} d\varpi \\
&\leq \ \| \mu - \mu^{*} \| + M_{\Gamma}L^{*} \| \mu - \mu^{*} \| \int_0^{\zeta} \Big(\frac{\zeta-\varpi}{L}\Big)^{1-\chi^{*}} d\varpi + p \| \mu - \mu^{*} \| \frac{\zeta^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&+ \ M_{\Gamma}L^{*} \| \mu - \mu^{*} \| \int_0^{L} \Big(\frac{L-\varpi}{L}\Big)^{1-\chi^{*}} d\varpi + p \| \mu - \mu^{*} \| \frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&\leq \ \| \mu - \mu^{*} \| + \frac{M_{\Gamma}L^{*}}{L^{1-\chi^{*}}} \frac{(\zeta)^{2-\chi^{*}}}{(2-\chi^{*})} \| \mu - \mu^{*} \| + 2p \| \mu - \mu^{*} \| \frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&+ \ \frac{M_{\Gamma}L^{*}}{L^{1-\chi^{*}}} \frac{(L)^{2-\chi^{*}}}{2-\chi^{*}} \| \mu - \mu^{*} \| \\
&\leq \ \| \mu - \mu^{*} \| + \frac{M_{\Gamma}LL^{*}}{2-\chi^{*}} \| \mu - \mu^{*} \| + 2p \| \mu - \mu^{*} \| \frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&+ \ \frac{M_{\Gamma}LL^{*}}{2-\chi^{*}} \| \mu - \mu^{*} \| \\
&\leq \ \| \mu - \mu^{*} \| + \frac{2M_{\Gamma}LL^{*}}{2-\chi^{*}} \| \mu - \mu^{*} \| + 2p \| \mu - \mu^{*} \| \frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&\leq \ \Big(1 + \frac{2M_{\Gamma}LL^{*}}{2-\chi^{*}} + 2p\frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}\Big) \| \mu - \mu^{*} \| .
\end{aligned}$$

We put $\sigma = 1 + \frac{2M_\Gamma LL^*}{2-\chi^*} + 2p\frac{L^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}$ so that we have

$$\| C\mu - C\mu^* \| \leq \sigma \| \mu - \mu^* \| .$$

By induction, we can prove that

$$\| C^n\mu - C^n\mu^* \| \leq \frac{\sigma^n}{n!} \| \mu - \mu^* \|,$$

where $C^n = C \circ C \circ C \circ C \circ .... \circ C$ "n times".
We have $\lim_{n\to\infty} \frac{\sigma^n}{n!} = 0$. Then, for sufficiently large n, we get $\frac{\sigma^n}{n!} < 1$.
According to Theorem 2.7, the operator C has a unique fixed point which is the unique solution of the ITVP (1). □

5. Numerical examples

Example 5.1. *Consider the following ITVP*

$$\begin{cases} D^{\chi(\zeta)}\mu(\zeta) = \eta(\zeta, \mu(\zeta)), & \zeta \in A = [0, 1], \\ \mu(0) = \mu(1) = 0, \end{cases} \tag{6}$$

where $\chi(\zeta) = 1 + \frac{\zeta}{2}$, *and* $\eta(\zeta, \mu) = \zeta^2 + \frac{1}{3}\mu$.
Clearly $\chi(\zeta)$ *is a continuous function on* $[0, 1]$ *and*, $1 < \chi(\zeta) < 1 + \frac{1}{2} = \frac{3}{2} = \chi^* < 2$.
In addition, $\eta(\zeta, \mu)$ *is a continuous function on* $A \times R$, *and*

$$\begin{aligned} \zeta^\sigma \mid \eta(\zeta, \mu) - \eta(\zeta, y) \mid = \zeta^\sigma \mid \zeta^2 + \frac{1}{3}\mu - \zeta^2 - \frac{1}{3}y \mid &= \zeta^\sigma \mid \frac{1}{3}(\mu - y) \mid \\ &\leq \frac{1}{3} \mid \mu - y \mid, \end{aligned}$$

so (AS1) satisfied for $p = \frac{1}{3}$ *and* $\sigma \in (0, 1)$.
By Theorem(4.1) the equation (6) has a unique solution.

