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On co-intersection graph of ideals of a commutative ring

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Abstract. This research article focuses on the co-intersection graph of a commutative ring $\mathcal{R}$. The co-intersection graph of $\mathcal{R}$ is a simple graph with vertices that are non-trivial ideals of $\mathcal{R}$, and two distinct vertices I and J are adjacent if and only if $I + J \neq \mathcal{R}$. The objective of the research article is to characterize the Artinian commutative rings $\mathcal{R}$, determining whether their associated co-intersection graph is tree, bipartite, planar, outerplanar, or toroidal. In essence, the article explores the inherent properties of the co-intersection graph and investigates how specific characteristics of Artinian commutative rings influence the resulting graph structures.

1. Introduction

The investigation of graphs associated with algebraic structures is a rapidly growing field, with a focus on classifying the graphs of algebraic structures and vice versa. Researchers are particularly interested in understanding the relationship between the algebraic structure and the graph-theoretic properties of the corresponding graph. When a combinatorial object is assigned to an algebraic structure, it often leads to intriguing problems in both algebra and combinatorics. Currently, one of the most active areas of research in this field is the study of graphs associated with commutative rings. Commutative rings are a fundamental algebraic structure in mathematics and have many applications in diverse areas such as coding theory, cryptography, and algebraic geometry. Algebraic structures and graph theory intertwine in a rapidly evolving field, focusing on classifying graphs corresponding to algebraic structures and attributing algebraic properties to them. A notable example is the *zero-divisor graph* associated with a commutative ring $\mathcal{R}$, where vertices represent ring elements, introduced by Beck [9] and, redefined by Anderson and Livingston [5], this graph captures connectivity: vertices u and v are adjacent if uv equals zero. This concept offers insights into commutative rings and zero-divisors, with applications in algebraic geometry, topology, and coding theory. Extensively explored, as evidenced by references [4, 16–19]. The *cozero-divisor graph* of a ring $\mathcal{R}$ with unity, which was introduced by Afkhami et al. in [1], is an undirected simple graph whose vertex set is the set of all non-zero and non-unit elements of $\mathcal{R}$, and two

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distinct vertices x and y are adjacent if and only if $x \notin \mathcal{R}y$ and $y \notin \mathcal{R}x$. Some of the works associated with the cozero-divisor graph on the rings can be found in [2, 3].

Intersection graph theory is one of the traditional areas of study in graph theory. Wide-ranging applications of intersection graph theory can be found in many fields of human endeavor, including common interest networks, secure sensor networks, wireless communication networks, and collaboration networks. A notable result says that "every simple graph is an intersection graph". Note that intersection graph of a ring is a particular case of intersection graphs. Let $\mathcal{L}(R)$ denote the set of all non-trivial left ideals of a ring R. The *intersection graph of ideals of a ring* $\mathcal{R}$ is an undirected simple graph whose vertices are in a one-to-one correspondence with $\mathcal{L}(\mathcal{R})$ and two distinct vertices are joined by an edge if and only if the corresponding left ideals of $\mathcal{R}$ have a non-zero intersection. For more details on the intersection graph of ideals of rings, one may refer to the survey article [10].

Inspired by the intersection graph of ideals, Hoseini and Y. Talebi [11] introduced the notion of co-intersection graph, which is represented as $\Omega(\mathcal{R})$, specifically for commutative ring $\mathcal{R}$. In the *co-intersection graph* $\Omega(\mathcal{R})$, the vertices correspond to non-trivial ideals of the commutative ring $\mathcal{R}$. Notably, two distinct vertices, represented as I and J, are deemed adjacent only when their sum, $I + J$, does not encompass the entire ring $\mathcal{R}$. This elegant construction sheds light on the connectivity and relationships among non-trivial ideals in the given commutative ring. Few works on co-intersection graphs in algebraic structures can be found in [6, 12, 13]. Keep in mind that many ring theoretical properties are reflected in a ring's ideal structure.

2. Preliminaries

We provide an overview of the terminologies and ideas in this section. These are necessary for understanding the sections that follow.

A mathematical structure called a graph $G = (V, E)$ is made up of a set of vertices $V(G)$ (nodes) and a set of edges $E(G)$ that connect these vertices. Graphs are frequently used to show the connections between different entities or objects. In a graph, a *path* is a series of edges that join a series of vertices without having the vertices twice. A closed path that begins and ends at the same vertex in a graph is called a *cycle*. A graph that has a path connecting each pair of vertices is said to be a *connected graph*. A *disconnected graph* is a graph that is not connected. The *complete graph* K_n is a graph with n vertices in which every pair of distinct vertices is connected by an edge. A graph with vertices that can be divided into two disjoint sets of m elements and n elements such that each edge connects a vertex in one set to a vertex in the other set is called a *bipartite graph* $K_{m,n}$. A *tree* is a connected graph that does not contain any cycle.

