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Inner-star and star-inner matrices

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Abstract. In order to generalize the concepts of the 1MP and MP1 inverses and to solve some systems of matrix equations, we present the inner-star and star-inner matrices as two new classes of rectangular matrices. We prove different characterizations and representations of these new matrices. We obtain the relation between the set of all inner-star (or star-inner) matrices of a given rectangular matrix A and the set of all 1MP (MP1) inverses of $(A^\dagger)^*$, and also the relations between inner-star and star-inner matrices with corresponding nonsingular matrices. Applications of inner-star and star-inner matrices in defining two new partial orders and in solving several systems of linear equations are proposed.

1. Introduction

Standardly, the symbols A^*, $\mathrm{rank}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, represent the conjugate transpose, the rank, the null space and the range (column space) of $A \in \mathbb{C}^{m\times n}$, where $\mathbb{C}^{m\times n}$ denotes the set of $m \times n$ complex matrices. We use the notation $\mathrm{ind}(A)$ for the index of $A \in \mathbb{C}^{n\times n}$. Also, we denote by $P_{H,G}$ the projector onto H along G, where H and G are complementary subspaces of $\mathbb{C}^{m\times 1} \cong \mathbb{C}^m$, and by P_H the orthogonal projector onto H.

Generalized inverses were intensively investigated since the fifties of the previous century until now, because they have applications in many areas such as Markov chains [8], robotics [14], chemical equations [45], etc. Let $A \in \mathbb{C}^{m\times n}$. The Moore-Penrose inverse of A is a unique matrix $X \in \mathbb{C}^{n\times m}$ (denoted by $A^\dagger$) satisfying [5]:

$$(1)\ AXA = A, \qquad (2)\ XAX = X, \qquad (3)\ (AX)^* = AX, \qquad (4)\ (XA)^* = XA.$$

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For $\alpha \subseteq \{1,2,3,4\}$, any matrix satisfying the equations contained in α is called an α-inverse of A, and $A\{\alpha\}$ represents the set of all α-inverses of A. Recall that an $\{1\}$-inverse (or inner inverse) and $\{2\}$-inverse (or outer inverse) of A are not uniquely determined in general. For a subspace T of $\mathbb{C}^n$ with dimension $s \le \operatorname{rank}(A)$ and a subspace S of $\mathbb{C}^m$ with dimension $m-s$, an outer inverse X of A with the range $\mathcal{R}(X) = T$ and null-space $\mathcal{N}(X) = S$ is unique (if it exists) and is denoted by $A^{(2)}_{T,S}$. It is well known that A has an outer inverse X with the range T and the null-space S if and only if $AT \oplus S = \mathbb{C}^m$ [5]. The symbol $\mathbb{C}^{m\times n}_{T,S}$ will be used for the set of all $A \in \mathbb{C}^{m\times n}$ such that T, S satisfy the above mentioned conditions and $A^{(2)}_{T,S}$ exists.

The Drazin inverse of $A \in \mathbb{C}^{n\times n}$ with $k = \operatorname{ind}(A)$ is a unique matrix $X \in \mathbb{C}^{n\times n}$ (denoted by A^D) [5, 8, 41] such that

$$A^{k+1}X = A^k, \quad XAX = X, \quad AX = XA.$$

If $\operatorname{ind}(A) = 1$, $A^D = A^\#$ is the group inverse of A. Some interesting results related to Drazin inverses were considered in [11–13].

Combining the Moore-Penrose inverse with certain generalized inverses, the following generalized inverses were presented as solutions of adequate system of equations:

-the core inverse and its dual of $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) \le 1$, are given by $A^{\#\!\!\!\bigcirc} = A^\# AA^\dagger$ and $A_{\#\!\!\!\bigcirc} = A^\dagger AA^\#$, respectively, [1];

-the DMP inverse and its dual of $A \in \mathbb{C}^{n\times n}$, respectively, are expressed by $A^{D,\dagger} = A^D AA^\dagger$ and $A^{\dagger,D} = A^\dagger AA^D$ [30];

-the MPCEP inverse of $A \in \mathbb{C}^{n\times n}$ is represented by $A^{\dagger,\dagger\!\!\!\bigcirc} = A^\dagger AA^{\dagger\!\!\!\bigcirc}$ [7], where $k = \operatorname{ind}(A)$ and $A^{\dagger\!\!\!\bigcirc} = A^D A^k (A^k)^\dagger$ is the core-EP inverse of A [29, 42, 43];

-the OMP inverse and MPO inverse (commonly called composite outer inverses) of $A \in \mathbb{C}^{m\times n}_{T,S}$ are defined in [38] as $A^{(2),\dagger}_{T,S} = A^{(2)}_{T,S} AA^\dagger$ and $A^{\dagger,(2)}_{T,S} = A^\dagger AA^{(2)}_{T,S}$, respectively;

-the GDMP inverse and its dual of $A \in \mathbb{C}^{n\times n}$ are introduced in [16], respectively, as $A^{GD,\dagger} = A^{GD} AA^\dagger$ and $A^{\dagger,GD} = A^\dagger AA^{GD}$, where A^{GD} is the G-Drazin inverse of A [9, 46].

-the GO-(T,S)-MP (or shortly GOMP) inverse and its dual of $A \in \mathbb{C}^{m\times n}_{T,S}$, respectively, are proposed in [35] by $A^{GO,\dagger}_{T,S} = A^{GO}_{T,S} AA^\dagger$ and $A^{\dagger,GO}_{T,S} = A^\dagger AA^{GO}_{T,S}$, for a G-outer (T,S)-inverse $A^{GO}_{T,S}$ of A [36].

-the 1MP inverse and MP1 inverse of $A \in \mathbb{C}^{m\times n}$, respectively, are presented by $A^{-,\dagger} = A^- AA^\dagger$ and $A^{\dagger,-} = A^\dagger AA^-$ [15], for $A^- \in A\{1\}$.

For significant properties of the core, DMP, MPCEP, composite outer, 1MP and MP1 inverses see [3, 17, 18, 20–23, 25, 26, 28, 31, 37, 47, 48, 50, 51].

Two kinds of partial orders defined by the 1MP and MP1 inverses are given in [15]. Let $A, B \in \mathbb{C}^{m\times n}$. Then $A \le^{-,\dagger} B$ if there exists an 1MP inverse $A^{-,\dagger}$ of A such that

$$AA^{-,\dagger} = BA^{-,\dagger} \qquad \text{and} \qquad A^{-,\dagger}A = A^{-,\dagger}B.$$

Similarly, $A \le^{\dagger,-} B$ if there exists an MP1 inverse $A^{\dagger,-}$ of A such that

$$AA^{\dagger,-} = BA^{\dagger,-} \qquad \text{and} \qquad A^{\dagger,-}A = A^{\dagger,-}B.$$

The definitions of the DMP and OMP inverses and the fact that the Moore–Penrose inverse $A^\dagger$ and the conjugate transpose A^* of a given square matrix A have certain same properties (for example $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$ and $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$), lead us to the definitions of new classes of matrices: the Drazin-star matrix of A introduced in [33] and the outer-star matrix of A presented in [34]. Let $A \in \mathbb{C}^{m\times n}_{T,S}$. The (T,S)-outer-star matrix of A (or the (T,S)-outer-star inverse of $(A^\dagger)^*$) is represented by

$$A^{(2,*)}_{T,S} = A^{(2)}_{T,S} AA^*$$

and presents the unique solution to the system

$$X(A^\dagger)^* X = X, \qquad AX = AA^{(2)}_{T,S} AA^* \qquad \text{and} \qquad X(A^\dagger)^* = A^{(2)}_{T,S} A.$$

The star-(T,S)-outer matrix of A (or the star-(T,S)-outer inverse of $(A^\dagger)^*$) is defined as

$$A^{(*,2)}_{T,S} = A^*AA^{(2)}_{T,S}.$$

In particular, when $m = n$ and $A^{(2)}_{T,S} = A^D$, the (T,S)-outer-star (or star-(T,S)-outer) matrix of A becomes the Drazin-star (star-Drazin) matrix $A^{D,*} = A^DAA^*$ ($A^{*,D} = A^*AA^D$) of A. Recall that the Drazin-star matrix of A (or the Drazin-star inverse of $(A^\dagger)^*$)) is a new class of the outer inverse of $(A^\dagger)^*$ because it is different from each of the Drazin inverse, Moore–Penrose inverse, DMP inverse and MPD inverse of $(A^\dagger)^*$ [33, Example 2.2]. Some extensions of Drazin-star matrices can be found in [2, 39, 40, 49].

