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On nonlocal terminal value problem for tempered fractional diffusion equation

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Abstract. In this paper, we are interested in studying the tempered fractional diffusion equation subject to a nonlocal terminal condition. The primary equation incorporates the tempered Caputo derivative, which serves as a generalized form of the traditional Caputo derivative. Our findings contribute by first establishing the well-posedness, highlighting the challenges added by the tempered kernel together with nonlocal conditions. Following this, we study the solution's continuity with respect to the tempered parameter, a crucial consideration for modeling, due to the challenges in accurately measuring this index. Lastly, we propose convergence results as parameters $b \to 0^+$, $a \to 0^+$, and $\alpha \to 1^-$, linking the current terminal fractional approach with traditional cases.

1. Introduction

Let Ω be a bounded region $\mathbb{R}^N$ with a smooth boundary denoted by $\partial\Omega$. Assume that T is a fixed positive constant. In this paper, our aim is to examine the following problem

$$\begin{cases} D_t^{\alpha,k} u - \Delta u = G(x,t), & \text{in} \quad \Omega \times (0,T], \\ u(x,t) = 0, & \text{on} \quad \partial\Omega \times (0,T], \end{cases} \tag{1}$$

with the nonlocal condition

$$au(x,0) + bu(x,T) = f(x), \quad x \in \Omega, \tag{2}$$

where a, b are two non-negative parameters, f will be specified later, $D_t^{\alpha,k}$ denotes the *tempered Caputo derivative* (see p. 430[19]) of order $\alpha \in (0,1)$

$$D_t^{\alpha,k} w(t) = \frac{e^{-kt}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{d}{ds}\Big(e^{ks} w(s)\Big) ds,$$

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where $k > 0$ is called tempered parameter.

In recent years, the field of fractional differential equations (FDEs) has become an area of growing interest, with applications extending across disciplines such as physics, biology, and environmental science [12]. Various types of FDEs are applied in these fields, with different fractional derivatives like the Caputo, Riemann-Liouville, and Caputo-Fabrizio derivatives proving particularly useful [7, 8, 11, 12, 16–18]. These models incorporate memory effects and appropriate kernel functions. The *tempered fractional derivative*, which was initially introduced in [13], is noteworthy in this work. An important point to note is that, when $k = 0$, the tempered Caputo derivative reduces to the standard Caputo operator [12]. As a result, it is obvious that the tempered fractional derivative represents a generalized form of the Caputo fractional derivative.

It is widely recognized that *tempered fractional calculus* was developed to effectively address complex applications in various fields, such as poroelasticity [3], geophysical fluid dynamics [9], and groundwater hydrology [10]. To achieve a better comprehension of how solutions behave in the context of the tempered fractional equations, significant research has been undertaken to address this problem from multiple viewpoints[2, 4, 5, 8–10, 13–15, 19]. Within this body of work, Li [5] proved the well-posedness of tempered fractional ordinary differential equations, discussing important factors like existence, uniqueness, and stability. In [19], M.A. Zaky explored the existence, uniqueness, and stability characteristics of solutions to the nonlinear tempered fractional differential equations

$$D_t^{\alpha,k} u = g(\tau, u(\tau)), \quad 0 \le \tau \le T \tag{3}$$

associated with nonlocal condition

$$au(0) + be^{kt} u(T) = c. \tag{4}$$

In this context, $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and a, b, c are real constants with the condition $a + b \neq 0$. In the case $a = 0$ and $b = 1$, the model (3)-(4) was examined by Shiri in [15], in which regularity results were established in weighted spaces, and the order of convergence was analyzed.

Inspired by the model described by equations (3) and (4), in this paper, we are strongly interested in investigating the model represented by equations (1) and (2). It is important to highlight that this model represents a generalized form of the classical (and fractional) heat equations $u_t - \Delta u = G(x, t)$ (and $D_t^{\alpha} u - \Delta u = G(x, t)$), which have been effectively employed to model heat transfer in various environments. Unlike the previously mentioned results [15, 19], this work extends beyond the investigation of the well-posedness of the problem to also examine the continuity of the solution with respect to the tempered parameter k, in addition to proposing several convergence results.

The main contributions presented in our findings are outlined in detail as follows.

- The first result is the well-posedness in a Hilbert scale space, provided that f, G are sufficiently smooth. In comparison to the model with the standard Caputo fractional derivative, it is evident that the tempered model presents greater complexities due to the additional kernel term e^{ks} in the derivative $D_t^{\alpha,k}$. This, coupled with the nonlocal condition, complicates the mild formulation, resulting in significant challenges when estimating the solution within the Hilbert scale space.

- The second focus of this paper is to investigate the continuity of the solution with respect to the tempered parameter k. It is undeniable that, in practice, this index can only be ascertained through experimentation. Consequently, their precise values are not available; instead, only perturbed values with associated errors are known. For this reason, it is essential to examine the continuity of the solution with respect to the parameters involved.

- The last purpose is to establish some convergence results as $b \to 0^+$, $a \to 0^+$, and $\alpha \to 1^-$. It should be noted that if $b = 0$ (or $a = 0$) then the nonlocal problem described by equations (1) and (2) reduces to the standard initial value problem (or final value problem). Owing to this reason, the two convergence results when $b \to 0^+$ and $a \to 0^+$ are crucial for demonstrating the connection between our model and traditional ones. As a consequence, the solution of the conventional models can be obtained as a limit of an approximate sequence of solutions to the nonlocal problem. Additionally, in the case $\alpha = 1$ and $k = 0$, the tempered fractional operator simplifies to the classical derivative with integer order. This observation underscores the importance of examining the convergence result when $\alpha \to 1^-$. Understanding this convergence is vital for establishing a clear connection between the tempered fractional framework and traditional differentiation.

We now present a concise outline of the paper. In Section 2, we present essential preliminaries, including the relevant Sobolev spaces and the notation employed throughout the paper. In Section 3, we dedicate our discussion to the well-posedness within a Hilbert scale space, given that f and G are adequately smooth. The continuity of solution in k is illustrated in Section 4. Section 5 is devoted to convergence results as the parameters $b \to 0^+, a \to 0^+$ and $\alpha \to 1^-$, which relate the present terminal fractional equations to traditional models.

2. Preliminaries

In this section, we present relevant Sobolev spaces and introduce the notation that will be consistently used throughout this paper. To begin with, let (λ_n, e_n) represent an eigenpair of the negative Laplacian operator. For $p \geq 0$, we define by

$$\mathbb{H}^p(\Omega) = \Big\{ v \in L^2(\Omega) \Big| \|v\|_{\mathbb{H}^p(\Omega)} := \sum_{n=1}^{\infty} \lambda_n^{2p} |\langle v, e_n \rangle|^2 < \infty \Big\}.$$

Let $\mathbb{B}$ be a Banach space, we denote by $L^q(0,T;\mathbb{B}), 1 \leq q \leq \infty$, the space of functions $w : (0,T) \to \mathbb{B}$ with the norm

$$\|w\|_{L^q(0,T;\mathbb{B})} := \begin{cases} \left(\int_0^T \|w(t)\|_{\mathbb{B}}^q dt \right)^{\frac{1}{q}}, & q < \infty, \\ \operatorname{ess\,sup}_{t \in (0,T)} \|w(t)\|_{\mathbb{B}}, & q = \infty. \end{cases}$$

Denote by $C^\theta([0,T];\mathbb{B})$, with $\theta > 0$, the space of functions $w : (0,T) \to \mathbb{B}$ satisfying

$$\|w\|_{C^\theta([0,T];\mathbb{B})} := \sup_{0 \leq t < s \leq T} \frac{\|w(\cdot,t) - w(\cdot,s)\|_{\mathbb{B}}}{|t-s|^\theta} < \infty.$$

Next, we compile essential background information and key concepts from fractional calculus. For $\alpha > 0$ and $\beta \in \mathbb{C}$, the two-parameter Mittag-Leffler function is defined by the following series expression

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad z \in \mathbb{C}.$$

Several key properties of this function are summarized in the following lemmas [12].

Lemma 2.1. *Given $\lambda > 0$, and $\alpha > 0$. Then, the following identities hold*

$$\partial_t E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0,$$

and

$$\partial_t (t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)) = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha), \quad t > 0.$$

Furthermore, for $0 < \alpha < 1$, there holds

$$\partial_t^\alpha E_{\alpha,1}(-\lambda t^\alpha) = -\lambda E_{\alpha,1}(-\lambda t^\alpha), t > 0.$$

Lemma 2.2. *For $0 < \alpha < 1$ and $\beta \in \mathbb{C}$, it holds that*

$$\frac{m_1}{1+|z|} \le \left|E_{\alpha,\beta}(z)\right| \le \frac{m_2}{1+|z|},$$

where m_1 and m_2 are positive constants depending only on α, β. In addition

$$\frac{m_1}{1+|z|} \le E_{\alpha,\gamma}(z) \le \frac{\overline{m}_1}{1+|z|}, \text{ where } \gamma \in (0,1]. \tag{5}$$

Lemma 2.3. *For $0 < \alpha < 1$ and $t > 0$, there holds*

$$\frac{E_{\alpha,1}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)} \le \frac{\overline{m}_1}{m_1} T^\alpha t^{-\alpha}.$$

Proof. As a consequence of (5), for $t > 0$, it is obvious to see that

$$\frac{E_{\alpha,1}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)} \le \frac{\overline{m}_1}{m_1} \frac{1+\lambda_j T^\alpha}{1+\lambda_j t^\alpha} \le \frac{\overline{m}_1}{m_1} T^\alpha t^{-\alpha}.$$

The proof is complete. □

3. Well-posedness of the problem

In this section, we examine the well-posedness of the following problem

$$\begin{cases} D_t^{\alpha,k} u - \Delta u = G(x,t), & \text{in } \Omega \times (0,T], \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T], \\ au(x,0) + bu(x,T) = f(x), & \text{in } \Omega, \end{cases} \tag{6}$$

where $a \ge 0$ and $b > 0$. Assume that Problem (6) has a unique solution denoted by u.

We first derive an explicit formula for the solution in the form of a Fourier series, given by $u(x,t) = \sum_{j=1}^\infty u_j(t) e_j(x)$, with $u_j(t) = \left\langle u(\cdot,t), e_j(\cdot)\right\rangle_{L^2(\Omega)}$. From the first equation of (6), taking the inner product of both sides of (6) with $e_j(x)$, we obtain

$$D_t^{\alpha,k} u_j(t) + \lambda_j u_j(t) = G_j(t), \;\; u_j(0) = \left\langle u_0, e_j\right\rangle_{L^2(\Omega)},$$

where $G_j(t) = \left\langle G(\cdot,t), e_j(\cdot)\right\rangle_{L^2(\Omega)}, u_0 = u(x,0)$. The theory of fractional ordinary differential equations [2] gives a unique function as follows

$$u_j^{\alpha,k}(t) = e^{-kt} E_{\alpha,1}(-\lambda_j t^\alpha) u_j(0) + \int_0^t (t-r)^{\alpha-1} e^{-k(t-r)} E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha) G_j(r) dr. \tag{7}$$

The nonlocal condition $au(x,0) + bu(x,T) = f(x)$ allows us to obtain

$$\begin{aligned} &au_j(0) + be^{-kT} E_{\alpha,1}(-\lambda_j T^\alpha) u_j(0) \\ &\qquad + b\int_0^T (T-r)^{\alpha-1} e^{-k(T-r)} E_{\alpha,\alpha}(-\lambda_j (T-r)^\alpha) G_j(r) dr = f_j. \end{aligned}$$

It follows that

$$u_j(0) = \frac{f_j - b\int_0^T (T-r)^{\alpha-1} e^{-k(T-r)} E_{\alpha,\alpha}(-\lambda_j (T-r)^\alpha) G_j(r) dr}{a + be^{-kT} E_{\alpha,1}(-\lambda_j T^\alpha)}. \tag{8}$$

Combining (7) and (8), we derive that

$$
\begin{aligned}
u_j^{\alpha,k}(t) &= \frac{e^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)} f_j \\
&- b\frac{e^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)} \int_0^T (T-r)^{\alpha-1}e^{-k(T-r)}E_{\alpha,\alpha}(-\lambda_j (T-r)^\alpha)G_j(r)dr \\
&+ \int_0^t (t-r)^{\alpha-1}e^{-k(t-r)}E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha)G_j(r)dr. \qquad (9)
\end{aligned}
$$

Consequently, the mild solution of Problem (6) is obtained as follows

$$
\begin{aligned}
u^{\alpha,k}(x,t) &= \sum_{j=1}^{\infty} \frac{e^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)} f_j e_j(x) \\
&- b\sum_{j=1}^{\infty}\Big(\frac{e^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)} \int_0^T (T-r)^{\alpha-1}e^{-k(T-r)}E_{\alpha,\alpha}(-\lambda_j (T-r)^\alpha)G_j(r)dr\Big)e_j(x) \\
&+ \sum_{j=1}^{\infty}\Big(\int_0^t (t-r)^{\alpha-1}e^{-k(t-r)}E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha)G_j(r)dr\Big)e_j(x). \qquad (10)
\end{aligned}
$$

In the following theorem, the well-posedness result in the Hilbert scale space is provided.

