Filomat 38:33 (2024), 11889–11912
https://doi.org/10.2298/FIL2433889D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

AMDS constacyclic codes and quantum AMDS codes

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Abstract. One significant application of almost maximum distance separable (briefly, AMDS) codes is in secure communication systems, such as secure messaging and encrypted data transmission. By incorporating AMDS codes into the data encoding process, information can be safeguarded from accidental errors that might occur during transmission or storage. In this paper, we study all AMDS constacyclic codes of length p^s over $\mathbb{F}_{p^m}$. We also provide some examples of AMDS constacyclic codes over finite fields. As an application, we establish all quantum AMDS (briefly, qAMDS) codes from repeated-root codes of prime power lengths over finite fields using the CSS and Hermitian constructions.

1. Introduction

Berlekamp [3] first studied constacyclic codes over finite fields, which have a rich algebraic structure and are generalizations of cyclic and negacyclic codes. A γ-constacyclic code of length n over $\mathbb{F}$ is an ideal of $\frac{\mathbb{F}[x]}{\langle x^n-\gamma\rangle}$. If $\gamma = 1$ ($\gamma = -1$), then an ideal of $\frac{\mathbb{F}[x]}{\langle x^n-\gamma\rangle}$ is called a *cyclic (negacyclic) code*. Cyclic codes over finite fields were first studied by Prange in 1957 [72]. After that, negacyclic codes over finite fields were considered by Berlekamp [4]. Berman [5] first initiated the case $(n, p) = p$, where n is the code length and p is the characteristic of the fields. If $(n, p) = p$, then codes over finite fields are so-called repeated-root codes. Such codes are also studied by some authors (for examples, Massey et al. [69], Roth and Seroussi [76], and van Lint [90]). Recently, Dinh, in a series of papers ([24], [27], [28], [29], [30]), determined the algebraic structures of constacyclic codes in terms of generator polynomials over $\mathbb{F}_{p^m}$ of length mp^s, where $m = 1, 2, 3, 4, 6$.

Due to the decoherence and other quantum noise in quantum information, quantum error-correcting (briefly, QEC) codes are proposed to prevent errors in quantum information. In 1995, Shor first introduced QEC codes [82]. Many good QEC codes were constructed from classical codes such as Hamming, BCH and Reed-Solomon codes [9, 45, 46]. The study of QEC codes has developed rapidly in recent years. After the publications of several foundation papers [2, 9, 63, 82, 86], which were the key theoretical development,

2020 *Mathematics Subject Classification*. Primary 94B15, 94B05; Secondary 11T71.
Keywords. Constacyclic codes, Hamming distance, AMDS codes, quantum AMDS codes.
Received: 26 October 2023; Revised: 06 May 2024; Accepted: 29 August 2024
Communicated by Paola Bonacini
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in rapid succession, QEC codes have been studied extensively. There have been many results on the structure, properties and operation of QEC codes [2, 10, 20, 41, 61]. In the last several years, CSS, Hermitian constructions are used to construct some classes of QEC codes.

The Singleton bound relates the parameters of a code as follows: $|\mathbf{C}| \leq p^{m(n-d_H(\mathbf{C})+1)}$ [77]. A code $\mathbf{C}$ satisfying $|\mathbf{C}| = p^{m(n-d_H(\mathbf{C})+1)}$ is called an *MDS code*. Maximum Distance Separable (MDS) codes are a fascinating and crucial concept in the realm of coding theory and information security. These codes play a pivotal role in ensuring the reliability and integrity of data transmission, making them indispensable in various applications. One of the primary applications of MDS codes is in the field of error correction. In scenarios where data transmission is susceptible to noise, interference, or corruption, MDS codes serve as a powerful tool for mitigating these issues ([36], [37]). They enable us to detect and correct errors, preserving the integrity of the information being transmitted. This is particularly critical in applications such as wireless communication, where signal degradation is common, and in data storage systems, where data corruption can lead to catastrophic losses. Furthermore, MDS codes find extensive use in cryptography. Protecting sensitive information from unauthorized access and ensuring data confidentiality is of paramount importance. MDS codes can be employed to construct secure cryptographic primitives, such as secret sharing schemes, where a secret is divided into shares distributed among multiple parties. Only when a sufficient number of shares are combined can the original secret be reconstructed. This adds an extra layer of security, making it challenging for adversaries to access confidential data (see, [92], [73], [87]).

Let $\mathbf{Q} = [[n, k, d]]_q$ be a QEC code. If $k = n - 2(d - 1)$, then C is called a quantum maximum distance separable (qMDS). QMDS codes are a class of quantum error-correcting codes that play a pivotal role in quantum information theory and quantum computing. They are quantum analogs of classical MDS codes and are designed to correct a maximum number of errors while preserving quantum information. In quantum systems, information is stored in quantum bits (or qubits), which are susceptible to various types of errors, such as bit flips, phase flips, and both combined (depolarizing errors). QMDS codes provide a powerful tool for protecting quantum states against these errors, ensuring the reliable storage and manipulation of quantum information. They achieve the highest possible error-correction capability for a given code length and dimension ([43, 51, 78, 79], [21, 48, 57],[57],[48],[21]).

MDS codes are specific types of error-correcting codes that have maximum possible distance between codewords, making them highly effective in correcting errors in data transmission and storage. However, not all code classes have the property of being MDS. In some cases, the parameters or characteristics of a given code class may not meet the requirements to qualify as MDS codes. Therefore, we pay attention to some codes very close to MDS, called almost maximum distance separable (briefly, AMDS) codes for designing codes that are optimized for specific applications. AMDS codes are a critical component in modern information theory and error correction. These codes play a pivotal role in ensuring the integrity and reliability of data transmission and storage systems. AMDS codes are characterized by their remarkable ability to correct errors while maximizing efficiency. They are particularly useful in scenarios where data integrity is paramount, such as in communication systems, data storage devices, and even in the field of cryptography [85]. These codes can recover lost or corrupted data with high precision, making them indispensable in safeguarding sensitive information and ensuring smooth data transmission in various applications. In the realm of telecommunications, AMDS codes are the backbone of error correction techniques in wireless networks and satellite communication systems. They enable data to be transmitted over long distances with minimal risk of corruption, ensuring seamless connectivity even in challenging environments. Furthermore, in data storage, AMDS codes are employed in technologies to safeguard against data loss due to hardware failures or errors. By utilizing AMDS codes, data redundancy can be efficiently managed, allowing for both fault tolerance and efficient use of storage resources. In the world of cybersecurity, AMDS codes are a valuable tool for protecting sensitive information. They are employed in encryption schemes, ensuring that even if an attacker gains access to encrypted data, the chances of them successfully deciphering it are extremely low (see, [7], [33], [34], [36], [37], [92], [73], [87]).

In this paper, we give the definition of quantum almost maximum distance separable codes (briefly, qAMDS) codes. If $\mathbf{Q} = [[n, k, d]]_q$ is a QEC code such that $k = n - 2d$, then C is called a qAMDS code. QAMDS codes have the quantum distance that is close to quantum distance of qMDS codes. We believe that QAMDS codes can be applied in quantum communication systems to protect the transmission of quantum information over long distances. By encoding quantum states using AMDS codes, the data remains secure even in the presence of channel noise or adversarial attacks, ensuring that confidential information remains confidential. QAMDS codes are pivotal in securing quantum networks, which are poised to play a significant role in the future of quantum communication. By protecting quantum information against various threats, these codes facilitate the development of quantum-enhanced secure communication networks. QAMDS codes are a powerful tool that combines error correction and security features in the realm of quantum information processing. Their applications extend beyond quantum communication, impacting various fields of quantum technology, and ensuring the reliability, integrity, and confidentiality of quantum data in diverse scenarios. As quantum technologies continue to advance, the significance of quantum AMDS codes in securing and advancing these technologies cannot be overstated.

Motivated by those researchers, in this research, we investigate AMDS constacyclic codes of length p^s over $\mathbb{F}_{p^m}$. We also give some examples to illustrate. As an important application, all qAMDS codes from repeated-root constacyclic codes of prime power lengths over finite fields using the CSS and Hermitian constructions are constructed.

The rest of our paper is organized as follows. Section 2 gives some preliminaries. Section 3 provides all AMDS constacyclic codes length p^s over $\mathbb{F}_{p^m}$. Section 4 focuses on constructing qAMDS codes of length p^s over $\mathbb{F}_{p^m}$. Finally, Section 5 gives some conclusions and some directions for future work.

2. Preliminaries

Let $\mathbb{F}_{p^m}$ be a finite field, where p is prime and $m \in \mathbb{N} \setminus \{0\}$. A code of length n over $\mathbb{F}_{p^m}$ is a nonempty subset $\mathbf{C}$ of $\mathbb{F}_{p^m}^n$. If a nonempty subset $\mathbf{C}$ is a vector space over $\mathbb{F}_{p^m}$, then $\mathbf{C}$ is called a *linear code*. For an invertible γ of $\mathbb{F}_{p^m}$, the γ-constacyclic (γ-twisted) shift τ_γ on $\mathbb{F}_{p^m}^n$ is the shift

$$\tau_\gamma(x_0, x_1, \ldots, x_{n-1}) = (\gamma x_{n-1}, x_0, x_1, \cdots, x_{n-2}).$$

If $\tau_\gamma(C) = C$, then $\mathbf{C}$ is a γ-constacyclic code.

Let $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ be a codeword. Then we have a bijective correspondence between $\mathbf{C}$ and the polynomial $c(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} \in \frac{\mathbb{F}_{p^m}[x]}{\langle x^n - \gamma \rangle}$. From this, a linear code $\mathbf{C}$ of length n over $\mathbb{F}_{p^m}$ is a γ-constacyclic code of length n over $\mathbb{F}_{p^m}$ if and only if $\mathbf{C}$ is an ideal of $\frac{\mathbb{F}_{p^m}[x]}{\langle x^n - \gamma \rangle}$ (cf. [71]).

Given n-tuples

$$u = (u_0, u_1, \ldots, u_{n-1}), v = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{F}_{p^m}^n,$$

the inner product (dot product) of two vectors u, v is expressed as follows:

$$u \cdot v = u_0 v_0 + u_1 v_1 + \cdots + u_{n-1} v_{n-1},$$

evaluated in $\mathbb{F}_{p^m}$. If $u \cdot v = 0$, then two vectors u, v are called *orthogonal*. Dual code of a linear code $\mathbf{C}$ over $\mathbb{F}_{p^m}$, denoted by $\mathbf{C}^\perp$, is defined as follows:

$$\mathbf{C}^\perp = \{u \in \mathbb{F}_{p^m}^n \mid u \cdot v = 0, \forall v \in \mathbf{C}\}.$$

The dual of a γ-constacyclic code is given in the following result.

Proposition 2.1 (cf. [26]) *The dual of a γ-constacyclic code is a γ^{-1}-constacyclic code.*

For a code $\mathbf{C}$ containing at least two words, the Hamming distance of the code C, denoted by $d_H(\mathbf{C})$, is

$$d_H(\mathbf{C}) = \min\{d_H(u, v), u, v \in \mathbf{C}, u \neq v\}.$$

The following result is given in [26].

Theorem 2.2. (cf. [26]) *Each γ-constacyclic code of length p^s over $\mathbb{F}_{p^m}$ is an ideal which has the form $\langle (x-\gamma_0)^j \rangle$, $j = 0, 1, \ldots, p^s$, of the chain ring R. Each $\mathbf{C}_j = \langle (x-\gamma_0)^j \rangle \subseteq R$ has $p^{m(p^s-j)}$ codewords and the dual of $\mathbf{C}_j^{\perp} = \langle (x-\gamma_0^{-1})^{p^s-j} \rangle$ has p^{mj} codewords.*

In [24, 26], the authors gave the Hamming distance of each code $\mathbf{C}_j$ in Theorem 2.2.