For complex matrices, the concept of the conjugate transpose essentially extends the notion of complex conjugation and it is basic conceptual object in matrix theory and linear algebra, which have many significant applications in the research areas of both theoretical and applied mathematics. The conjugate transpose is used to check special complex matrices, i.e. hermitian, skew hermitian, normal and unitary matrices. For real matrices, the conjugate transpose reduces to the matrix transpose. It is known that each square complex matrix A is similar to its transpose A^T, and A is similar to A^* if and only if A is similar to a real matrix (if and only if the Jordan blocks of the nonreal eigenvalues of A occur in conjugate pairs). It is interesting to consider some generalization of the conjugate transpose matrix.

Transpose-inverse terms $(A^{-1})^T$ (or, equivalently, $(A^T)^{-1}$), commonly abbreviated as A^{-T}, arise naturally in a variety of control system contexts, e.g., the relative gain array (RGA) [6] and formulations of the controllability Gramian [19]. As an example, classical solution methods for the discrete-time algebraic Riccati equation involve the symplectic form [24]: $\begin{bmatrix} F + GF^{-T}H & -GF^{-T} \\ -F^{-T}H & {}^{-T} \end{bmatrix}$. Note that $(A^\dagger)^*$ is an extension of the transpose-inverse A^{-T}.

The special solution of matrix equations has raised much interest among researchers due to the wide applications such as robust control, neural network, singular system control, model reduction and image processing. Reviewing the fundamental linear equation $Ax = b$, where A is a complex matrix, and x and b are complex column vectors, by using the Moore–Penrose inverse $A^\dagger$ and the conjugate transpose A^*, we consider the equation $(A^\dagger)^*x = b$. These operations are all involutory, i.e., $(A^\dagger)^\dagger = (A^*)^* = A$, thus each operation defines a mutual relation.

Motivated by recent research about outer-star and star-outer matrices, our aim is to present two new wider classes of rectangular matrices to extend the notions of the conjugate transpose matrix, the 1MP and MP1 inverses and solve certain systems of matrix equations. Precisely, the following investigation streams are studied in this manuscript.

(1) By some analog characterizations of the Moore–Penrose inverse $A^\dagger$ and the conjugate transpose of a given rectangular matrix A, the inner-star matrix is introduced based on an inner inverse of A and the conjugate transpose A^*. More precisely, the inner-star matrix is defined by writing A^* instead of $A^\dagger$ in the definition of the 1MP inverse of A [15]. In particular, for a partial isometry A (i.e. $A = AA^*A$ or equivalently $A^* = A^\dagger$), notice that the inner-star matrix coincides with the 1MP inverse. Especially, when $A^- = A^\dagger$, the inner-star matrix reduces to the conjugate transpose A^*. Thus, we propose a new wider class of rectangular matrices.

(2) Secondly, the concept of star-inner matrix is introduced to generalize the notion of the MP1 inverse.

(3) We establish a number of characterizations of the inner-star and star-inner matrices from algebraic and geometrical point of view.

(4) Various representations of inner-star and star-inner matrices are developed as well as their relations with corresponding nonsingular matrices.

(5) The relation between the set of all inner-star (or star-inner) matrices of A and the set of all 1MP (MP1) inverses of $(A^\dagger)^*$ is proved.

(6) Using inner-star and star-inner matrices, we define and investigate the inner-star and star-inner partial orders on $\mathbb{C}^{m\times n}$.

(7) We apply inner-star and star-inner matrices to solve several systems of linear equations and give a Cramer's rule for finding the solution. In particular, the equation $(A^\dagger)^*x = b$ is solved and its least-squares solutions are investigated.

The above listed items are organized in this paper as follows. In Section 2, combining an inner inverse and the conjugate transpose of a given rectangular matrix, the inner-star and star-inner matrices are defined and characterized. We verify that the assumption, A is a partial isometry, is a necessary and sufficient condition for the inner-star (or star-inner) matrix of A to be an inner or outer inverse of A. Different representations of inner-star and star-inner matrices involving the most general form, are proposed in Section 3. Two new partial orders on the set $\mathbb{C}^{m\times n}$ are introduced in Section 4 in terms of inner-star and star-inner matrices. Applying the inner-star and star-inner matrices, some systems of matrix equations are solved in Section 5.

2. Characterizations of inner-star and star-inner matrices

Solving corresponding systems of equations, we present two new classes of rectangular matrices as proper combinations of an inner inverse with the conjugate transpose of a given matrix. In this way, we extend the notions of 1MP and MP1 inverses as well as their special cases such as GDMP and GOMP inverses and their dual inverses.

Theorem 2.1. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$. Then*

(a) *$X = A^-AA^*$ is the unique solution to the system*

$$X(A^\dagger)^*X = X, \qquad AX = AA^* \qquad \text{and} \qquad X(A^\dagger)^* = A^-A. \tag{1}$$

(b) *$X = A^*AA^-$ is the unique solution to the system*

$$X(A^\dagger)^*X = X, \qquad XA = A^*A \qquad \text{and} \qquad (A^\dagger)^*X = AA^-.$$

Proof. (a) For $X = A^-AA^*$, notice that $AX = (AA^-A)A^* = AA^*$, $X(A^\dagger)^* = A^-AA^*(A^\dagger)^* = A^-(AA^\dagger A) = A^-A$ and $X(A^\dagger)^*X = A^-(AX) = A^-AA^* = X$, i.e. $X = A^-AA^*$ is the solution to (1).

If X satisfies (1), then $X = (X(A^\dagger)^*)X = A^-(AX) = A^-AA^*$ and it is the unique solution to (1).

Similarly, we verify the part (b). □

Definition 2.2. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$.*

(a) *The inner-star matrix of A is defined as*

$$A^{-,*} = A^-AA^*.$$

(b) *The star-inner matrix of A is defined as*

$$A^{*,-} = A^*AA^-.$$

We list some special cases of the inner-star and star-inner matrices. Note that

1) if A is a partial isometry, then $A^{-,*} = A^-AA^* = A^-AA^\dagger = A^{-,\dagger}$ and $A^{*,-} = A^*AA^- = A^\dagger AA^- = A^{\dagger,-}$;

2) choosing $A^- = A^\dagger$, we have $A^{\dagger,*} = A^\dagger AA^* = A^* = A^{*,\dagger}$;

3) for $A^- = A^{GO}_{T,S}$ [36], we define the GO-(T,S)-star and star-GO-(T,S) matrices, respectively, as

$$A^{GO,*}_{T,S} = A^{GO}_{T,S}AA^* \qquad \text{and} \qquad A^{*,GO}_{T,S} = A^*AA^{GO}_{T,S};$$

4) when $A^- = A^{GD}$ [46], we introduce the GD-star and star-GD matrices, respectively, as

$$A^{GD,*} = A^{GD}AA^* \qquad \text{and} \qquad A^{*,GD} = A^*AA^{GD}.$$

Example 2.3. *Consider the following complex 2×2 matrix*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then

$$A^{\dagger} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \quad A^* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad (A^{\dagger})^* = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}.$$

Since an arbitrary $A^- \in A\{1\}$ can be expressed by

$$A^- = \begin{bmatrix} a & d \\ 1-a & c \end{bmatrix},$$

for $a, d, c \in \mathbb{C}$, we obtain

$$A^{-,*} = \begin{bmatrix} 2a & 0 \\ 2(1-a) & 0 \end{bmatrix}, \quad A^{*,-} = \begin{bmatrix} 1 & d+c \\ 1 & d+c \end{bmatrix},$$

$$A^{-,\dagger} = \begin{bmatrix} a & 0 \\ 1-a & 0 \end{bmatrix}, \quad A^{\dagger,-} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}(d+c) \\ \frac{1}{2} & \frac{1}{2}(d+c) \end{bmatrix}.$$

Therefore, the inner-star matrix and star-inner matrix of A are different from the Moore-Penorse inverse, 1MP inverse and MP1 inverse of A in general.