Topological graph theory is concerned with finding ways to embed a graph onto a surface, with the minimum number of handles, called the genus of the graph. The *genus of a graph* G, denoted by $\gamma(G)$, is defined as the minimum integer k such that the graph can be drawn on a sphere with k handles without any edge-crossing except at the vertices. The goal is to draw the graph in a way that the edges only intersect at their vertices. A planar graph is one with genus 0, while a toroidal graph has genus 1. Euler's formula states that for a connected graph with n vertices, e edges, and genus γ, the equation $n - e + f = 2 - 2\gamma$ holds true, where f is the number of faces created when the graph is embedded in S_γ. This formula, along with combinatorial identities and inequalities, can be used to determine if certain embeddings exist. For more information on embedding graphs onto surfaces we refer the reader to [22].

This research delves into the properties of the co-intersection graph, denoted as $\Omega(\mathcal{R})$, associated with an Artinian commutative ring $\mathcal{R}$. The focus lies on understanding the structural characteristics of this graph, where the vertices correspond to nontrivial ideals of the ring. The adjacency between two distinct vertices, represented by ideals I and J, is determined by the condition that their sum, denoted as $I + J$, does not equal the entire ring $\mathcal{R}$. The primary objective is to characterize Artinian commutative rings by analyzing the graph $\Omega(\mathcal{R})$. Specifically, the research aims to determine the Artinian commutative ring when the co-intersection graph is tree, or bipartite, or planar, or outerplanar, or toroidal. In essence, the article explores the intrinsic properties of $\Omega(\mathcal{R})$ and investigates how specific features of Artinian commutative rings influence the resulting graph structures.

3. When a co-intersection graph is tree or bipartite

Remember that the number of edges in a simple connected graph with n vertices falls between $n-1$ and $\binom{n}{2}$. Keep in mind that G is complete if $|E(G)| = \binom{n}{2}$, and tree if $|E(G)| = n-1$. The authors of [11] characterized all Artinian commutative ring whose co-intersection graph is complete. The following is the corresponding outcome.

Proposition 3.1. [11] *The co-intersection graph of a ring $\mathcal{R}$ is complete if and only if $\mathcal{R}$ has unique maximal left ideal. In other words, $\Omega(\mathcal{R})$ is complete if and only if $\mathcal{R}$ is a local ring.*

Now it is natural to characterize all Artinian commutative rings whose co-intersection graph is a tree. In what follows, a ring with a unique maximal ideal is referred to as a local ring. If a ring fulfills the descending chain condition of ideals, then it is an Artinian ring. Notice that Artinian rings play an important role in algebraic geometry, for example in deformation theory. The decomposition theorem on Artinian commutative rings states that every Artinian commutative ring can be decomposed as a direct product of local rings. This decomposition is unique up to isomorphism.

Theorem 3.2. *Let $\mathcal{R}$ be an Artinian commutative ring with unity. Then $\Omega(\mathcal{R})$ is a tree if and only if $\mathcal{R}$ is a local with at most two non-trivial ideals.*

Proof. Suppose $\Omega(\mathcal{R})$ is tree. Given that $\mathcal{R}$ is an Artinian commutative ring with unity. We can employ the structure theorem $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$, where each $\mathcal{R}_i$ is an Artinian local ring for $1 \leq i \leq n$. When $n \geq 3$, the collection of non-trivial ideals $\{\mathcal{R}_1 \times (0) \times (0) \times (0) \times \cdots \times (0), (0) \times \mathcal{R}_2 \times (0) \times (0) \times \cdots \times (0), (0) \times (0) \times \mathcal{R}_3 \times (0) \times \cdots \times (0)$ forms a triangle in $\Omega(\mathcal{R})$, a configuration that contradicts the assumption of a tree. Therefore, it can be inferred that $n \leq 2$.

Consider the case where $n = 2$, meaning $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2$. Suppose both $\mathcal{R}_1$ and $\mathcal{R}_2$ are fields; in this scenario, $\Omega(\mathcal{R})$ would contain two isolated vertices, leading to a contradiction of connectedness. Therefore, at least one of the $\mathcal{R}_i$ is not a field. Let's assume that $\mathcal{R}_2$ is not a field and has a non-trivial ideal $\mathfrak{m}$. In this case, the set of non-trivial ideals $\{\mathcal{R}_1 \times (0), (0) \times \mathfrak{m}, \mathcal{R}_1 \times \mathfrak{m}\}$ form a triangle in $\Omega(\mathcal{R})$, which results in a contradiction. A similar contradiction would arise if R_1 is not a field.

In the case when $n = 1$, indicating $\mathcal{R}$ is an Artinian local ring. Assume that $\mathcal{R}$ has at least three non-trivial ideals. According to Proposition 3.1, this implies that $\Omega(\mathcal{R})$ contains a triangle, resulting in a contradiction. Therefore, it can be concluded that $\mathcal{R}$ is an Artinian local ring with at most two non-trivial ideals. On the flip side, if $\mathcal{R}$ is an Artinian local ring with at most two non-trivial ideals, it follows from Proposition 3.1 that $\Omega(\mathcal{R})$ forms K_2 and is consequently a tree, which leads to the completion of the proof. □

Next we are interested in characterizing bipartite nature of co-intersection graph of an Artinian ring. The following result plays a key role in bipartite characterization.

Proposition 3.3. [21] *An undirected graph is bipartite if and only if it does not contain an odd cycle.*

Theorem 3.4. *Let $\mathcal{R}$ be an Artinian commutative ring with unity. Then $\Omega(\mathcal{R})$ is a bipartite graph if and only if one of the following holds:*

(1) $\mathcal{R} \cong \mathbb{F}_1 \times \mathbb{F}_2$, where $\mathbb{F}_1$ and $\mathbb{F}_2$ are fields.

