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Multivalued Suzuki-generalized Ćirić type nonlinear contractions with an application to fractal space

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Abstract. In this article, we introduce the idea of relation theoretic multivalued Suzuki-generalized nonlinear contractions and utilized the same to prove some fixed point results in a $\mathcal{R}$-complete partial metric space. Our newly proved results are sharpened versions of several known results of the existing literature, which is substantiated by an example. Moreover, the stability of fixed point sets of the multivalued contractions is also discussed. Further, we give an application to the iterated function space.

1. Introduction

In mathematics, Banach contraction principle (BCP) [1] is one of the most fundamental and useful tool for solving existence problems in many branches of mathematics. The principle was established in the Ph.D. thesis of a Polish mathematician S. Banach in 1922. He proved that a contraction mapping on a complete metric space has a unique fixed point. To generalize this result, several researchers like Kannan [2], Chatterjea [3] and Reich [4] have constructed relatively weaker contractive type mappings in the context of several ambient spaces, see [5, 6]. They have attempted to replace the contraction condition with some more generic conditions in order to include a wider class of continuous and discontinuous mappings. In this continuation, Ćirić [7] obtained a new contractive condition known as generalized contraction. A self-mapping T defined on a metric space (X, d) is called a nonlinear contraction if $d(Tx, Ty) \leq \phi(d(x, y))$ for a suitable auxiliary function $\phi : [0, \infty) \to [0, \infty)$. In 1968, Browder [8] imposed some conditions like right continuity and monotonocity on the auxiliary function and improved some existing fixed point theorems. In this sequel, many authors generalized Browder fixed point theorem by slightly varying the conditions on the auxiliary function ϕ. In 1969, Boyd and Wong [9] introduced the following class of auxiliary function:

$$\Phi = \{\psi : [0, \infty) \to [0, \infty) : \phi(t) < t \text{ for each } t > 0 \text{ and } \limsup_{r \to t^+} \phi(r) > t \text{ for each } t > 0\}.$$

Definition 1.1. *A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a comparison function if*

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(i) ϕ *is increasing,*
(ii) $\lim_{n\to\infty} \phi^n(t) = 0$ *for all* $t > 0$.

In 1975, Matkowski [10] initiated the concept of nonlinear contraction via a comparison function. We denote the class of auxiliary function due to Matkowski as:

$$\Phi' = \{\phi : [0,\infty) \to [0,\infty) : \phi \text{ is a comparison function}\}.$$

It can be noticed here that the classes of auxiliary function due to Boyd-Wong and Matkowski are independent. For instance, consider the following two functions:

$$\phi_1(t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{1}{n+1}, & \text{if } t \in (\frac{1}{n+1}, \frac{1}{n}], n = 1, 2, 3, \dots \\ 1, & \text{if } t > 1. \end{cases}$$

$$\phi_2(t) = \begin{cases} \frac{t}{5}, & \text{if } t < 2, \\ \frac{1}{t}, & \text{if } t \leq 2. \end{cases}$$

Notice that $\phi_1 \notin \Phi$ but $\phi_1 \in \Phi'$ whereas $\phi_2 \notin \Phi'$ but $\phi_2 \in \Phi$. Recently, in 2018, Pant [11] furnished some fixed point results in a metric space under Suzuki type generalized ϕ-contraction (Boyd and Wong type).

Definition 1.2. *Let* (X, d) *be a metric space. A mapping* $T : X \to X$ *is called Suzuki type generalized* ϕ*-contraction if for all* $x, y \in X$*, there exist* $\phi \in \Phi$ *such that*

$$\tfrac{1}{2} d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq \phi(m(x, y)),$$

where $m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.