We present necessary and sufficient conditions for a rectangular matrix to be an inner-star matrix in Theorem 2.4.

Theorem 2.4. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$. For $X \in \mathbb{C}^{n\times m}$, the following statements are equivalent:*

(i) $X = A^-AA^*$;

(ii) $X(A^{\dagger})^*X = X$, $(A^{\dagger})^*X(A^{\dagger})^* = (A^{\dagger})^*$, $AX = AA^*$ *and* $X(A^{\dagger})^* = A^-A$;

(iii) $X(A^{\dagger})^*X = X$, $(A^{\dagger})^*X(A^{\dagger})^* = (A^{\dagger})^*$, $(A^{\dagger})^*X = AA^{\dagger}$ *and* $X(A^{\dagger})^* = A^-A$;

(iv) $X(A^{\dagger})^*X = X$, $(A^{\dagger})^*X = AA^{\dagger}$ *and* $X(A^{\dagger})^* = A^-A$;

(v) $A^-AX = X$ *and* $AX = AA^*$;

(vi) $A^-AX = X$ *and* $A^{\dagger}AX = A^*$;

(vii) $A^-AX = X$ *and* $(A^{\dagger})^*X = AA^{\dagger}$;

(viii) $XAA^{\dagger} = X$ *and* $X(A^{\dagger})^* = A^-A$;

(ix) $XAA^{\dagger} = X$ *and* $XA = A^-AA^*A$;

(x) $A^-AXAA^{\dagger} = X$ *and* $AX(A^{\dagger})^* = A$;

(xi) $XAA^{\dagger} = X$ *and* $X(A^{\dagger})^*A^{\dagger} = A^{-,\dagger}$.

Proof. (i) ⇒ (ii): By Theorem 2.1, $X = A^-AA^*$ satisfies $X(A^{\dagger})^*X = X$, $AX = AA^*$ and $X(A^{\dagger})^* = A^-A$. Further,

$$(A^{\dagger})^*X(A^{\dagger})^* = (A^{\dagger}AA^{\dagger})^*A^-A = (A^{\dagger})^*A^{\dagger}(AA^-A) = (A^{\dagger})^*A^{\dagger}A = (A^{\dagger})^*.$$

(ii) ⇒ (iii): The assumption $AX = AA^*$ gives

$$(A^{\dagger})^*X = (A^{\dagger})^*A^{\dagger}(AX) = (A^{\dagger})^*A^{\dagger}AA^* = (A^{\dagger})^*A^* = AA^{\dagger}.$$

(iii) ⇒ (iv): This implication is obvious.

(iv) ⇒ (i): Notice that the equalities $X(A^{\dagger})^*X = X$, $(A^{\dagger})^*X = AA^{\dagger}$ and $X(A^{\dagger})^* = A^-A$ imply

$$X = X(A^{\dagger})^*X = XAA^{\dagger} = X(A^{\dagger})^*A^* = A^-AA^*.$$

(i) ⇒ (v): Since $X = A^-AA^*$, then $AX = AA^*$ and $A^-AX = A^-AA^* = X$.
(v) ⇒ (vi): The equality $AX = AA^*$ yields $A^\dagger AX = A^\dagger AA^* = A^*$.
(vi) ⇒ (vii): Multiplying $A^\dagger AX = A^*$ by $(A^\dagger)^*$ from the left hand side, we get $(A^\dagger)^*X = (A^\dagger)^*A^* = AA^\dagger$.
(vii) ⇒ (i): Applying $A^-AX = X$ and $(A^\dagger)^*X = AA^\dagger$, we have

$$X = A^-AX = A^-AA^\dagger AX = A^-AA^*(A^\dagger)^*X = A^-AA^*AA^\dagger = A^-AA^*.$$

(i) ⇒ (viii): If $X = A^-AA^*$, then $X(A^\dagger)^* = A^-A$ and $XAA^\dagger = A^-AA^*AA^\dagger = A^-AA^* = X$.
(viii) ⇒ (ix): The hypothesis $X(A^\dagger)^* = A^-A$ implies $XA = X(A^\dagger)^*A^*A = A^-AA^*A$.
(ix) ⇒ (i): Using $XAA^\dagger = X$ and $XA = A^-AA^*A$, we obtain $X = (XA)A^\dagger = A^-AA^*AA^\dagger = A^-AA^*$.
Analogously, the rest can be shown. □

As Theorem 2.4, we prove the next result involving equivalent conditions for a rectangular matrix to be the star-inner matrix.

Theorem 2.5. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$. For $X \in \mathbb{C}^{n\times m}$, the following statements are equivalent:*

(i) $X = A^*AA^-$;

(ii) $X(A^\dagger)^*X = X$, $(A^\dagger)^*X(A^\dagger)^* = (A^\dagger)^*$, $XA = A^*A$ *and* $(A^\dagger)^*X = AA^-$;

(iii) $X(A^\dagger)^*X = X$, $(A^\dagger)^*X(A^\dagger)^* = (A^\dagger)^*$, $X(A^\dagger)^* = A^\dagger A$ *and* $(A^\dagger)^*X = AA^-$;

(iv) $X(A^\dagger)^*X = X$, $X(A^\dagger)^* = A^\dagger A$ *and* $(A^\dagger)^*X = AA^-$;

(v) $XAA^- = X$ *and* $XA = A^*A$;

(vi) $XAA^- = X$ *and* $XAA^\dagger = A^*$;

(vii) $XAA^- = X$ *and* $X(A^\dagger)^* = A^\dagger A$;

(viii) $A^\dagger AX = X$ *and* $(A^\dagger)^*X = AA^-$;

(ix) $A^\dagger AX = X$ *and* $AX = AA^*AA^-$;

(x) $A^\dagger AXAA^- = X$ *and* $(A^\dagger)^*XA = A$;

(xi) $A^\dagger AX = X$ *and* $A^\dagger(A^\dagger)^*X = A^{\dagger,-}$.

Note that the inner-star and star-inner matrices are both inner and outer inverses of $(A^\dagger)^*$. It is interesting to find ranges and null spaces for inner-star and star-inner matrices as well as for projections determined by them.

Lemma 2.6. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$. Then:*

(i) $(A^\dagger)^*A^{-,*}$ *is the orthogonal projector onto* $\mathcal{R}(A)$;

(ii) $A^{-,*}(A^\dagger)^*$ *is a projector onto* $\mathcal{R}(A^-A)$ *along* $\mathcal{N}(A)$;

(iii) $A^{-,*} = [(A^\dagger)^*]^{(2)}_{\mathcal{R}(A^-A),\mathcal{N}(A^*)} = [(A^\dagger)^*]^{(1,2,3)}_{\mathcal{R}(A^-A),\mathcal{N}(A^*)}$;

(iv) $(A^\dagger)^*A^{*,-}$ *is a projector onto* $\mathcal{R}(A)$ *along* $\mathcal{N}(AA^-)$;

(v) $A^{*,-}(A^\dagger)^*$ *is the orthogonal projector onto* $\mathcal{R}(A^*)$;

(vi) $A^{*,-} = [(A^\dagger)^*]^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(AA^-)} = [(A^\dagger)^*]^{(1,2,4)}_{\mathcal{R}(A^*),\mathcal{N}(AA^-)}$.