Example 5.2. *Consider the following ITVP*

$$\begin{cases} D^{\chi(\zeta)}\mu(\zeta) = \eta(\zeta, \mu(\zeta)), & \zeta \in A = [0, 1], \\ \mu(0) = \mu(1) = 0, \end{cases} \tag{7}$$

where $\chi(\zeta) = \exp(\zeta) - \zeta$ *and* $\eta(\zeta, \mu) = \frac{\exp(-\zeta)}{(\exp(\exp(\frac{\zeta^2}{1+\zeta}))+4\exp(2\zeta)+1)(1+\mu)}$.
Clearly $\chi(\zeta)$ *is a continuous function on* $[0, 1]$ *and*, $1 < \chi(\zeta) < \exp(1) - 1 = \chi^* < 2$.
Also, $\eta(\zeta, \mu)$ *is a continuous function on* $A \times R$, *and*

$$\begin{aligned} \zeta^\sigma \mid \eta(\zeta, \mu) - \eta(\zeta, y) \mid &= \zeta^\sigma \mid \frac{\exp(-\zeta)}{(\exp(\exp(\frac{\zeta^2}{1+\zeta})) + 4\exp(2\zeta) + 1)}\Big(\frac{1}{1+\mu} - \frac{1}{1+y}\Big) \mid \\ &\leq \zeta^\sigma \frac{\exp(-\zeta) \mid \mu - y \mid}{(\exp(\exp(\frac{\zeta^2}{1+\zeta})) + 4\exp(2\zeta) + 1)(1+\mu)(1+y)} \\ &\leq \zeta^\sigma \frac{\exp(-\zeta)}{(\exp(\exp(\frac{\zeta^2}{1+\zeta})) + 4\exp(2\zeta) + 1)} \mid \mu - y \mid \\ &\leq \frac{\exp(-1)}{(\exp(\exp(\frac{1}{2})) + 4\exp(2) + 1)} \mid \mu - y \mid, \end{aligned}$$

so $(AS1)$ *satisfied for* $p = \frac{\exp(-1)}{(\exp(\exp(\frac{1}{2}))+4\exp(2)+1)}$ *and* $\sigma \in (0,1)$.
By Theorem(4.1) the equation (7) has a unique solution.

Numerical results
Now, we present the numerical solution $\mu(\zeta)$ for $\chi(\zeta) = \exp(\zeta) - \zeta$ with $\zeta \in [0,1]$ and $\mu_i(\zeta)$ for $\chi(\zeta_i) = \exp(\zeta_i) - \zeta_i$ where ζ_i is fixed.
In Figure (1), we plot the solution μ depending on ζ.