(2) $\mathcal{R}$ is an Artinian local ring with exactly two non-trivial ideals.

Proof. Assume that $\Omega(\mathcal{R})$ is a bipartite graph. Since $\mathcal{R}$ is Artinian, by the structure theorem, $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$, where $\mathcal{R}_i$ is an Artinian local ring for each $1 \leq i \leq n$. For $n \geq 3$, the set of non-trivial ideals $\{\mathcal{R}_1 \times (0) \times (0) \times (0) \times \cdots \times (0), (0) \times \mathcal{R}_2 \times (0) \times (0) \times \cdots \times (0), (0) \times (0) \times \mathcal{R}_3 \times (0) \times \cdots \times (0)\}$ in $\Omega(\mathcal{R})$ create a cycle of length 3, contradicting Proposition 3.3. Thus $n \leq 2$.

Consider the case where $n = 2$, implying $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2$. Suppose, one of the $\mathcal{R}_i$ is not a field say $\mathcal{R}_2$ with non-trivial ideal $\mathfrak{m}$. Then the subgraph in $\Omega(\mathcal{R})$ formed by the set of non-trivial ideals $\{\mathcal{R}_1 \times (0), (0) \times \mathfrak{m}, \mathcal{R}_1 \times \mathfrak{m}\}$ constitutes a triangle, leading to a contradiction according to Proposition 3.3. Hence, each $\mathcal{R}_i$, where $i = 1, 2$, is a field. Consequently, $\mathcal{R} \cong \mathbb{F}_1 \times \mathbb{F}_2$, where $\mathbb{F}_1$ and $\mathbb{F}_2$ are fields. Ultimately, when $n = 1$, it implies that $\mathcal{R}$ is an Artinian local ring. Assuming $\mathcal{R}$ possesses at least three non-trivial ideals, Proposition 3.1 indicates that $\Omega(\mathcal{R})$ forms a triangle, contradicting Proposition 3.3. Therefore, $\mathcal{R}$ has at most two non-trivial ideals. If $\mathcal{R}$ has at most one non-trivial ideal, then $\Omega(\mathcal{R})$ reduces to a single isolated vertex, resulting in a contradiction. Consequently, it can be concluded that $\mathcal{R}$ is an Artinian local ring with precisely two non-trivial ideals.

In the reverse direction, if $\mathcal{R} \cong \mathbb{F}_1 \times \mathbb{F}_2$, where $\mathbb{F}_1$ and $\mathbb{F}_2$ are fields. Then $\Omega(\mathcal{R})$ consists of two isolated vertices, forming a bipartite graph. Additionally, when $\mathcal{R}$ is an Artinian local ring with precisely two non-trivial ideals, the application of Proposition 3.1 establishes that $\Omega(\mathcal{R})$ forms K_2 and is also a bipartite graph. This concludes the proof. □

4. When a co-intersection graph is planar or outer planar

Planar embeddings have a long and rich history, entwining with enumeration, chromatic graph theory, algorithmic analysis, and many other areas. Recall that if a graph can be drawn in the plane with all of its edges intersecting only at vertices, then it is said to be planar. There is a long history of studying planar zero-divisor graphs; interested readers may refer to Chapter 4 in [4]. The Artinian commutative ring $\mathcal{R}$ with unity that satisfies specific properties, like being a planar or outer planar with respect to its $\Omega(\mathcal{R})$ structure, is characterized in this section.

Let us recall the famous Kuratowski's Theorem.

Proposition 4.1. [21] (Kuratowski's Theorem) *A graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.*

The following result provides a response to the question that "When a co-intersection graph is planar?"

Theorem 4.2. *Let $\mathcal{R}$ be an Artinian commutative ring with unity. Then the associated graph $\Omega(\mathcal{R})$ is planar if and only if one of the following criteria is satisfied:*

(1) $\mathcal{R} \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, where $\mathbb{F}_1$, $\mathbb{F}_2$ and $\mathbb{F}_3$ are fields.
(2) $\mathcal{R} \cong \mathbb{F}_1 \times \mathcal{R}_2$, where $\mathbb{F}_1$ is a field and $\mathcal{R}_2$ is a local ring having unique non-trivial ideal.
(3) $\mathcal{R}$ is a local ring having at most four non-trivial ideals.

Proof. Assume that $\Omega(\mathcal{R})$ is planar. Since $\mathcal{R}$ is Artinian, $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$, where $\mathcal{R}_i$ is an Artinian local ring for each $1 \leq i \leq n$. For $n \geq 4$, examining the subgraph within $\Omega(\mathcal{R})$ generated by the set of non-trivial ideals $\{\mathcal{R}_1 \times (0) \times (0) \times (0) \times \cdots \times (0),\ (0) \times \mathcal{R}_2 \times (0) \times (0) \times \cdots \times (0),\ (0) \times (0) \times \mathcal{R}_3 \times (0) \times \cdots \times (0),\ (0) \times (0) \times (0) \times \mathcal{R}_4 \times \cdots \times (0),\ \mathcal{R}_1 \times \mathcal{R}_2 \times (0) \times (0) \times \cdots \times (0)\}$ reveals an isomorphism with K_5, contradicting Proposition 4.1. Consequently, this implies that $n \leq 3$.