In 1969 one more famous generalization of the Banach contraction principle for set valued mapping in metric spaces was established by Nadler [12] which is stated as follows:

Theorem 1.3. *[12] Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$ *a set valued mapping, where* $CB(X)$ *is the collection of all closed and bounded subsets of* X*. If there exists* $\alpha \in (0, 1)$ *such that*

$$H(Tx, Ty) \leq \alpha d(x, y) \text{ for all } x, y \in X$$

where H *is Hausdorff metric on* $CB(X)$*, then* T *has a fixed point.*

The concept of multivalued mapping has been improved many times since then in the literature. Recently, Andres et al. investigated the existence of coupled fixed point theorems for multivalued contractions in complete metric space, see [13].

On the other hand, fixed point theorems for monotone single-valued mappings in a metric space endowed with a partial ordering have been widely investigated, see ([14, 15]). In 2015, Alam and Imdad [16] established a profound generalization of the Banach contraction principle with an amorphous binary relation instead of partial order. Soon after, various relation-theoretic results were proposed by several researchers, see ([17–20]).

In this paper, we initiate the concept of relation-theoretic multivalued Suzuki-generalized Ciric type Matkowski contractions and utilize the same to prove some fixed point theorems in a $\mathcal{R}$-complete partial metric space endowed with a certain binary relation. Some examples are also presented in support of our results. Further, we discussed the stability of fixed point sets of the multivalued contractions. At the end, we state an application of our result to construct multivalued fractals.

2. Partial Metric Space and Set theoretic Distance

2.1. Partial metric notions

One of the generalizations of metric spaces namely partial metric space was introduced by Mathews [5] in 1994, wherein the distance of self point need not be zero along with modified triangle inequality. Thereafter, Matthews proved the partial metric version of Banach fixed point theorem. Mathews defined the partial metric space as follows,

Definition 2.1. *[5] Let X be a non-empty set and $p : X \times X \to [0, \infty)$ a mapping satisfying the following conditions:*

(i) $x = y \iff p(x, x) = p(x, y) = p(y, y)$,
(ii) $p(x, x) \leq p(x, y)$,
(iii) $p(x, y) = p(y, x)$,
(iv) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z), \quad \forall x, y, z \in X$.

Then the mapping is known as partial metric and the pair (X, p) is called partial metric space.

Remark 2.2. *Let p be the partial metric on X. Then the mapping $d_p : X \times X \to [0, \infty)$ defined by*

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \quad \forall x, y \in X,$$

is a metric on X and hence (X, d_p) is a metric space.

Definition 2.3. *[5] Let (X, p) be a partial metric space,*
(i) a sequence $\{x_n\}$ is convergent to a point $x \in X$ if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$,
(ii) a sequence $\{x_n\}$ is Cauchy if $\lim_{m,n\to\infty} p(x_m, x_n)$ is exist and finite,
(iii) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point in $x \in X$ and $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Lemma 2.4. *[5] Let (X, p) be a partial metric space.*
(i) A sequence (X, p) is Cauchy in (X, p) if and only if it is Cauchy in (X, d_p).
(ii) (X, p) is complete if and only if (X, d_p) is complete, and

$$\lim_{n\to\infty} d_p(x_n, x) = 0 \iff p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

2.2. Set Theoretic Distances

For a metric space (X, p), $C(X)$ denotes the set of all compact subsets of X. The distance between a point $x \in X$ and the set $A \in C(X)$ is defined by

$$\mathcal{D}_p(x, A) = \inf \{p(x, a), \forall a \in A\}.$$

Now we defined distance between two sets in a metric space as

$$\rho(A, B) = \sup \{\mathcal{D}_p(a, B), \ \forall a \in A\}.$$

The Hausdorff distance on $C(X)$ is denoted by $\mathcal{H}_p$ and it is defined by

$$\mathcal{H}_p(A, B) = \max\{ \rho(A, B), \rho(B, A)\} \ \forall A, B \in C(X).$$

Note that $(C(X), \mathcal{H}_p)$ is indeed a metric space.