Theorem 2.3. (cf. [24, 26]) *Let $\mathbf{C}$ be a γ-constacyclic code of length p^s over $\mathbb{F}_{p^m}$, then $\mathbf{C} = \langle (x-\gamma_0)^j \rangle \subseteq R$, for $j \in \{0, 1, \ldots, p^s\}$, and its Hamming distance $d_H(C)$ is determined by:*

$$d_H(\mathbf{C}) = \begin{cases} 1, & \text{if } j = 0 \\ \ell + 2, & \text{if } \ell p^{s-1} + 1 \leq j \leq (\ell+1) p^{s-1} \\ & \quad\quad \text{where } 0 \leq \ell \leq p-2 \\ (\ell+1)p^k, & \text{if } p^s - p^{s-k} + (\ell-1)p^{s-k-1} + 1 \leq j \leq p^s - p^{s-k} + \ell p^{s-k-1} \\ & \quad\quad \text{where } 1 \leq \ell \leq p-1, \text{ and } 1 \leq k \leq s-1 \\ 0, & \text{if } j = p^s. \end{cases}$$

Quantum error correction (QEC) is a set of methods to protect quantum information from unwanted environmental interactions (decoherence) and other forms of quantum noise. Quantum error correction is the key factor of quantum information technology, quantum computer and quantum communication. In less than 25 years, the subject that previously had many quantum theorists doubting its practicality has become a relatively extensive and well-developed theoretical field of study. In fact, there have been numerous published studies as well as thousands of international conferences held. It is well-known that classical error-correction codes (which are widely used in computer and communication error correction nowadays) cannot be used in the case of quantum information. Therefore, we need to construct quantum error-correcting codes (briefly, QEC codes) that preserve Quantum Information.

Let $H_q(C) = H_q(C) \otimes \cdots \otimes H_q(C)$ (n times) be a q-dimensional Hilbert vector space (where $q = p^m$). Then $H_q^n(C)$ is said to be a q^n-dimensional Hilbert space. The definition of QEC codes is given in [74].

Definition 2.4. [74] *A quantum code of length n and dimension t over $\mathbb{F}_q$ is defined to be a q^t dimensional subspace of $H_q^n(C)$ and simply denoted by $[[n, t, d]]_q$, where d is the distance of the quantum code.*

3. AMDS Constacyclic Codes

In this section, we construct AMDS constacyclic codes from repeated-root constacyclic codes of length p^s over $\mathbb{F}_{p^m}$. Let $\mathbf{C}_j = \langle (x-\gamma_0)^j \rangle \subseteq R$ be a γ-constacyclic code of length p^s over $\mathbb{F}_{p^m}$, where $0 \leq j \leq p^s$. Then $|\mathbf{C}_j| = p^{m(p^s-j)}$. From the Singleton Bound, $\mathbf{C}_j$ is an AMDS constacyclic code if and only if $p^s - d_H(\mathbf{C}_j) = p^s - j$, i.e., $j = d_H(\mathbf{C}_j)$.

The Hamming distance $d_H(\mathbf{C}_j)$ for all $j \in \{0, 1, \ldots, p^s\}$ is given in Theorem 2.3. Applying this theorem, we determine all AMDS constacyclic codes of length p^s over $\mathbb{F}_{p^m}$ in the following theorem.

Theorem 3.1. *Let $\mathbf{C}_j = \langle (x-\gamma_0)^j \rangle \subseteq R$ be a γ-constacyclic code of length p^s over $\mathbb{F}_{p^m}$, where $0 \leq j \leq p^s$. Then $\mathbf{C}$ is an AMDS constacyclic code when $s \geq 2$, $j = 2$.*

Proof. We divide j into 4 cases, namely, $j = 0, \beta p^{s-1} + 1 \leq j \leq (\beta+1)p^{s-1}, p^s - p^{s-k} + (t-1)p^{s-k-1} + 1 \leq j \leq p^s - p^{s-k} + tp^{s-k-1}$, and $j = p^s$.

Case 1: Suppose that $j = 0$. This implies that $d_H(\mathbf{C}_j) = 1$. Hence, $j \neq d_H(\mathbf{C}_j)$.

Case 2: We consider the case $\beta p^{s-1} + 1 \leq j \leq (\beta+1)p^{s-1}$, where $0 \leq \beta \leq p-2$. Theorem 2.3 implies that $d_H(\mathbf{C}_j) = \beta + 2$. We see that $j \geq \beta p^{s-1} + 1$. If $s = 1$, then $j \leq (\beta+1) < (\beta+2)$. Hence, $\mathbf{C}$ is not an AMDS code when $s = 1$. If $s \geq 2$, then $j = 2 = (\beta+2)$ when $\beta = 0$. Thus, $\mathbf{C}$ is an AMDS code when $j = 2$ and $\beta = 0$.

Case 3: Assume that $p^s - p^{s-k} + (t-1)p^{s-k-1} + 1 \leq j \leq p^s - p^{s-k} + tp^{s-k-1}$, where $1 \leq k \leq s-1$. If $s = 1$, then $j \neq t+1$. Hence, **C** is not an AMDS code. We observe that $j \geq p^s - p^{s-k} + (t-1)p^{s-k-1} + 1$. Since $p^s - p^{s-k} + (t-1)p^{s-k-1} + 1 = p^{s-k}(p^k - 1) + (t-1)p^{s-k-1} + 1$, we see that $j \geq p^{s-k}(p^k - 1) + (t-1)p^{s-k-1} + 1 \geq p(p^k - 1) + (t-1)p^{s-k-1} + 1$. Thus, $j \geq (t+1)(p^k - 1) + t - 1 + 1 = (t+1)p^k - 1$. Hence, $j \neq d_H(\mathbf{C_j})$. Therefore, **C** is not an AMDS code when $s \geq 2$ and $p^s - p^{s-k} + (t-1)p^{s-k-1} + 1 \leq j \leq p^s - p^{s-k} + tp^{s-k-1}$.

Case 4: Assume that $j = p^s$. Then $d_H(\mathbf{C}_j) = 0$. Hence, $j = p^s > d_H(\mathbf{C}_j) = 0$ for any $s \geq 1$. Thus, **C** is not an AMDS code.

Remark 3.2. Let $\mathbf{C}_j = \langle (x-\gamma_0)^j \rangle \subseteq R$ be a γ-constacyclic code of length p^s over $\mathbb{F}_{p^m}$, where $0 \leq j \leq p^s$. In [32, Theorem 3.2], the authors gave all MDS codes of length p^s over $\mathbb{F}_{p^m}$. Following that, we see that $\mathbf{C}_j$ is an MDS constacyclic code if and only if one of the following conditions holds:

- If $j = 0$, then $d_H(\mathbf{C}_j) = 1$ for any $s \geq 1$.
- If $s = 1$, then $0 \leq j \leq p-1$. In such case, $d_H(\mathbf{C}_j) = j+1$.
- If $s \geq 2$, then
  - $j = 1$, $d_H(\mathbf{C}_j) = 2$,
  - $j = p^s - 1$, $d_H(\mathbf{C}_j) = p^s$.

Combining with Theorem 3.1, we conclude that

- If $j = 0$ and $d_H(\mathbf{C}_j) = 1$ for any $s \geq 1$, then $\mathbf{C}_j$ is an MDS code and $\mathbf{C}_j$ is not an AMDS code.
- If $s = 1$, $0 \leq j \leq p-1$ and $d_H(\mathbf{C}_j) = j+1$, then $\mathbf{C}_j$ is an MDS code and $\mathbf{C}_j$ is not an AMDS code.
- If $s \geq 2$, $j = 1$, and $d_H(\mathbf{C}_j) = 2$, then $\mathbf{C}_j$ is an MDS code and $\mathbf{C}_j$ is not an AMDS code.
- If $s \geq 2$, $j = p^s - 1$ and $d_H(\mathbf{C}_j) = p^s$, then $\mathbf{C}_j$ is an MDS code and $\mathbf{C}_j$ is not an AMDS code.
- If $s \geq 2$, $j = 2$ and $d_H(\mathbf{C}_j) = 2$, then $\mathbf{C}_j$ is an AMDS code and $\mathbf{C}_j$ is not an MDS code.
- If $2 < j < p^s - 1$, $s \geq 2$, and $d_H(\mathbf{C}_j) \neq 2$, then $\mathbf{C}_j$ is not an AMDS code and $\mathbf{C}_j$ is not an MDS code.
- If $j = p^s$ and any $s \geq 1$, then $\mathbf{C}_j$ is not an AMDS code and $\mathbf{C}_j$ is not an MDS code.

To finish this section, we give some examples of AMDS constacyclic codes to illustrate our results.

Example 3.3. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_8[x]}{\langle x^4 - 1 \rangle}$, where $0 \leq j \leq 4$ be cyclic codes of length 4 over $\mathbb{F}_8$. Here, $\gamma = \gamma_0 = 1, p = 2, s = 2$ and $m = 3$. We give all Hamming distances of such codes. We also list all AMDS cyclic codes.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	MDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	Yes
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	Yes
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	2	Yes	No
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	4	No	Yes
4	$\mathbf{C}_4 = \langle 0 \rangle$	0	No	No

Table 1: AMDS cyclic codes of length 4 over $\mathbb{F}_8$.

Example 3.4. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_8[x]}{\langle x^8-1 \rangle}$, where $0 \le j \le 8$ be cyclic codes of length 8 over $\mathbb{F}_8$. Here, $\gamma = \gamma_0 = 1, p = 2, s = 3$ and $m = 3$. We give all AMDS cyclic codes of length 8 over $\mathbb{F}_8$.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	MDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	Yes
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	Yes
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	2	Yes	No
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	2	No	No
4	$\mathbf{C}_4 = \langle (x-1)^4 \rangle$	2	No	No
5	$\mathbf{C}_5 = \langle (x-1)^5 \rangle$	4	No	No
6	$\mathbf{C}_6 = \langle (x-1)^6 \rangle$	8	No	No
7	$\mathbf{C}_7 = \langle (x-1)^7 \rangle$	8	No	Yes
8	$\mathbf{C}_8 = \langle 0 \rangle$	0	No	No

Table 2: MDS cyclic codes of length 8 over $\mathbb{F}_8$.

Example 3.5. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_8[x]}{\langle x^{16}-1 \rangle}$, where $0 \le j \le 16$ be cyclic codes of length 16 over $\mathbb{F}_8$. Here, $\gamma = \gamma_0 = 1, p = 2, s = 4$ and $m = 3$. We give all AMDS cyclic codes of length 16 over $\mathbb{F}_8$.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	MDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	Yes
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	Yes
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	2	Yes	No
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	2	No	No
4	$\mathbf{C}_4 = \langle (x-1)^4 \rangle$	2	No	No
5	$\mathbf{C}_5 = \langle (x-1)^5 \rangle$	2	No	No
6	$\mathbf{C}_6 = \langle (x-1)^6 \rangle$	2	No	No
7	$\mathbf{C}_7 = \langle (x-1)^7 \rangle$	2	No	No
8	$\mathbf{C}_8 = \langle (x-1)^8 \rangle$	2	No	No
9	$\mathbf{C}_9 = \langle (x-1)^9 \rangle$	4	No	No
10	$\mathbf{C}_{10} = \langle (x-1)^{10} \rangle$	4	No	No
11	$\mathbf{C}_{11} = \langle (x-1)^{11} \rangle$	4	No	No
12	$\mathbf{C}_{12} = \langle (x-1)^{12} \rangle$	4	No	No
13	$\mathbf{C}_{13} = \langle (x-1)^{13} \rangle$	8	No	No
14	$\mathbf{C}_{14} = \langle (x-1)^{14} \rangle$	8	No	No
15	$\mathbf{C}_{15} = \langle (x-1)^{15} \rangle$	16	No	Yes
16	$\mathbf{C}_{16} = \langle 0 \rangle$	0	No	No

Table 3: AMDS cyclic codes of length 16 over $\mathbb{F}_8$.