Proof. (i) By Theorem 2.4, $(A^\dagger)^* A^{-,*} = AA^\dagger$ is the orthogonal projector onto $\mathcal{R}(A)$.

(ii) Applying Theorem 2.1, we deduce that $A^{-,*}(A^\dagger)^* = A^- A$ is a projector. Furthermore, $\mathcal{R}(A^{-,*}(A^\dagger)^*) = \mathcal{R}(A^- A)$ and $\mathcal{N}(A^{-,*}(A^\dagger)^*) = \mathcal{N}(A^- A) = \mathcal{N}(A)$.

(iii) Because $A^{-,*}(A^\dagger)^* A^{-,*} = A^{-,*}$, we have $\mathcal{R}(A^{-,*}) = \mathcal{R}(A^{-,*}(A^\dagger)^*) = \mathcal{R}(A^- A)$ and $\mathcal{N}(A^{-,*}) = \mathcal{N}((A^\dagger)^* A^{-,*}) = \mathcal{N}(A^*)$.

The rest follows similarly. □

We now characterize the inner-star and star-inner matrices from a geometrical point of view. By Theorem 2.1 and Theorem 2.7, we observe that algebraic and geometrical approaches are equivalent.

Theorem 2.7. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$. Then*

(i) *$X = A^{-,*}$ is the unique solution to*

$$(A^\dagger)^* X = P_{\mathcal{R}(A)} \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^- A); \tag{2}$$

(ii) *$X = A^{-,*}$ is the unique solution to*

$$X(A^\dagger)^* = P_{\mathcal{R}(A^- A),\mathcal{N}(A)} \quad \text{and} \quad \mathcal{R}(X^*) \subseteq \mathcal{R}(A); \tag{3}$$

(iii) *$X = A^{*,-}$ is the unique solution to*

$$(A^\dagger)^* X = P_{\mathcal{R}(A),\mathcal{N}(AA^-)} \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^*);$$

(iv) *$X = A^{*,-}$ is the unique solution to*

$$X(A^\dagger)^* = P_{\mathcal{R}(A^*)} \quad \text{and} \quad \mathcal{R}(X^*) \subseteq \mathcal{R}((AA^-)^*).$$

Proof. (i) For $X = A^{-,*}$, notice that, by Lemma 2.6, (2) is satisfied.

Let X and Y be two matrices for which (2) holds. Then $(A^\dagger)^*(X - Y) = P_{\mathcal{R}(A)} - P_{\mathcal{R}(A)} = 0$ gives that $\mathcal{R}(X - Y) \subseteq \mathcal{N}((A^\dagger)^*) = \mathcal{N}(A) \subseteq \mathcal{N}(A^- A)$. Thus, $\mathcal{R}(X - Y) \subseteq \mathcal{R}(A^- A) \cap \mathcal{N}(A^- A) = \{0\}$, that is, $X = Y$. It follows that (2) has the unique solution $A^{-,*}$.

(ii) If $X = A^{-,*}$, we see that $\mathcal{R}(X^*) = \mathcal{R}((A^- AA^*)^*) \subseteq \mathcal{R}(A)$. According to Lemma 2.6, $X(A^\dagger)^* = P_{\mathcal{R}(A^- A),\mathcal{N}(A)}$, which implies that (3) holds.

Assume that X and Y satisfy (3). By $(X - Y)(A^\dagger)^* = P_{\mathcal{R}(A^- A),\mathcal{N}(A)} - P_{\mathcal{R}(A^- A),\mathcal{N}(A)} = 0$, we have $A^\dagger(X^* - Y^*) = 0$, i.e. $\mathcal{R}(X^* - Y^*) \subseteq \mathcal{N}(A^\dagger) = \mathcal{N}(A^*) = \mathcal{R}(A)^\perp$. Now, $\mathcal{R}(X^* - Y^*) \subseteq \mathcal{R}(A)^\perp \cap \mathcal{R}(A) = \{0\}$ yields $X = Y$ and so $A^{-,*}$ presents the unique solution to (3).

The parts (iii) and (iv) follow similarly. □

We prove that an equivalent condition for the inner-star (or star-inner) matrix of A to be an inner or outer inverse of A, is that A is a partial isometry.

Theorem 2.8. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$. Then the following statements are equivalent:*

(i) $AA^* A = A$;

(ii) $AA^{*,-} A = A$;

(iii) $AA^{-,*} A = A$;

(iv) $A^{*,-} AA^{*,-} = A^{*,-}$;

(v) $A^{-,*} AA^{-,*} = A^{-,*}$.

Proof. (i) ⇔ (ii): We observe that $AA^{*,-}A = A$ if and only if $AA^*AA^-A = A$ which is equivalent to $AA^*A = A$.
(i) ⇔ (iv): Since $A^{*,-}AA^{*,-} = A^*AA^-AA^*AA^- = A^*AA^*AA^-$, we have

$$\begin{aligned} A^{*,-}AA^{*,-} = A^{*,-} \quad &\Leftrightarrow \quad A^*AA^*AA^- = A^*AA^- \\ &\Leftrightarrow \quad A^*AA^*A = A^*A \\ &\Leftrightarrow \quad AA^*A = A. \end{aligned}$$

Similarly, we can finish this proof. □

Theorem 2.9. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$. Then the following statements are equivalent:*

(i) *$A^{-,*}A$ is Hermitian if and only if $A = A(A^-A)^*$;*

(ii) *$AA^{*,-}$ is Hermitian if and only if $A = (AA^-)^*A$;*

(iv) *$A^{-,*}(A^\dagger)^*$ is Hermitian if and only if $A^{-,*} = A^*$;*

(v) *$(A^\dagger)^*A^{*,-}$ is Hermitian if and only if $A^{*,-} = A^*$.*

Proof. (i) If $A^{-,*}A = (A^{-,*}A)^*$, then $A^-AA^*A = A^*A(A^-A)^*$ which gives

$$A = (A^\dagger)^*A^*A = (A^\dagger)^*A^\dagger AA^-AA^*A = (A^\dagger)^*A^-AA^*A = (A^\dagger)^*A^*A(A^-A)^* = A(A^-A)^*.$$

Conversely, $A = A(A^-A)^*$ yields $(A^{-,*}A)^* = A^*A(A^-A)^* = A^*A = (A^*A)^* = A^{-,*}A$.
(iii) Recall that, by Theorem 2.4, $A^{-,*} \in (A^\dagger)^*\{1,2,3\}$. Thus, $A^{-,*} \in (A^\dagger)^*\{4\}$ if and only if $A^{-,*} = [(A^\dagger)^*]^\dagger = A^*$.
The rest can be verified analogously. □

3. Representations of inner-star and star-inner matrices

Different representations of inner-star and star-inner matrices are established in this section.

Let us recall that a singular value decomposition (or SVD) of $A \in \mathbb{C}^{m\times n}$ with $\mathrm{rank}(A) = k$ is expressed as follows [32]:

$$A = U\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}V^*, \tag{4}$$

where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, and $D \in \mathbb{C}^{k\times k}$ is a positive definite diagonal matrix. In addition, an arbitrary inner inverse A^- of A is represented by

$$A^- = V\begin{bmatrix} D^{-1} & A_1 \\ A_2 & A_3 \end{bmatrix}U^*, \tag{5}$$

where $A_1 \in \mathbb{C}^{k\times(m-k)}$, $A_2 \in \mathbb{C}^{(n-k)\times k}$ and $A_3 \in \mathbb{C}^{(n-k)\times(m-k)}$ are arbitrary matrices.