Theorem 3.1. *Let $f \in \mathbb{H}^\ell(\Omega)$ and $G \in L^\infty(0,T;\mathbb{H}^{\ell-\theta}(\Omega))$ for any $\ell > 0$ and $0 \le \theta \le \ell$. Then, the following boundedness result holds true*

$$
\begin{aligned}
\big\|u^{\alpha,k}(\cdot,t)\big\|_{\mathbb{H}^\ell(\Omega)} &\le \frac{1}{b}\frac{\overline{m}_1}{m_1}e^{k(T-t)}T^\alpha t^{-\alpha}\big\|f\big\|_{\mathbb{H}^\ell(\Omega)} + \frac{e^{k(T-t)}\overline{m}_1^2 T^{2\alpha-\theta\alpha}}{m_1(\alpha-\theta\alpha)}t^{-\alpha}\big\|G\big\|_{L^\infty(0,T;\mathbb{H}^{\ell-\theta}(\Omega))} \\
&+ \frac{\overline{m}_1 T^{\alpha-\theta\alpha}}{\alpha-\theta\alpha}\big\|G\big\|_{L^\infty(0,T;\mathbb{H}^{\ell-\theta}(\Omega))}.
\end{aligned}
$$

Proof. From (10), it is obvious that

$$
u^{\alpha,k}(x,t) = \mathbb{Q}^1_{a,b,\alpha,k}(x,t) + \mathbb{Q}^2_{a,b,\alpha,k}(x,t) + \mathbb{Q}^3_{a,b,\alpha,k}(x,t),
$$

where the three terms in the right hand side possess the following forms

$$
\begin{aligned}
\mathbb{Q}^1_{a,b,\alpha,k}(x,t) &= \sum_{j=1}^{\infty} \frac{e^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)} f_j e_j(x), \\
\mathbb{Q}^2_{a,b,\alpha,k}(x,t) &= \sum_{j=1}^{\infty}\Big[\frac{-be^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)} \\
&\qquad \times \int_0^T (T-r)^{\alpha-1}e^{-k(T-r)}E_{\alpha,\alpha}(-\lambda_j (T-r)^\alpha)G_j(r)dr\Big]e_j(x)
\end{aligned}
$$

and

$$
\mathbb{Q}^3_{a,b,\alpha,k}(x,t) = \sum_{j=1}^{\infty}\Big[\int_0^t (t-r)^{\alpha-1}e^{-k(t-r)}E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha)G_j(r)dr\Big]e_j(x).
$$

Step 1. Estimate of $\mathbb{Q}^1_{a,b,\alpha,k}$. The norm of the term $\mathbb{Q}^1_{a,b,\alpha,k}$ can be bounded above as

$$\begin{aligned}\left\|\mathbb{Q}^1_{a,b,\alpha,k}(\cdot,t)\right\|^2_{\mathbb{H}^\ell(\Omega)} &= \sum_{j=1}^{\infty}\lambda_j^{2\ell}\Big(\frac{e^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)}\Big)^2 f_j^2\\ &\le \frac{1}{b^2}\left(\frac{\overline{m}_1}{m_1}\right)^2 e^{2k(T-t)}T^{2\alpha}t^{-2\alpha}\sum_{j=1}^{\infty}\lambda_j^{2\ell}f_j^2,\end{aligned}$$

where we note that $b>0$. This implies that

$$\left\|\mathbb{Q}^1_{a,b,\alpha,k}(\cdot,t)\right\|_{\mathbb{H}^\ell(\Omega)} \le \frac{1}{b}\frac{\overline{m_1}}{m_1}e^{k(T-t)}T^\alpha t^{-\alpha}\left\|f\right\|_{\mathbb{H}^\ell(\Omega)}.$$

Step 2. Estimate of $\mathbb{Q}^2_{a,b,\alpha,k}$. Initially, using Lemma 2.3, we have the following observation

$$\frac{be^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)} \le \frac{e^{k(T-t)}\overline{m}_1}{m_1}T^\alpha t^{-\alpha}, \tag{11}$$

where we note that $a\ge 0$ and $b>0$. This implies that

$$\begin{aligned}&\left\|\mathbb{Q}^2_{a,b,\alpha,k}(\cdot,t)\right\|^2_{\mathbb{H}^\ell(\Omega)}\\ &= \sum_{j=1}^{\infty}\lambda_j^{2\ell}\Big[\frac{-be^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)}\int_0^T (T-r)^{\alpha-1}e^{-k(T-r)}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j(r)dr\Big]^2\\ &\le \Big(\frac{e^{k(T-t)}\overline{m}_1}{m_1}T^\alpha\Big)^2 t^{-2\alpha}\sum_{j=1}^{\infty}\lambda_j^{2\ell}\Big(\int_0^T (T-r)^{\alpha-1}e^{-k(T-r)}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)dr\Big)\\ &\qquad \Big(\int_0^T (T-r)^{\alpha-1}e^{-k(T-r)}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j^2(r)dr\Big),\end{aligned} \tag{12}$$

where we have used the Hölder inequality in (12). Using (5), we derive that

$$E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha) \le \frac{\overline{m}_1}{1+\lambda_j(T-r)^\alpha} \le \overline{m}_1\lambda_j^{-\theta}(T-r)^{-\theta\alpha},\ \text{where } 0<\theta<\alpha<1. \tag{13}$$

Hence, due to the fact that $e^{-k(T-r)}\le 1$, we infer that

$$\begin{aligned}\int_0^T (T-r)^{\alpha-1}e^{-k(T-r)}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)dr &\le \int_0^T (T-r)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)dr\\ &\le \overline{m}_1\lambda_j^{-\theta}\int_0^T (T-r)^{\alpha-\theta\alpha-1}dr = \frac{\overline{m}_1T^{\alpha-\theta\alpha}}{\alpha-\theta\alpha}\lambda_j^{-\theta}\end{aligned}$$

and

$$\int_0^T (T-r)^{\alpha-1}e^{-k(T-r)}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j^2(r)dr \le \overline{m}_1\lambda_j^{-\theta}\int_0^T (T-r)^{\alpha-\theta\alpha-1}G_j^2(r)dr.$$

As a consequence, for any $\ell \geq 0$, one has

$$\begin{aligned}
&\sum_{j=1}^{\infty} \lambda_j^{2\ell}\Big(\int_0^T (T-r)^{\alpha-1} e^{-k(T-r)} E_{\alpha,\alpha}(-\lambda_j (T-r)^{\alpha}) dr\Big) \\
&\qquad\qquad \times \Big(\int_0^T (T-r)^{\alpha-1} e^{-k(T-r)} E_{\alpha,\alpha}(-\lambda_j (T-r)^{\alpha}) G_j^2(r) dr\Big) \\
&\qquad \leq \frac{(\overline{m}_1)^2 T^{\alpha-\theta\alpha}}{\alpha-\theta\alpha} \int_0^T (T-r)^{\alpha-\theta\alpha-1}\Big(\sum_{j=1}^{\infty} \lambda_j^{2\ell-2\theta} G_j^2(r)\Big) dr \\
&\qquad = \frac{(\overline{m}_1)^2 T^{\alpha-\theta\alpha}}{\alpha-\theta\alpha} \int_0^T (T-r)^{\alpha-\theta\alpha-1} \big\|G(\cdot,r)\big\|^2_{\mathbb{H}^{\ell-\theta}(\Omega)} dr. \qquad (14)
\end{aligned}$$

Combining (12) and (14), we get

$$\big\|\mathbb{Q}^2_{a,b,\alpha,k}(\cdot,t)\big\|^2_{\mathbb{H}^{\ell}(\Omega)} \leq \frac{e^{2k(T-t)}\overline{m}_1^4 T^{3\alpha-\theta\alpha}}{m_1^2(\alpha-\theta\alpha)} t^{-2\alpha} \int_0^T (T-r)^{\alpha-\theta\alpha-1}\big\|G(\cdot,r)\big\|^2_{\mathbb{H}^{\ell-\theta}(\Omega)} dr.$$

It is obvious to see that

$$\begin{aligned}
\int_0^T (T-r)^{\alpha-\theta\alpha-1}\big\|G(\cdot,r)\big\|^2_{\mathbb{H}^{\ell-\theta}(\Omega)} dr &\leq \big\|G\big\|^2_{L^{\infty}(0,T;\mathbb{H}^{\ell-\theta}(\Omega))}\Big(\int_0^T (T-r)^{\alpha-\theta\alpha-1} dr\Big) \\
&= \frac{T^{\alpha-\theta\alpha}}{\alpha-\theta\alpha}\big\|G\big\|^2_{L^{\infty}(0,T;\mathbb{H}^{\ell-\theta}(\Omega))}. \qquad (15)
\end{aligned}$$

From two latter observations, we obtain

$$\big\|\mathbb{Q}^2_{a,b,\alpha,k}(\cdot,t)\big\|_{\mathbb{H}^{\ell}(\Omega)} \leq \frac{e^{k(T-t)}\overline{m}_1^2 T^{2\alpha-\theta\alpha}}{m_1(\alpha-\theta\alpha)} t^{-\alpha}\big\|G\big\|_{L^{\infty}(0,T;\mathbb{H}^{\ell-\theta}(\Omega))}.$$

Step 3. Estimate of $\mathbb{Q}^3_{a,b,\alpha,k}$. In view of Parseval's equality, we find that

$$\big\|\mathbb{Q}^3_{a,b,\alpha,k}(\cdot,t)\big\|^2_{\mathbb{H}^{\ell}(\Omega)} = \sum_{j=1}^{\infty} \lambda_j^{2\ell}\Big[\int_0^t (t-r)^{\alpha-1} e^{-k(t-r)} E_{\alpha,\alpha}(-\lambda_j (t-r)^{\alpha}) G_j(r) dr\Big]^2. \qquad (16)$$

Using Hölder inequality, one has

$$\begin{aligned}
\big\|\mathbb{Q}^3_{a,b,\alpha,k}(\cdot,t)\big\|^2_{\mathbb{H}^{\ell}(\Omega)} &\leq \sum_{j=1}^{\infty} \lambda_j^{2\ell}\Big(\int_0^t (t-r)^{\alpha-1} e^{-k(t-r)} E_{\alpha,\alpha}(-\lambda_j (t-r)^{\alpha}) dr\Big) \\
&\qquad \Big(\int_0^t (t-r)^{\alpha-1} e^{-k(t-r)} E_{\alpha,\alpha}(-\lambda_j (t-r)^{\alpha}) |G_j(r)|^2 dr\Big).
\end{aligned}$$

By a similar techniques as in (14) and (15), we also obtain that

$$\begin{aligned}
\big\|\mathbb{Q}^3_{a,b,\alpha,k}(\cdot,t)\big\|^2_{\mathbb{H}^{\ell}(\Omega)} &\leq \frac{(\overline{m}_1)^2 t^{\alpha-\theta\alpha}}{\alpha-\theta\alpha} \int_0^t (t-r)^{\alpha-\theta\alpha-1}\big\|G(\cdot,r)\big\|^2_{\mathbb{H}^{\ell-\theta}(\Omega)} dr \\
&\leq \frac{(\overline{m}_1)^2 t^{2\alpha-2\theta\alpha}}{(\alpha-\theta\alpha)^2}\big\|G\big\|^2_{L^{\infty}(0,T;\mathbb{H}^{\ell-\theta}(\Omega))}. \qquad (17)
\end{aligned}$$

Consequently, we arrive at the following estimate

$$\big\|\mathbb{Q}^3_{a,b,\alpha,k}(\cdot,t)\big\|_{\mathbb{H}^{\ell}(\Omega)} \leq \frac{\overline{m}_1 T^{\alpha-\theta\alpha}}{\alpha-\theta\alpha}\big\|G\big\|_{L^{\infty}(0,T;\mathbb{H}^{\ell-\theta}(\Omega))}.$$

Combining three steps, we deduce that

$$\begin{aligned}\left\|u^{\alpha,k}(\cdot,t)\right\|_{\mathbb{H}^{\ell}(\Omega)} &\le \left\|\mathbb{Q}^1_{a,b,\alpha,k}(\cdot,t)\right\|_{\mathbb{H}^{\ell}(\Omega)} + \left\|\mathbb{Q}^2_{a,b,\alpha,k}(\cdot,t)\right\|_{\mathbb{H}^{\ell}(\Omega)} + \left\|\mathbb{Q}^3_{a,b,\alpha,k}(\cdot,t)\right\|_{\mathbb{H}^{\ell}(\Omega)} \\ &\le \frac{1}{b}\frac{\overline{m}_1}{m_1} e^{k(T-t)} T^{\alpha} t^{-\alpha}\left\|f\right\|_{\mathbb{H}^{\ell}(\Omega)} + \frac{e^{k(T-t)}\overline{m}_1^2 T^{2\alpha-\theta\alpha}}{m_1(\alpha-\theta\alpha)} t^{-\alpha}\left\|G\right\|_{L^{\infty}(0,T;\mathbb{H}^{\ell-\theta}(\Omega))} \\ &+ \frac{\overline{m}_1 T^{\alpha-\theta\alpha}}{\alpha-\theta\alpha}\left\|G\right\|_{L^{\infty}(0,T;\mathbb{H}^{\ell-\theta}(\Omega))}.\end{aligned}$$

The proof is complete. □

4. Continuity results of the mild solution

In this section, we demonstrate the continuity results of the mild solution with respect to the tempered parameter k.