Example 3.6. Let $\mathbf{C}_j = \langle (x-\gamma_0)^j \rangle$ of $\frac{\mathbb{F}_5[x]}{\langle x^{25}-2 \rangle}$, where $\gamma_0^{25} = 2, 0 \le j \le 25$ be 2-constacyclic codes of length 25 over $\mathbb{F}_5$. From $2^{25} = 2 \in \mathbb{F}_5$, we have $\gamma_0 = 2$. Here, $p = 5, s = 2$ and $m = 1$. Applying Theorem 2.3, we can compute all Hamming distances of 2-constacyclic codes $d_H(\mathbf{C}_j)$. Using Theorem 3.1, all AMDS 2-constacyclic codes of length 25 over $\mathbb{F}_5$ are determined in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	MDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	Yes
1	$\mathbf{C}_1 = \langle (x-2) \rangle$	2	No	Yes
2	$\mathbf{C}_2 = \langle (x-2)^2 \rangle$	2	Yes	No
3	$\mathbf{C}_3 = \langle (x-2)^3 \rangle$	2	No	No
4	$\mathbf{C}_4 = \langle (x-2)^4 \rangle$	2	No	No
5	$\mathbf{C}_5 = \langle (x-2)^5 \rangle$	2	No	No
6	$\mathbf{C}_6 = \langle (x-2)^6 \rangle$	3	No	No
7	$\mathbf{C}_7 = \langle (x-2)^7 \rangle$	3	No	No
8	$\mathbf{C}_8 = \langle (x-2)^8 \rangle$	3	No	No
9	$\mathbf{C}_9 = \langle (x-2)^9 \rangle$	3	No	No
10	$\mathbf{C}_{10} = \langle (x-2)^{10} \rangle$	3	No	No
11	$\mathbf{C}_{11} = \langle (x-2)^{11} \rangle$	4	No	No
12	$\mathbf{C}_{12} = \langle (x-2)^{12} \rangle$	4	No	No
13	$\mathbf{C}_{13} = \langle (x-2)^{13} \rangle$	4	No	No
14	$\mathbf{C}_{14} = \langle (x-2)^{14} \rangle$	4	No	No
15	$\mathbf{C}_{15} = \langle (x-2)^{15} \rangle$	4	No	No
16	$\mathbf{C}_{16} = \langle (x-2)^{16} \rangle$	5	No	No
17	$\mathbf{C}_{17} = \langle (x-2)^{17} \rangle$	5	No	No
18	$\mathbf{C}_{18} = \langle (x-2)^{18} \rangle$	5	No	No
19	$\mathbf{C}_{19} = \langle (x-2)^{19} \rangle$	5	No	No
20	$\mathbf{C}_{20} = \langle (x-2)^{20} \rangle$	5	No	No
21	$\mathbf{C}_{21} = \langle (x-2)^{21} \rangle$	10	No	No
22	$\mathbf{C}_{22} = \langle (x-2)^{22} \rangle$	15	No	No
23	$\mathbf{C}_{23} = \langle (x-2)^{23} \rangle$	20	No	No
24	$\mathbf{C}_{24} = \langle (x-2)^{24} \rangle$	25	No	Yes
25	$\mathbf{C}_{25} = \langle 0 \rangle$	0	No	No

Table 4: AMDS 2-constacyclic codes of length 25 over $\mathbb{F}_5$.

Example 3.7. Let $\mathbf{C}_j = \langle (x-\gamma_0)^j \rangle$ of $\frac{\mathbb{F}_9[x]}{\langle x^9-2 \rangle}$, where $\gamma_0^9 = 2, 0 \leq j \leq 9$ be 2-constacyclic codes of length 9 over $\mathbb{F}_9$. From $2^9 = 2 \in \mathbb{F}_9$, we have $\gamma_0 = 2$. Here, $p = 3, s = 2$ and $m = 2$. We list all Hamming distances of such codes. We also give all AMDS 2-constacyclic codes.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	MDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	Yes
1	$\mathbf{C}_1 = \langle (x-2) \rangle$	2	No	Yes
2	$\mathbf{C}_2 = \langle (x-2)^2 \rangle$	2	Yes	No
3	$\mathbf{C}_3 = \langle (x-2)^3 \rangle$	2	No	No
4	$\mathbf{C}_4 = \langle (x-2)^4 \rangle$	3	No	No
5	$\mathbf{C}_5 = \langle (x-2)^5 \rangle$	3	No	No
6	$\mathbf{C}_6 = \langle (x-2)^6 \rangle$	3	No	No
7	$\mathbf{C}_7 = \langle (x-2)^7 \rangle$	6	No	No
8	$\mathbf{C}_8 = \langle (x-2)^8 \rangle$	9	No	Yes
9	$\mathbf{C}_9 = \langle 0 \rangle$	0	No	No

Table 5: AMDS 2-constacyclic codes of length 9 over $\mathbb{F}_9$.

Example 3.8. Let $\mathbf{C}_j = \langle (x-\gamma_0)^j \rangle$ of $\frac{\mathbb{F}_{81}[x]}{\langle x^9-2 \rangle}$, where $\gamma_0^9 = 2, 0 \leq j \leq 9$ be 2-constacyclic codes of length 9 over $\mathbb{F}_{81}$. From $2^9 = \gamma = 2 \in \mathbb{F}_{81}$, we have $\gamma_0 = 2$. Here, $p = 3, s = 2$ and $m = 4$. We list all Hamming distances of such codes. We also give all AMDS constacyclic codes.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	MDs code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	Yes
1	$\mathbf{C}_1 = \langle (x-2) \rangle$	2	No	Yes
2	$\mathbf{C}_2 = \langle (x-2)^2 \rangle$	2	Yes	No
3	$\mathbf{C}_3 = \langle (x-2)^3 \rangle$	2	No	No
4	$\mathbf{C}_4 = \langle (x-2)^4 \rangle$	3	No	No
5	$\mathbf{C}_5 = \langle (x-2)^5 \rangle$	3	No	No
6	$\mathbf{C}_6 = \langle (x-2)^6 \rangle$	3	No	No
7	$\mathbf{C}_7 = \langle (x-2)^7 \rangle$	6	No	No
8	$\mathbf{C}_8 = \langle (x-2)^8 \rangle$	9	No	Yes
9	$\mathbf{C}_9 = \langle 0 \rangle$	0	No	No

Table 6: AMDS 2-constacyclic codes of length 9 over $\mathbb{F}_{81}$.

Example 3.9. Let $\mathbf{C}_j = \langle (x-\gamma_0)^j \rangle$ of $\frac{\mathbb{F}_7[x]}{\langle x^7-3 \rangle}$, where $\gamma_0^7 = 3, 0 \leq j \leq 7$ be 3-constacyclic codes of length 7 over $\mathbb{F}_7$. Here, $p = 7, s = 1$ and $m = 1$. It is easy to see that $3^7 = 3 \in \mathbb{F}_7$. Therefore, $\gamma_0 = 3$. Applying Theorem 2.3, we can compute all Hamming distances of 3-constacyclic codes $d_H(\mathbf{C}_j)$. Using Theorem 3.1, all AMDS 3-constacyclic codes of length 7 are determined in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	MDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	Yes
1	$\mathbf{C}_1 = \langle (x-3) \rangle$	2	No	Yes
2	$\mathbf{C}_2 = \langle (x-3)^2 \rangle$	3	No	Yes
3	$\mathbf{C}_3 = \langle (x-3)^3 \rangle$	4	No	Yes
4	$\mathbf{C}_4 = \langle (x-3)^4 \rangle$	5	No	Yes
5	$\mathbf{C}_5 = \langle (x-3)^5 \rangle$	6	No	Yes
6	$\mathbf{C}_6 = \langle (x-3)^6 \rangle$	7	No	Yes
7	$\mathbf{C}_7 = \langle 0 \rangle$	0	No	No

Table 7: AMDS 3-constacyclic codes of length 7 over $\mathbb{F}_7$.

Example 3.10. Let $\mathbf{C}_j = \langle (x-\gamma_0)^j \rangle$ of $\frac{\mathbb{F}_{49}[x]}{\langle x^7-3 \rangle}$, where $\gamma_0^7 = 3, 0 \leq j \leq 7$ be 3-constacyclic codes of length 7 over $\mathbb{F}_{49}$. Here, $p = 7, s = 1$ and $m = 2$. It is easy to see that $3^7 = 3 \in \mathbb{F}_{49}$. Therefore, $\gamma_0 = 3$. Applying Theorem 2.3, we can compute all Hamming distances of 3-constacyclic codes $d_H(\mathbf{C}_j)$. Using Theorem 3.1, all AMDS 3-constacyclic codes of length 7 are determined in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	MDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	Yes
1	$\mathbf{C}_1 = \langle (x-3) \rangle$	2	No	Yes
2	$\mathbf{C}_2 = \langle (x-3)^2 \rangle$	3	No	Yes
3	$\mathbf{C}_3 = \langle (x-3)^3 \rangle$	4	No	Yes
4	$\mathbf{C}_4 = \langle (x-3)^4 \rangle$	5	No	Yes
5	$\mathbf{C}_5 = \langle (x-3)^5 \rangle$	6	No	Yes
6	$\mathbf{C}_6 = \langle (x-3)^6 \rangle$	7	No	Yes
7	$\mathbf{C}_7 = \langle 0 \rangle$	0	No	No

Table 8: MDS 3- constacyclic codes of length 7 over $\mathbb{F}_{49}$.

Example 3.11. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_{19}[x]}{\langle x^{19}-1 \rangle}$, where $0 \leq j \leq 19$ be cyclic codes of length 19 over $\mathbb{F}_{19}$ which are the ideals. Here, $p = 19, s = 1$ and $m = 1$. Applying Theorem 2.3, we can compute all Hamming distances of cyclic codes $d_H(\mathbf{C}_j)$. Using Theorem 3.1, all AMDS cyclic codes of length 19 are determined in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	MDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	Yes
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	Yes
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	3	No	Yes
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	4	No	Yes
4	$\mathbf{C}_4 = \langle (x-1)^4 \rangle$	5	No	Yes
5	$\mathbf{C}_5 = \langle (x-1)^5 \rangle$	6	No	Yes
6	$\mathbf{C}_6 = \langle (x-1)^6 \rangle$	7	No	Yes
7	$\mathbf{C}_7 = \langle (x-1)^7 \rangle$	8	No	Yes
8	$\mathbf{C}_8 = \langle (x-1)^8 \rangle$	9	No	Yes
9	$\mathbf{C}_9 = \langle (x-1)^9 \rangle$	10	No	Yes
10	$\mathbf{C}_{10} = \langle (x-1)^{10} \rangle$	11	No	Yes
11	$\mathbf{C}_{11} = \langle (x-1)^{11} \rangle$	12	No	Yes
12	$\mathbf{C}_{12} = \langle (x-1)^{12} \rangle$	13	No	Yes
13	$\mathbf{C}_{13} = \langle (x-1)^{13} \rangle$	14	No	Yes
14	$\mathbf{C}_{14} = \langle (x-1)^{14} \rangle$	15	No	Yes
15	$\mathbf{C}_{15} = \langle (x-1)^{15} \rangle$	16	No	Yes
16	$\mathbf{C}_{16} = \langle (x-1)^{16} \rangle$	17	No	Yes
17	$\mathbf{C}_{17} = \langle (x-1)^{17} \rangle$	18	No	Yes
18	$\mathbf{C}_{18} = \langle (x-1)^{18} \rangle$	19	No	Yes
19	$\mathbf{C}_{19} = \langle 0 \rangle$	0	No	No

Table 9: AMDS cyclic codes of length 19 over $\mathbb{F}_{19}$.