Denote by $A\{-,*\} = \{A^-AA^* : A^- \in A\{1\}\}$ and $A\{*,-\} = \{A^*AA^- : A^- \in A\{1\}\}$ the sets of all inner-star and star-inner matrices of A, respectively. These sets can be described in the following way.

Lemma 3.1. *Let $A \in \mathbb{C}^{m\times n}$ with $\mathrm{rank}(A) = k$ be given by (4). Then*

$$A\{-,*\} = \left\{V\begin{bmatrix} D & 0 \\ A_2D^2 & 0 \end{bmatrix}U^* : A_2 \in \mathbb{C}^{(n-k)\times k}\right\}$$

and

$$A\{*,-\} = \left\{V\begin{bmatrix} D & D^2A_1 \\ 0 & 0 \end{bmatrix}U^* : A_1 \in \mathbb{C}^{k\times(m-k)}\right\}.$$

Proof. Using (4), (5) and

$$A^* = V \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

this proof follows by definitions of inner-star and star-inner matrices. □

Theorem 3.2. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$. Then*

$$A\{-,*\} = \{A^{-,*} + (I - A^-A)YAA^* : Y \in \mathbb{C}^{n\times m}\}$$

and

$$A\{*,-\} = \{A^{*,-} + A^*AY(I - AA^-) : Y \in \mathbb{C}^{n\times m}\}.$$

Proof. It is well-know that

$$A\{1\} = \{A^- + Y - A^-AYAA^- : \ Y \in \mathbb{C}^{n\times m}\}.$$

The rest is clear. □

Write $A\{-,\dagger\} = \{A^-AA^\dagger : A^- \in A\{1\}\}$ and $A\{\dagger,-\} = \{A^\dagger AA^- : A^- \in A\{1\}\}$ for the sets of all 1MP and MP1 inverses of A, respectively. Now, we prove that the set of all inner-star (or star-inner) matrices of A coincides with the set of all 1MP (MP1) inverses of $(A^\dagger)^*$.

Corollary 3.3. *Let $A \in \mathbb{C}^{m\times n}$. Then*

$$A\{-,*\} = (A^\dagger)^*\{-,\dagger\}$$

and

$$A\{*,-\} = (A^\dagger)^*\{\dagger,-\}.$$

Proof. For $A^- \in A\{1\}$, by Theorem 2.4, we know that $A^{-,*} \in (A^\dagger)^*\{1\}$ and

$$A^{-,*} = A^{-,*}AA^\dagger = A^{-,*}(A^\dagger)^*A^* = A^{-,*}(A^\dagger)^*[(A^\dagger)^*]^\dagger \in (A^\dagger)^*\{-,\dagger\}.$$

So, $A\{-,*\} \subseteq (A^\dagger)^*\{-,\dagger\}$.

If $[(A^\dagger)^*]^- \in (A^\dagger)^*\{1\}$, then $[(A^\dagger)^*]^-(A^\dagger)^*A^\dagger \in A\{1\}$ by

$$A[(A^\dagger)^*]^-(A^\dagger)^*A^\dagger A = A[(A^\dagger)^*]^-(A^\dagger)^* = AA^*(A^\dagger)^*[(A^\dagger)^*]^-(A^\dagger)^* = AA^*(A^\dagger)^* = A.$$

Therefore,

$$[(A^\dagger)^*]^{-,\dagger} = [(A^\dagger)^*]^-(A^\dagger)^*A^* = \left([(A^\dagger)^*]^-(A^\dagger)^*A^\dagger\right)AA^* \in A\{-,*\},$$

i.e. $(A^\dagger)^*\{-,\dagger\} \subseteq A\{-,*\}$. Hence, $A\{-,*\} = (A^\dagger)^*\{-,\dagger\}$.

Similarly, we verify the rest. □

It can be interesting to find the most general representation of the inner-star matrix.

Theorem 3.4. *Let $A \in \mathbb{C}^{m\times n}$, $A^- \in A\{1\}$ and $Z \in \mathbb{C}^{n\times m}$. Then the following statements are equivalent:*

(i) $A^{-,*} = ZAA^*$;

(ii) $A^-A = ZA$;

(iii) $\mathcal{R}(A^-A) = \mathcal{R}(ZA)$ *and* $AZA = A$;

(iv) $Z = A^- + Y(I - AA^\dagger)$, *for arbitrary* $Y \in \mathbb{C}^{n\times m}$.

Proof. (i) ⇒ (ii): The equality $A^{-,*} = A^-AA^*$ implies that $A^{-,*} = ZAA^*$ if and only if $A^-AA^* = ZAA^*$. Hence, $A^-A = (A^-AA^*)(A^\dagger)^* = ZAA^*(A^\dagger)^* = ZA$.

(ii) ⇒ (iii): From $A^-A = ZA$, we deduce that $\mathcal{R}(A^-A) = \mathcal{R}(ZA)$ and $AZA = AA^-A = A$.

(iii) ⇒ (i): Since $\mathcal{R}(A^-A) = \mathcal{R}(ZA)$ and $AZA = A$, we obtain $ZA = A^-AH = A^-A(A^-AH) = A^-(AZA) = A^-A$, for some $H \in \mathbb{C}^{n\times n}$. So, $A^{-,*} = (A^-A)A^* = ZAA^*$.

(ii) ⇒ (iv): All solutions to $ZA = A^-A$ are a sum of a particular solution to $ZA = A^-A$ and the general solution to $ZA = 0$. Therefore, by [5, p. 52], the general solution to $ZA = A^-A$ is expressed by $Z = A^- + Y(I - AA^\dagger)$, for arbitrary $Y \in \mathbb{C}^{n\times m}$.

(iv) ⇒ (i): For $Z = A^- + Y(I - AA^\dagger)$, we get $ZAA^* = A^-AA^* = A^{-,*}$. □

In the same way as Theorem 3.4, we can get maximal classes for the representation of the star-inner matrix.

Theorem 3.5. *Let $A \in \mathbb{C}^{m\times n}$, $A^- \in A\{1\}$ and $Z \in \mathbb{C}^{n\times m}$. Then the following statements are equivalent:*

(i) $A^{*,-} = A^*AZ$;

(ii) $AA^- = AZ$;

(iii) $\mathcal{N}(AA^-) = \mathcal{N}(AZ)$ *and* $AZA = A$;

(iv) $Z = A^- + (I - A^\dagger A)Y$, *for arbitrary* $Y \in \mathbb{C}^{n\times m}$.

The basic idea of these Cramer's rules is to construct a nonsingular bordered matrix by adjoining certain matrices to the original matrix. In particular, for a given matrix A, an associated bordered matrix $\begin{bmatrix} A & E \\ F & G \end{bmatrix}$ is an expanded matrix that contains A as its leading principal submatrix. We can relate the inner-star matrix with an adequate nonsingular bordered matrix.