Theorem 4.1. *Suppose that* $0 < \alpha < \theta < 1$ *and* $G \in L^{\infty}(0,T;\mathbb{H}^{\ell-\theta}(\Omega))$. *Then, the following continuity in the tempered parameter holds true*

$$\begin{aligned}\left\|u^{\alpha,k}(\cdot,t) - u^{\alpha,k'}(\cdot,t)\right\|_{\mathbb{H}^{\ell}(\Omega)} &\le \mathscr{D}_1 t^{-\alpha}|k-k'|\left\|f\right\|_{\mathbb{H}^{\ell+1}(\Omega)} \\ &+ \mathscr{D}_3 t^{-\alpha}|k-k'|\Big(\|G\|_{L^{\infty}(0,T;\mathbb{H}^{\ell-\theta}(\Omega))} + \|G\|_{L^{\infty}(0,T;\mathbb{H}^{\ell+1-\theta}(\Omega))}\Big) \\ &+ \mathscr{D}_4|k-k'|\left\|G\right\|_{L^{\infty}(0,T;\mathbb{H}^{\ell-\theta}(\Omega))},\end{aligned}$$

where $\mathscr{D}_1$, $\mathscr{D}_3$ *and* $\mathscr{D}_4$ *are positive constants.*

Proof. To begin with, we follow from (10) to divide the deviation into some parts as follows

$$\begin{aligned}u^{\alpha,k}(x,t) - u^{\alpha,k'}(x,t) &= \mathbb{Q}^1_{a,b,\alpha,k}(x,t) - \mathbb{Q}^1_{a,b,\alpha,k'}(x,t) \\ &+ \mathbb{Q}^2_{a,b,\alpha,k}(x,t) - \mathbb{Q}^2_{a,b,\alpha,k'}(x,t) + \mathbb{Q}^3_{a,b,\alpha,k}(x,t) - \mathbb{Q}^3_{a,b,\alpha,k'}(x,t).\end{aligned}$$

In what follows, we find upper bounds for the three terms in the right hand side.
Step 1. Estimate of $\left\|\mathbb{Q}^1_{a,b,\alpha,k}(\cdot,t) - \mathbb{Q}^1_{a,b,\alpha,k'}(\cdot,t)\right\|_{\mathbb{H}^{\ell}(\Omega)}$. First of all, it can be seen that

$$\begin{aligned}&\mathbb{Q}^1_{a,b,\alpha,k}(x,t) - \mathbb{Q}^1_{a,b,\alpha,k'}(x,t) \\ &= \sum_{j=1}^{\infty}\left(\frac{e^{-kt}E_{\alpha,1}(-\lambda_j t^{\alpha})}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^{\alpha})} - \frac{e^{-k't}E_{\alpha,1}(-\lambda_j t^{\alpha})}{a+be^{-k'T}E_{\alpha,1}(-\lambda_j T^{\alpha})}\right) f_j e_j(x).\end{aligned}$$

In addition, we note that the kernel inside the series has the following explicit formula

$$\begin{aligned}&\frac{e^{-kt}E_{\alpha,1}(-\lambda_j t^{\alpha})}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^{\alpha})} - \frac{e^{-k't}E_{\alpha,1}(-\lambda_j t^{\alpha})}{a+be^{-k'T}E_{\alpha,1}(-\lambda_j T^{\alpha})} \\ &= \frac{a\Big(e^{-kt} - e^{-k't}\Big)E_{\alpha,1}(-\lambda_j t^{\alpha})}{\Big(a+be^{-kT}E_{\alpha,1}(-\lambda_j T^{\alpha})\Big)\Big(a+be^{-k'T}E_{\alpha,1}(-\lambda_j T^{\alpha})\Big)} \\ &+ \frac{be^{-(k+k')T}\Big(e^{k(T-t)} - e^{k'(T-t)}\Big)E_{\alpha,1}(-\lambda_j t^{\alpha})E_{\alpha,1}(-\lambda_j T^{\alpha})}{\Big(a+be^{-kT}E_{\alpha,1}(-\lambda_j T^{\alpha})\Big)\Big(a+be^{-k'T}E_{\alpha,1}(-\lambda_j T^{\alpha})\Big)} \\ &= B_1 + B_2. \qquad (18)\end{aligned}$$

By using the fundamental inequality $|e^{-a} - e^{-b}| \le |a-b|$, for any $a, b \ge 0$, we can bound the first quantity as follows

$$|B_1| \le \frac{ae^{(k+k')T}|k-k'|tE_{\alpha,1}(-\lambda_j t^\alpha)}{b^2|E_{\alpha,1}(-\lambda_j T^\alpha)|^2} \le \frac{ae^{(k+k')T}|k-k'|T}{b^2}\frac{\overline{m_1}}{m_1}T^\alpha t^{-\alpha}\frac{1+\lambda_j T^\alpha}{m_1}$$
$$\le \frac{T^{\alpha+1}ae^{(k+k')T}\overline{m_1}}{b^2 m_1^2}|k-k'|t^{-\alpha}(1+\lambda_j T^\alpha), \tag{19}$$

where we have used Lemma 2.3 and the inequality $E_{\alpha,1}(-\lambda_j T^\alpha) \ge \frac{m_1}{1+\lambda_j T^\alpha}$. For the second quantity, it is obvious to see that

$$|B_2| \le \frac{bE_{\alpha,1}(-\lambda_j t^\alpha)e^{-(k+k')T}\left|e^{k(T-t)} - e^{k'(T-t)}\right|}{b^2 e^{-(k+k')T}E_{\alpha,1}(-\lambda_j T^\alpha)} = \frac{1}{b}\frac{E_{\alpha,1}(-\lambda_j t^\alpha)\left|e^{k(T-t)} - e^{k'(T-t)}\right|}{E_{\alpha,1}(-\lambda_j T^\alpha)}.$$

The inequality $|e^a - e^b| \le |a-b|e^{\max(a,b)}$, for any $a, b \ge 0$, allows us to obtain

$$|B_2| \le \frac{E_{\alpha,1}(-\lambda_j t^\alpha)}{bE_{\alpha,1}(-\lambda_j T^\alpha)}|k(T-t) - k'(T-t)|e^{\max(k,k')(T-t)} \le \frac{T}{b}\frac{\overline{m_1}}{m_1}T^\alpha t^{-\alpha}|k-k'|e^{\max(k,k')T}$$
$$\le \frac{T^{\alpha+1}\overline{m_1}}{bm_1}t^{-\alpha}|k-k'|e^{\max(k,k')T} \le \frac{T^{\alpha+1}\overline{m_1}}{bm_1}t^{-\alpha}|k-k'|e^{\max(k,k')T}(1+\lambda_j T^\alpha). \tag{20}$$

Combining (18), (19), (20), we deduce that

$$\left|\frac{e^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a + be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)} - \frac{e^{-k't}E_{\alpha,1}(-\lambda_j t^\alpha)}{a + be^{-k'T}E_{\alpha,1}(-\lambda_j T^\alpha)}\right|$$
$$\le \frac{\overline{m}_1 T^{\alpha+1}}{bm_1}\Big(\frac{ae^{(k+k')T}}{bm_1} + e^{\max(k,k')T}\Big)t^{-\alpha}|k-k'|(1+\lambda_j T^\alpha). \tag{21}$$

This implies that

$$\left\|\mathbb{Q}^1_{a,b,\alpha,k}(\cdot,t) - \mathbb{Q}^1_{a,b,\alpha,k'}(\cdot,t)\right\|^2_{\mathbb{H}^\ell(\Omega)}$$
$$= \sum_{j=1}^\infty \lambda_j^{2\ell}\left|\frac{e^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a + be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)} - \frac{e^{-k't}E_{\alpha,1}(-\lambda_j t^\alpha)}{a + be^{-k'T}E_{\alpha,1}(-\lambda_j T^\alpha)}\right|^2 f_j^2$$
$$\le \left(\frac{\overline{m}_1 T^{\alpha+1}}{bm_1}\Big(\frac{ae^{(k+k')T}}{bm_1} + e^{\max(k,k')T}\Big)\right)^2 t^{-2\alpha}|k-k'|^2\sum_{j=1}^\infty \lambda_j^{2\ell}(1+\lambda_j T^\alpha)^2 f_j^2.$$

Due to the fact that $1 \le \lambda_1^{-1}\lambda_j$, we derive

$$\sum_{j=1}^\infty \lambda_j^{2\ell}(1+\lambda_j T^\alpha)^2 f_j^2 \le \left(T^\alpha + \lambda_1^{-1}\right)^2\sum_{j=1}^\infty \lambda_j^{2\ell+2}f_j^2 = \left(T^\alpha + \lambda_1^{-1}\right)^2\|f\|^2_{\mathbb{H}^{\ell+1}(\Omega)}.$$

From the above observations, we deduce that

$$\left\|\mathbb{Q}^1_{a,b,\alpha,k}(\cdot,t) - \mathbb{Q}^1_{a,b,\alpha,k'}(\cdot,t)\right\|_{\mathbb{H}^\ell(\Omega)} \le \mathscr{D}_1 t^{-\alpha}|k-k'|\|f\|_{\mathbb{H}^{\ell+1}(\Omega)} \tag{22}$$

where $\mathscr{D}_1$ is a positive constant which depends on $\overline{m}_1, m_1, a, T, b, \lambda_1, k, k', \alpha$.

Step 2. Estimate of $\left\|\mathbb{Q}^2_{a,b,\alpha,k}(\cdot,t)-\mathbb{Q}^2_{a,b,\alpha,k'}(\cdot,t)\right\|_{\mathbb{H}^\ell(\Omega)}$. It is obvious to see that

$$\begin{aligned}\mathbb{Q}^2_{a,b,\alpha,k}(t)-\mathbb{Q}^2_{a,b,\alpha,k'}(t)&=\sum_{j=1}^{\infty}\Big(\frac{-be^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)}\\&\times\int_0^T (T-r)^{\alpha-1}\Big(e^{-k(T-r)}-e^{-k'(T-r)}\Big)E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j(r)dr\Big)e_j(x)\\&-\sum_{j=1}^{\infty}\Big(\frac{be^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)}-\frac{be^{-k't}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-k'T}E_{\alpha,1}(-\lambda_j T^\alpha)}\Big)\\&\times\Big(\int_0^T (T-r)^{\alpha-1}e^{-k'(T-r)}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j(r)dr\Big)e_j(x).\end{aligned}\tag{23}$$

For the sake of convenience, from now on, we define by

$$\begin{aligned}\mathscr{B}_1(x,t):=\sum_{j=1}^{\infty}&\Big(\frac{-be^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)}\\&\times\int_0^T (T-r)^{\alpha-1}\Big(e^{-k(T-r)}-e^{-k'(T-r)}\Big)E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j(r)dr\Big)e_j(x),\end{aligned}$$

and

$$\begin{aligned}\mathscr{B}_2(x,t):=-\sum_{j=1}^{\infty}&\Big(\frac{be^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)}-\frac{be^{-k't}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-k'T}E_{\alpha,1}(-\lambda_j T^\alpha)}\Big)\\&\times\Big(\int_0^T (T-r)^{\alpha-1}e^{-k'(T-r)}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j(r)dr\Big)e_j(x).\end{aligned}$$

By Parseval's equality, the first term can be estimated as

$$\begin{aligned}\left\|\mathscr{B}_1(\cdot,t)\right\|^2_{\mathbb{H}^\ell(\Omega)}=\sum_{j=1}^{\infty}\lambda_j^{2\ell}&\Big(\frac{-be^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)}\Big)^2\\&\Big(\int_0^T (T-r)^{\alpha-1}\Big(e^{-k(T-r)}-e^{-k'(T-r)}\Big)E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j(r)dr\Big)^2.\end{aligned}$$

Looking at (11), we know that

$$\lambda_j^{2\ell}\Big(\frac{-be^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)}\Big)^2\le\lambda_j^{2\ell}\Big(\frac{T^\alpha e^{k(T-t)}\overline{m}_1}{m_1}\Big)^2 t^{-2\alpha}.$$

Using the inequality $|e^{-a}-e^{-b}|\le|a-b|$ for any $a,b\ge 0$ and the Hölder inequality, we find that

$$\begin{aligned}&\Big(\int_0^T (T-r)^{\alpha-1}\Big(e^{-k(T-r)}-e^{-k'(T-r)}\Big)E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j(r)dr\Big)^2\\&\le|k-k'|^2\Big(\int_0^T (T-r)^{\alpha}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j(r)dr\Big)^2\\&\le|k-k'|^2\Big(\int_0^T (T-r)^{\alpha}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)dr\Big)\Big(\int_0^T (T-r)^{\alpha}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j^2(r)dr\Big).\end{aligned}$$