Example 3.12. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_{23}[x]}{\langle x^{23}-1 \rangle}$, where $0 \leq j \leq 23$ cyclic codes of length 23 over $\mathbb{F}_{23}$. Here, $p = 23, s = 1$ and $m = 1$. Applying Theorem 2.3, we can compute all Hamming distances of cyclic codes

$d_H(\mathbf{C}_j)$. Using Theorem 3.1, all AMDS cyclic codes of length 23 are determined in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	MDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	Yes
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	Yes
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	3	No	Yes
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	4	No	Yes
4	$\mathbf{C}_4 = \langle (x-1)^4 \rangle$	5	No	Yes
5	$\mathbf{C}_5 = \langle (x-1)^5 \rangle$	6	No	Yes
6	$\mathbf{C}_6 = \langle (x-1)^6 \rangle$	7	No	Yes
7	$\mathbf{C}_7 = \langle (x-1)^7 \rangle$	8	No	Yes
8	$\mathbf{C}_8 = \langle (x-1)^8 \rangle$	9	No	Yes
9	$\mathbf{C}_9 = \langle (x-1)^9 \rangle$	10	No	Yes
10	$\mathbf{C}_{10} = \langle (x-1)^{10} \rangle$	11	No	Yes
11	$\mathbf{C}_{11} = \langle (x-1)^{11} \rangle$	12	No	Yes
12	$\mathbf{C}_{12} = \langle (x-1)^{12} \rangle$	13	No	Yes
13	$\mathbf{C}_{13} = \langle (x-1)^{13} \rangle$	14	No	Yes
14	$\mathbf{C}_{14} = \langle (x-1)^{14} \rangle$	15	No	Yes
15	$\mathbf{C}_{15} = \langle (x-1)^{15} \rangle$	16	No	Yes
16	$\mathbf{C}_{16} = \langle (x-1)^{16} \rangle$	17	No	Yes
17	$\mathbf{C}_{17} = \langle (x-1)^{17} \rangle$	18	No	Yes
18	$\mathbf{C}_{18} = \langle (x-1)^{18} \rangle$	19	No	Yes
19	$\mathbf{C}_{19} = \langle (x-1)^{19} \rangle$	20	No	Yes
20	$\mathbf{C}_{20} = \langle (x-1)^{20} \rangle$	21	No	Yes
21	$\mathbf{C}_{21} = \langle (x-1)^{21} \rangle$	22	No	Yes
22	$\mathbf{C}_{22} = \langle (x-1)^{22} \rangle$	23	No	Yes
23	$\mathbf{C}_{23} = \langle 0 \rangle$	0	No	No

Table 10: AMDS cyclic codes of length 23 over $\mathbb{F}_{23}$.

4. Quantum AMDS Repeated-root Constacyclic Codes over Finite Fields

AMDS codes have a wide range of applications in various fields, owing to their exceptional error-correction capabilities and efficiency in data transmission and storage. AMDS codes are widely employed in data storage systems such as hard drives, solid-state drives, and optical media. They help safeguard data integrity by correcting errors that may occur during the reading or writing processes. In wireless communication systems, where channel conditions can be volatile, AMDS codes play a crucial role in error correction. They enhance the reliability of data transmission over noisy channels, ensuring that information reaches its destination accurately. Satellite communication relies heavily on error-correcting codes like AMDS codes to counteract the effects of signal interference, atmospheric conditions, and other forms of distortion. This ensures the integrity of data transmitted to and from satellites. AMDS codes are also used in secure communication systems to protect sensitive information from eavesdropping and data tampering. By adding redundancy and error correction, these codes enhance the security of encrypted communications. Next, we study qAMDS codes from repeated-root constacyclic codes of length p^s over $\mathbb{F}_{p^m}$.

In 1995, Shor pioneered the introduction of Quantum Error Correction (QEC) codes [82]. Subsequently, Calderbank et al. [9] employed classical codes over $GF(4)$ to explore and identify specific QEC codes. In 1998, Calderbank, Rains, Shor, and Steane [10] introduced a novel approach for constructing QEC codes derived from classical error-correcting codes. More recently, researchers have developed QEC codes over finite fields and various classes of finite rings [2, 10, 20, 41, 61]. However, there has been a notable absence of investigations into qMDS codes derived from repeated-root constacyclic codes of length p^s over $\mathbb{F}_{p^m}$ using the CSS and Hermitian constructions. In this section, we shall embark on the task of constructing qMDS codes employing repeated-root constacyclic codes of length p^s over $\mathbb{F}_{p^m}$ through the CSS and Hermitian construction methods.

We recall a construction of QEC codes, the so-called CSS construction.

Theorem 4.1. *(CSS construction) [9] Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two linear codes over $\mathbb{F}_q$. They have the parameters $[n,k_1,d_1]_q$ and $[n,k_2,d_2]_q$ such that $\mathbf{C}_2 \subseteq \mathbf{C}_1$, respectively. Then there exists a QEC code with the parameters $[[n,k_1-k_2,\min\{d_1,d_2^{\perp}\}]]_q$, where $d_2^{\perp}$ is the Hamming distance of the dual code $\mathbf{C}_2^{\perp}$. Moreover, if $\mathbf{C}_2 = \mathbf{C}_1^{\perp}$, then there exists a QEC code having the parameters $[[n,2k_1-n,d_1]]_q$.*

The quantum Singleton bound is a crucial concept in quantum coding theory. It establishes an upper limit on the number of errors that a quantum code can correct. This limit is based on the code's parameters, such as its length and Hamming distance. Codes that achieve the Singleton bound are said to be "optimal" in the sense that they provide the best possible error correction performance for a given set of parameters. Understanding and characterizing the quantum Singleton bound for codes over finite fields is essential for designing efficient quantum error-correcting codes that can protect quantum information from errors and noise. In 1997, Knill and Laflamme [60] introduced the binary version of the quantum Singleton bound, a fundamental result in the field of quantum error-correcting codes. A year later, in 1998, Calderbank, Rains, Shor, and Sloane [10] expanded upon the work of Knill and Laflamme by investigating the quantum Singleton bound in a broader context. Specifically, they extended the quantum Singleton bound for all codes over finite fields. This extension is significant because it allows researchers to explore the quantum Singleton bound's implications for a wider range of quantum error-correcting codes. We recall the quantum Singleton bound in the following theorem.

Theorem 4.2. (Quantum Singleton Bound) [44, Theorem 1] *Let $\mathbf{Q} = [[n,k,d]]_q$ be a QEC code. Then* $2(d-1) \leq n-k$.

Definition 4.3. *Let $\mathbf{Q} = [[n,k,d]]_q$ be a QEC code. If $2(d-1) = n-k$, then $\mathbf{Q}$ is called a qMDS code. If $2d = n-k$, then $\mathbf{Q}$ is called a qAMDS code.*

We will now proceed with the construction of qAMDS codes by utilizing repeated-root constacyclic codes with a length of p^s over the finite field $\mathbb{F}_{p^m}$. To accomplish this, we must first enumerate all linear AMDS repeated-root constacyclic codes with a length of p^s over $\mathbb{F}_{p^m}$ that satisfy the condition $\mathbf{C}^{\perp} \subseteq C$. Let $\mathbf{C}_j$ as $\langle (x-\gamma_0)^j \rangle \subseteq R = \dfrac{\mathbb{F}p^m[x]}{\langle x^{p^s}-\gamma \rangle}$ be a γ-constacyclic code of length p^s over $\mathbb{F}_{p^m}$, where j belonging to the set $0,1,\ldots,p^s$. The dual of $\mathbf{C}_j$ can be expressed as the γ^{-1}-constacyclic code, denoted as $\mathbf{C}_j^{\perp} = \langle (x-\gamma_0^{-1})^{p^s-j} \rangle$. According to [29, Proposition 2.5], when $\gamma \neq \gamma^{-1}$, it follows that $\mathbf{C}_j^{\perp} \not\subseteq \mathbf{C}_j$ for all $j \in 0,1,\ldots,p^s$. In cases where $\gamma = \gamma^{-1}$, both $\mathbf{C}_j$ and $\mathbf{C}_j^{\perp}$ are ideals within the chain ring R. Consequently, if $\gamma = \gamma^{-1}$ and $0 \leq j \leq \frac{p^s}{2}$, then $\mathbf{C}_j^{\perp} \subseteq \mathbf{C}_j$. We will determine all qAMDS codes that can be constructed from $\mathbf{C}_j$ by employing the CSS construction, which is detailed in the following theorem.

Theorem 4.4. *Let $\mathbf{C}_j = \langle (x-\gamma_0)^j \rangle \subseteq R$ be a γ-constacyclic code of length p^s over $\mathbb{F}_{p^m}$, for $j \in \{0,1,\ldots,p^s\}$. Suppose that $\gamma = \gamma^{-1}$. If $s \geq 2$ and $j = 2$, then there exists a qAMDS code with parameters $[[p^s,p^s-4,2]]_q$.*

Proof. Let $\mathbf{C}_j = [p^s,k_j,d_H(\mathbf{C}_j)]_q$ be an AMDS constacyclic code such that $\mathbf{C}_j^{\perp} \subseteq \mathbf{C}_j$. Then we have $k_j = p^s - d_H(\mathbf{C}_j)$ and $0 \leq j \leq \frac{p^s}{2}$. From $\mathbf{C}_j^{\perp} \subseteq \mathbf{C}_j$, by applying Theorem 4.1 (the CSS construction), a quantum code $\mathbf{D}_j$ with parameters $[[p^s,2k_j-p^s,d_H(\mathbf{C}_j)]]_q$ is existed. Since $k_j = p^s-d_H(\mathbf{C}_j)$, $2k_j-p^s = p^s-2d_H(\mathbf{C}_j)$. By using Definition 4.3, $\mathbf{D}_j$ is a qAMDS code with parameters $[[p^s,2k_j-p^s,d_H(\mathbf{C}_j)]]_q$. Hence, if $\mathbf{C}_j = [p^s,k_j,d_H(\mathbf{C}_j)]_q$ is an AMDS constacyclic code and $\mathbf{C}_j^{\perp} \subseteq \mathbf{C}_j$, a qAMDS code with parameters $[[p^s,p^s-2d_H(\mathbf{C}_j),d_H(\mathbf{C}_j)]]_q$ is existed. If $s \geq 2$ and $j = 2$, we have $d_H(\mathbf{C}_1) = 2$. Applying Theorem 3.1, $\mathbf{C}_1$ is an AMDS constacyclic code. Since $j = 2$ and $\gamma = \gamma^{-1}$, we have $\mathbf{C}_1^{\perp} \subseteq \mathbf{C}_1$. Hence, there exists a qAMDS code with parameters $[[p^s,p^s-4,2]]_q$ is existed. □

Remark 4.5. In [32, Theorem 4.3], the authors provided all qMDS codes constructed from $\mathbf{C}_j$ using the CSS construction. Let $\mathbf{C}_j = \langle (x-\gamma_0)^j \rangle \subseteq R$ be a γ-constacyclic code of length p^s over $\mathbb{F}_{p^m}$, for $j \in \{0, 1, \ldots, p^s\}$ and $\gamma = \gamma^{-1}$. Then the following statements hold:

- If $j = 0$ and for any $s \geq 1$, then a qMDS code with parameters $[[p^s, p^s, 1]]_q$ is existed.
- If $s = 1$ and $1 \leq j < \frac{p}{2}$, then a qMDS code with parameters $[[p, p-2j, j+1]]_q$ is existed.
- If $s \geq 2$ and $j = 1$, then a qMDS code with parameters $[[p^s, p^s-2, 2]]_q$ is existed.