Theorem 3.6. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$. For the full column rank matrices E and F^* such that*

$$\mathcal{R}(I - AA^*) \subseteq \mathcal{R}(E) \qquad \text{and} \qquad \mathcal{R}(A^-A) \subseteq \mathcal{N}(F)$$

$$(\text{or} \qquad \mathcal{R}(E) \subseteq \mathcal{N}(A^*) \qquad \text{and} \qquad \mathcal{N}(F) \subseteq \mathcal{N}(I - A^{-,*}A)),$$

the bordered matrix

$$M = \begin{bmatrix} A & E \\ F & 0 \end{bmatrix}$$

is nonsingular and

$$M^{-1} = \begin{bmatrix} A^{-,*} & (I - A^{-,*}A)F^\dagger \\ E^\dagger(I - AA^*) & -E^\dagger(A - AA^*A)F^\dagger \end{bmatrix}. \tag{6}$$

Proof. Note that $\mathcal{R}(I - AA^*) \subseteq \mathcal{R}(E) = \mathcal{R}(EE^\dagger) = \mathcal{N}(I - EE^\dagger)$ implies $(I - EE^\dagger)(I - AA^*) = 0$, that is, $I - AA^* = EE^\dagger(I - AA^*)$. Since $\mathcal{R}(A^{-,*}) = \mathcal{R}(A^-A) \subseteq \mathcal{N}(F)$, we get $FA^{-,*} = 0$. Denote by N the right hand side of (6). From

$$\begin{aligned} MN &= \begin{bmatrix} AA^* + EE^\dagger(I - AA^*) & A(I - A^{-,*}A)F^\dagger - EE^\dagger(I - AA^*)AF^\dagger \\ FA^{-,*} & F(I - A^{-,*}A)F^\dagger \end{bmatrix} \\ &= \begin{bmatrix} AA^* + I - AA^* & (I - AA^*)AF^\dagger - (I - AA^*)AF^\dagger \\ 0 & FF^\dagger \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ &= I. \end{aligned}$$

we conclude that M is nonsingular and $M^{-1} = N$. □

As Theorem 3.6, we can prove the next result involving the block $(A^{\dagger})^*$ instead of A in matrix M of Theorem 3.6.

Theorem 3.7. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$. For the full column rank matrices E and F^* such that*

$$\mathcal{R}(E) = \mathcal{N}(A^*) \qquad \text{and} \qquad \mathcal{R}(A^-A) = \mathcal{N}(F),$$

the bordered matrix

$$M = \begin{bmatrix} (A^{\dagger})^* & E \\ F & 0 \end{bmatrix}$$

is nonsingular and

$$M^{-1} = \begin{bmatrix} A^{-,*} & (I - A^-A)F^{\dagger} \\ E^{\dagger}(I - AA^{\dagger}) & 0 \end{bmatrix}.$$

Similar results can be verified for star-inner matrices.

Theorem 3.8. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$.*

(i) *For the full column rank matrices E and F^* such that*

$$\mathcal{R}(I - AA^{*,-}) \subseteq \mathcal{R}(E) \qquad \text{and} \qquad \mathcal{R}(A^*) \subseteq \mathcal{N}(F)$$

$$(\text{or} \qquad \mathcal{R}(E) \subseteq \mathcal{N}(AA^-) \qquad \text{and} \qquad \mathcal{N}(F) \subseteq \mathcal{N}(I - A^{*,-}A)),$$

the bordered matrix

$$M = \begin{bmatrix} A & E \\ F & 0 \end{bmatrix}$$

is nonsingular and

$$M^{-1} = \begin{bmatrix} A^{*,-} & (I - A^*A)F^{\dagger} \\ E^{\dagger}(I - AA^{*,-}) & -E^{\dagger}(A - AA^*A)F^{\dagger} \end{bmatrix}.$$

(ii) *For the full column rank matrices E and F^* such that*

$$\mathcal{R}(E) = \mathcal{N}(AA^-) \qquad \text{and} \qquad \mathcal{R}(A^*) = \mathcal{N}(F),$$

the bordered matrix

$$M = \begin{bmatrix} (A^{\dagger})^* & E \\ F & 0 \end{bmatrix}$$

is nonsingular and

$$M^{-1} = \begin{bmatrix} A^{*,-} & (I - A^{\dagger}A)F^{\dagger} \\ E^{\dagger}(I - AA^-) & 0 \end{bmatrix}.$$

4. Inner-star and star-inner partial orders

The definitions of two new binary relations on $\mathbb{C}^{m\times n}$ are presented in terms of the inner-star and star-inner matrices in this section.

Definition 4.1. *Let $A, B \in \mathbb{C}^{m\times n}$. Then we say that*

(a) *A is below B under the inner-star relation (denoted by $A \leq^{-,*} B$) if there exist $A^- \in A\{1\}$ such that*

$$(A^{\dagger})^*A^{-,*} = (B^{\dagger})^*A^{-,*} \qquad \text{and} \qquad A^{-,*}(A^{\dagger})^* = A^{-,*}(B^{\dagger})^*;$$

(b) *A is below B under the star-inner relation (denoted by* $A \leq^{*,-} B$*) if there exist* $A^- \in A\{1\}$ *such that*

$$(A^\dagger)^* A^{*,-} = (B^\dagger)^* A^{*,-} \qquad \text{and} \qquad A^{*,-}(A^\dagger)^* = A^{*,-}(B^\dagger)^*.$$

The inner-star relation $\leq^{-,*}$ can be characterized in the following way.

Theorem 4.2. *Let* $A, B \in \mathbb{C}^{m\times n}$*. Then the following statements are equivalent:*

(i) $A \leq^{-,*} B$*;*

(ii) *there exists* $A^- \in A\{1\}$ *such that* $AA^\dagger = (B^\dagger)^* A^- AA^*$ *and* $A^-A = A^-AA^*(B^\dagger)^*$*;*

(iii) *there exists* $A^- \in A\{1\}$ *such that* $(A^\dagger)^* = (B^\dagger)^* A^- A$ *and* $A^* = B^\dagger AA^*$*;*

(iv) *there exists* $A^- \in A\{1\}$ *such that* $(A^\dagger)^* = (B^\dagger)^* A^- A$ *and* $A^\dagger A = B^\dagger A$*.*

Proof. (i) ⇒ (ii): The hypothesis $A \leq^{-,*} B$ implies $(A^\dagger)^* A^{-,*} = (B^\dagger)^* A^{-,*}$ and $A^{-,*}(A^\dagger)^* = A^{-,*}(B^\dagger)^*$ for some $A^- \in A\{1\}$. Using $A^{-,*} = A^-AA^*$, we have $AA^\dagger = (B^\dagger)^* A^- AA^*$ and $A^-A = A^-AA^*(B^\dagger)^*$.

(ii) ⇒ (iii): From $AA^\dagger = (B^\dagger)^* A^- AA^*$, we get $(A^\dagger)^* = AA^\dagger(A^\dagger)^* = (B^\dagger)^* A^- AA^*(A^\dagger)^* = (B^\dagger)^* A^- A$. Since $A^-A = A^-AA^*(B^\dagger)^*$, it follows $A = AA^-A = AA^-AA^*(B^\dagger)^* = AA^*(B^\dagger)^*$ which yields $A^* = B^\dagger AA^*$.

(iii) ⇒ (iv): Multiplying $A^* = B^\dagger AA^*$ by $(A^\dagger)^*$ from the right hand side, we obtain $A^\dagger A = B^\dagger AA^\dagger A = B^\dagger A$.

(iv) ⇒ (i): Assume that $(A^\dagger)^* = (B^\dagger)^* A^- A$ and $A^\dagger A = B^\dagger A$, for some $A^- \in A\{1\}$. Then, for $A^{-,*} = A^-AA^*$,

$$(A^\dagger)^* A^{-,*} = AA^\dagger = (A^\dagger)^* A^* = (B^\dagger)^* A^- AA^* = (B^\dagger)^* A^{-,*}$$

and

$$A^{-,*}(A^\dagger)^* = A^-A = A^-A(A^\dagger A)^* = A^-A(B^\dagger A)^* = A^{-,*}(B^\dagger)^*.$$

□

As Theorem 4.2, we prove the next result related to the star-inner relation.

Theorem 4.3. *Let* $A, B \in \mathbb{C}^{m\times n}$*. Then the following statements are equivalent:*

(i) $A \leq^{*,-} B$*;*

(ii) *there exists* $A^- \in A\{1\}$ *such that* $AA^- = (B^\dagger)^* A^* AA^-$ *and* $A^\dagger A = A^*AA^-(B^\dagger)^*$*;*

(iii) *there exists* $A^- \in A\{1\}$ *such that* $A^* = A^*AB^\dagger$ *and* $(A^\dagger)^* = AA^-(B^\dagger)^*$*;*

(iv) *there exists* $A^- \in A\{1\}$ *such that* $AA^\dagger = AB^\dagger$ *and* $(A^\dagger)^* = AA^-(B^\dagger)^*$*.*

Under the assumptions $A \leq^{-,*} B$ or $A \leq^{*,-} B$, the next inclusions are satisfied.