On the other hand, in view of (13), one has

$$\int_0^T (T-r)^\alpha E_{\alpha,\alpha}(-\lambda_j (T-r)^\alpha) dr \le \overline{m}_1 \lambda_j^{-\theta} \int_0^T (T-r)^{\alpha-\theta\alpha} dr$$
$$= \overline{m}_1 \lambda_j^{-\theta} \frac{T^{\alpha-\theta\alpha+1}}{\alpha-\theta\alpha+1},$$

where we remind that $0 < \alpha < \theta$. By collecting the previous evaluations, one gets

$$\begin{aligned}\Big\|\mathscr{B}_1(\cdot,t)\Big\|^2_{\mathbb{H}^\ell(\Omega)} &\le |k-k'|^2 \overline{m}_1 \frac{T^{\alpha-\theta\alpha+1}}{\alpha-\theta\alpha+1}\Big(\frac{T^\alpha e^{k(T-t)}\overline{m}_1}{m_1}\Big)^2 t^{-2\alpha} \\ &\quad\times\Big(\int_0^T (T-r)^\alpha \Big(\sum_{j=1}^\infty \lambda_j^{2\ell-\theta} E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha) G_j^2(r)\Big) dr\Big) \\ &\le \overline{M}_1 |k-k'|^2 t^{-2\alpha}\Big(\int_0^T (T-r)^{\alpha-\theta\alpha}\Big\|G(\cdot,r)\Big\|^2_{\mathbb{H}^{\ell-\theta}(\Omega)} dr\Big) \\ &\le (\overline{M}_2)^2 |k-k'|^2 t^{-2\alpha}\Big\|G\Big\|^2_{L^\infty(0,T;\mathbb{H}^{\ell-\theta}(\Omega))}.\end{aligned}$$

Here $\overline{M}_1 > 0$ depends on $\overline{m}_1, m_1, \alpha, T, k, \theta$, and $\overline{M}_2 > 0$ depends on $\overline{M}_1$, T, α, θ. Consequently,

$$\Big\|\mathscr{B}_1(\cdot,t)\Big\|_{\mathbb{H}^\ell(\Omega)} \le \overline{M}_2 |k-k'| t^{-\alpha}\Big\|G\Big\|_{L^\infty(0,T;\mathbb{H}^{\ell-\theta}(\Omega))}. \tag{24}$$

Our next aim is to bound the second quantity $\mathscr{B}_2(\cdot,t)$ in the space $\mathbb{H}^\ell(\Omega)$. By using Parseval's equality, one gets

$$\begin{aligned}\Big\|\mathscr{B}_2(\cdot,t)\Big\|^2_{\mathbb{H}^\ell(\Omega)} &= \sum_{j=1}^\infty \lambda_j^{2\ell}\Big(\frac{be^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)} - \frac{be^{-k't}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-k'T}E_{\alpha,1}(-\lambda_j T^\alpha)}\Big)^2 \\ &\quad\times\Big(\int_0^T (T-r)^{\alpha-1} e^{-k'(T-r)} E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha) G_j(r) dr\Big)^2.\end{aligned}$$

In view of (21), one can see that

$$\Big|\frac{be^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)} - \frac{be^{-k't}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-k'T}E_{\alpha,1}(-\lambda_j T^\alpha)}\Big| \le \mathscr{D}_1^* t^{-\alpha}|k-k'|(1+\lambda_j T^\alpha)$$

where $\mathscr{D}_1^* > 0$ depends on $m_1, \overline{m}_1, T, \alpha, a, b, k, k'$. This leads to

$$\begin{aligned}\Big\|\mathscr{B}_2(\cdot,t)\Big\|^2_{\mathbb{H}^\ell(\Omega)} &\le \sum_{j=1}^\infty \lambda_j^{2\ell} (\mathscr{D}_1^*)^2 t^{-2\alpha}|k-k'|^2 (1+\lambda_j T^\alpha)^2 \\ &\quad\times\Big(\int_0^T (T-r)^{\alpha-1} e^{-k'(T-r)} E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha) G_j(r) dr\Big)^2 \\ &\le 2(\mathscr{D}_1^*)^2 t^{-2\alpha}|k-k'|^2 \sum_{j=1}^\infty \lambda_j^{2\ell}\Big(\int_0^T (T-r)^{\alpha-1} e^{-k'(T-r)} E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha) G_j(r) dr\Big)^2 \\ &\quad+ 2(\mathscr{D}_1^*)^2 T^{2\alpha} t^{-2\alpha}|k-k'|^2 \sum_{j=1}^\infty \lambda_j^{2\ell+2}\Big(\int_0^T (T-r)^{\alpha-1} e^{-k'(T-r)} E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha) G_j(r) dr\Big)^2.\end{aligned}$$

Combining two estimates (16) and (17) and noting that $0 < \alpha < \theta < 1$, we find that

$$\sum_{j=1}^{\infty} \lambda_j^{2\ell} \Big(\int_0^T (T-r)^{\alpha-1} e^{-k'(T-r)} E_{\alpha,\alpha}(-\lambda_j (T-r)^\alpha) G_j(r) dr \Big)^2$$
$$\leq \frac{(\overline{m}_1)^2 T^{2\alpha-2\theta\alpha}}{(\alpha-\theta\alpha)^2} \big\|G\big\|^2_{L^\infty(0,T;\mathbb{H}^{\ell-\theta}(\Omega))}$$

and

$$\sum_{j=1}^{\infty} \lambda_j^{2\ell+2} \Big(\int_0^T (T-r)^{\alpha-1} e^{-k'(T-r)} E_{\alpha,\alpha}(-\lambda_j (T-r)^\alpha) G_j(r) dr \Big)^2$$
$$\leq \frac{(\overline{m}_1)^2 T^{2\alpha-2\theta\alpha}}{(\alpha-\theta\alpha)^2} \big\|G\big\|^2_{L^\infty(0,T;\mathbb{H}^{\ell+1-\theta}(\Omega))}.$$

Form three latter observations, one finds that

$$\big\|\mathscr{B}_2(\cdot,t)\big\|^2_{\mathbb{H}^\ell(\Omega)} \leq (\mathscr{D}_2)^2 t^{-2\alpha} |k-k'|^2 \Big(\big\|G\big\|^2_{L^\infty(0,T;\mathbb{H}^{\ell-\theta}(\Omega))} + \big\|G\big\|^2_{L^\infty(0,T;\mathbb{H}^{\ell+1-\theta}(\Omega))} \Big)$$

where $\mathscr{D}_2 > 0$ depends on $m_1, \overline{m}_1, T, \alpha, \theta, a, b, k, k'$. Thus, we obtain

$$\big\|\mathscr{B}_2(\cdot,t)\big\|_{\mathbb{H}^\ell(\Omega)} \leq \mathscr{D}_2 t^{-\alpha} |k-k'| \Big(\big\|G\big\|_{L^\infty(0,T;\mathbb{H}^{\ell-\theta}(\Omega))} + \big\|G\big\|_{L^\infty(0,T;\mathbb{H}^{\ell+1-\theta}(\Omega))} \Big). \tag{25}$$

Combining (23), (24) and (25), we deduce that

$$\begin{aligned} \big\|\mathbb{Q}^2_{a,b,\alpha,k}(\cdot,t) - \mathbb{Q}^2_{a,b,\alpha,k'}(\cdot,t)\big\|_{\mathbb{H}^\ell(\Omega)} &\leq \big\|\mathscr{B}_1(\cdot,t)\big\|_{\mathbb{H}^\ell(\Omega)} + \big\|\mathscr{B}_2(\cdot,t)\big\|_{\mathbb{H}^\ell(\Omega)} \\ &\leq \mathscr{D}_3 t^{-\alpha} |k-k'| \Big(\big\|G\big\|_{L^\infty(0,T;\mathbb{H}^{\ell-\theta}(\Omega))} + \big\|G\big\|_{L^\infty(0,T;\mathbb{H}^{\ell+1-\theta}(\Omega))} \Big). \end{aligned} \tag{26}$$

Here, $\mathscr{D}_3 > 0$ depends on $m_1, \overline{m}_1, T, \alpha, \theta, a, b, k, k'$.
Step 3. Estimate of $\big\|\mathbb{Q}^3_{a,b,\alpha,k}(\cdot,t) - \mathbb{Q}^3_{a,b,\alpha,k'}(\cdot,t)\big\|_{\mathbb{H}^\ell(\Omega)}$. First of all, we have the following difference

$$\begin{aligned} \mathbb{Q}^3_{a,b,\alpha,k}(x,t) - \mathbb{Q}^3_{a,b,\alpha,k'}(x,t) &= \sum_{j=1}^{\infty} \Big[\int_0^t (t-r)^{\alpha-1} e^{-k(t-r)} E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha) G_j(r) dr \Big] e_j(x) \\ &\quad - \sum_{j=1}^{\infty} \Big[\int_0^t (t-r)^{\alpha-1} e^{-k'(t-r)} E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha) G_j(r) dr \Big] e_j(x) \\ &= \sum_{j=1}^{\infty} \Big[\int_0^t (t-r)^{\alpha-1} \Big(e^{-k(t-r)} - e^{-k'(t-r)} \Big) E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha) G_j(r) dr \Big] e_j(x) \\ &\leq \sum_{j=1}^{\infty} \Big[|k-k'| \int_0^t (t-r)^{\alpha} E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha) G_j(r) dr \Big] e_j(x). \end{aligned}$$

By Hölder's inequality, it is obvious to see that

$$\begin{aligned} &|k-k'|^2 \Big(\int_0^t (t-r)^{\alpha} E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha) G_j(r) dr \Big)^2 \\ &\leq |k-k'|^2 \Big(\int_0^t (t-r)^{\alpha} E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha) dr \Big) \Big(\int_0^t (t-r)^{\alpha} E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha) G_j^2(r) dr \Big) \\ &\leq |k-k'|^2 \overline{m}_1 \lambda_j^{-\theta} \frac{t^{\alpha-\theta\alpha+1}}{\alpha-\theta\alpha+1} \Big(\int_0^t (t-r)^{\alpha} E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha) G_j^2(r) dr \Big). \end{aligned}$$

To together with Parsval's equality allows us to obtain

$$\begin{aligned}
&\left\|\mathbb{Q}^3_{a,b,\alpha,k}(\cdot,t)-\mathbb{Q}^3_{a,b,\alpha,k'}(\cdot,t)\right\|^2_{\mathbb{H}^\ell(\Omega)}\\
&=\sum_{j=1}^{\infty}\lambda_j^{2\ell}\Big[\int_0^t (t-r)^{\alpha-1}\Big(e^{-k(t-r)}-e^{-k'(t-r)}\Big)E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha)G_j(r)dr\Big]^2\\
&\le \sum_{j=1}^{\infty}\lambda_j^{2\ell}|k-k'|^2\overline{m_1}\lambda_j^{-\theta}\frac{t^{\alpha-\theta\alpha+1}}{\alpha-\theta\alpha+1}\int_0^t (t-r)^{\alpha}E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha)G_j^2(r)dr\\
&\le |k-k'|^2\overline{m}_1\frac{t^{\alpha-\theta\alpha+1}}{\alpha-\theta\alpha+1}\Big(\int_0^t (t-r)^{\alpha}\Big(\sum_{j=1}^{\infty}\lambda_j^{2\ell-\theta}E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha)G_j^2(r)dr\Big)\Big)\\
&\le |k-k'|^2\overline{m}_1^2\frac{t^{\alpha-\theta\alpha+1}}{\alpha-\theta\alpha+1}\Big(\int_0^t (t-r)^{\alpha-\theta\alpha}\left\|G(\cdot,r)\right\|^2_{\mathbb{H}^{\ell-\theta}(\Omega)}dr\Big)\\
&\le |k-k'|^2\overline{m}_1^2\Big(\frac{t^{\alpha-\theta\alpha+1}}{\alpha-\theta\alpha+1}\Big)^2\left\|G\right\|^2_{L^\infty(0,T;H^{\ell-\theta}(\Omega))}\\
&\le |k-k'|^2\overline{m}_1^2\Big(\frac{T^{\alpha-\theta\alpha+1}}{\alpha-\theta\alpha+1}\Big)^2\left\|G\right\|^2_{L^\infty(0,T;H^{\ell-\theta}(\Omega))}.
\end{aligned}$$

Hence, we have the following bound

$$\left\|\mathbb{Q}^3_{a,b,\alpha,k}(\cdot,t)-\mathbb{Q}^3_{a,b,\alpha,k'}(\cdot,t)\right\|_{\mathbb{H}^\ell(\Omega)}\le \mathscr{D}_4|k-k'|\left\|G\right\|_{L^\infty(0,T;H^{\ell-\theta}(\Omega))}, \tag{27}$$

where $\mathscr{D}_4$ depends on $\overline{m}_1, T, \alpha, \theta$. Combining (22), (26), and (27), we obtain

$$\begin{aligned}
\left\|u^{\alpha,k}(\cdot,t)-u^{\alpha,k'}(\cdot,t)\right\|_{\mathbb{H}^\ell(\Omega)}&\le \mathscr{D}_1 t^{-\alpha}|k-k'|\left\|f\right\|_{\mathbb{H}^{\ell+1}(\Omega)}\\
&\quad+\mathscr{D}_3 t^{-\theta}|k-k'|\Big(\left\|G\right\|_{L^\infty(0,T;\mathbb{H}^{\ell-\theta}(\Omega))}+\left\|G\right\|_{L^\infty(0,T;\mathbb{H}^{\ell+1-\theta}(\Omega))}\Big)\\
&\quad+\mathscr{D}_4|k-k'|\left\|G\right\|_{L^\infty(0,T;H^{\ell-\theta}(\Omega))}.
\end{aligned}$$

The proof is complete. □

5. Convergence of the mild solution

This section outlines convergence results as the parameters $b\to 0^+, a\to 0^+$ and $\alpha\to 1^-$, bridging the current terminal fractional framework with conventional models.