Combining with Theorem 4.4, we conclude that

- If $j = 0$ and for any $s \geq 1$, then a qMDS code with parameters $[[p^s, p^s, 1]]_q$ is existed and there is no qAMDS code.
- If $s = 1$ and $1 \leq j < \frac{p}{2}$, then a qMDS code with parameters $[[p, p-2j, j+1]]_q$ is existed and there is no qAMDS code.
- If $s \geq 2$ and $j = 1$, then a qMDS code with parameters $[[p^s, p^s-2, 2]]_q$ is existed and there is no qAMDS code.
- If $s \geq 2$ and $j = 2$, then a qAMDS code with parameters $[[p^s, p^s-4, 2]]_q$ is existed and there is no qMDS code.
- If $s \geq 2$ and $j > 2$, then there is no qAMDS code and there is no qMDS code.
- If $s = 1$ and $j \geq \frac{p}{2}$, there is no qAMDS code and there is no qMDS code.

We give some qAMDS codes in the following examples. We can compare our qAMDS codes and known families of QEC codes in [47] to see that our qAMDS codes are coincided in the sense that their parameters are same from all the known ones.

Example 4.6. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_8[x]}{\langle x^8-1 \rangle}$, where $0 \leq j \leq 8$ be cyclic codes of length 8 over $\mathbb{F}_8$. Hence, $\mathbf{C}_j^\perp = \langle (x-1)^{p^s-j} \rangle$, where $0 \leq j \leq 8$. Here, $p = 2, s = 3$ and $m = 3$. If $0 \leq j \leq 4$, then $\mathbf{C}_j^\perp \subseteq \mathbf{C}_j$. Applying Theorem 4.3, we give all qAMDS codes constructed from $\mathbf{C}_j$ using the CSS construction.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[8,8,1]]_8$
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	No	$[[8,6,2]]_8$
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	2	Yes	$[[8,4,2]]_8$	No
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	2	No	No	No
4	$\mathbf{C}_4 = \langle (x-1)^4 \rangle$	2	No	No	No
5	$\mathbf{C}_5 = \langle (x-1)^5 \rangle$	4	No	No	No
6	$\mathbf{C}_6 = \langle (x-1)^6 \rangle$	8	No	No	No
7	$\mathbf{C}_7 = \langle (x-1)^7 \rangle$	8	No	No	No
8	$\mathbf{C}_8 = \langle 0 \rangle$	0	No	No	No

Table 11: qAMDS codes of length 8 over $\mathbb{F}_8$.

Example 4.7. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_8[x]}{\langle x^{16}-1 \rangle}$, where $0 \leq j \leq 16$ be cyclic codes of length 16 over $\mathbb{F}_8$. Hence, $\mathbf{C}_j^\perp = \langle (x-1)^{p^s-j} \rangle$, where $0 \leq j \leq 16$. Here, $p = 2, s = 4$ and $m = 3$. If $0 \leq j \leq 8$, then $\mathbf{C}_j^\perp \subseteq \mathbf{C}_j$. Applying

Theorem 4.3, we give all qAMDS codes constructed from $\mathbf{C}_j$ using the CSS construction.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[16, 16, 1]]_8$
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	No	$[[16, 14, 2]]_8$
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	2	Yes	$[[16, 12, 2]]_8$	No
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	2	No	No	No
4	$\mathbf{C}_4 = \langle (x-1)^4 \rangle$	2	No	No	No
5	$\mathbf{C}_5 = \langle (x-1)^5 \rangle$	2	No	No	No
6	$\mathbf{C}_6 = \langle (x-1)^6 \rangle$	2	No	No	No
7	$\mathbf{C}_7 = \langle (x-1)^7 \rangle$	2	No	No	No
8	$\mathbf{C}_8 = \langle (x-1)^8 \rangle$	2	No	No	No
9	$\mathbf{C}_9 = \langle (x-1)^9 \rangle$	4	No	No	No
10	$\mathbf{C}_{10} = \langle (x-1)^{10} \rangle$	4	No	No	No
11	$\mathbf{C}_{11} = \langle (x-1)^{11} \rangle$	4	No	No	No
12	$\mathbf{C}_{12} = \langle (x-1)^{12} \rangle$	4	No	No	No
13	$\mathbf{C}_{13} = \langle (x-1)^{13} \rangle$	8	No	No	No
14	$\mathbf{C}_{14} = \langle (x-1)^{14} \rangle$	8	No	No	No
15	$\mathbf{C}_{15} = \langle (x-1)^{15} \rangle$	16	No	No	No
16	$\mathbf{C}_{16} = \langle 0 \rangle$	0	No	No	No

Table 12: qAMDS codes of length 16 over $\mathbb{F}_8$.

Example 4.8. Let $\mathbf{C}_j = \langle (x+1)^i \rangle$ of $\frac{\mathbb{F}_5[x]}{\langle x^{25}+1 \rangle}$, where $0 \leq j \leq 25$ be negacyclic codes of length 25 over $\mathbb{F}_5$. Here, $p = 5, s = 2$ and $m = 1$. If $0 \leq j \leq 12$, then $\mathbf{C}_j^{\perp} \subseteq \mathbf{C}_j$. By using Theorem 4.3, we can determine all qAMDS codes constructed from $\mathbf{C}_j$ using the CSS construction in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[25, 25, 1]]_5$
1	$\mathbf{C}_1 = \langle (x+1) \rangle$	2	No	No	$[[25, 23, 2]]_5$
2	$\mathbf{C}_2 = \langle (x+1)^2 \rangle$	2	Yes	$[[25, 23, 2]]_5$	No
3	$\mathbf{C}_3 = \langle (x+1)^3 \rangle$	2	No	No	No
4	$\mathbf{C}_4 = \langle (x+1)^4 \rangle$	2	No	No	No
5	$\mathbf{C}_5 = \langle (x+1)^5 \rangle$	2	No	No	No
6	$\mathbf{C}_6 = \langle (x+1)^6 \rangle$	3	No	No	No
7	$\mathbf{C}_7 = \langle (x+1)^7 \rangle$	3	No	No	No
8	$\mathbf{C}_8 = \langle (x+1)^8 \rangle$	3	No	No	No
9	$\mathbf{C}_9 = \langle (x+1)^9 \rangle$	3	No	No	No
10	$\mathbf{C}_{10} = \langle (x+1)^{10} \rangle$	3	No	No	No
11	$\mathbf{C}_{11} = \langle (x+1)^{11} \rangle$	4	No	No	No
12	$\mathbf{C}_{12} = \langle (x+1)^{12} \rangle$	4	No	No	No
13	$\mathbf{C}_{13} = \langle (x+1)^{13} \rangle$	4	No	No	No
14	$\mathbf{C}_{14} = \langle (x+1)^{14} \rangle$	4	No	No	No
15	$\mathbf{C}_{15} = \langle (x+1)^{15} \rangle$	4	No	No	No
16	$\mathbf{C}_{16} = \langle (x+1)^{16} \rangle$	5	No	No	No
17	$\mathbf{C}_{17} = \langle (x+1)^{17} \rangle$	5	No	No	No
18	$\mathbf{C}_{18} = \langle (x+1)^{18} \rangle$	5	No	No	No
19	$\mathbf{C}_{19} = \langle (x+1)^{19} \rangle$	5	No	No	No
20	$\mathbf{C}_{20} = \langle (x+1)^{20} \rangle$	5	No	No	No
21	$\mathbf{C}_{21} = \langle (x+1)^{21} \rangle$	10	No	No	No
22	$\mathbf{C}_{22} = \langle (x+1)^{22} \rangle$	15	No	No	No
23	$\mathbf{C}_{23} = \langle (x+1)^{23} \rangle$	20	No	No	No
24	$\mathbf{C}_{24} = \langle (x+1)^{24} \rangle$	25	No	No	No
25	$\mathbf{C}_{25} = \langle 0 \rangle$	0	No	No	No

Table 13: qAMDS codes of length 25 over $\mathbb{F}_5$.

Example 4.9. Let $\mathbf{C}_j = \langle (x+1)^j \rangle$ of $\frac{\mathbb{F}_9[x]}{\langle x^9+1 \rangle}$, where $0 \leq j \leq 9$ be negacyclic codes of length 9 over $\mathbb{F}_9$. Hence, $\mathbf{C}_j^{\perp} = \langle (x+1)^{p^s-j} \rangle$, where $0 \leq j \leq 9$. Here, $p = 3, s = 2$ and $m = 2$. If $0 \leq j \leq 4$, then $\mathbf{C}_j^{\perp} \subseteq \mathbf{C}_j$. Using Theorem 4.3, we also give all qAMDS codes constructed from $\mathbf{C}_j$ using the CSS construction.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[9, 9, 1]]_3$
1	$\mathbf{C}_1 = \langle (x+1) \rangle$	2	No	No	$[[9, 7, 2]]_3$
2	$\mathbf{C}_2 = \langle (x+1)^2 \rangle$	2	Yes	$[[9, 5, 2]]_3$	No
3	$\mathbf{C}_3 = \langle (x+1)^3 \rangle$	2	No	No	No
4	$\mathbf{C}_4 = \langle (x+1)^4 \rangle$	3	No	No	No
5	$\mathbf{C}_5 = \langle (x+1)^5 \rangle$	3	No	No	No
6	$\mathbf{C}_6 = \langle (x+1)^6 \rangle$	3	No	No	No
7	$\mathbf{C}_7 = \langle (x+1)^7 \rangle$	6	No	No	No
8	$\mathbf{C}_8 = \langle (x+1)^8 \rangle$	9	No	No	No
9	$\mathbf{C}_{9,0} = \langle 0 \rangle$	0	No	No	No

Table 14: qAMDS codes of length 9 over $\mathbb{F}_9$.

Example 4.10. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_{81}[x]}{\langle x^9-1 \rangle}, 0 \leq j \leq 9$ be cyclic codes of length 9 over $\mathbb{F}_{81}$. Hence, $\mathbf{C}_j^{\perp} = \langle (x-1)^{p^s-j} \rangle$, where $0 \leq j \leq 9$. Here, $p = 3, s = 2$ and $m = 4$. If $0 \leq j \leq 4$, then $\mathbf{C}_j^{\perp} \subseteq \mathbf{C}_j$. Using Theorem

4.3, we list all qMDS codes constructed from $\mathbf{C}_j$ using the CSS construction.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qAMDS code
0	$\mathbf{C}_0 = \langle 1\rangle$	1	No	No	$[[9,9,1]]_9$
1	$\mathbf{C}_1 = \langle (x-1)\rangle$	2	No	No	$[[9,7,2]]_9$
2	$\mathbf{C}_2 = \langle (x-1)^2\rangle$	2	Yes	$[[9,5,2]]_9$	No
3	$\mathbf{C}_3 = \langle (x-1)^3\rangle$	2	No	No	No
4	$\mathbf{C}_4 = \langle (x-1)^4\rangle$	3	No	No	No
5	$\mathbf{C}_5 = \langle (x-1)^5\rangle$	3	No	No	No
6	$\mathbf{C}_6 = \langle (x-1)^6\rangle$	3	No	No	No
7	$\mathbf{C}_7 = \langle (x-1)^7\rangle$	6	No	No	No
8	$\mathbf{C}_8 = \langle (x-1)^8\rangle$	9	Yes	No	No
9	$\mathbf{C}_9 = \langle 0\rangle$	0	No	No	No

Table 15: qAMDS codes of length 9 over $\mathbb{F}_{81}$.