Theorem 4.4. *Let* $A, B \in \mathbb{C}^{m\times n}$*.*

(a) *If* $A \leq^{-,*} B$*, then* $B\{1\} \cdot B \cdot A\{1\} \subseteq A\{1\}$*.*

(b) *If* $A \leq^{*,-} B$*, then* $A\{1\} \cdot B \cdot B\{1\} \subseteq A\{1\}$*.*

Proof. (a) By Theorem 4.2, $A \leq^{-,*} B$ gives $A^* = B^\dagger AA^*$, which implies $A = AA^*(B^\dagger)^*$. For arbitrary $A^- \in A\{1\}$ and $B^- \in B\{1\}$, notice that $B^-BA^- \in A\{1\}$ by

$$\begin{aligned} AB^-BA^-A &= AA^*(B^\dagger)^* B^-BA^-A = AA^*(B^\dagger)^* B^\dagger(BB^-B)A^-A \\ &= (AA^*(B^\dagger)^*)A^-A = AA^-A = A. \end{aligned}$$

So, $B\{1\} \cdot B \cdot A\{1\} \subseteq A\{1\}$.

The part (b) can be verified similarly. □

Theorem 4.5. *The inner-star relation $\leq^{-,*}$ and the star-inner relation $\leq^{*,-}$ are partial orders on the set $\mathbb{C}^{m\times n}$.*

Proof. We will only show that the relation $\leq^{-,*}$ is a partial order, because we can check similarly that the relation $\leq^{*,-}$ is a partial order. It is clear $\leq^{-,*}$ is reflexive.

To verify that $\leq^{-,*}$ is antisymmetric, let $A, B \in \mathbb{C}^{m\times n}$ satisfy $A \leq^{-,*} B$ and $B \leq^{-,*} A$. Then, by Theorem 4.2,

$$B = (B^*)^* = (A^\dagger BB^*)^* = BB^*(A^\dagger)^* = BB^*(B^\dagger)^*A^-A = BA^-A$$

and

$$(A^\dagger)^* = (B^\dagger)^*A^-A = (A^\dagger)^*B^-(BA^-A) = (A^\dagger)^*B^-B = (B^\dagger)^*,$$

for some $A^- \in A\{1\}$ and $B^- \in B\{1\}$. Therefore, $A^\dagger = B^\dagger$, which gives $A = (A^\dagger)^\dagger = (B^\dagger)^\dagger = B$.

In order to prove that the relation $\leq^{-,*}$ is transitive, assume that $A, B, C \in \mathbb{C}^{m\times n}$, $A \leq^{-,*} B$ and $B \leq^{-,*} C$. Applying Theorem 4.2, there exist $A^- \in A\{1\}$ and $B^- \in B\{1\}$ such that $AA^\dagger = (B^\dagger)^*A^-AA^*$ (or equivalently $(A^\dagger)^* = (B^\dagger)^*A^-A$), $B^\dagger B = C^\dagger B$, $A^\dagger A = B^\dagger A$ and $(B^\dagger)^* = (C^\dagger)^*B^-B$. Now, we get

$$\begin{aligned} C^\dagger A &= C^\dagger(AA^\dagger)A = C^\dagger(B^\dagger)^*A^-AA^*A = (C^\dagger B)B^\dagger(B^\dagger)^*A^-AA^*A \\ &= B^\dagger BB^\dagger(B^\dagger)^*A^-AA^*A = B^\dagger((B^\dagger)^*A^-AA^*)A = (B^\dagger A)A^\dagger A \\ &= A^\dagger AA^\dagger A = A^\dagger A \end{aligned}$$

and

$$(A^\dagger)^* = (B^\dagger)^*A^-A = (C^\dagger)^*(B^-BA^-)A.$$

Recall that, by Theorem 4.4, $B^-BA^- \in A\{1\}$. According to Theorem 4.2, we conclude that $A \leq^{-,*} C$. □

Theorem 4.6. *Let $A, B \in \mathbb{C}^{m\times n}$. Then the following statements are equivalent:*

(i) *$A \leq^{-,*} B$;*

(ii) *A and B are represented by* (4) *and*

$$B = U\begin{bmatrix} D & D^2A_2^*B_4^\dagger B_4 \\ 0 & (B_4^*)^\dagger \end{bmatrix}V^*,$$

where $\operatorname{rank}(A) = k$, *$A_2 \in \mathbb{C}^{(n-k)\times k}$ and $B_4 \in \mathbb{C}^{(n-k)\times(m-k)}$ are arbitrary matrices.*

Proof. (i) ⇒ (ii): Let A be represented as in (4). Using Lemma 3.1,

$$A^{-,*} = V\begin{bmatrix} D & 0 \\ A_2D^2 & 0 \end{bmatrix}U^*,$$

where $A_2 \in \mathbb{C}^{(n-k)\times k}$ is an arbitrary matrix. Notice that

$$A^\dagger = V\begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* \quad \text{and} \quad (A^\dagger)^* = U\begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^*.$$

Suppose that

$$(B^\dagger)^* = U\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}V^*$$

is partitioned according to the partition of A. From

$$V\begin{bmatrix} I & 0 \\ A_2D & 0 \end{bmatrix}V^* = A^{-,*}(A^\dagger)^* = A^{-,*}(B^\dagger)^* = V\begin{bmatrix} DB_1 & DB_2 \\ A_2D^2B_1 & A_2D^2B_2 \end{bmatrix}V^*,$$

we get $B_1 = D^{-1}$ and $B_2 = 0$. Now

$$U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^* = (A^\dagger)^*A^{-,*} = (B^\dagger)^*A^{-,*} = U\begin{bmatrix} I & 0 \\ B_3D + B_4A_2D^2 & 0 \end{bmatrix}U^*,$$

which gives $B_3 = -B_4A_2D$. Because $(B^\dagger)^* = U\begin{bmatrix} D^{-1} & 0 \\ -B_4A_2D & B_4 \end{bmatrix}V^*$, we have

$$B^\dagger = V\begin{bmatrix} D^{-1} & -(B_4A_2D)^* \\ 0 & B_4^* \end{bmatrix}U^*. \tag{7}$$

According to [10, Lemma 6], we obtain

$$B = (B^\dagger)^\dagger = U\begin{bmatrix} D & D^2A_2^*B_4^\dagger B_4 \\ 0 & (B_4^*)^\dagger \end{bmatrix}V^*.$$

(ii) ⇒ (i): For B expressed as in (ii), we observe that $B^\dagger$ is given by (7). By elementary computations, we verify that $(A^\dagger)^*A^{-,*} = (B^\dagger)^*A^{-,*}$ and $A^{-,*}(A^\dagger)^* = A^{-,*}(B^\dagger)^*$, i.e. $A \leq^{-,*} B$. □

Theorem 4.7. *Let $A, B \in \mathbb{C}^{m\times n}$. Then the following statements are equivalent:*

(i) $A \leq^{*,-} B$;

(ii) *A and B, respectively, are represented by* (4) *and*

$$B = U\begin{bmatrix} D & 0 \\ B_4B_4^\dagger A_1^*D^2 & (B_4^\dagger)^* \end{bmatrix}V^*,$$

where $\operatorname{rank}(A) = k$, *$A_1 \in \mathbb{C}^{k\times(m-k)}$ and $B_4 \in \mathbb{C}^{(n-k)\times(m-k)}$ are arbitrary matrices.*

By Corollary 3.3, we obtain the next result.