5.1. Convergence of the mild solution when $b\to 0^+$

It should be noted that, when $b=0$, Problem (6) becomes the following initial value problem

$$\begin{cases}
D_t^{\alpha,k}u_a^*-\Delta u_a^*=G(x,t), & \text{in}\quad \Omega\times(0,T],\\
u_a^*(x,t)=0, & \text{on}\quad \partial\Omega\times(0,T],\\
au_a^*(x,0)=f(x), & \text{in}\quad \Omega.
\end{cases}\tag{28}$$

To avoid any confusion, we denote by $u_{a,b}^{\alpha,k}$ and u_a^* the mild solutions to Problem (6) and Problem (28) respectively. In the following theorem, we show that $u_{a,b}^{\alpha,k}$ tends to u_a^* when $b\to 0^+$.

Theorem 5.1. *Assume that $a>0$. Let $f\in\mathbb{H}^{\ell-1}(\Omega)$ and $G\in L^\infty(0,T;\mathbb{H}^{\ell-\theta}(\Omega))$. Then, there holds*

$$\left\|u_{a,b}^{\alpha,k}(\cdot,t)-u_a^*(\cdot,t)\right\|_{\mathbb{H}^\ell(\Omega)}\le \frac{b}{a^2}\frac{\overline{m}_1}{T^\alpha}\left\|f\right\|_{\mathbb{H}^{\ell-1}(\Omega)}+\frac{b}{a}\frac{\overline{m}_1T^{\alpha-\alpha\theta}}{(\alpha-\alpha\theta)}\left\|G\right\|_{L^\infty(0,T;\mathbb{H}^{\ell-\theta}(\Omega))}.$$

Proof. From (7), it can be seen the mild solution of Problem (28) is as follows

$$u_a^*(x,t) = \sum_{j=1}^{\infty} \Big(e^{-kt} E_{\alpha,1}(-\lambda_j t^\alpha) \frac{f_j}{a} + \int_0^t (t-r)^{\alpha-1} e^{-k(t-r)} E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha) G_j(r) dr \Big) e_j(x). \tag{29}$$

Combining (10) and (29), we get

$$\begin{aligned} u_{a,b}^{\alpha,k}(x,t) - u_a^*(x,t) &= \sum_{j=1}^{\infty} \Big(\frac{e^{-kt} E_{\alpha,1}(-\lambda_j t^\alpha)}{a + be^{-kT} E_{\alpha,1}(-\lambda_j T^\alpha)} - \frac{e^{-kt} E_{\alpha,1}(-\lambda_j t^\alpha)}{a} \Big) f_j e_j(x) \\ &- b \sum_{j=1}^{\infty} \Big(\frac{e^{-kt} E_{\alpha,1}(-\lambda_j t^\alpha)}{a + be^{-kT} E_{\alpha,1}(-\lambda_j T^\alpha)} \int_0^T (T-r)^{\alpha-1} e^{-k(T-r)} E_{\alpha,\alpha}(-\lambda_j (T-r)^\alpha) G_j(r) dr \Big) e_j(x) \\ &= \mathscr{Q}_1 + \mathscr{Q}_2. \end{aligned} \tag{30}$$

Using Parseval's equality, the norm of $\mathscr{Q}_1$ can be rewritten as

$$\begin{aligned} \big\| \mathscr{Q}_1 \big\|^2_{\mathbb{H}^\ell(\Omega)} &= \sum_{j=1}^{\infty} \lambda_j^{2\ell} \Big(\frac{e^{-kt} E_{\alpha,1}(-\lambda_j t^\alpha)}{a + be^{-kT} E_{\alpha,1}(-\lambda_j T^\alpha)} - \frac{e^{-kt} E_{\alpha,1}(-\lambda_j t^\alpha)}{a} \Big)^2 |f_j|^2 \\ &= b^2 e^{-2k(T+t)} \sum_{j=1}^{\infty} \lambda_j^{2\ell} \frac{|E_{\alpha,1}(-\lambda_j t^\alpha)|^2 |E_{\alpha,1}(-\lambda_j T^\alpha)|^2}{a^2 \Big(a + be^{-kT} E_{\alpha,1}(-\lambda_j T^\alpha) \Big)^2} f_j^2. \end{aligned} \tag{31}$$

Using Lemma 2.2 and noting that $E_{\alpha,1}(-z) \le 1$ for $z > 0$, we arrive at

$$\frac{|E_{\alpha,1}(-\lambda_j t^\alpha)|^2 |E_{\alpha,1}(-\lambda_j T^\alpha)|^2}{a^2 \Big(a + be^{-kT} E_{\alpha,1}(-\lambda_j T^\alpha) \Big)^2} \le \frac{1}{a^4} \frac{(\overline{m}_1)^2}{\lambda_j^2 T^{2\alpha}},$$

where we note that the denominator component on the left hand side is larger than a^4. Therefore, we have from (31) that

$$\big\| \mathscr{Q}_1 \big\|^2_{\mathbb{H}^\ell(\Omega)} \le \frac{b^2}{a^4} \frac{(\overline{m}_1)^2}{T^{2\alpha}} \sum_{j=1}^{\infty} \lambda_j^{2\ell-2} f_j^2,$$

which allows us to obtain

$$\big\| \mathscr{Q}_1 \big\|_{\mathbb{H}^\ell(\Omega)} \le \frac{b}{a^2} \frac{\overline{m}_1}{T^\alpha} \big\| f \big\|_{\mathbb{H}^{\ell-1}(\Omega)}. \tag{32}$$

Using Parseval's equality, one gets the following bound

$$\begin{aligned} &\big\| \mathscr{Q}_2 \big\|^2_{\mathbb{H}^\ell(\Omega)} \\ &= \sum_{j=1}^{\infty} \lambda_j^{2\ell} \Big[\frac{-be^{-kt} E_{\alpha,1}(-\lambda_j t^\alpha)}{a + be^{-kT} E_{\alpha,1}(-\lambda_j T^\alpha)} \int_0^T (T-r)^{\alpha-1} e^{-k(T-r)} E_{\alpha,\alpha}(-\lambda_j (T-r)^\alpha) G_j(r) dr \Big]^2 \\ &\le \frac{b^2}{a^2} \sum_{j=1}^{\infty} \lambda_j^{2\ell} \Big[\int_0^T (T-r)^{\alpha-1} e^{-k(T-r)} E_{\alpha,\alpha}(-\lambda_j (T-r)^\alpha) G_j(r) dr \Big]^2 \end{aligned} \tag{33}$$

where we note that

$$\frac{be^{-kt} E_{\alpha,1}(-\lambda_j t^\alpha)}{a + be^{-kT} E_{\alpha,1}(-\lambda_j T^\alpha)} \le \frac{b}{a}.$$

Combining (14), (15) and (33), we obtain the following bound

$$\left\|\mathcal{Q}_2\right\|^2_{\mathbb{H}^\ell(\Omega)} \leq \frac{b^2}{a^2}\frac{(\overline{m}_1)^2T^{2\alpha-2\theta\alpha}}{(\alpha-\theta\alpha)^2}\left\|G\right\|^2_{L^\infty(0,T;\mathbb{H}^{\ell-\theta}(\Omega))}. \tag{34}$$

By collecting three previous results (30), (32) and (34), we derive that

$$\begin{aligned}\left\|u^{\alpha,k}_{a,b}(\cdot,t)-u^*_a(\cdot,t)\right\|_{\mathbb{H}^\ell(\Omega)} &\leq \left\|\mathcal{Q}_1\right\|_{\mathbb{H}^\ell(\Omega)}+\left\|\mathcal{Q}_2\right\|_{\mathbb{H}^\ell(\Omega)}\\ &\leq \frac{b}{a^2}\frac{\overline{m}_1}{T^\alpha}\left\|f\right\|_{\mathbb{H}^{\ell-1}(\Omega)}+\frac{b}{a}\frac{\overline{m}_1T^{\alpha-\theta\alpha}}{\alpha-\theta\alpha}\left\|G\right\|_{L^\infty(0,T;\mathbb{H}^{\ell-\theta}(\Omega))}.\end{aligned}$$

The proof is complete. □

5.2. *Convergence of the mild solution when $a\to 0^+$*

Noting that, when $a=0$, Problem (6) becomes the following terminal value problem

$$\begin{cases} D^{\alpha,k}_t w^*_b - \Delta w^*_b = G(x,t), & \text{in} \quad \Omega\times(0,T],\\ w^*_b(x,t)=0, & \text{on} \quad \partial\Omega\times(0,T],\\ bw^*_b(x,T)=f(x), & \text{in} \quad \Omega.\end{cases} \tag{35}$$

In the following theorem, we show that, when $a\to 0^+$, the solution $u^{\alpha,k}_{a,b}$ of Problem (6) tends to the solution w^*_b of Problem (35).

Theorem 5.2. *Let $b>0$. Assume that $f\in\mathbb{H}^{\ell+\frac{1}{2}}(\Omega)$ and $G\in L^\infty(0,T;\mathbb{H}^{\frac{2\ell-2\theta+1}{2}}(\Omega))$. Then*

$$\begin{aligned}&\left\|u^{\alpha,k}_{a,b}(\cdot,t)-w^*_b(\cdot,t)\right\|_{L^p(0,T;\mathbb{H}^\ell(\Omega))}\\ &\qquad\leq C_4\Big(\frac{\sqrt{a}}{\sqrt{2}b\sqrt{b}}+\frac{\sqrt{a}}{\sqrt{2}\sqrt{b}}\Big)\Big(\left\|f\right\|_{\mathbb{H}^{\ell+\frac{1}{2}}(\Omega)}+\left\|G\right\|_{L^\infty(0,T;\mathbb{H}^{\frac{2\ell-2\theta+1}{2}}(\Omega))}\Big),\end{aligned}$$

where $C_4=C_4(m_1,\overline{m}_1,k,T,\alpha,\lambda_1,\theta,p)$ and $1<p<\frac{1}{\alpha}$.

Proof. The mild solution to Problem (35) is given by

$$\begin{aligned}w^*_b(x,t)&=\sum_{j=1}^\infty\frac{e^{k(T-t)}E_{\alpha,1}(-\lambda_jt^\alpha)}{bE_{\alpha,1}(-\lambda_jT^\alpha)}f_je_j(x)\\ &-\sum_{j=1}^\infty\Big(\frac{e^{k(T-t)}E_{\alpha,1}(-\lambda_jt^\alpha)}{E_{\alpha,1}(-\lambda_jT^\alpha)}\int_0^T(T-r)^{\alpha-1}e^{-k(T-r)}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j(r)dr\Big)e_j(x)\\ &+\sum_{j=1}^\infty\Big(\int_0^t(t-r)^{\alpha-1}e^{-k(t-r)}E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha)G_j(r)dr\Big)e_j(x).\end{aligned} \tag{36}$$

Combining (36) and (9), we get

$$\begin{aligned}u^{\alpha,k}_{a,b}(x,t)-w^*_b(x,t)&=\sum_{j=1}^\infty\Big(\frac{e^{-kt}E_{\alpha,1}(-\lambda_jt^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_jT^\alpha)}-\frac{e^{k(T-t)}E_{\alpha,1}(-\lambda_jt^\alpha)}{bE_{\alpha,1}(-\lambda_jT^\alpha)}\Big)f_je_j(x)\\ &-\sum_{j=1}^\infty\Big(\Big(b\frac{e^{-kt}E_{\alpha,1}(-\lambda_jt^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_jT^\alpha)}-\frac{e^{k(T-t)}E_{\alpha,1}(-\lambda_jt^\alpha)}{E_{\alpha,1}(-\lambda_jT^\alpha)}\Big)\\ &\qquad\int_0^T(T-r)^{\alpha-1}e^{-k(T-r)}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j(r)dr\Big)e_j(x)\\ &=\mathcal{Q}_3(x,t)+\mathcal{Q}_4(x,t).\end{aligned} \tag{37}$$