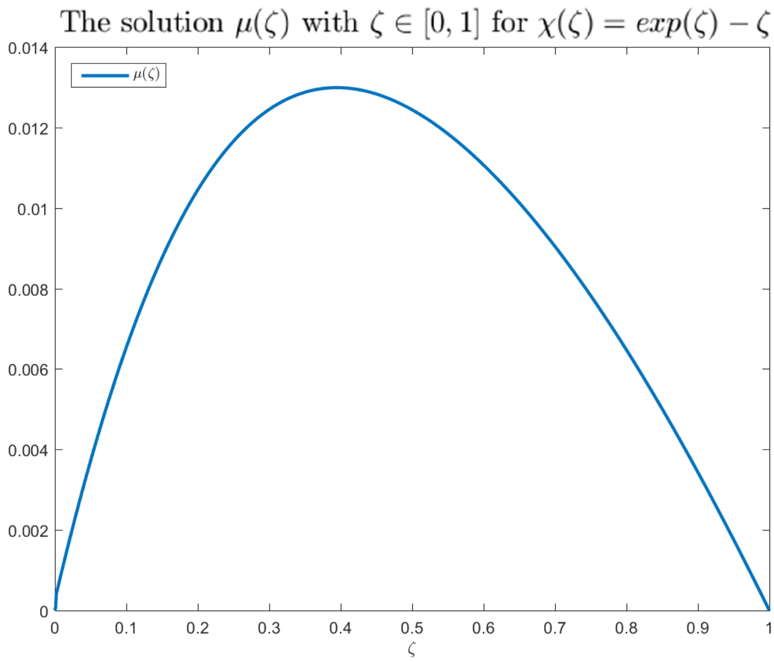
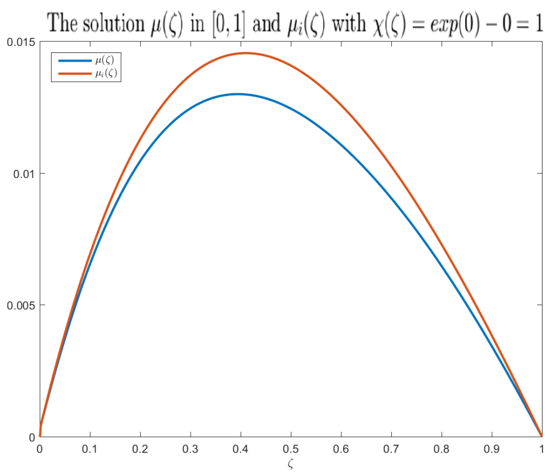
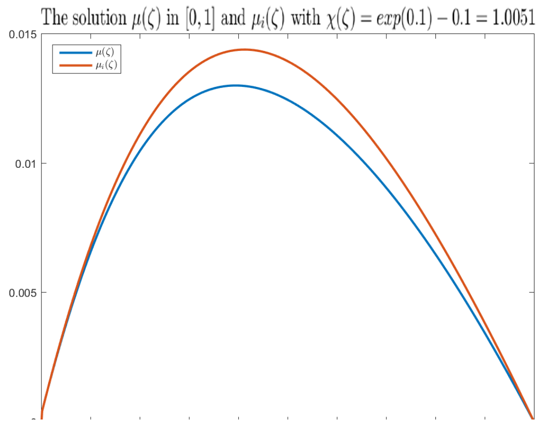
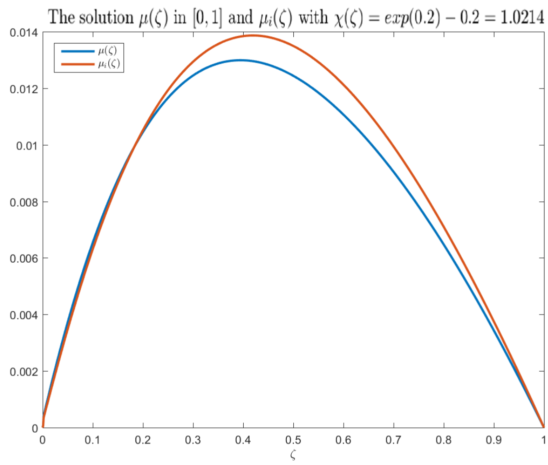


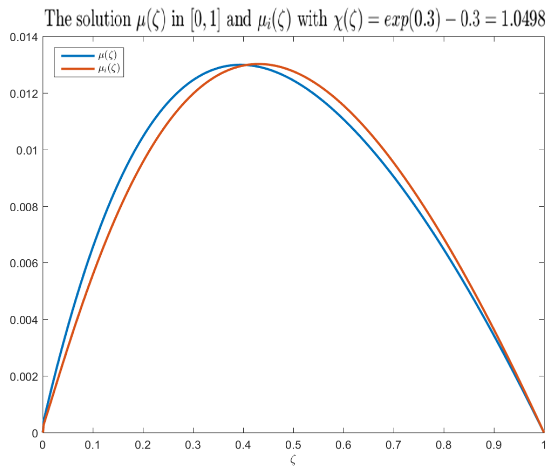
Figure 1: The solution $\mu(\zeta)$ in $[0,1]$ with $\chi(\zeta) = \exp(\zeta) - \zeta$.