Lemma 2.5. *Let (X, p) be a partial metric space, $A, B \in C(X)$ then for any $x \in A$ there exists $y \in B$ such that $p(x, y) \leq \mathcal{H}_p(A, B)$.*

Proof. Let $A, B \in C(X)$ and $x \in A$. We know that

$$\mathcal{H}_p(A,B) = max\left\{\sup_{x\in A}\mathcal{D}(x,B), \sup_{y\in B}\mathcal{D}(y,A)\right\}.$$

From the definition, $q = \mathcal{D}_p(x,B) = \inf\{p(x,b) : b \in B\} \leq \mathcal{H}_p(A,B)$. Then there exists a sequence $\{y_n\}$ in B such that $p(x, y_n) \to q$ as $n \to \infty$. Since B is compact, $\{y_n\}$ has a convergent subsequence $\{y_{n_k}\}$. Hence there exists $y \in X$ such that $y_{n_k} \to y$ as $k \to \infty$. As B is compact, it is closed and $y \in B$. Now, $\lim_{n\to\infty} p(x, y_n) = q$ implies that $\lim_{k\to\infty} p(x, y_{n_k}) = q$ i.e., $p(x, y) = q = \mathcal{D}_p(x,B) \leq \mathcal{H}_p(A,B)$. Hence the proof is completed. □

Lemma 2.6. *Let $A, B \in C(X)$ of a partial metric space (X,p) and $T : A \to C(B)$ be a multivalued mapping. Then for $a, b \in A$ and $x \in Ta$, there exists a $y \in Tb$ such that $p(x, y) \leq \mathcal{H}_p(Ta, Tb)$.*

3. Relation Theoratic Notions and Auxiliary Results

Throughout this paper, $\mathbb{N}$ stands for the set of all natural numbers and $\mathbb{N}_0$ stands for the set of all whole numbers, *i.e.*, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Definition 3.1. *[21] Let X be a nonempty set. A subset $\mathcal{R}$ of X^2 is called a binary relation on X. If $(x, y) \in \mathcal{R}$, then we say that "x is related to y" or "x relates to y" under $\mathcal{R}$. The subsets, X^2 and $\emptyset$ of X^2 are called the universal relation and empty relation, respectively.*

Definition 3.2. *[21–25] A binary relation $\mathcal{R}$ defined on a nonempty set X is called*

(i) *amorphous if $\mathcal{R}$ has no specific property,*
(ii) *reflexive if $(x, x) \in \mathcal{R}$ $\forall$ $x \in X$,*
(iii) *symmetric if $(x, y) \in \mathcal{R}$ implies $(y, x) \in \mathcal{R}$,*
(iv) *anti-symmetric if $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ implies $x = y$,*
(v) *transitive if $(x, y) \in \mathcal{R}$ and $(y, w) \in \mathcal{R}$ implies $(x, w) \in \mathcal{R}$,*
(vi) *a partial order if $\mathcal{R}$ is reflexive, anti-symmetric and transitive.*

Definition 3.3. *[16] Let $\mathcal{R}$ be a binary relation on a non-empty set X and $x, y \in X$. We say that x and y are $\mathcal{R}$-comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. It is denoted by $[x, y] \in \mathcal{R}$.*

Definition 3.4. *[16] Given a non-empty set X and a binary relation $\mathcal{R}$ on X, a sequence $\{x_n\} \subset X$ is termed as $\mathcal{R}$-preserving if*

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall\ n \in \mathbb{N}_0.$$

Definition 3.5. *[26] Given a partial metric space (X,p), a binary relation $\mathcal{R}$ defined on X is called p-self-closed if for an $\mathcal{R}$-preserving sequence $\{x_n\} \subset X$ converging to $x \in X$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}$.*

Definition 3.6. *[26] Let $\mathcal{R}$ be a binary relation defined on a nonempty set X. We say that (X,p) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X converges.*

Definition 3.7. *[27] Let (X,p) be a partial metric space endowed with a binary relation $\mathcal{R}$ and $T : X \to C(X)$ then*

$$X(T;\mathcal{R}) = \{x \in X \text{ such that } (x, y) \in \mathcal{R} \text{ for some } y \in Tx\}.$$