Let us focus the term $\mathscr{Q}_3$. Using Parseval's equality, one gets the following equality

$$\begin{aligned}\left\|\mathscr{Q}_3\right\|^2_{\mathbb{H}^\ell(\Omega)} &= \sum_{j=1}^{\infty} \lambda_j^{2\ell}\Big(\frac{e^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)} - \frac{e^{k(T-t)}E_{\alpha,1}(-\lambda_j t^\alpha)}{bE_{\alpha,1}(-\lambda_j T^\alpha)}\Big)^2 |f_j|^2 \\ &= \sum_{j=1}^{\infty} \lambda_j^{2\ell} \frac{a^2 e^{2k(T-t)}|E_{\alpha,1}(-\lambda_j t^\alpha)|^2}{\Big(a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)\Big)^2 b^2 |E_{\alpha,1}(-\lambda_j T^\alpha)|^2} f_j^2.\end{aligned}$$

Using Cauchy inequality and Lemma 2.2, we find that

$$\begin{aligned}\Big(a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)\Big)^2 b^2|E_{\alpha,1}(-\lambda_j T^\alpha)|^2 &\geq 2ab^3e^{-kT}|E_{\alpha,1}(-\lambda_j T^\alpha)|^3 \\ &\geq 2ab^3e^{-kT}\Big(\frac{m_1}{1+\lambda_j T^\alpha}\Big)^3.\end{aligned}$$

Using Lemma 2.2 again, one gets $E_{\alpha,1}(-\lambda_j t^\alpha) \leq \frac{\overline{m}_1}{1+\lambda_j t^\alpha}$. This implies that

$$\begin{aligned}\left\|\mathscr{Q}_3\right\|^2_{\mathbb{H}^\ell(\Omega)} &\leq \frac{\overline{m}_1^2}{m_1^3}\frac{a}{2b^3} e^{2k(T-t)+kT} \sum_{j=1}^{\infty} \lambda_j^{2\ell} \frac{(1+\lambda_j T^\alpha)^3}{(1+\lambda_j t^\alpha)^2} f_j^2 \\ &\leq \frac{\overline{m}_1^2}{m_1^3}\frac{a}{2b^3} e^{2k(T-t)+kT} T^{2\alpha}\Big(T^\alpha + \lambda_1^{-1}\Big) t^{-2\alpha} \sum_{j=1}^{\infty} \lambda_j^{2\ell+1} f_j^2, \end{aligned} \tag{38}$$

where we note that $\frac{1+\lambda_j T^\alpha}{1+\lambda_j t^\alpha} \leq T^\alpha t^{-\alpha}$. The latter inequality allows us to obtain that

$$\left\|\mathscr{Q}_3(\cdot,t)\right\|_{\mathbb{H}^\ell(\Omega)} \leq C_1(m_1,\overline{m}_1,k,T,\alpha,\lambda_1)\frac{\sqrt{a}}{\sqrt{2}b\sqrt{b}} t^{-\alpha}\left\|f\right\|_{\mathbb{H}^{\ell+\frac{1}{2}}(\Omega)}. \tag{39}$$

Let us now return to the term $\mathscr{Q}_4$. Indeed, we get

$$\begin{aligned}&\Big(b\frac{e^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)} - \frac{e^{k(T-t)}E_{\alpha,1}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)}\Big)^2 \\ &\qquad = \frac{a^2 e^{2k(T-t)}|E_{\alpha,1}(-\lambda_j t^\alpha)|^2}{\Big(a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)\Big)^2|E_{\alpha,1}(-\lambda_j T^\alpha)|^2}.\end{aligned}$$

Using the techniques as in (38), we obtain

$$\begin{aligned}\left\|\mathscr{Q}_4(\cdot,t)\right\|^2_{\mathbb{H}^\ell(\Omega)} &\leq \frac{\overline{m}_1^2}{m_1^3}\frac{a}{2b} e^{2k(T-t)+kT} T^{2\alpha}\Big(T^\alpha + \lambda_1^{-1}\Big) t^{-2\alpha} \\ &\quad \times \sum_{j=1}^{\infty} \lambda_j^{2\ell+1}\Big(\int_0^T (T-r)^{\alpha-1} e^{-k(T-r)} E_{\alpha,\alpha}(-\lambda_j (T-r)^\alpha) G_j(r) dr\Big)^2. \end{aligned} \tag{40}$$

In the same way as (14) and Hölder inequality, we find that

$$\begin{aligned}
&\sum_{j=1}^{\infty} \lambda_j^{2\ell+1}\Big(\int_0^T (T-r)^{\alpha-1} e^{-k(T-r)} E_{\alpha,\alpha}(-\lambda_j (T-r)^{\alpha}) G_j(r) dr \Big)^2 \\
&\leq \sum_{j=1}^{\infty} \lambda_j^{2\ell+1}\Big(\int_0^T (T-r)^{\alpha-1} e^{-k(T-r)} E_{\alpha,\alpha}(-\lambda_j (T-r)^{\alpha}) dr \Big) \\
&\qquad\qquad \times \Big(\int_0^T (T-r)^{\alpha-1} e^{-k(T-r)} E_{\alpha,\alpha}(-\lambda_j (T-r)^{\alpha}) |G_j(r)|^2 dr \Big) \\
&\leq \frac{|\overline{m}_1|^2 T^{\alpha-\theta\alpha}}{\alpha-\theta\alpha} \int_0^T (T-r)^{\alpha-\theta\alpha-1} \big\| G(\cdot,r) \big\|^2_{\mathbb{H}^{\frac{2\ell-2\theta+1}{2}}(\Omega)} dr.
\end{aligned} \tag{41}$$

Combining (40) and (41), we arrive at

$$\big\| \mathcal{Q}_4(\cdot,t) \big\|^2_{\mathbb{H}^{\ell}(\Omega)} \leq C_2(m_1, \overline{m}_1, k, T, \alpha, \theta, \lambda_1) \left(\frac{\sqrt{a}}{\sqrt{2}\sqrt{b}} t^{-\alpha} \right)^2 \int_0^T (T-r)^{\alpha-\theta\alpha-1} \big\| G(\cdot,r) \big\|^2_{\mathbb{H}^{\frac{2\ell-2\theta+1}{2}}(\Omega)} dr.$$

It is not difficult to verify that

$$\int_0^T (T-r)^{\alpha-\theta\alpha-1} \big\| G(\cdot,r) \big\|^2_{\mathbb{H}^{\frac{2\ell-2\theta+1}{2}}(\Omega)} dr \leq \frac{T^{\alpha-\theta\alpha}}{\alpha-\theta\alpha} \big\| G \big\|^2_{L^{\infty}(0,T;\mathbb{H}^{\frac{2\ell-2\theta+1}{2}}(\Omega))}.$$

From two latter inequalities above, we confirm that

$$\big\| \mathcal{Q}_4(\cdot,t) \big\|_{\mathbb{H}^{\ell}(\Omega)} \leq C_3 \frac{\sqrt{a}}{\sqrt{2}\sqrt{b}} t^{-\alpha} \big\| G \big\|_{L^{\infty}(0,T;\mathbb{H}^{\frac{2\ell-2\theta+1}{2}}(\Omega))} \tag{42}$$

where $C_3 = C_3(m_1, \overline{m}_1, k, T, \alpha, \theta, \lambda_1)$. Combining (37), (39) and (42), we deduce that for any $0 < \theta < \alpha$

$$\begin{aligned}
\big\| u^{\alpha,k}_{a,b}(\cdot,t) - w^*_b(\cdot,t) \big\|_{\mathbb{H}^{\ell}(\Omega)} &\leq \big\| \mathcal{Q}_3(\cdot,t) \big\|_{\mathbb{H}^{\ell}(\Omega)} + \big\| \mathcal{Q}_4(\cdot,t) \big\|_{\mathbb{H}^{\ell}(\Omega)} \\
&\leq C_1(m_1, \overline{m}_1, k, T, \alpha, \lambda_1) \frac{\sqrt{a}}{\sqrt{2b}\sqrt{b}} t^{-\alpha} \big\| f \big\|_{\mathbb{H}^{\ell+\frac{1}{2}}(\Omega)} \\
&\quad + C_3 \frac{\sqrt{a}}{\sqrt{2}\sqrt{b}} t^{-\alpha} \big\| G \big\|_{L^{\infty}(0,T;\mathbb{H}^{\frac{2\ell-2\theta+1}{2}}(\Omega))}.
\end{aligned}$$

Since the fact that $1 < p < \frac{1}{\alpha}$, we obtain that the following bound

$$\begin{aligned}
&\big\| u^{\alpha,k}_{a,b}(\cdot,t) - w^*_b(\cdot,t) \big\|_{L^p(0,T;\mathbb{H}^{\ell}(\Omega))} \\
&\qquad \leq C_4 \Big(\frac{\sqrt{a}}{\sqrt{2b}\sqrt{b}} + \frac{\sqrt{a}}{\sqrt{2}\sqrt{b}} \Big) \Big(\big\| f \big\|_{\mathbb{H}^{\ell+\frac{1}{2}}(\Omega)} + \big\| G \big\|_{L^{\infty}(0,T;\mathbb{H}^{\frac{2\ell-2\theta+1}{2}}(\Omega))} \Big)
\end{aligned}$$

where $C_4 = C_4(m_1, \overline{m}_1, k, T, \alpha, \lambda_1, \theta, p)$. □

5.3. *Convergence of the mild solution when $\alpha \to 1^-$.*

In this subsection, we continue to investigate another convergence result, namely, the convergence of the mild solution to nonlocal problem (6) when $\alpha \to 1^-$.

Theorem 5.3. *Let $a > 0$ and $b \geq 0$. Let us assume that $f \in \mathbb{H}^{\ell-\rho}(\Omega)$ and $G \in L^2(0,T;\mathbb{H}^{\ell-\rho-\frac{1}{2}}(\Omega)) \cap L^\infty(0,T;\mathbb{H}^{\ell-\rho}(\Omega))$. Then*

$$\begin{aligned}&\left\|u^{\alpha,k}(\cdot,t)-v(\cdot,t)\right\|_{\mathbb{H}^\ell(\Omega)}\\ &\leq C_0\mathbf{V}_\rho(\alpha,1)\Big(t^{\frac{-\rho-1}{2}}+t^{-\alpha\rho}\Big)\Big(\|f\|_{\mathbb{H}^{\ell-\rho}(\Omega)}+\|G\|_{L^2(0,T;\mathbb{H}^{\ell-\rho-\frac{1}{2}}(\Omega))}+\|G\|_{L^\infty(0,T;\mathbb{H}^{\ell-\rho}(\Omega))}\Big),\end{aligned} \tag{43}$$

where

$$\mathbf{V}_\rho(\alpha,1)=(1-\alpha)^{\frac{\rho}{2}}+\Big((T^*)^{1-\alpha}-1\Big)T^{\frac{\rho}{2}}+(1-\alpha)^{\frac{\alpha(1-\rho)}{2}}+1-\alpha+\Big((T^*)^{1-\alpha}-1\Big)^{\frac{1-\rho}{2}}T^{\frac{\alpha(1-\rho)}{2}}.$$

Here $\ell \geq 0$, $0 < \rho < \frac{2\alpha}{3}$, $\rho < 2\alpha-1$ and $T^* = \max(T,1)$. The function v on the left hand side of (43) is defined by

$$\begin{aligned}v(x,t)&=\sum_{j=1}^\infty\frac{e^{-kt}e^{-\lambda_jt}}{a+be^{-kT}e^{-\lambda_jT}}f_je_j(x)\\ &\quad-b\sum_{j=1}^\infty\Big(\frac{e^{-kt}e^{-\lambda_jt}}{a+be^{-kT}e^{-\lambda_jT}}\int_0^Te^{-k(T-r)}e^{-\lambda_j(T-r)}G_j(r)dr\Big)e_j(x)\\ &\quad+\sum_{j=1}^\infty\Big(\int_0^te^{-k(t-r)}e^{-\lambda_j(t-r)}G_j(r)dr\Big)e_j(x).\end{aligned} \tag{44}$$

Remark 5.4. *Let s be a positive number such that*

$$0 < s < \min\Big(\frac{2}{\rho+1},\ \frac{1}{\alpha\rho}\Big).$$

Then, we deduce from (43) that

$$\begin{aligned}&\left\|u^{\alpha,k}-v\right\|_{L^s(0,T;\mathbb{H}^\ell(\Omega))}\\ &\leq C_0\mathbf{V}_\rho(\alpha,1)\Big(\|f\|_{\mathbb{H}^{\ell-\rho}(\Omega)}+\|G\|_{L^2(0,T;\mathbb{H}^{\ell-\rho-\frac{1}{2}}(\Omega))}+\|G\|_{L^\infty(0,T;\mathbb{H}^{\ell-\rho}(\Omega))}\Big).\end{aligned}$$

To prove the above theorem, we first prepare several key inequalities in the following lemmas.