Consider the case where $n = 3$, meaning $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_3$ where $\mathcal{R}_i$ being an Artinian local ring for $i = 1, 2, 3$. Suppose, for at least one i, say $i = 3$, $\mathcal{R}_3$ is not a field. Let $\mathfrak{m}$ be a non-trivial ideal of $\mathcal{R}_3$. Then the subgraph within $\Omega(\mathcal{R})$ formed by the set of non-trivial ideals $S = \{\mathcal{R}_1 \times (0) \times (0),\ (0) \times \mathcal{R}_2 \times (0), (0) \times (0) \times \mathcal{R}_3,\ (0) \times (0) \times \mathfrak{m}, (0) \times \mathcal{R}_2 \times \mathfrak{m}\}$ corresponds to K_5. According to Proposition 4.1, this leads to a contradiction. Therefore, it follows that $\mathcal{R}_3$ must be a field. Likewise, we can demonstrate that $\mathcal{R}_1$ and $\mathcal{R}_2$ are fields, leading to the conclusion that $\mathcal{R} \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$.

Consider the scenario where $n = 2$, meaning $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2$ where both $\mathcal{R}_1$ and $\mathcal{R}_2$ are local rings. Assume that neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ are fields, and each has non-trivial ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively. Now $\Omega(\mathcal{R})$ contains a subgraph generated by the set of non-trivial ideals $\{(0)\times\mathcal{R}_2,\ \mathfrak{m}_1\times(0),\ (0)\times\mathfrak{m}_2,\ \mathfrak{m}_1\times\mathfrak{m}_2,\ \mathfrak{m}_1\times\mathcal{R}_2\}$ which forms K_5. However, this contradicts Proposition 4.1. Therefore, it follows that at least one of the $\mathcal{R}_i$ must be a field; let us assume that $\mathcal{R}_1$ is a field, and $\mathcal{R}_2$ is an Artinian local ring with non-trivial ideal $\mathfrak{m}_2$. Suppose $\mathcal{R}_2$ has at least two non-trivial ideals namely $\mathfrak{m}_2$ and $\mathfrak{m}_2'$. Then $\Omega(\mathcal{R})$ exhibits a K_5 structure formed by the set of non-trivial ideals $\{\mathcal{R}_1 \times (0),\ (0) \times \mathfrak{m}_2,\ (0) \times \mathfrak{m}_2',\ \mathcal{R}_1 \times \mathfrak{m}_2,\ \mathcal{R}_1 \times \mathfrak{m}_2'\}$, leading to a contradiction. Thus, it can be concluded that $\mathcal{R}_2$ is a local ring with at most one non-trivial ideal.

For the case when $n = 1$, implying $\mathcal{R}$ is an Artinian local ring, assume that $\mathcal{R}$ has at least five non-trivial ideals. Using Proposition 3.1, it follows that $\Omega(\mathcal{R})$ contains a K_5, contradicting Proposition 4.1. Therefore, $\mathcal{R}$ must be an Artinian local ring with at most four non-trivial ideals.

The reverse implication is supported by Figure 1(a) and Figure 1(b). Additionally, when $\mathcal{R}$ is an Artinian local ring with at most four non-trivial ideals, it follows from Proposition 3.1 that $\Omega(\mathcal{R})$ forms a K_4 and is consequently a planar graph. This concludes the proof.

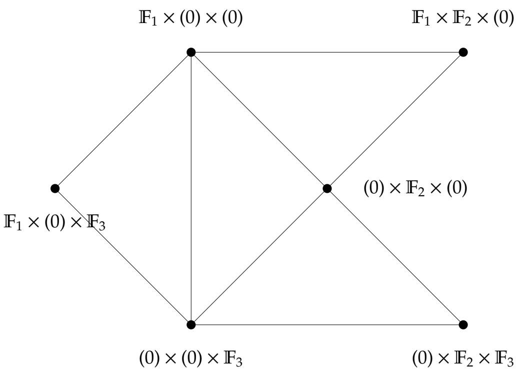

Figure 1(a). $\Omega(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)$

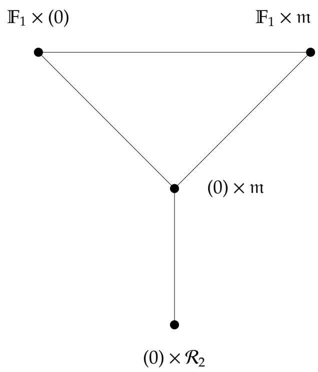

Figure 1(b). $\Omega(\mathbb{F}_1 \times \mathcal{R}_2)$
with atmost one non-trivial ideal of $\mathcal{R}_2$