Definition 3.8. *For a nonempty set X with a map $T : X \to C(X)$. Any binary relation $\mathcal{R}$ on X is T-p-closed if $\forall$ $x, y \in X$,*

$$(x, y) \in \mathcal{R},\ a \in Tx \text{ and } b \in Ty \text{ with } p(a, b) \leq p(x, y) \implies (a, b) \in \mathcal{R}.$$

Definition 3.9. *Let (X, p) be a partial metric space endowed with a binary relation $\mathcal{R}$ with $x \in X$. Then $T : X \to C(X)$ is called $\mathcal{R}_{\mathcal{H}_p}$-continuous at x if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{p} x$, we obtain $T(x_n) \xrightarrow{\mathcal{H}_p} T(x)$, i.e., $\lim_{n\to\infty} \mathcal{H}_p(Tx_n, Tx) = 0$. Furthermore, T is called $\mathcal{R}_{\mathcal{H}_p}$-continuous if it is $\mathcal{R}_{\mathcal{H}_p}$-continuous at each point of X.*

Lemma 3.10. *[28] Let $\phi \in \Phi'$. Then for all $t > 0$, we have $\phi(t) < t$.*

Definition 3.11. *[29] Given $N \in \mathbb{N}_0$, $N \geq 2$, a binary relation $\mathcal{R}$ defined on a non-empty set X is called N-transitive if for any $x_0, x_1, x_2, .., x_N \in X$,*

$$(x_{i-1}, x_i) \in \mathcal{R} \text{ for each } \ i(1 \leq i \leq N) \Longrightarrow (x_0, x_N) \in \mathcal{R}.$$

Notice that notion of 2-transitivity coincides with transitivity. Following Turinici[30], $\mathcal{R}$ is called finitely transitive if it is N-transitive for some $N \geq 2$.

Definition 3.12. *[30] A binary relation $\mathcal{R}$ defined on a nonempty set X is called locally finitely transitive if for each denumerable subset E of X, there exists $N = N(E) \geq 2$, such that $\mathcal{R}|_E$ is N-transitive.*

Definition 3.13. *[20] Let X be a nonempty set and T a self-mapping on X. A binary relation $\mathcal{R}$ on X is called locally finitely T-transitive if for each denumerable subset E of $T(X)$, there exists $N = N(E) \geq 2$, such that $\mathcal{R}|_E$ is N-transitive.*

The following result eastablishes the superiority of the idea of 'locally finitely T-transitivity'over other varients of 'transitivity':

Proposition 3.14. *[20] Let X be a nonempty set, $\mathcal{R}$ a binary relation on X and T a self-mapping on X. Then*

(i) *$\mathcal{R}$ is T-transitive $\Leftrightarrow$ $\mathcal{R}|_{T(X)}$ is transitive,*
(ii) *$\mathcal{R}$ is locally finitely T-transitive $\Leftrightarrow$ $\mathcal{R}|_{T(X)}$ is locally finitely transitive,*
(iii) *$\mathcal{R}$ is transitive $\Longrightarrow$ $\mathcal{R}$ is finitely transitive $\Longrightarrow$ $\mathcal{R}$ is locally transitive $\Longrightarrow$ $\mathcal{R}$ is locally finitely T-transitive,*
(iv) *$\mathcal{R}$ is transitive $\Longrightarrow$ $\mathcal{R}$ is T-transitive $\Longrightarrow$ $\mathcal{R}$ is locally finitely T-transitive.*

Given a binary relation $\mathcal{R}$ and a self-mapping T on a nonempty set X, we use the following notations:

(*i*) $\mathcal{N}(x, y) := max\Big\{p(x, y), \mathcal{D}_p(x, Tx), \mathcal{D}_p(y, Ty)\Big\}$,
(*ii*) $\mathcal{M}(x, y) := max\Big\{p(x, y), \mathcal{D}_p(x, Tx), \mathcal{D}_p(y, Ty), \frac{1}{2}\{\mathcal{D}_p(x, Ty) + \mathcal{D}_p(y, Tx)\}\Big\}$.