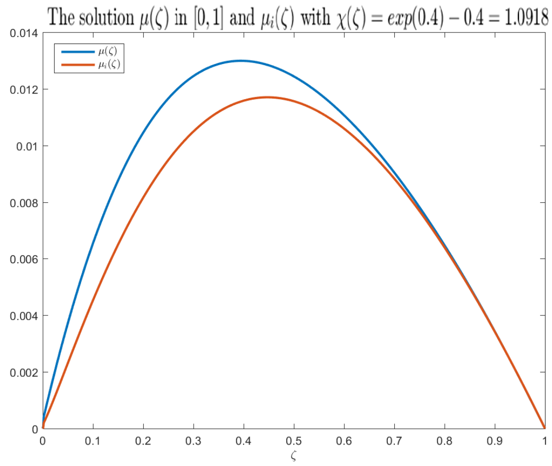
The following figures present a comparison between the solution μ and the various solutions μ_i, each with a different ζ.

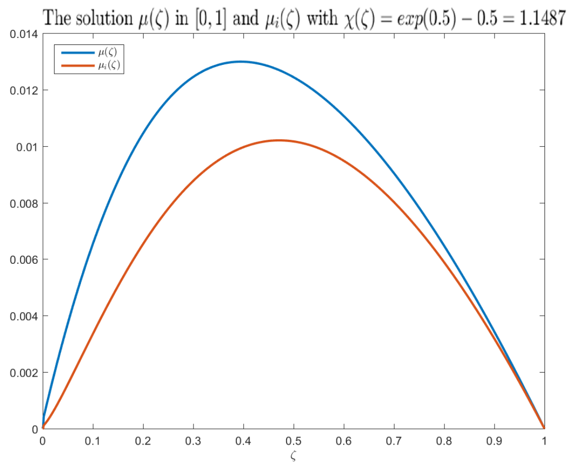


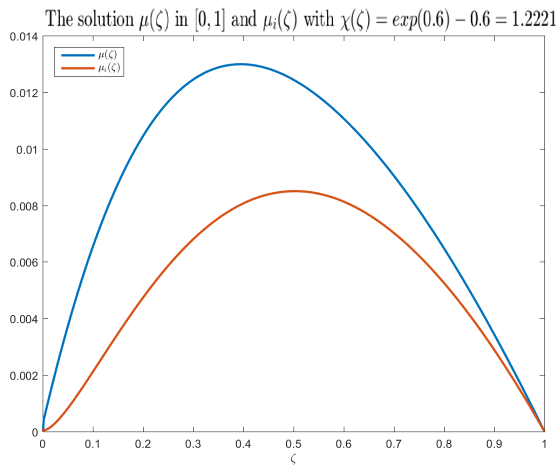

The solution $\mu(\zeta)$ in $[0,1]$ and $\mu_i(\zeta)$ with $\chi(\zeta) = exp(0.1) - 0.1 = 1.0051$
$\mu(\zeta)$
$\mu_i(\zeta)$
0.015
0.01
0.005

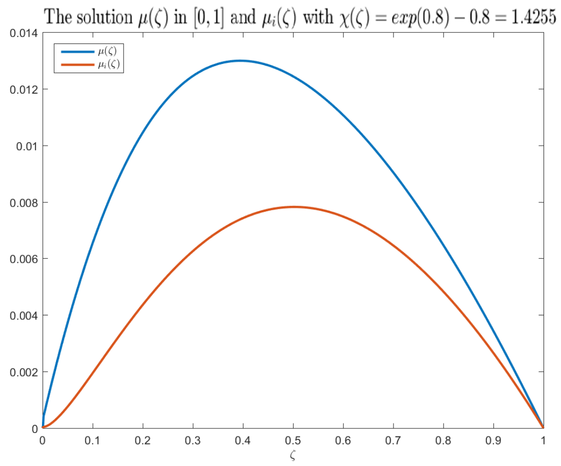

The solution $\mu(\zeta)$ in $[0,1]$ and $\mu_i(\zeta)$ with $\chi(\zeta) = exp(0.2) - 0.2 = 1.0214$
$\mu(\zeta)$
$\mu_i(\zeta)$
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
ζ