Example 4.11. Let $\mathbf{C}_j = \langle (x+1)^j\rangle$ of $\frac{\mathbb{F}_7[x]}{\langle x^7+1\rangle}$, where $0 \le j \le 7$ be negacyclic codes of length 7 over $\mathbb{F}_7$. Hence, $\mathbf{C}_j^{\perp} = \langle (x+1)^{p^s-j}\rangle$, where $0 \le j \le 7$. Here, $p = 7, s = 1$ and $m = 1$. If $0 \le j \le 3$, then $\mathbf{C}_j^{\perp} \subseteq \mathbf{C}_j$. Using Theorem 4.3, we give all qAMDS codes constructed from $\mathbf{C}_j$ using the CSS construction in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1\rangle$	1	No	No	$[[7,7,1]]_7$
1	$\mathbf{C}_1 = \langle (x+1)\rangle$	2	No	No	$[[7,5,2]]_7$
2	$\mathbf{C}_2 = \langle (x+1)^2\rangle$	3	No	No	$[[7,3,3]]_7$
3	$\mathbf{C}_3 = \langle (x+1)^3\rangle$	4	No	No	$[[7,1,4]]_7$
4	$\mathbf{C}_4 = \langle (x+1)^4\rangle$	5	No	No	No
5	$\mathbf{C}_5 = \langle (x+1)^5\rangle$	6	No	No	No
6	$\mathbf{C}_6 = \langle (x+1)^6\rangle$	7	No	No	No
7	$\mathbf{C}_7 = \langle 0\rangle$	0	No	No	No

Table 16: qAMDS codes of length 7 over $\mathbb{F}_7$.

Example 4.12. Let $C_j = \langle (x+1)^j\rangle$ of $\frac{\mathbb{F}_{49}[x]}{\langle x^7+1\rangle}, 0 \le j \le 7$ be negacyclic codes of length 7 over $\mathbb{F}_{49}$. Hence, $\mathbf{C}_j^{\perp} = \langle (x+1)^{p^s-j}\rangle$, where $0 \le j \le 7$. Here, $p = 7, s = 1$ and $m = 2$. If $0 \le j \le 3$, then $\mathbf{C}_j^{\perp} \subseteq \mathbf{C}_j$. Using Theorem 4.3 again, we give all qAMDS codes constructed from $\mathbf{C}_j$ using the CSS construction in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1\rangle$	1	No	No	$[[7,7,1]]_7$
1	$\mathbf{C}_1 = \langle (x+1)\rangle$	2	No	No	$[[7,5,2]]_7$
2	$\mathbf{C}_2 = \langle (x+1)^2\rangle$	3	No	No	$[[7,3,3]]_7$
3	$\mathbf{C}_3 = \langle (x+1)^3\rangle$	4	No	No	$[[7,1,4]]_7$
4	$\mathbf{C}_4 = \langle (x+1)^4\rangle$	5	No	No	No
5	$\mathbf{C}_5 = \langle (x+1)^5\rangle$	6	No	No	No
6	$\mathbf{C}_6 = \langle (x+1)^6\rangle$	7	No	No	No
7	$\mathbf{C}_7 = \langle 0\rangle$	0	No	No	No

Table 17: qAMDS codes of length 7 over $\mathbb{F}_{49}$.

Example 4.13. Let $\mathbf{C}_j = \langle (x-1)^j\rangle$ of $\frac{\mathbb{F}_{17}[x]}{\langle x^{17}-1\rangle}$, where $0 \le j \le 17$ be cyclic codes of length 17 over $\mathbb{F}_{17}$. Hence, $\mathbf{C}_j^{\perp} = \langle (x-1)^{p^s-j}\rangle$, where $0 \le j \le 17$. Here, $q = p = 17, s = 1$ and $m = 1$. If $0 \le j \le 8$, then $\mathbf{C}_j^{\perp} \subseteq \mathbf{C}_j$. Applying Theorem 4.3, all qAMDS codes constructed from $\mathbf{C}_j$ using the CSS construction are determined in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[17,17,1]]_{17}$
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	No	$[[17,15,2]]_{17}$
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	3	No	No	$[[17,13,3]]_{17}$
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	4	No	No	$[[17,11,4]]_{17}$
4	$\mathbf{C}_4 = \langle (x-1)^4 \rangle$	5	No	No	$[[17,9,5]]_{17}$
5	$\mathbf{C}_5 = \langle (x-1)^5 \rangle$	6	No	No	$[[17,7,6]]_{17}$
6	$\mathbf{C}_6 = \langle (x-1)^6 \rangle$	7	No	No	$[[17,5,7]]_{17}$
7	$\mathbf{C}_7 = \langle (x-1)^7 \rangle$	8	No	No	$[[17,3,8]]_{17}$
8	$\mathbf{C}_8 = \langle (x-1)^8 \rangle$	9	No	No	$[[17,1,9]]_{17}$
9	$\mathbf{C}_9 = \langle (x-1)^9 \rangle$	10	No	No	No
10	$\mathbf{C}_{10} = \langle (x-1)^{10} \rangle$	11	No	No	No
11	$\mathbf{C}_{11} = \langle (x-1)^{11} \rangle$	12	No	No	No
12	$\mathbf{C}_{12} = \langle (x-1)^{12} \rangle$	13	No	No	No
13	$\mathbf{C}_{13} = \langle (x-1)^{13} \rangle$	14	No	No	No
14	$\mathbf{C}_{14} = \langle (x-1)^{14} \rangle$	15	No	No	No
15	$\mathbf{C}_{15} = \langle (x-1)^{15} \rangle$	16	No	No	No
16	$\mathbf{C}_{16} = \langle (x-1)^{16} \rangle$	17	No	No	No
17	$\mathbf{C}_{17} = \langle 0 \rangle$	0	No	No	No

Table 18: qAMDS codes of length 17 over $\mathbb{F}_{17}$.

Example 4.14. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_{19}[x]}{\langle x^{19}-1 \rangle}$, where $0 \le j \le 19$ be cyclic codes of length 19 over $\mathbb{F}_{19}$. Here, $p = 19, s = 1$ and $m = 1$. Applying Theorem 2.3, we can compute all Hamming distances of cyclic codes $d_H(\mathbf{C}_j)$. Using Theorem 3.2, all AMDS cyclic codes of length 19 are determined. If $0 \le j \le 8$, then $\mathbf{C}_j^{\perp} \subseteq \mathbf{C}_j$. Using Theorem 4.3 again, we give all qAMDS codes constructed from $\mathbf{C}_j$ using the CSS construction in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[19,19,1]]_{19}$
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	No	$[[19,17,2]]_{19}$
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	3	No	No	$[[19,15,3]]_{19}$
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	4	No	No	$[[19,13,4]]_{19}$
4	$\mathbf{C}_4 = \langle (x-1)^4 \rangle$	5	No	No	$[[19,11,5]]_{19}$
5	$\mathbf{C}_5 = \langle (x-1)^5 \rangle$	6	No	No	$[[19,9,6]]_{19}$
6	$\mathbf{C}_6 = \langle (x-1)^6 \rangle$	7	No	No	$[[19,7,7]]_{19}$
7	$\mathbf{C}_7 = \langle (x-1)^7 \rangle$	8	No	No	$[[19,5,8]]_{19}$
8	$\mathbf{C}_8 = \langle (x-1)^8 \rangle$	9	No	No	$[[19,3,9]]_{19}$
9	$\mathbf{C}_9 = \langle (x-1)^9 \rangle$	10	No	No	$[[19,1,10]]_{19}$
10	$\mathbf{C}_{10} = \langle (x-1)^{10} \rangle$	11	No	No	No
11	$\mathbf{C}_{11} = \langle (x-1)^{11} \rangle$	12	No	No	No
12	$\mathbf{C}_{12} = \langle (x-1)^{12} \rangle$	13	No	No	No
13	$\mathbf{C}_{13} = \langle (x-1)^{13} \rangle$	14	No	No	No
14	$\mathbf{C}_{14} = \langle (x-1)^{14} \rangle$	15	No	No	No
15	$\mathbf{C}_{15} = \langle (x-1)^{15} \rangle$	16	No	No	No
16	$\mathbf{C}_{16} = \langle (x-1)^{16} \rangle$	17	No	No	No
17	$\mathbf{C}_{17} = \langle (x-1)^{17} \rangle$	18	No	No	No
18	$\mathbf{C}_{18} = \langle (x-1)^{18} \rangle$	19	No	No	No
19	$\mathbf{C}_{19} = \langle 0 \rangle$	0	No	No	No

Table 19: qAMDS cyclic codes of length 19 over $\mathbb{F}_{19}$.

Example 4.15. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_{23}[x]}{\langle x^{23}-1 \rangle}$, where $0 \le j \le 23$ be cyclic codes of length 23 over $\mathbb{F}_{23}$. Here, $p = 23, s = 1$ and $m = 1$. Applying Theorem 2.3, we can compute all Hamming distances of cyclic codes $d_H(\mathbf{C}_j)$. Using Theorem 3.2, all AMDS cyclic codes of length 23 are determined. If $0 \le j \le 11$, then $\mathbf{C}_j^{\perp} \subseteq \mathbf{C}_j$. Using Theorem 4.3 again, we give all qAMDS codes constructed from $\mathbf{C}_j$ using the CSS construction in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[23, 23, 1]]_{23}$
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	No	$[[23, 21, 2]]_{23}$
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	3	No	No	$[[23, 19, 3]]_{23}$
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	4	No	No	$[[23, 17, 4]]_{23}$
4	$\mathbf{C}_4 = \langle (x-1)^4 \rangle$	5	No	No	$[[23, 15, 5]]_{23}$
5	$\mathbf{C}_5 = \langle (x-1)^5 \rangle$	6	No	No	$[[23, 13, 6]]_{23}$
6	$\mathbf{C}_6 = \langle (x-1)^6 \rangle$	7	No	No	$[[23, 11, 7]]_{23}$
7	$\mathbf{C}_7 = \langle (x-1)^7 \rangle$	8	No	No	$[[23, 9, 8]]_{23}$
8	$\mathbf{C}_8 = \langle (x-1)^8 \rangle$	9	No	No	$[[23, 7, 9]]_{23}$
9	$\mathbf{C}_9 = \langle (x-1)^9 \rangle$	10	No	No	$[[23, 5, 10]]_{23}$
10	$\mathbf{C}_{10} = \langle (x-1)^{10} \rangle$	11	No	No	$[[23, 3, 11]]_{23}$
11	$\mathbf{C}_{11} = \langle (x-1)^{11} \rangle$	12	No	No	$[[23, 1, 12]]_{23}$
12	$\mathbf{C}_{12} = \langle (x-1)^{12} \rangle$	13	No	No	No
13	$\mathbf{C}_{13} = \langle (x-1)^{13} \rangle$	14	No	No	No
14	$\mathbf{C}_{14} = \langle (x-1)^{14} \rangle$	15	No	No	No
15	$\mathbf{C}_{15} = \langle (x-1)^{15} \rangle$	16	No	No	No
16	$\mathbf{C}_{16} = \langle (x-1)^{16} \rangle$	17	No	No	No
17	$\mathbf{C}_{17} = \langle (x-1)^{17} \rangle$	18	No	No	No
18	$\mathbf{C}_{18} = \langle (x-1)^{18} \rangle$	19	No	No	No
19	$\mathbf{C}_{19} = \langle (x-1)^{19} \rangle$	20	No	No	No
20	$\mathbf{C}_{20} = \langle (x-1)^{20} \rangle$	21	No	No	No
21	$\mathbf{C}_{21} = \langle (x-1)^{21} \rangle$	22	No	No	No
22	$\mathbf{C}_{22} = \langle (x-1)^{22} \rangle$	23	No	No	No
23	$\mathbf{C}_{23} = \langle 0 \rangle$	0	No	No	No

Table 20: qAMDS cyclic codes of length 23 over $\mathbb{F}_{23}$.