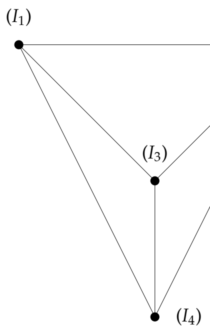

Figure 1(c). $\Omega(\mathcal{R})$
with atmost four non-trivial ideals

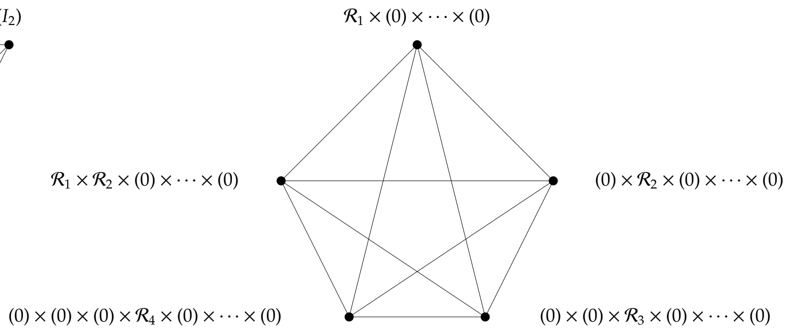

Figure 1(d).
Forbidden induced subgraph of $\Omega(\mathcal{R})$, for $n \geq 4$

□

A graph that can be drawn in the plane with all of its vertices lying on the outer boundary and no edges crossing anywhere other than at their endpoints is known as an outerplanar graph. Thus, it follows that all outerplanar graphs are planar. A well-known result states that a graph is outerplanar if and only if it contains no subgraph that is a subdivision of either K_4 or $K_{2,3}$. Combining these facts with proof of Theorem 4.2 yields the following result.

Corollary 4.3. *Let $\mathcal{R}$ be an Artinian commutative ring with unity. Then $\Omega(\mathcal{R})$ is an outerplanar graph if and only one of the following conditions is met:*

(1) $\mathcal{R} \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, where $\mathbb{F}_1$, $\mathbb{F}_2$ and $\mathbb{F}_3$ are fields.

(2) $\mathcal{R} \cong \mathbb{F}_1 \times \mathcal{R}_2$, where $\mathbb{F}_1$ is a field and $\mathcal{R}_2$ is a local ring having atmost one non-trivial ideal.

(3) $\mathcal{R}$ is a local ring having atmost three non-trivial ideals.

5. When a co-intersection graph is toroidal

This section's goal is to investigate the problem of embedding a co-intersection graph on the orientable surfaces of one handle. Over the past several years, the topological structures are widely investigated. More specifically, the graph embedding of graphs arising from algebraic ones. For research on graph embedding of graphs from algebraic structure, one may refer to [7, 8, 15, 20]. Recall that a graph G is called toroidal if $\gamma(G) = 1$. In this section, we will analyze the Artinian commutative rings $\mathcal{R}$ and determine the conditions under which the genus of $\Omega(\mathcal{R})$ is equal to one.

We now present a few findings that will help to prove the section's main finding.

Proposition 5.1. [22] *Let $n \geq 3$. Then*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

In particular, $\gamma(K_n) = 1$ if $n = 5, 6, 7$ and $\gamma(K_8) = 2$.

Proposition 5.2. [22] *Let $n, m \geq 2$. Then*

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

In particular, $\gamma(K_{4,4}) = \gamma(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$. Also, $\gamma(K_{5,4}) = \gamma(K_{6,4}) = \gamma(K_{3,m}) = 2$, if $m = 7, 8, 9, 10$.

Proposition 5.3. [14, Proposition 4.4.4] *Let G be a connected graph with $3 \leq n$ vertices and m edges. Then*

$$\gamma(G) \geq \left\lceil \frac{m}{6} - \frac{n}{2} + 1 \right\rceil.$$

We are now able to describe Artinian commutative rings $\mathcal{R}$ in which the co-intersection graph has genus one.

Theorem 5.4. *Let $\mathcal{R}$ be an Artinian commutative ring with unity. Then $\Omega(\mathcal{R})$ is toroidal if and only if one of the following holds:*

(1) $\mathcal{R} \cong \mathbb{F}_1 \times \mathcal{R}_2$, where $\mathbb{F}_1$ is a field and $\mathcal{R}_2$ is a local ring having two or three non-trivial ideals.

(2) $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2$, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are local rings having exactly one non-trivial ideals.

(3) $\mathcal{R}$ is an Artinian local ring having at least five and at most seven non-trivial ideals.

Proof. Suppose that the genus of $\Omega(\mathcal{R})$ is one, where $\mathcal{R}$ is an Artinian commutative ring with unity. Let $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$, where each $\mathcal{R}_i$ is an Artinian local ring for $1 \leq i \leq n$. For $n \geq 5$, the subgraph in $\Omega(\mathcal{R})$ formed by the set of non-trivial ideals $\{\mathcal{R}_1 \times (0) \times (0) \times (0) \times \cdots \times (0),\ (0) \times \mathcal{R}_2 \times (0) \times (0) \times \cdots \times (0),\ (0) \times (0) \times \mathcal{R}_3 \times (0) \times \cdots \times (0),\ (0) \times (0) \times (0) \times \mathcal{R}_4 \times (0) \times \cdots \times (0),\ (0) \times (0) \times (0) \times (0) \times \mathcal{R}_5 \times (0) \times \cdots \times (0),\ \mathcal{R}_1 \times \mathcal{R}_2 \times (0) \times (0) \times \cdots \times (0),\ (0) \times \mathcal{R}_2 \times \mathcal{R}_3 \times (0) \times (0) \times \cdots \times (0),\ (0) \times \mathcal{R}_2 \times (0) \times \mathcal{R}_4 \times (0) \times (0) \times \cdots \times (0)\}$ is K_8. This, however, contradicts Proposition 5.1. Consequently, it follows that $n \leq 4$.