Lemma 5.5 (see [6]). *Let $0 < \alpha < 1$. Then, for any $\mu > 0$, we get*

$$|t^{\alpha-1}-1| \leq t^{\alpha-1}\Big(C_\mu(1-\alpha)^\mu t^{-\mu}+(T^*)^{1-\alpha}-1\Big).$$

Lemma 5.6 (see [1]). *Let $3/4 \leq \alpha < 1$ and $\alpha \leq \beta \leq 1$. Then, there exists a constant C independent of α,β,z such that, for any $z < 0$,*

$$\Big|E_{\alpha,\beta}(z)-e^z\Big| \leq \frac{C}{1+|z|}(1-\alpha). \tag{45}$$

Proof. [Proof of Theorem 5.3] To begin with, we design an estimate for the error

$$\mathscr{K}_j^*(\alpha,t)=E_{\alpha,1}(-\lambda_jt^\alpha)-e^{-\lambda_jt}.$$

In view of the inequality (45), we get

$$\begin{aligned}\Big|E_{\alpha,1}(-\lambda_jt^\alpha)-e^{-\lambda_jt^\alpha}\Big| &\leq \frac{C}{1+\lambda_jt^\alpha}(1-\alpha)\\ &\leq \frac{C}{(1+\lambda_jt^\alpha)^\rho}(1-\alpha) \leq Ct^{-\alpha\rho}\lambda_j^{-\rho}(1-\alpha),\end{aligned} \tag{46}$$

for any $0 < \rho < 1$. Using the inequality $|e^{-a} - e^{-b}| \le C_{\delta,\delta'} \max(a^{-\delta}, b^{-\delta})|a - b|^{\delta'}$ for any $a, b > 0$ and $\delta, \delta' > 0$, we find that

$$\left|e^{-\lambda_j t} - e^{-\lambda_j t^\alpha}\right| \le C_{\delta,\delta'} \lambda_j^{-\delta} \max(t^{-\delta}, t^{-\alpha\delta}) \lambda_j^{\delta'} |t - t^\alpha|^{\delta'}$$
$$\le C(T, \alpha, \delta, \delta') t^{-\delta} \lambda_j^{-(\delta-\delta')} t^{\delta'} |t^{\alpha-1} - 1|^{\delta'}.$$

Using Lemma 5.5 and the inequality $(a + b)^{\delta'} \le C_{\delta'} a^{\delta'} + C_{\delta'} b^{\delta'}$, we obtain

$$|t^{\alpha-1} - 1|^{\delta'} \le t^{(\alpha-1)\delta'} \Big(C_\mu (1-\alpha)^\mu t^{-\mu} + (T^*)^{1-\alpha} - 1\Big)^{\delta'}$$
$$\le C(\mu, \delta') t^{(\alpha-1-\mu)\delta'} \Big((1-\alpha)^{\mu\delta'} + \Big((T^*)^{1-\alpha} - 1\Big)^{\delta'} t^{\mu\delta'}\Big).$$

By choosing $\mu = \alpha, \quad \delta' = \frac{1-\rho}{2}$ and $\delta = \frac{1+\rho}{2}$, we see that

$$\left|e^{-\lambda_j t} - e^{-\lambda_j t^\alpha}\right| \le C(\alpha, T, \rho) \lambda_j^{-\rho} t^{\frac{-\rho-1}{2}} \Big[(1-\alpha)^{\frac{\alpha(1-\rho)}{2}} + \Big((T^*)^{1-\alpha} - 1\Big)^{\frac{1-\rho}{2}} t^{\frac{\alpha(1-\rho)}{2}}\Big]. \tag{47}$$

Combining (46) and (47), we arrive at

$$\left|\mathscr{K}_j^*(\alpha, t)\right| \le \left|E_{\alpha,1}(-\lambda_j t^\alpha) - e^{-\lambda_j t^\alpha}\right| + \left|e^{-\lambda_j t} - e^{-\lambda_j t^\alpha}\right|$$
$$\le C(\alpha, T, \rho) \lambda_j^{-\rho} \Big(t^{\frac{-\rho-1}{2}} + t^{-\alpha\rho}\Big) \Big[(1-\alpha)^{\frac{\alpha(1-\rho)}{2}} + 1 - \alpha + \Big((T^*)^{1-\alpha} - 1\Big)^{\frac{1-\rho}{2}} t^{\frac{\alpha(1-\rho)}{2}}\Big], \tag{48}$$

for any $0 < \rho < 1$. In a similar way, we also get the following result

$$\left|E_{\alpha,\alpha}(-\lambda_j t^\alpha) - e^{-\lambda_j t}\right|$$
$$\le C(\alpha, T, \rho) \lambda_j^{-\rho} \Big(t^{\frac{-\rho-1}{2}} + t^{-\alpha\rho}\Big) \Big[(1-\alpha)^{\frac{\alpha(1-\rho)}{2}} + 1 - \alpha + \Big((T^*)^{1-\alpha} - 1\Big)^{\frac{1-\rho}{2}} t^{\frac{\alpha(1-\rho)}{2}}\Big]. \tag{49}$$

From (9) and (44), we get the following equality

$$u^{\alpha,k}(x, t) - v(x, t) = \sum_{j=1}^\infty \Big(\frac{e^{-kt} E_{\alpha,1}(-\lambda_j t^\alpha)}{a + be^{-kT} E_{\alpha,1}(-\lambda_j T^\alpha)} - \frac{e^{-kt} e^{-\lambda_j t}}{a + be^{-kT} e^{-\lambda_j T}}\Big) f_j e_j(x)$$
$$- \sum_{j=1}^\infty \Big(\frac{be^{-kt} E_{\alpha,1}(-\lambda_j t^\alpha)}{a + be^{-kT} E_{\alpha,1}(-\lambda_j T^\alpha)} \int_0^T e^{-k(T-r)} \Big((T-r)^{\alpha-1}$$
$$\times E_{\alpha,\alpha}(-\lambda_j (T-r)^\alpha) - e^{-\lambda_j (T-r)}\Big) G_j(r) dr\Big) e_j(x)$$
$$- b \sum_{j=1}^\infty \Big(\frac{e^{-kt} E_{\alpha,1}(-\lambda_j t^\alpha)}{a + be^{-kT} E_{\alpha,1}(-\lambda_j T^\alpha)} - \frac{e^{-kt} e^{-\lambda_j t}}{a + be^{-kT} e^{-\lambda_j T}}\Big) \Big(\int_0^T e^{-k(T-r)} e^{-\lambda_j (T-r)} G_j(r) dr\Big) e_j(x)$$
$$+ \sum_{j=1}^\infty \Big(\int_0^t e^{-k(t-r)} \Big((t-r)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j (t-r)^\alpha) - e^{-\lambda_j (t-r)}\Big) G_j(r) dr\Big) e_j(x)$$
$$= \sum_{i=1}^4 \mathscr{Q}_i(x, t). \tag{50}$$

We begin with the first term $\mathscr{Q}_1(x, t)$. For the sake of convenience, from now on, we denote by

$$J_{1,k}(\alpha, j, t) = \frac{e^{-kt} E_{\alpha,1}(-\lambda_j t^\alpha)}{a + be^{-kT} E_{\alpha,1}(-\lambda_j T^\alpha)}, \quad J_{1,k}^*(j, t) = \frac{e^{-kt} e^{-\lambda_j t}}{a + be^{-kT} e^{-\lambda_j T}}.$$

From the two above formulations, it is easy to see that

$$\begin{aligned}
&J_{1,k}(\alpha,j,t)-J^*_{1,k}(j,t)\\
&=\frac{ae^{-kt}}{\Big(a+be^{-kT}E_{\alpha,1}(-\lambda_jT^\alpha)\Big)\Big(a+be^{-kT}e^{-\lambda_jT}\Big)}\Big(E_{\alpha,1}(-\lambda_jt^\alpha)-e^{-\lambda_jt}\Big)\\
&+\frac{be^{-k(T+t)}}{\Big(a+be^{-kT}E_{\alpha,1}(-\lambda_jT^\alpha)\Big)\Big(a+be^{-kT}e^{-\lambda_jT}\Big)}\Big(E_{\alpha,1}(-\lambda_jt^\alpha)e^{-\lambda_jT}-e^{-\lambda_jt}E_{\alpha,1}(-\lambda_jT^\alpha)\Big)\\
&=\mathscr{M}_1(\alpha,j,t)+\mathscr{M}_2(\alpha,j,t).
\end{aligned}$$

Using the bound (48), we find that

$$\begin{aligned}
&|\mathscr{M}_1(\alpha,j,t)|\\
&\le\frac{e^{-kt}}{a}C(\alpha,T,\rho)\lambda_j^{-\rho}\Big(t^{\frac{-\rho-1}{2}}+t^{-\alpha\rho}\Big)\Big[(1-\alpha)^{\frac{\alpha(1-\rho)}{2}}+1-\alpha+\Big((T^*)^{1-\alpha}-1\Big)^{\frac{1-\rho}{2}}t^{\frac{\alpha(1-\rho)}{2}}\Big]\\
&\le C(\alpha,T,\rho,a)\lambda_j^{-\rho}\Big(t^{\frac{-\rho-1}{2}}+t^{-\alpha\rho}\Big)\Big[(1-\alpha)^{\frac{\alpha(1-\rho)}{2}}+1-\alpha+\Big((T^*)^{1-\alpha}-1\Big)^{\frac{1-\rho}{2}}T^{\frac{\alpha(1-\rho)}{2}}\Big].
\end{aligned}$$

In view of (49) and by some simple caculations, we derive that

$$\begin{aligned}
|\mathscr{M}_2(\alpha,j,t)|&\le\frac{be^{-k(T+t)}}{a+be^{-kT}e^{-\lambda_jT}}e^{-\lambda_jT}\frac{\Big|E_{\alpha,1}(-\lambda_jt^\alpha)-e^{-\lambda_jt}\Big|}{a}+\frac{b}{a^2}\Big|E_{\alpha,1}(-\lambda_jT^\alpha)-e^{-\lambda_jT}\Big|\\
&\le\frac{\Big|E_{\alpha,1}(-\lambda_jt^\alpha)-e^{-\lambda_jt}\Big|}{a}+\frac{b}{a^2}\Big|E_{\alpha,1}(-\lambda_jT^\alpha)-e^{-\lambda_jT}\Big|\\
&\le C(\alpha,T,\rho,a,b)\lambda_j^{-\rho}\Big(t^{\frac{-\rho-1}{2}}+t^{-\alpha\rho}\Big)\Big[(1-\alpha)^{\frac{\alpha(1-\rho)}{2}}+1-\alpha+\Big((T^*)^{1-\alpha}-1\Big)^{\frac{1-\rho}{2}}T^{\frac{\alpha(1-\rho)}{2}}\Big].
\end{aligned}$$

By collecting three latter bounds as above, one gets

$$\begin{aligned}
&|J_{1,k}(\alpha,j,t)-J^*_{1,k}(j,t)|\le|\mathscr{M}_1(\alpha,j,t)|+|\mathscr{M}_2(\alpha,j,t)|\\
&\le C(\alpha,T,\rho,a,b)\lambda_j^{-\rho}\Big(t^{\frac{-\rho-1}{2}}+t^{-\alpha\rho}\Big)\Big[(1-\alpha)^{\frac{\alpha(1-\rho)}{2}}+1-\alpha+\Big((T^*)^{1-\alpha}-1\Big)^{\frac{1-\rho}{2}}T^{\frac{\alpha(1-\rho)}{2}}\Big].
\end{aligned}$$

This implies that

$$\begin{aligned}
\Big\|\mathscr{Q}_1(\cdot,t)\Big\|^2_{\mathbb{H}^\ell(\Omega)}&=\sum_{j=1}^\infty\lambda_j^{2\ell}\Big(\frac{e^{-kt}E_{\alpha,1}(-\lambda_jt^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_jT^\alpha)}-\frac{e^{-kt}e^{-\lambda_jt}}{a+be^{-kT}e^{-\lambda_jT}}\Big)^2|f_j|^2\\
&\le C(\alpha,T,\rho,a,b)\Big(t^{\frac{-\rho-1}{2}}+t^{-\alpha\rho}\Big)^2\Big[(1-\alpha)^{\frac{\alpha(1-\rho)}{2}}\\
&\qquad+1-\alpha+\Big((T^*)^{1-\alpha}-1\Big)^{\frac{1-\rho}{2}}T^{\frac{\alpha(1-\rho)}{2}}\Big]^2\Big(\sum_{j=1}^\infty\lambda_j^{2\ell-2\rho}f_j^2\Big).
\end{aligned}$$

Hence, since the assumption $f\in\mathbb{H}^{\ell-\rho}(\Omega)$, we arrive at

$$\begin{aligned}
\Big\|\mathscr{Q}_1(\cdot,t)\Big\|_{\mathbb{H}^\ell(\Omega)}&\le C(\alpha,T,\rho,a,b)\Big(t^{\frac{-\rho-1}{2}}+t^{-\alpha\rho}\Big)\\
&\times\Big[(1-\alpha)^{\frac{\alpha(1-\rho)}{2}}+1-\alpha+\Big((T^*)^{1-\alpha}-1\Big)^{\frac{1-\rho}{2}}T^{\frac{\alpha(1-\rho)}{2}}\Big]\|f\|_{\mathbb{H}^{\ell-\rho}(\Omega)}. \qquad (51)
\end{aligned}$$