Next, we construct qAMDS codes from repeated-root codes of length p^s over $\mathbb{F}_{p^m}$ using the Hermitian construction. Let $q = p^m$ and $\mathbb{F}_{q^2}$ be a finite field of q^2 elements. If $u = (u_0, u_1, \ldots, u_{n-1}), v = (v_0, v_1, \cdots, v_{n-1})$ are two vectors of $\mathbb{F}_{q^2}$, then Hermitian inner product of u and v is

$$u \circ_{\mathbb{F}_{q^2}} v = u_0\bar{v}_0 + u_1\bar{v}_1 + \cdots + u_{n-1}\bar{v}_{n-1},$$

where $\bar{v}_i = v_i^q$. The Hermitian dual code of $\mathbf{C}$ is defined as

$$C^{\perp_H} = \{u \in \mathbb{F}_{q^2}^n | \quad \sum_{i=0}^{n-1} u_i\bar{v}_i = 0, \forall v \in C\}.$$

If $C^{\perp_H} \subseteq C$, then $\mathbf{C}$ is said to be *Hermitian dual-containing*.

Other than the CSS construction, the so-called Hermitian construction is also an important construction, which is given in [1].

Theorem 4.16. (Hermitian construction) [1] *If* $\mathbf{C}$ *is a* q^2*-ary* $[n, k, d_H]$ *linear code such that* $\mathbf{C}^{\perp_H} \subseteq \mathbf{C}$*, then a* q*-ary quantum code with parameters* $[[n, 2k-n, \ge d_H]]_q$ *is existed.*

We have the following result.

Lemma 4.17. *Let $\mathbf{C}_j = \langle (x-\gamma_0)^j \rangle$ be a γ-constacyclic code of length p^s over $\mathbb{F}_{q^2}$, where $0 \le j \le p^s$. If $\gamma_0^{-q} = \gamma_0$ and $0 \le j \le \frac{p^s}{2}$, then $\mathbf{C}_j^{\perp_H} \subseteq \mathbf{C}_j$.*

Proof. We have $\mathbf{C}_j^q = \langle (x-\gamma_0^q)^j \rangle$ and $\mathbf{C}_j^{\perp_H} = (\mathbf{C}_j^q)^{\perp}$. Then $\mathbf{C}_j^{\perp_H} = \langle (x-\gamma_0^{-q})^{p^s-j} \rangle$. If $\gamma_0^{-q} = \gamma_0$ and $0 \le j \le \frac{p^s}{2}$, then $\mathbf{C}_j^{\perp_H} \subseteq \mathbf{C}_j$, as required. □

The Hermitian construction is a mathematical framework used in the theory of quantum error-correcting codes. In quantum information theory, quantum error-correcting codes play a crucial role in protecting quantum information from errors and decoherence. The Hermitian construction is a method to construct quantum stabilizer codes, a class of quantum error-correcting codes. The Hermitian construction is a valuable tool for constructing stabilizer codes, which are essential for quantum error correction. These stabilizer codes can detect errors in a quantum state by measuring stabilizer generators and correct errors by applying appropriate quantum operations based on the measurement outcomes. Once the stabilizer generators are determined using the Hermitian construction, they provide a set of mutually commuting observables. Measuring these observables allows for the detection of errors in the quantum state. Subsequent operations based on the measurement results can correct errors, preserving the encoded quantum information. We now construct qAMDS codes from $\mathbf{C}_j$ using the Hermitian construction.

Theorem 4.18. *Let $\mathbf{C}_j = \langle (x-\gamma_0)^j \rangle$ be a γ-constacyclic code of length p^s over $\mathbb{F}_{q^2}$, where $0 \le j \le p^s$. If $\gamma_0^{-q} = \gamma_0$, $s \ge 2$ and $j = 2$, then there exists a qAMDS code with parameters $[[p^s, p^s-4, 2]]_q$.*

Proof. Let $\mathbf{C}_j = [p^s, k_j, d_H(\mathbf{C}_j)]_q$ be an AMDS constacyclic code such that $\mathbf{C}_j^{\perp_H} \subseteq \mathbf{C}_j$. Then $k_j = p^s - d_H(\mathbf{C}_j)$ and $0 \le j \le \frac{p^s}{2}$. From $\mathbf{C}_j^{\perp_H} \subseteq \mathbf{C}_j$, by Theorem 4.14 (the Hermitian construction), a quantum code $\mathbf{Q}_j$ with parameters $[[p^s, 2k_j - p^s, d^\star]]_q$, where $d^\star \ge d_H(\mathbf{C}_j)$ is existed. Using Theorem 4.2 for $\mathbf{Q_j}$, $2k_j - p^s + 2d^\star \le p^s$. Hence, $d^\star \le p^s - k_j = d_H(\mathbf{C}_j)$. Thus, $d^\star = d_H(\mathbf{C}_j)$. Therefore, if $\mathbf{C}_j = [p^s, k_j, d_H(\mathbf{C}_j)]_q$ is an AMDS constacyclic code and $\mathbf{C}_j^{\perp_H} \subseteq \mathbf{C}_j$, a qAMDS code with parameters $[[p^s, p^s - 2d_H(\mathbf{C}_j), d_H(\mathbf{C}_j)]]_q$ is existed. If $s \ge 2$ and $j = 2$, $d_H(\mathbf{C}_2) = 2$. Applying Theorem 3.1, we can see that $\mathbf{C}_2$ is an AMDS constacyclic code. As $j = 2$ and $\gamma_0^{-q} = \gamma_0$, we have $\mathbf{C}_2^{\perp_H} \subseteq \mathbf{C}_2$. Since a qAMDS code with parameters $[[p^s, p^s - 2d_H(\mathbf{C}_j), d_H(\mathbf{C}_j)]]_q$ is existed, we have a qAMDS code with parameters $[[p^s, p^s-4, 2]]_q$. □

Using the Hermitian construction, we give some examples of qAMDS codes. We can compare our qAMDS codes and known families of QEC codes in [47] to see that our qAMDS codes are coincided in the sense that their parameters are same from all the known ones.

Example 4.19. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_{16}[x]}{\langle x^8-1 \rangle}$, where $0 \le j \le 8$ be consider cyclic codes of length 8 over $\mathbb{F}_{16}$. Hence, $\mathbf{C}_j^{\perp_H} = \langle (x-1)^{p^s-j} \rangle$, where $0 \le j \le 8$. Here, $\gamma = \gamma_0 = 1, q = 4, p = 2, s = 3$ and $m = 4$. If $0 \le j \le 4$, then $\mathbf{C}_j^{\perp_H} \subseteq \mathbf{C}_j$. We give all qAMDS codes constructed from $\mathbf{C_j}$ using the Hermitian construction.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[8,8,1]]_4$
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	No	$[[8,6,2]]_4$
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	2	Yes	$[[8,4,2]]_4$	No
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	2	No	No	No
4	$\mathbf{C}_4 = \langle (x-1)^4 \rangle$	2	No	No	No
5	$\mathbf{C}_5 = \langle (x-1)^5 \rangle$	4	No	No	No
6	$\mathbf{C}_6 = \langle (x-1)^6 \rangle$	8	No	No	No
7	$\mathbf{C}_7 = \langle (x-1)^7 \rangle$	8	No	No	No
8	$\mathbf{C}_8 = \langle 0 \rangle$	0	No	No	No

Table 21: qAMDS codes of length 8 over $\mathbb{F}_{16}$.

Example 4.20. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_{16}[x]}{\langle x^{16}-1 \rangle}$, where $0 \le j \le 16$ be cyclic codes of length 16 over $\mathbb{F}_{16}$. Hence, $\mathbf{C}_j^{\perp_H} = \langle (x-1)^{p^s-j} \rangle$, where $0 \le j \le 16$. Here, $\gamma = \gamma_0 = 1, q = 4, p = 2, s = 4$ and $m = 2$. If $0 \le j \le 8$, then $\mathbf{C}_j^{\perp_H} \subseteq \mathbf{C}_j$. We give all qAMDS codes constructed from $\mathbf{C}_j$ using the Hermitian construction.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[16,16,1]]_4$
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	No	$[[16,14,2]]_4$
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	2	Yes	$[[16,12,2]]_4$	No
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	2	No	No	No
4	$\mathbf{C}_4 = \langle (x-1)^4 \rangle$	2	No	No	No
5	$\mathbf{C}_5 = \langle (x-1)^5 \rangle$	2	No	No	No
6	$\mathbf{C}_6 = \langle (x-1)^6 \rangle$	2	No	No	No
7	$\mathbf{C}_7 = \langle (x-1)^7 \rangle$	2	No	No	No
8	$\mathbf{C}_8 = \langle (x-1)^8 \rangle$	2	No	No	No
9	$\mathbf{C}_9 = \langle (x-1)^9 \rangle$	4	No	No	No
10	$\mathbf{C}_{10} = \langle (x-1)^{10} \rangle$	4	No	No	No
11	$\mathbf{C}_{11} = \langle (x-1)^{11} \rangle$	4	No	No	No
12	$\mathbf{C}_{12} = \langle (x-1)^{12} \rangle$	4	No	No	No
13	$\mathbf{C}_{13} = \langle (x-1)^{13} \rangle$	8	No	No	No
14	$\mathbf{C}_{14} = \langle (x-1)^{14} \rangle$	8	No	No	No
15	$\mathbf{C}_{15} = \langle (x-1)^{15} \rangle$	16	No	No	No
16	$\mathbf{C}_{16} = \langle 0 \rangle$	0	No	No	No

Table 22: qAMDS codes of length 16 over $\mathbb{F}_{16}$.

Example 4.21. Let $\mathbf{C}_j = \langle (x+1)^j \rangle$ of $\frac{\mathbb{F}_9[x]}{\langle x^9+1 \rangle}$, where $0 \le j \le 9$ be negacyclic codes of length 9 over $\mathbb{F}_9$. Hence, $\mathbf{C}_j^{\perp_H} = \langle (x+1)^{p^s-j} \rangle$, where $0 \le j \le 9$. Here, $\gamma = \gamma_0 = -1, q = p = 3, s = 2$ and $m = 2$. If $0 \le j \le 4$, then $\mathbf{C}_j^{\perp_H} \subseteq \mathbf{C}_j$. We give all qAMDS codes constructed from $\mathbf{C_j}$ using the Hermitian construction.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qAMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[9,9,1]]_3$
1	$\mathbf{C}_1 = \langle (x+1) \rangle$	2	No	No	$[[9,7,2]]_3$
2	$\mathbf{C}_2 = \langle (x+1)^2 \rangle$	2	Yes	$[[9,5,2]]_3$	No
3	$\mathbf{C}_3 = \langle (x+1)^3 \rangle$	2	No	No	No
4	$\mathbf{C}_4 = \langle (x+1)^4 \rangle$	3	No	No	No
5	$\mathbf{C}_5 = \langle (x+1)^5 \rangle$	3	No	No	No
6	$\mathbf{C}_6 = \langle (x+1)^6 \rangle$	3	No	No	No
7	$\mathbf{C}_7 = \langle (x+1)^7 \rangle$	6	No	No	No
8	$\mathbf{C}_8 = \langle (x+1)^8 \rangle$	9	No	No	No
9	$\mathbf{C}_9 = \langle 0 \rangle$	0	No	No	No

Table 23: qAMDS codes of length 9 over $\mathbb{F}_9$.