Consider the case where $n = 4$, implying $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_3 \times \mathcal{R}_4$. Assume that at least one of $\mathcal{R}_i$, say $\mathcal{R}_1$, is not a field with nonzero maximal ideal $\mathfrak{m}$. In such a scenario, the subgraph in $\Omega(\mathcal{R})$ generated by the set of non-trivial ideals $\{\mathcal{R}_1 \times (0) \times (0) \times (0),\ (0) \times \mathcal{R}_2 \times (0) \times (0),\ (0) \times (0) \times \mathcal{R}_3 \times (0),\ (0) \times (0) \times (0) \times \mathcal{R}_4,\ \mathfrak{m} \times (0) \times (0) \times (0),\ \mathfrak{m} \times \mathcal{R}_2 \times (0) \times (0),\ \mathfrak{m} \times \mathcal{R}_2 \times \mathcal{R}_3 \times (0),\ \mathfrak{m} \times (0) \times \mathcal{R}_3 \times (0)\}$ is isomorphic to K_8. However, this contradicts Proposition 5.1. Therefore, it can be concluded that each $\mathcal{R}_i$ is a field for each $i \in \{1, 2, 3, 4\}$. That is $\mathcal{R} \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_4$, where each $\mathbb{F}_i$ is a field. Upon examining $\Omega(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_4)$ we observe that it has 14 vertices and 66 edges. By applying Proposition 5.3, it follows that $\gamma(\Omega(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_4)) \geq 5$, leading to a contradiction.

Consider the case when $n = 3$, indicating that $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_3$. Let us suppose that none of the $\mathcal{R}_i$ is a field with non-trivial ideal $\mathfrak{m}_i$. Then the subgraph in $\Omega(\mathcal{R})$ generated by the set of non-trivial ideals $\{\mathcal{R}_1 \times (0) \times (0),\ (0) \times \mathcal{R}_2 \times (0), (0) \times (0) \times \mathcal{R}_3,\ \mathfrak{m}_1 \times (0) \times (0), (0) \times \mathfrak{m}_2 \times (0), (0) \times (0) \times \mathfrak{m}_3, \mathfrak{m}_1 \times \mathfrak{m}_2 \times (0), (0) \times \mathfrak{m}_2 \times \mathfrak{m}_3\}$ is claimed to be isomorphic to K_8. However, by Lemma 5.1, this leads to a contradiction. Consequently, it can be concluded that at least one of the $\mathcal{R}_i$ is a field. Assume, without loss of generality, that $\mathcal{R}_1$ is a field. Let $\mathcal{R}_2$ and $\mathcal{R}_3$ be local rings with non-trivial ideals $\mathfrak{m}_2$ and $\mathfrak{m}_3$, respectively. Then the subgraph in $\Omega(\mathcal{R})$ generated by the set of non-trivial ideals $\{\mathcal{R}_1 \times (0) \times (0),\ (0) \times \mathcal{R}_2 \times (0), (0) \times (0) \times \mathcal{R}_3,\ (0) \times \mathfrak{m}_2 \times (0), (0) \times (0) \times \mathfrak{m}_3, (0) \times \mathfrak{m}_2 \times \mathfrak{m}_3, \mathcal{R}_1 \times \mathfrak{m}_2 \times (0), \mathcal{R}_1 \times (0) \times \mathfrak{m}_3\}$ is K_8, a contradiction. Thus, it can be inferred that at least two of the $\mathcal{R}_i$'s, say $\mathcal{R}_1$ and $\mathcal{R}_2$ are fields making $\mathcal{R} \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathcal{R}_3$. Suppose $\mathcal{R}_3$ has at least two non-trivial ideals $\mathfrak{m}_3$ and $\mathfrak{m}_3'$. Then the subgraph in $\Omega(\mathcal{R})$ generated by the set of non-trivial ideals $\{\mathcal{R}_1 \times (0) \times (0),\ (0) \times \mathcal{R}_2 \times (0),\ (0) \times (0) \times \mathcal{R}_3,\ (0) \times (0) \times \mathfrak{m}_3,\ (0) \times (0) \times \mathfrak{m}_3',\ \mathcal{R}_1 \times (0) \times \mathfrak{m}_3,\ \mathcal{R}_1 \times (0) \times \mathfrak{m}_3',\ (0) \times \mathcal{R}_2 \times \mathfrak{m}_3\}$ is K_8, leading to a contradiction. Hence $\mathcal{R}_3$ has a unique non-trivial ideal. Then $\Omega(\mathcal{R})$ has 34 edges and 10 vertices, referring to Lemma 5.3, $\gamma(\Omega(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathcal{R}_3)) \geq 2$, leads to a contradiction. Hence $\mathcal{R}_3$ is also a field. Then according to Theorem 4.2, $\gamma(\Omega(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) = 0$, which is again a contradiction.