Remark 3.15. *Observe that $\mathcal{N}(x, y) \leq \mathcal{M}(x, y)$ $(\forall x, y \in X)$.*

In view of symmetry of metric p, the following conclusion is immediate.

Proposition 3.16. *If (X, p) is a partial metric space, $\mathcal{R}$ is a binary relation on X, T is a mapping from X to $C(X)$ and $\phi \in \Phi'$, then the following contractivity conditions are equivalent:*

$$(I) \ \frac{1}{2}\mathcal{D}_p(x, Tx) \leq p(x, y) \Longrightarrow \mathcal{H}_p(Tx, Ty) \leq \phi(\mathcal{M}(x, y)) \ \ \forall\, x, y \in X \text{ with } \ (x, y) \in \mathcal{R},$$

$$(II) \ \frac{1}{2}\mathcal{D}_p(x, Tx) \leq p(x, y) \Longrightarrow \mathcal{H}_p(Tx, Ty) \leq \phi(\mathcal{M}(x, y)) \ \ \forall\, x, y \in X \text{ with } \ [x, y] \in \mathcal{R}.$$

4. Main Results

In this section, we first define the relation-theoretic multivalued Suzuki-generalized Ćirić type Matkowski contractions, then we prove some fixed point results in relational partial metric space under the same contraction.

Definition 4.1. *Let (X, p) be a partial metric space endowed with a binary relation $\mathcal{R}$. Let $T : X \to C(X)$ be a multivalued mapping. Then, T is said to be relation-theoretic multivalued Suzuki-generalized Ćirić type Matkowski contraction if there exists $\phi \in \Phi'$ such that*

$$\frac{1}{2}\mathcal{D}_p(x, Tx) \le p(x, y) \Longrightarrow \mathcal{H}_p(Tx, Ty) \le \phi(\mathcal{M}(x, y)) \ \forall \ x, y \in X \ \ with \ \ (x, y) \in \mathcal{R}.$$

Theorem 4.2. *Let (X, p) be a partial metric space endowed with a binary relation $\mathcal{R}$. Let $T : X \to C(X)$ be a multivalued mapping. Suppose that the following conditions hold:*

(a) *(X, p) is $\mathcal{R}$-complete,*
(b) *$X(T; \mathcal{R})$ is non-empty,*
(c) *$\mathcal{R}$ is T-p-closed and locally T-transitive,*
(d) *T is relation-theoretic multivalued Suzuki-generalized Ćirić type Matkowski contraction,*
(*e*) *either T is $\mathcal{R}_{\mathcal{H}_p}$-continuous or $\mathcal{R}$ is p-self-closed.*

Then T has a fixed point.

Proof. Let us assume that T has no fixed point. Then $\mathcal{D}_p(x, Tx) > 0$ for all $x \in X$. By condition (b), choose $x_0 \in X(T; \mathcal{R})$ such that $(x_0, x_1) \in \mathcal{R}$ for some $x_1 \in Tx_0$. So now we have $0 < \mathcal{D}_p(x_1, Tx_1) \le \mathcal{H}_p(Tx_0, Tx_1)$ and $\frac{1}{2}\mathcal{D}_p(x_0, Tx_0) \le \mathcal{D}_p(x_0, Tx_0) \le p(x_0, x_1)$. Then by contractivity condition (d), Lemma 2.5 and increasing property of ϕ, we have