Example 4.22. Let $\mathbf{C}_j = \langle (x-1)^i \rangle$ of $\frac{\mathbb{F}_{81}[x]}{\langle x^9-1 \rangle}$, where $0 \le j \le 9$ be cyclic codes of length 9 over $\mathbb{F}_{81}$. Hence, $\mathbf{C}_j^{\perp_H} = \langle (x-1)^{p^s-j} \rangle$, where $0 \le j \le 9$. Here, $\gamma = \gamma_0 = 1, q = 9, p = 3, s = 2$ and $m = 4$. If $0 \le j \le 4$, then

$\mathbf{C}_j^{\perp_H} \subseteq \mathbf{C}_j$. We list all qAMDS codes constructed from $\mathbf{C}_j$ using the Hermitian construction.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[9,9,1]]_3$
1	$\mathbf{C}_1 = \langle (x+1) \rangle$	2	No	No	$[[9,7,2]]_3$
2	$\mathbf{C}_2 = \langle (x+1)^2 \rangle$	2	Yes	$[[9,5,2]]_3$	No
3	$\mathbf{C}_3 = \langle (x+1)^3 \rangle$	2	No	No	No
4	$\mathbf{C}_4 = \langle (x+1)^4 \rangle$	3	No	No	No
5	$\mathbf{C}_5 = \langle (x+1)^5 \rangle$	3	No	No	No
6	$\mathbf{C}_6 = \langle (x+1)^6 \rangle$	3	No	No	No
7	$\mathbf{C}_7 = \langle (x+1)^7 \rangle$	6	No	No	No
8	$\mathbf{C}_8 = \langle (x+1)^8 \rangle$	9	No	No	No
9	$\mathbf{C}_{9,0} = \langle 0 \rangle$	0	No	No	No

Table 24: qAMDS codes of length 9 over $\mathbb{F}_{81}$.

Example 4.23. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_{49}[x]}{\langle x^7-1 \rangle}$, where $0 \leq j \leq 7$ be cyclic codes of length 7 over $\mathbb{F}_{49}$. Hence, $\mathbf{C}_j^{\perp_H} = \langle (x-1)^{p^s-j} \rangle$, where $0 \leq j \leq 7$. Here, $\gamma = \gamma_0 = 1, q = p = 7, s = 1$ and $m = 2$. If $0 \leq j \leq 3$, then $\mathbf{C}_j^{\perp_H} \subseteq \mathbf{C}_j$. Applying Theorem 4.18, all qAMDS codes constructed from $\mathbf{C}_j$ using the Hermitian construction are determined in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[7,7,1]]_7$
1	$\mathbf{C}_1 = \langle (x+1) \rangle$	2	No	No	$[[7,5,2]]_7$
2	$\mathbf{C}_2 = \langle (x+1)^2 \rangle$	3	No	No	$[[7,3,3]]_7$
3	$\mathbf{C}_3 = \langle (x+1)^3 \rangle$	4	No	No	$[[7,1,4]]_7$
4	$\mathbf{C}_4 = \langle (x+1)^4 \rangle$	5	No	No	No
5	$\mathbf{C}_5 = \langle (x+1)^5 \rangle$	6	No	No	No
6	$\mathbf{C}_6 = \langle (x+1)^6 \rangle$	7	No	No	No
7	$\mathbf{C}_7 = \langle 0 \rangle$	0	No	No	No

Table 25: qAMDS codes of length 7 over $\mathbb{F}_{49}$.

Example 4.24. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_{289}[x]}{\langle x^{17}-1 \rangle}$, where $0 \leq j \leq 17$ be cyclic codes of length 17 over $\mathbb{F}_{289}$. Hence, $\mathbf{C}_j^{\perp_H} = \langle (x-1)^{p^s-j} \rangle$, where $0 \leq j \leq 17$. Here, $\gamma = \gamma_0 = 1, q = 17, p = 17, s = 1$ and $m = 2$. If $0 \leq j \leq 8$, then $\mathbf{C}_j^{\perp_H} \subseteq \mathbf{C}_j$. Applying Theorem 4.18, all qAMDS codes constructed from $\mathbf{C}_j$ using the Hermitian construction are determined in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[17,17,1]]_{17}$
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	No	$[[17,15,2]]_{17}$
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	3	No	No	$[[17,13,3]]_{17}$
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	4	No	No	$[[17,11,4]]_{17}$
4	$\mathbf{C}_4 = \langle (x-1)^4 \rangle$	5	No	No	$[[17,9,5]]_{17}$
5	$\mathbf{C}_5 = \langle (x-1)^5 \rangle$	6	No	No	$[[17,7,6]]_{17}$
6	$\mathbf{C}_6 = \langle (x-1)^6 \rangle$	7	No	No	$[[17,5,7]]_{17}$
7	$\mathbf{C}_7 = \langle (x-1)^7 \rangle$	8	No	No	$[[17,3,8]]_{17}$
8	$\mathbf{C}_8 = \langle (x-1)^8 \rangle$	9	No	No	$[[17,1,9]]_{17}$
9	$\mathbf{C}_9 = \langle (x-1)^9 \rangle$	10	No	No	No
10	$\mathbf{C}_{10} = \langle (x-1)^{10} \rangle$	11	No	No	No
11	$\mathbf{C}_{11} = \langle (x-1)^{11} \rangle$	12	No	No	No
12	$\mathbf{C}_{12} = \langle (x-1)^{12} \rangle$	13	No	No	No
13	$\mathbf{C}_{13} = \langle (x-1)^{13} \rangle$	14	No	No	No
14	$\mathbf{C}_{14} = \langle (x-1)^{14} \rangle$	15	No	No	No
15	$\mathbf{C}_{15} = \langle (x-1)^{15} \rangle$	16	No	No	No
16	$\mathbf{C}_{16} = \langle (x-1)^{16} \rangle$	17	No	No	No
17	$\mathbf{C}_{17} = \langle 0 \rangle$	0	No	No	No

Table 26: qAMDS codes of length 17 over $\mathbb{F}_{289}$.

Example 4.25. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_{361}[x]}{\langle x^{19}-1 \rangle}$, where $0 \leq j \leq 19$ be cyclic codes of length 19 over $\mathbb{F}_{361}$. Here, $p = 19, s = 1$ and $m = 2$. Applying Theorem 2.3, we can compute all Hamming distances of cyclic codes $d_H(\mathbf{C}_j)$. Using Theorem 3.1, all AMDS cyclic codes of length 19 are determined. If $0 \leq j \leq 9$, then $\mathbf{C}_j^{\perp} \subseteq \mathbf{C}_j$. Using Theorem 4.18 again, we give all qAMDS codes constructed from $\mathbf{C}_j$ using the Hermitian construction in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[19,19,1]]_{19}$
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	No	$[[19,17,2]]_{19}$
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	3	No	No	$[[19,15,3]]_{19}$
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	4	No	No	$[[19,13,4]]_{19}$
4	$\mathbf{C}_4 = \langle (x-1)^4 \rangle$	5	No	No	$[[19,11,5]]_{19}$
5	$\mathbf{C}_5 = \langle (x-1)^5 \rangle$	6	No	No	$[[19,9,6]]_{19}$
6	$\mathbf{C}_6 = \langle (x-1)^6 \rangle$	7	No	No	$[[19,7,7]]_{19}$
7	$\mathbf{C}_7 = \langle (x-1)^7 \rangle$	8	No	No	$[[19,5,8]]_{19}$
8	$\mathbf{C}_8 = \langle (x-1)^8 \rangle$	9	No	No	$[[19,3,9]]_{19}$
9	$\mathbf{C}_9 = \langle (x-1)^9 \rangle$	10	No	No	$[[19,1,10]]_{19}$
10	$\mathbf{C}_{10} = \langle (x-1)^{10} \rangle$	11	No	No	No
11	$\mathbf{C}_{11} = \langle (x-1)^{11} \rangle$	12	No	No	No
12	$\mathbf{C}_{12} = \langle (x-1)^{12} \rangle$	13	No	No	No
13	$\mathbf{C}_{13} = \langle (x-1)^{13} \rangle$	14	No	No	No
14	$\mathbf{C}_{14} = \langle (x-1)^{14} \rangle$	15	No	No	No
15	$\mathbf{C}_{15} = \langle (x-1)^{15} \rangle$	16	No	No	No
16	$\mathbf{C}_{16} = \langle (x-1)^{16} \rangle$	17	No	No	No
17	$\mathbf{C}_{17} = \langle (x-1)^{17} \rangle$	18	No	No	No
18	$\mathbf{C}_{18} = \langle (x-1)^{18} \rangle$	19	No	No	No
19	$\mathbf{C}_{19} = \langle 0 \rangle$	0	No	No	No

Table 27: qAMDS codes of length 19 over $\mathbb{F}_{361}$.

Example 4.26. Let $\mathbf{C}_j = \langle (x-1)^j \rangle$ of $\frac{\mathbb{F}_{529}[x]}{\langle x^{23}-1 \rangle}$, where $0 \le j \le 23$ be cyclic codes of length 23 over $\mathbb{F}_{529}$. Here, $p = 23, s = 1$ and $m = 2$. Applying Theorem 2.3, we can compute all Hamming distances of cyclic codes $d_H(\mathbf{C}_j)$. Using Theorem 3.1, all AMDS cyclic codes of length 23 are determined. If $0 \le j \le 11$, then $\mathbf{C}_j^{\perp} \subseteq \mathbf{C}_j$. Using Theorem 4.18 again, we give all qAMDS codes constructed from $\mathbf{C}_j$ using the Hermitian construction in the following table.

j	$\mathbf{C}_j$	$d_H(\mathbf{C}_j)$	AMDS code	qAMDS code	qMDS code
0	$\mathbf{C}_0 = \langle 1 \rangle$	1	No	No	$[[23, 23, 1]]_{23}$
1	$\mathbf{C}_1 = \langle (x-1) \rangle$	2	No	No	$[[23, 21, 2]]_{23}$
2	$\mathbf{C}_2 = \langle (x-1)^2 \rangle$	3	No	No	$[[23, 19, 3]]_{23}$
3	$\mathbf{C}_3 = \langle (x-1)^3 \rangle$	4	No	No	$[[23, 17, 4]]_{23}$
4	$\mathbf{C}_4 = \langle (x-1)^4 \rangle$	5	No	No	$[[23, 15, 5]]_{23}$
5	$\mathbf{C}_5 = \langle (x-1)^5 \rangle$	6	No	No	$[[23, 13, 6]]_{23}$
6	$\mathbf{C}_6 = \langle (x-1)^6 \rangle$	7	No	No	$[[23, 11, 7]]_{23}$
7	$\mathbf{C}_7 = \langle (x-1)^7 \rangle$	8	No	No	$[[23, 9, 8]]_{23}$
8	$\mathbf{C}_8 = \langle (x-1)^8 \rangle$	9	No	No	$[[23, 7, 9]]_{23}$
9	$\mathbf{C}_9 = \langle (x-1)^9 \rangle$	10	No	No	$[[23, 5, 10]]_{23}$
10	$\mathbf{C}_{10} = \langle (x-1)^{10} \rangle$	11	No	No	$[[23, 3, 11]]_{23}$
11	$\mathbf{C}_{11} = \langle (x-1)^{11} \rangle$	12	No	No	$[[23, 1, 12]]_{23}$
12	$\mathbf{C}_{12} = \langle (x-1)^{12} \rangle$	13	No	No	No
13	$\mathbf{C}_{13} = \langle (x-1)^{13} \rangle$	14	No	No	No
14	$\mathbf{C}_{14} = \langle (x-1)^{14} \rangle$	15	No	No	No
15	$\mathbf{C}_{15} = \langle (x-1)^{15} \rangle$	16	No	No	No
16	$\mathbf{C}_{16} = \langle (x-1)^{16} \rangle$	17	No	No	No
17	$\mathbf{C}_{17} = \langle (x-1)^{17} \rangle$	18	No	No	No
18	$\mathbf{C}_{18} = \langle (x-1)^{18} \rangle$	19	No	No	No
19	$\mathbf{C}_{19} = \langle (x-1)^{19} \rangle$	20	No	No	No
20	$\mathbf{C}_{20} = \langle (x-1)^{20} \rangle$	21	No	No	No
21	$\mathbf{C}_{21} = \langle (x-1)^{21} \rangle$	22	No	No	No
22	$\mathbf{C}_{22} = \langle (x-1)^{22} \rangle$	23	No	No	No
23	$\mathbf{C}_{23} = \langle 0 \rangle$	0	No	No	No

Table 28: qAMDS codes of length 23 over $\mathbb{F}_{529}$.

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