Now, consider $n = 2$, that is $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2$. We can analyze the following cases:
Case(i): Suppose both $\mathcal{R}_1$ and $\mathcal{R}_2$ are fields. In this scenario, according to Theorem 4.2, $\Omega(\mathcal{R})$ is a planar graph, which leads to a contradiction.
Case(ii): Suppose $\mathcal{R}_1$ a field and $\mathcal{R}_2$ is not a field. If $\mathcal{R}_2$ has at least four non-trivial ideals denoted as $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_3$, and $\mathfrak{m}_4$. Then $\Omega(\mathcal{R})$ has K_8 generated by the set of non-trivial ideals $\{(0) \times \mathfrak{m}_1,\ (0) \times \mathfrak{m}_2,\ (0) \times \mathfrak{m}_3,\ (0) \times \mathfrak{m}_4,\ (\mathcal{R}_1) \times \mathfrak{m}_1,\ \mathcal{R}_1 \times \mathfrak{m}_2,\ \mathcal{R}_1 \times \mathfrak{m}_3,\ \mathcal{R}_1 \times \mathfrak{m}_4\}$, a contradiction. Thus, $\mathcal{R}_2$ has at most three non-trivial ideals. If $\mathcal{R}_2$ has exactly one non-trivial ideal then by Theorem 4.2, $\gamma(\Omega(\mathcal{R})) = 0$, a contradiction. Hence, $\mathcal{R}_2$ has at least two and at most three non-trivial ideals.
Case(iii): Suppose both $\mathcal{R}_1$ and $\mathcal{R}_2$ are not fields with non-trivial ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively. Suppose one of the $\mathcal{R}_i$, say $\mathcal{R}_1$, has at least two non-trivial ideals $\mathfrak{m}_1$ and $\mathfrak{m}_1'$, then $\Omega(\mathcal{R})$ has K_8 generated by the set of non-trivial ideals $\{(0) \times \mathcal{R}_2,\ \mathfrak{m}_1 \times \mathcal{R}_2,\ \mathfrak{m}_1' \times \mathcal{R}_2,\ \mathfrak{m}_1 \times \mathfrak{m}_2,\ \mathfrak{m}_1' \times \mathfrak{m}_2,\ \mathfrak{m}_1 \times (0),\ \mathfrak{m}_1' \times (0),\ (0) \times \mathfrak{m}_2\}$, a contradiction. Hence each $\mathcal{R}_i$ has exactly one non-trivial ideal.

Finally for $n = 1$, $\mathcal{R}$ is a local ring. Suppose $\mathcal{R}$ has at least eight non-trivial ideals, then by Proposition 3.1, $\Omega(\mathcal{R})$ has K_8, a contradiction by Proposition 5.1. Hence, $\mathcal{R}$ has at most eight non-trivial ideals. If $\mathcal{R}$ has at least four non-trivial ideals, then by Theorem 4.2, $\gamma(\Omega(\mathcal{R})) = 0$, a contradiction. Thus, $\mathcal{R}$ has at least five and at most seven non-trivial ideals.

The reverse implication is evident from the graphical representations in Figure 2(a), Figure 2(b), and Figure 2(c). Additionally, when $\mathcal{R}$ is an Artinian local ring with at least five and at most seven non-trivial ideals, the proof is completed by employing Proposition 3.1 and Proposition 5.1, which establish that $\gamma(\Omega(\mathcal{R})) = 1$. □

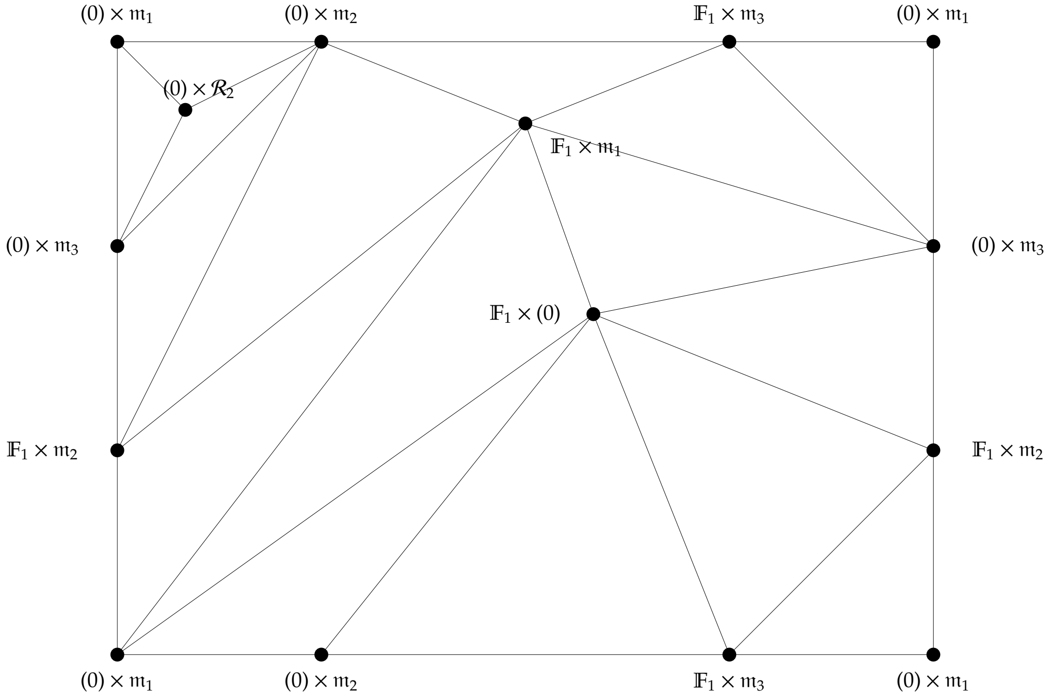

Figure 2 (a). **Toroidal embedding of** $\Omega(\mathbb{F}_1 \times \mathcal{R}_2)$
where $\mathbb{F}_1$ is a field and $\mathcal{R}_2$ is an Artinian local ring with three non-trivial ideals.