Theorem 4.8. *Let $A, B \in \mathbb{C}^{m\times n}$. Then*

(i) *$A \leq^{-,*} B$ if and only if $(A^\dagger)^* \leq^{-,\dagger} (B^\dagger)^*$;*

(ii) *$A \leq^{*,-} B$ if and only if $(A^\dagger)^* \leq^{\dagger,-} (B^\dagger)^*$.*

5. Applications of inner-star and star-inner matrices

Solvability of some systems of linear equations can be proved applying inner-star and star-inner matrices.

Theorem 5.1. *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$ and $A^- \in A\{1\}$. Then the equation*

$$Ax = AA^*b \tag{8}$$

is consistent and its general solution is

$$x = A^{-,*}b + (I - A^-A)y, \tag{9}$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. For x given by (9), we have

$$Ax = AA^{-,*}b = AA^-AA^*b = AA^*b,$$

i.e. (8) holds.

If x presents a solution of (8), we get $A^-Ax = A^-AA^*b = A^{-,*}b$. Thus,

$$x = A^{-,*}b + x - A^-Ax = A^{-,*}b + (I - A^-A)x,$$

which implies that x has the form (9). □

We show that $A^{-,*}b$ is the unique solution in $\mathcal{R}(A^-A)$ of the equation (8).

Theorem 5.2. *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$ and $A^- \in A\{1\}$. Then $A^{-,*}b$ is the unique solution in $\mathcal{R}(A^-A)$ of the equation* (8).

Proof. Firstly, by Theorem 5.1, notice that $x = A^{-,*}b \in \mathcal{R}(A^-A)$ is a solution of (8).

To check that $x = A^{-,*}b$ presents the unique solution of (8) in $\mathcal{R}(A^-A)$, suppose that $x_1 \in \mathcal{R}(A^-A)$ is one more solution of (8). Hence

$$x - x_1 \in \mathcal{R}(A^-A) \cap \mathcal{N}(A^-A) = \{0\}$$

gives $x = x_1$. $\square$

Theorem 5.3. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$. Then $A^{-,*}b$ is the unique solution in $\mathcal{R}(A^-A)$ of the equation*

$$(A^\dagger)^*x = b, \qquad b \in \mathcal{R}(A). \tag{10}$$

Proof. It is well-known that $x = A^{-,*}b \in \mathcal{R}(A^-A)$. Since $b \in \mathcal{R}(A) = \mathcal{R}(AA^\dagger)$, then $b = AA^\dagger b$ gives

$$(A^\dagger)^*x = (A^\dagger)^*A^{-,*}b = AA^\dagger b = b.$$

So, $x = A^{-,*}b$ is a solution of (10). As in the proof of Theorem 5.2, we show that $x = A^{-,*}b$ is the unique solution of (10) in $\mathcal{R}(A^-A)$. $\square$

If we omit the assumption $b \in \mathcal{R}(A)$ in the equation (10), it is interesting to find an approximation solution of the equation $(A^\dagger)^*x = b$. In statistical applications, an approximation solution which is often used, is the least-squares solution. A vector x is a least-squares solution of $(A^\dagger)^*x = b$ if it minimizing the Euclidean norm $\|(A^\dagger)^*x - b\|$. According to [5] and Lemma 2.6, using the relation between $\{1,3\}$-inverses and the least-squares solutions, $\|(A^\dagger)^*x - b\|$ is smallest for $A^{-,*}b = [(A^\dagger)^*]^{(1,2,3)}_{\mathcal{R}(A^-A),\mathcal{N}(A^*)}b$, i.e. the least-square solution of $(A^\dagger)^*x = b$ is given by $A^{-,*}b$, and the general least-squares solution has the form (9).

We verify the next result in a similar way as Theorem 5.1.

Theorem 5.4. *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$ and $A^- \in A\{1\}$. Then the equation*

$$(A^\dagger)^*x = AA^-b, \tag{11}$$

is consistent and its general solution is

$$x = A^{*,-}b + (I - A^\dagger A)y,$$

for arbitrary $y \in \mathbb{C}^n$.

As a consequence of Theorem 5.4, we check the following result when $b \in \mathcal{R}(A)$.

Corollary 5.5. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$. Then the equation* (10) *is consistent and its general solution is*

$$x = A^*b + (I - A^\dagger A)y,$$

for arbitrary $y \in \mathbb{C}^n$.

Using Theorem 5.4, we prove the next theorem similarly as Theorem 5.2.

Theorem 5.6. *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$ and $A^- \in A\{1\}$. Then $A^{*,-}b$ is the unique solution in $\mathcal{R}(A^*)$ of the equation* (11).

Corollary 5.7. *Let $A \in \mathbb{C}^{m\times n}$ and $A^- \in A\{1\}$. Then A^*b is the unique solution in $\mathcal{R}(A^*)$ of the equation* (10).

Recall that, the unique solution of $Mx = b$, where $M \in \mathbb{C}^{n\times n}$ is nonsingular and $b \in \mathbb{C}^n$, is $x = M^{-1}b$. Denote by $M(j \to b)$ the matrix obtained from M replacing the j-th column of M by b. The Cramer's rule for the solution $x = (x_1, x_2, \dots, x_n)^\top$ of nonsingular equation $Mx = b$ is [4, 44]

$$x_j = \frac{\det(M(j \to b))}{\det(M)}, \quad j = \overline{1, n}. \tag{12}$$

We can now propose a Cramer's rule for finding the unique solution of (10) in $\mathcal{R}(A^- A)$.

Theorem 5.8. *Let $A \in \mathbb{C}^{m\times n}$, $A^- \in A\{1\}$ and $b \in \mathcal{R}(A)$. Suppose that E and F^* are full column rank matrices such that*

$$\mathcal{R}(E) = \mathcal{N}(A^*) \qquad and \qquad \mathcal{R}(A^- A) = \mathcal{N}(F).$$

Then the unique solution $x = (x_1, x_2, \dots, x_n)^\top$ of (10) in $\mathcal{R}(A^- A)$ can be expressed componentwise by

$$x_l = \det\left(\begin{bmatrix} (A^\dagger)^*(l \to b) & E \\ F(l \to 0) & 0 \end{bmatrix}\right) / \det\left(\begin{bmatrix} (A^\dagger)^* & E \\ F & 0 \end{bmatrix}\right), \quad l = \overline{1, n}.$$

Proof. By Theorem 5.3, the unique solution $x = A^{-,*}b \in \mathcal{R}(A^- A) = \mathcal{N}(F)$ of (10) in $\mathcal{R}(A^- A)$ satisfies $Fx = 0$ and

$$\begin{bmatrix} (A^\dagger)^* & E \\ F & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

Applying Theorem 3.7 and the Cramer rule (12), we get components of $x = A^{-,*}b$. □

Example 5.9. *Let A be given as in Example 2.3, $b = [1\ 2]^T$ and $y = [y_1\ y_2]^T$. By elementary computations,*

$$\begin{aligned} x &= A^{-,*}b + (I - A^- A)y \\ &= \begin{bmatrix} 2a + (1-a)y_1 - ay_2 \\ 2(1-a) + (a-1)y_1 + ay_2 \end{bmatrix}, \end{aligned}$$

gives

$$Ax = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = AA^*b.$$

So, we confirm Theorem 5.1. Notice that the above x represents the general least-squares solution form of $(A^\dagger)^*x = b$. When

$$\begin{aligned} x &= A^{*,-}b + (I - A^\dagger A)y \\ &= \begin{bmatrix} 1 + 2(d+c) + \frac{1}{2}y_1 - \frac{1}{2}y_2 \\ 1 + 2(d+c) - \frac{1}{2}y_1 + \frac{1}{2}y_2 \end{bmatrix}, \end{aligned}$$

it follows

$$(A^\dagger)^*x = \begin{bmatrix} 1 + 2(d+c) \\ 0 \end{bmatrix} = AA^-b.$$

Hence, Theorem 5.4 is confirmed.

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