$$\begin{aligned}
\mathcal{D}_p(x_1, Tx_1) &\le \mathcal{H}_p(Tx_0, Tx_1) \\
&\le \phi(\mathcal{M}(x_0, x_1)) \\
&= \phi(max\{p(x_0, x_1), \mathcal{D}_p(x_0, Tx_0), \mathcal{D}_p(x_1, Tx_1), \frac{1}{2}\{\mathcal{D}_p(x_0, Tx_1) + \mathcal{D}_p(x_1, Tx_0)\}\}) \\
&= \phi(max\{p(x_0, x_1), p(x_0, x_1), \mathcal{D}_p(x_1, Tx_1), \frac{1}{2}\{\mathcal{D}_p(x_0, Tx_1) + p(x_1, x_1)\}\}) \\
&\le \phi(max\{p(x_0, x_1), \mathcal{D}_p(x_1, Tx_1), \frac{1}{2}\{p(x_0, x_1) + \mathcal{D}_p(x_1, Tx_1)\}\}) \\
&= \phi\Big(max\{p(x_0, x_1), \mathcal{D}_p(x_1, Tx_1)\}\Big)
\end{aligned}$$

so that

$$\mathcal{D}_p(x_1, Tx_1) \quad \le \quad \phi(max\{p(x_0, x_1), \mathcal{D}_p(x_1, Tx_1)\}). \tag{1}$$

In case if $max\{p(x_0, x_1), \mathcal{D}_p(x_1, Tx_1)\} = \mathcal{D}_p(x_1, Tx_1)$ then using Lemma 3.10 and equation (1), we obtain $\mathcal{D}_p(x_1, Tx_1) < \mathcal{D}_p(x_1, Tx_1)$, which is a contradiction and hence (1) reduces to

$$\mathcal{D}_p(x_1, Tx_1) \quad \le \quad \phi(p(x_0, x_1)). \tag{2}$$

Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that $p(x_1, x_2) = \mathcal{D}_p(x_1, Tx_1)$. Hence from (2) and property of ϕ, we have

$$p(x_1, x_2) < p(x_0, x_1).$$

Then by the Definition 3.8 we have $(x_1, x_2) \in \mathcal{R}$. Now again as $(x_1, x_2) \in \mathcal{R}$ we have $0 < \mathcal{D}_p(x_2, Tx_2) \le \mathcal{H}_p(Tx_1, Tx_2)$ and $\frac{1}{2}\mathcal{D}_p(x_1, Tx_1) \le \mathcal{D}_p(x_1, Tx_1) \le p(x_1, x_2)$. Then by contractivity condition (d), Lemma 2.5 and increasing property of ϕ, we have

$$\mathcal{D}_p(x_2, Tx_2) \le \mathcal{H}_p(Tx_1, Tx_2) \le \phi(p(x_1, x_2)) < p(x_1, x_2). \tag{3}$$

Since Tx_2 is compact, there exists $x_3 \in Tx_2$ such that $p(x_2, x_3) = \mathcal{D}_p(x_2, Tx_2)$. Hence from (2) and property of ϕ, we have

$$p(x_2, x_3) < p(x_1, x_2).$$

Then by the Definition 3.8 we have $(x_2, x_3) \in \mathcal{R}$. Continuing the process, we can contruct a sequence $\{x_n\}$ such that for all $n \geq 0$,

$$x_{n+1} \in Tx_n \ \ , \ (x_n, x_{n+1}) \in \mathcal{R}. \tag{4}$$

and

$$p(x_n, x_{n+1}) \leq \phi(p(x_{n-1}, x_n)). \tag{5}$$

Thus applying inductivity on euqation (5), we get

$$p(x_n, x_{n+1}) \leq \phi(p(x_{n-1}, x_n)) \leq \cdots \leq \phi^n(p(x_0, x_1)). \tag{6}$$

Tending $n \to \infty$ and using the property of ϕ, we get

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{7}$$

Fix $\epsilon > 0$. In view of (7), we can write

$$p(x_n, x_{n+1}) < \epsilon - \phi(\epsilon). \tag{8}$$

Now we will show that $\{x_n\}$ is a Cauchy sequence. Using (7) and increasing property of ϕ, we get

$$\begin{aligned} p(x_n, x_{n+2}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) - p(x_{n+1}, x_{n+1}) \\ &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) \\ &< \epsilon - \phi(\epsilon) + p(x_{n+1}, x_{n+2}) \\ &\leq \epsilon - \phi(\epsilon) + \phi(p(x_n, x_{n+1})) \\ &< \epsilon - \phi(\epsilon) + \phi(\epsilon - \phi(\epsilon)) \\ &\leq \epsilon - \phi(\epsilon) + \phi(\epsilon) \\ &= \epsilon. \end{aligned}$$