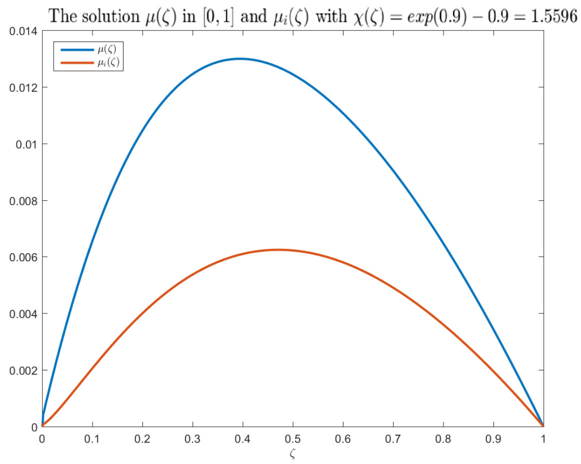

The solution $\mu(\zeta)$ in $[0,1]$ and $\mu_i(\zeta)$ with $\chi(\zeta) = exp(0.3) - 0.3 = 1.0498$
$\mu(\zeta)$
$\mu_i(\zeta)$
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
ζ

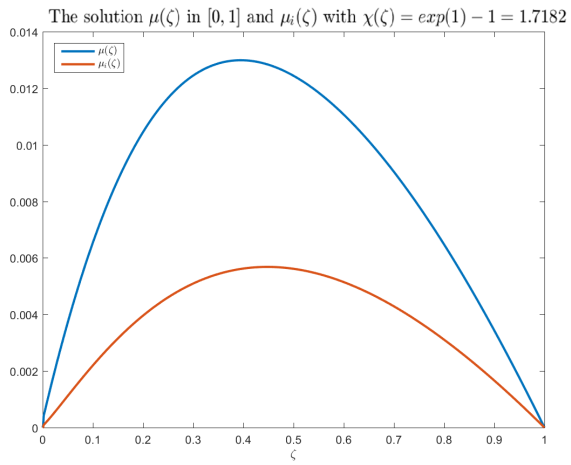

The solution $\mu(\zeta)$ in $[0,1]$ and $\mu_i(\zeta)$ with $\chi(\zeta) = exp(0.4) - 0.4 = 1.0918$
$\mu(\zeta)$
$\mu_i(\zeta)$
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
ζ


The solution $\mu(\zeta)$ in $[0,1]$ and $\mu_i(\zeta)$ with $\chi(\zeta) = exp(0.5) - 0.5 = 1.1487$
$\mu(\zeta)$
$\mu_i(\zeta)$
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
ζ


The solution $\mu(\zeta)$ in $[0,1]$ and $\mu_i(\zeta)$ with $\chi(\zeta) = exp(0.6) - 0.6 = 1.2221$
$\mu(\zeta)$
$\mu_i(\zeta)$
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
ζ







In this table, we present the $Norm_i = \max_{\zeta\in[0,1]} \left|\mu(\zeta) - \mu_i(\zeta)\right|$ for $\chi(\zeta) \in [1, 2]$.

ζ	0	0.1	0.2	0.3	0.4
$\chi(\zeta)$	1	1.005	1.021	1.049	1.091
$Norm_i$	1.63×10^{-6}	1.49×10^{-6}	1.08×10^{-6}	1×10^{-6}	2.3×10^{-6}
ζ	0.5	0.6	0.8	0.9	1
$\chi(\zeta)$	1.148	1.222	1.425	1.559	1.718
$Norm_i$	3.92×10^{-6}	5.8×10^{-6}	6.26×10^{-6}	7.09×10^{-6}	7.44×10^{-6}

We observe that when ζ approaches to $0,4(\chi = 1,091)$ the $Norm_i$ is decreasing and when ζ approaches to $1(\chi = 1,718)$ is creasing.

6. Conclusion

In this paper we have presented results on the existence and uniqueness of solutions to initial and terminal value problem ITVP (1) for non linear fractional differential equation of variable order $\chi(\zeta)$ where $1 < \chi(\zeta) < 2$.
Our results are based on the fixed point theorem technique (Theorem 3.3, Theorem 4.1). Finally, we illustrated the theoretical results with numerical examples.
Considering the scarcity of studies on fractional calculus of variable order, the results we obtained are very important and can be applied in various sciences.

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