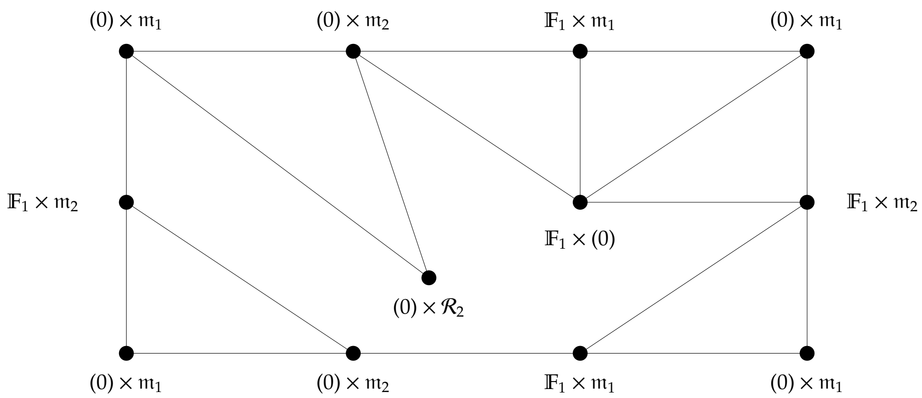

Figure 2 (b). **Toroidal embedding of** $\Omega(\mathbb{F}_1 \times \mathcal{R}_2)$
where $\mathbb{F}_1$ is a field and $\mathcal{R}_2$ is an Artinian local ring with exactly two non-trivial ideals.

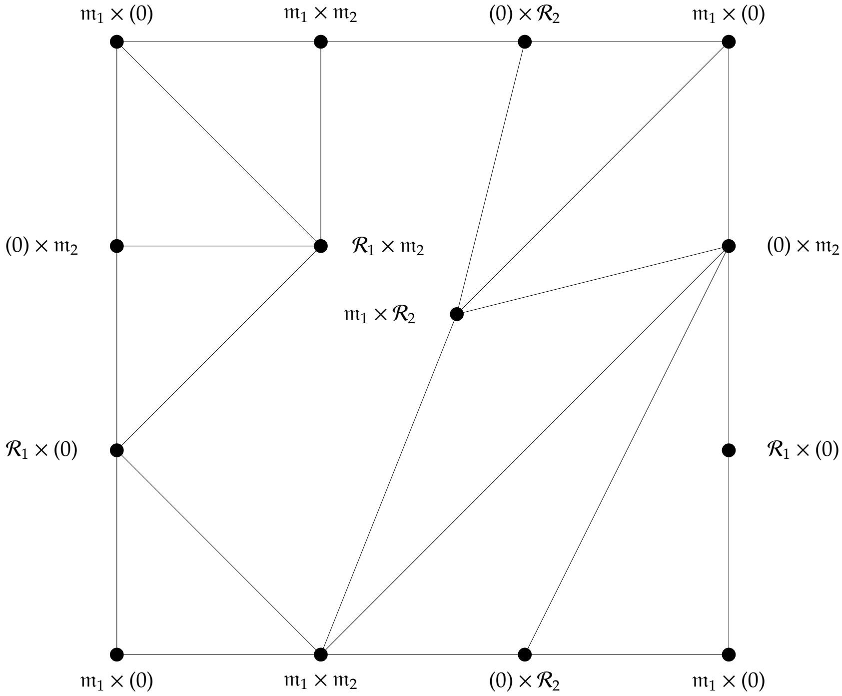

Figure 2 (c). **Toroidal embedding of** $\Omega(\mathcal{R}_1 \times \mathcal{R}_2)$
where $\mathcal{R}_1$ and $\mathcal{R}_2$ are local rings with exactly one non-trivial ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively

6. Conclusion

This research article explores the interesting domain of Artinian commutative rings, specifically focusing on the co-intersection graph $\Omega(\mathcal{R})$ associated with such rings. By characterizing the structural properties of this graph, the study sheds light on the intricate relationships between non-trivial ideals of the ring and the resulting connectivity patterns. The investigation successfully determines whether $\Omega(\mathcal{R})$ exhibits planar, outerplanar, tree, or bipartite characteristics, offering valuable insights into the graph-theoretic aspects of Artinian commutative rings. Moreover, the pursuit of identifying rings with a genus of one adds an additional layer of complexity to the analysis, highlighting the nuanced interplay between algebraic structures and graph theory. Overall, this research contributes to a deeper understanding of the inherent properties of co-intersection graphs and their connection to the algebraic nature of Artinian commutative rings.

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