Let us continue to estimate the term $\left\|\mathscr{Q}_3(\cdot,t)\right\|_{\mathbb{H}^\ell(\Omega)}$. By using Hölder inequality, we obtain

$$\Big(\int_0^T e^{-k(T-r)}e^{-\lambda_j(T-r)}G_j(r)dr\Big)^2 \le \Big(\int_0^T e^{-2k(T-r)}e^{-2\lambda_j(T-r)}dr\Big)\Big(\int_0^T G_j^2(r)dr\Big) = \frac{1-e^{-2T(k+\lambda_j)}}{2(k+\lambda_j)}\Big(\int_0^T G_j^2(r)dr\Big) \le \frac{1}{2}\int_0^T \lambda_j^{-1}G_j^2(r)dr.$$

Thus, we derive that

$$\begin{aligned}\left\|\mathscr{Q}_3(\cdot,t)\right\|^2_{\mathbb{H}^\ell(\Omega)} &= b^2\sum_{j=1}^\infty \lambda_j^{2\ell}\Big(\frac{e^{-kt}E_{\alpha,1}(-\lambda_j t^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_j T^\alpha)} - \frac{e^{-kt}e^{-\lambda_j t}}{a+be^{-kT}e^{-\lambda_j T}}\Big)^2 \\ &\quad\times\Big(\int_0^T e^{-k(T-r)}e^{-\lambda_j(T-r)}G_j(r)dr\Big)^2 \\ &\le C(\alpha,T,\rho,a,b)\Big(t^{\frac{-\rho-1}{2}}+t^{-\alpha\rho}\Big)^2\Big(\int_0^T\sum_{j=1}^\infty \lambda_j^{2\ell-2\rho-1}G_j^2(r)dr\Big) \\ &\quad\times\Big[(1-\alpha)^{\frac{\alpha(1-\rho)}{2}}+1-\alpha+\Big((T^*)^{1-\alpha}-1\Big)^{\frac{1-\rho}{2}}T^{\frac{\alpha(1-\rho)}{2}}\Big]^2.\end{aligned}$$

It follows that

$$\begin{aligned}\left\|\mathscr{Q}_3(\cdot,t)\right\|_{\mathbb{H}^\ell(\Omega)} &\le C(\alpha,T,\rho,a,b)\Big(t^{\frac{-\rho-1}{2}}+t^{-\alpha\rho}\Big) \\ &\quad\times\Big[(1-\alpha)^{\frac{\alpha(1-\rho)}{2}}+1-\alpha+\Big((T^*)^{1-\alpha}-1\Big)^{\frac{1-\rho}{2}}T^{\frac{\alpha(1-\rho)}{2}}\Big]\left\|G\right\|_{L^2(0,T;\mathbb{H}^{\ell-\rho-\frac{1}{2}}(\Omega))}.\end{aligned} \tag{52}$$

We continute to estimate the term $\mathbb{Q}_4(x,t)$. Initially, we find that

$$\begin{aligned}\Big|(t-r)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha)-e^{-\lambda_j(t-r)}\Big| &\le (t-r)^{\alpha-1}\Big|E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha)-e^{-\lambda_j(t-r)}\Big| \\ &\quad+\Big|(t-r)^{\alpha-1}-1\Big|e^{-\lambda_j(t-r)} \\ &= \mathscr{M}_3(\alpha,j,t)+\mathscr{M}_4(\alpha,j,t).\end{aligned} \tag{53}$$

Denote by

$$\mathbf{E}_\rho(\alpha,1) = (1-\alpha)^{\frac{\alpha(1-\rho)}{2}}+1-\alpha+\Big((T^*)^{1-\alpha}-1\Big)^{\frac{1-\rho}{2}}T^{\frac{\alpha(1-\rho)}{2}}.$$

Using (49), we obtain

$$\mathscr{M}_3(\alpha,j,t) \le C(\alpha,T,\rho)\mathbf{E}_\rho(\alpha,1)\lambda_j^{-\rho}(t-r)^{\alpha-1}\Big((t-r)^{\frac{-\rho-1}{2}}+(t-r)^{-\alpha\rho}\Big). \tag{54}$$

In view of Lemma 5.5, we get

$$\Big|(t-r)^{\alpha-1}-1\Big| \le C_\rho(t-r)^{\alpha-1-\frac{\rho}{2}}\Big((1-\alpha)^{\frac{\rho}{2}}+\Big((T^*)^{1-\alpha}-1\Big)T^{\frac{\rho}{2}}\Big).$$

Using the inequality $e^{-z}\le C_\rho z^{-\rho}$, we have $e^{-\lambda_j(t-r)}\le C_\rho\lambda_j^{-\rho}(t-r)^{-\rho}$. Thus, the following inequality holds true

$$\mathscr{M}_4(\alpha,j,t) \le C_\rho\lambda_j^{-\rho}(t-r)^{\alpha-1-\frac{3\rho}{2}}\Big((1-\alpha)^{\frac{\rho}{2}}+\Big((T^*)^{1-\alpha}-1\Big)T^{\frac{\rho}{2}}\Big). \tag{55}$$

Combining (53), (54), (55), we arrive at

$$\Big|(t-r)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha)-e^{-\lambda_j(t-r)}\Big|$$
$$\leq C(\alpha,T,\rho)\mathbf{V}_\rho(\alpha,1)\lambda_j^{-\rho}\Big((t-r)^{\alpha-\frac{\rho}{2}-\frac{3}{2}}+(t-r)^{\alpha-1-\alpha\rho}+(t-r)^{\alpha-1-\frac{3\rho}{2}}\Big), \tag{56}$$

where

$$\mathbf{V}_\rho(\alpha,1)=(1-\alpha)^{\frac{\rho}{2}}+\Big((T^*)^{1-\alpha}-1\Big)T^{\frac{\rho}{2}}+(1-\alpha)^{\frac{\alpha(1-\rho)}{2}}+1-\alpha+\Big((T^*)^{1-\alpha}-1\Big)^{\frac{1-\rho}{2}}T^{\frac{\alpha(1-\rho)}{2}}.$$

Using Parseval's equality, one gets

$$\begin{aligned}\Big\|\mathscr{Q}_4(\cdot,t)\Big\|^2_{\mathbb{H}^\ell(\Omega)}&=\sum_{j=1}^\infty\lambda_j^{2\ell}\Big(\int_0^t e^{-k(t-r)}\Big((t-r)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha)-e^{-\lambda_j(t-r)}\Big)G_j(r)dr\Big)^2\\&\leq\sum_{j=1}^\infty\lambda_j^{2\ell}\Big(\int_0^t\Big|(t-r)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha)-e^{-\lambda_j(t-r)}\Big|dr\Big)\\&\qquad\Big(\int_0^t\Big|(t-r)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha)-e^{-\lambda_j(t-r)}\Big|G_j^2(r)dr\Big).\end{aligned}$$

This inequality together with (56) allows us to get that

$$\begin{aligned}\Big\|\mathscr{Q}_4(\cdot,t)\Big\|^2_{\mathbb{H}^\ell(\Omega)}&\leq C(\alpha,T,\rho)|\mathbf{V}_\rho(\alpha,1)|^2\Big\|G\Big\|^2_{L^\infty(0,T;\mathbb{H}^{\ell-\rho}(\Omega))}\\&\Big(\int_0^t(t-r)^{\alpha-\frac{\rho}{2}-\frac{3}{2}}dr+\int_0^t(t-r)^{\alpha-1-\alpha\rho}dr+\int_0^t(t-r)^{\alpha-1-\frac{3\rho}{2}}dr\Big)^2\end{aligned}\tag{57}$$

where we note that

$$\sum_{j=1}^\infty\lambda_j^{2\ell-2\rho}G_j^2(r)\leq\Big\|G\Big\|^2_{L^\infty(0,T;\mathbb{H}^{\ell-\rho}(\Omega))}.$$

Additionally, it is easy to verify that

$$\begin{aligned}&\int_0^t(t-r)^{\alpha-\frac{\rho}{2}-\frac{3}{2}}dr+\int_0^t(t-r)^{\alpha-1-\alpha\rho}dr+\int_0^t(t-r)^{\alpha-1-\frac{3\rho}{2}}dr\\&\qquad=C(\alpha,\rho)\Big(t^{\alpha-\frac{\rho}{2}-\frac{1}{2}}+t^{\alpha-\alpha\rho}+t^{\alpha-\frac{3\rho}{2}}\Big),\end{aligned}\tag{58}$$

where we remind that $\rho<\frac{2\alpha}{3}$ and $\rho<2\alpha-1$. Hence, from (57) and (58), we obtain the following bound

$$\begin{aligned}\Big\|\mathscr{Q}_4(\cdot,t)\Big\|_{\mathbb{H}^\ell(\Omega)}&\leq C(\alpha,T,\rho)\mathbf{V}_\rho(\alpha,1)\Big(t^{\alpha-\frac{\rho}{2}-\frac{1}{2}}+t^{\alpha-\alpha\rho}+t^{\alpha-\frac{3\rho}{2}}\Big)\Big\|G\Big\|_{L^\infty(0,T;\mathbb{H}^{\ell-\rho}(\Omega))}\\&\leq C(\alpha,T,\rho)\mathbf{V}_\rho(\alpha,1)\Big(T^{\alpha-\frac{\rho}{2}-\frac{1}{2}}+T^{\alpha-\alpha\rho}+T^{\alpha-\frac{3\rho}{2}}\Big)\Big\|G\Big\|_{L^\infty(0,T;\mathbb{H}^{\ell-\rho}(\Omega))}.\end{aligned}\tag{59}$$

Lastly, we focus on the term $\mathscr{Q}_2(x,t)$. By using Parseval's equality, we have

$$\begin{aligned}\Big\|\mathscr{Q}_2(\cdot,t)\Big\|^2_{\mathbb{H}^\ell(\Omega)}&=\sum_{j=1}^\infty\lambda_j^{2\ell}\Big(\frac{be^{-kt}E_{\alpha,1}(-\lambda_jt^\alpha)}{a+be^{-kT}E_{\alpha,1}(-\lambda_jT^\alpha)}\int_0^T e^{-k(T-r)}\\&\qquad\times\Big((T-r)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)-e^{-\lambda_j(T-r)}\Big)G_j(r)dr\Big)^2\\&\leq\frac{b^2}{a^2}\sum_{j=1}^\infty\lambda_j^{2\ell}\Big(\int_0^T e^{-k(T-r)}\Big((T-r)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)-e^{-\lambda_j(T-r)}\Big)G_j(r)dr\Big)^2.\end{aligned}\tag{60}$$

By employing the same proof as in (59), we arrive at

$$\left\|\mathscr{Q}_2(\cdot,t)\right\|_{\mathbb{H}^\ell(\Omega)} \le \frac{b}{a}C(\alpha,T,\rho)\mathbf{V}_\rho(\alpha,1)\Big(T^{\alpha-\frac{\rho}{2}-\frac{1}{2}}+T^{\alpha-\alpha\rho}+T^{\alpha-\frac{3\rho}{2}}\Big)\left\|G\right\|_{L^\infty(0,T;\mathbb{H}^{\ell-\rho}(\Omega))}. \tag{61}$$

Combining (51), (52), (59), (61), we conclude that

$$\begin{aligned}&\left\|u^{\alpha,k}(\cdot,t)-v(\cdot,t)\right\|_{\mathbb{H}^\ell(\Omega)} \le \sum_{j=1}^{4}\left\|\mathscr{Q}_j(\cdot,t)\right\|_{\mathbb{H}^\ell(\Omega)}\\ &\le C_0\mathbf{V}_\rho(\alpha,1)\Big(t^{\frac{-\rho-1}{2}}+t^{-\alpha\rho}\Big)\Big(\left\|f\right\|_{\mathbb{H}^{\ell-\rho}(\Omega)}+\left\|G\right\|_{L^2(0,T;\mathbb{H}^{\ell-\rho-\frac{1}{2}}(\Omega))}+\left\|G\right\|_{L^\infty(0,T;\mathbb{H}^{\ell-\rho}(\Omega))}\Big),\end{aligned}$$

where we note

$$t^{\alpha-\frac{\rho}{2}-\frac{1}{2}}+t^{\alpha-\alpha\rho}+t^{\alpha-\frac{3\rho}{2}} \le T^{\alpha-\frac{\rho}{2}-\frac{1}{2}}+T^{\alpha-\alpha\rho}+T^{\alpha-\frac{3\rho}{2}}.$$

Here, C_0 depends on α, T, ρ, a, b. □

6. Conclusion

This study focuses on the tempered fractional diffusion equation subject to a nonlocal terminal condition. The fractional operator used in this model is the tempered Caputo derivative, which extends the traditional Caputo derivative. We begin by demonstrating the well-posedness while highlighting the complexities introduced by the tempered kernel and the nonlocal condition. Next, we examine the continuity of the solution in relation to the tempered parameter, which is vital in modeling purpose. Lastly, we present numerous convergence results, relating the terminal fractional model to established classical equations.

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