On using locally T-transitivity property of $\mathcal{R}$, we obtain

$$\begin{aligned} p(x_n, x_{n+3}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+3}) - p(x_{n+1}, x_{n+1}) \\ &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+3}) \\ &< \epsilon - \phi(\epsilon) + p(x_{n+1}, x_{n+3}) \\ &\leq \epsilon - \phi(\epsilon) + \phi(p(x_n, x_{n+2})) \\ &\leq \epsilon - \phi(\epsilon) + \phi(\epsilon) \\ &= \epsilon. \end{aligned}$$

Continuing this process, we can write

$$p(x_n, x_{n+k}) < \epsilon \ \ \forall \, k \in \mathbb{N},$$

which shows that the sequence $\{x_n\}$ is Cauchy. Since $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}_0$, owing to Lemma 2.4, $\{x_n\}$ is a Cauchy sequence in both (X, p) and (X, d_p). Since (X, p) is $\mathcal{R}$-complete, so is (X, d_p). Then there always exists $x \in X$ such that $x_n \to x$. Then by $\mathcal{R}_{\mathcal{H}_p}$-continuity of T, we have

$$\lim_{n\to\infty} \mathcal{H}_p(Tx_n, Tx) = 0. \tag{9}$$

Then

$$\begin{aligned} \mathcal{D}_p(x, Tx) = \inf_{y\in Tx} p(x, y) &\leq p(x, x_{n+1}) + \inf_{y\in Tx} p(x_{n+1}, y) - p(x_{n+1}, x_{n+1}) \\ &\leq p(x, x_{n+1}) + \inf_{y\in Tx} p(x_{n+1}, y) \\ &\leq p(x, x_{n+1}) + \mathcal{H}_p(Tx_n, Tx) \end{aligned}$$

as $n \to \infty$ and using equation (9) we get $\lim_{n\to\infty} \mathcal{D}_p(x, Tx) = 0$. Since $Tx \in C(X)$ is compact then Tx is closed, which amount to say that $x \in Tx$. This implies that x is a fixed point of T.
Alternately, assume that $\mathcal{R}$ is p-self-closed. As $\{x_n\}$ is $\mathcal{R}$-preserving such that $x_n \xrightarrow{p} z$, the p-self-closedness of $\mathcal{R}$ guarantees the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, z] \in \mathcal{R}$ $(\forall\, k \in \mathbb{N}_0)$.
Now, we claim that (for all $k \in \mathbb{N}_0$),

$$\frac{1}{2}\mathcal{D}_p(x_{n_k}, Tx_{n_k+1}) \leq p(x_{n_k}, z) \text{ or } \frac{1}{2}\mathcal{D}_p(x_{n_k+1}, Tx_{n_k+1}) \leq p(x_{n_k+1}, z). \tag{10}$$

Arguing by contradiction, we assume that (for some $k_1 \in \mathbb{N}_0$)

$$\frac{1}{2}\mathcal{D}_p(x_{n_{k_1}}, Tx_{n_{k_1}}) > p(x_{n_{k_1}}, z) \text{ and } \frac{1}{2}\mathcal{D}_p(x_{n_{k_1}+1}, Tx_{n_{k_1}+1}) > p(x_{n_{k_1}+1}, z).$$

Applying the triangle inequality of partial metric, we obtain

$$\begin{aligned} \mathcal{D}_p(x_{n_{k_1}}, Tx_{n_{k_1}}) \leq p(x_{n_{k_1}}, x_{n_{k_1}+1}) &\leq p(x_{n_{k_1}}, z) + p(z, x_{n_{k_1}+1}) - p(z,z) \\ &\leq p(x_{n_{k_1}}, z) + p(z, x_{n_{k_1}+1}) \\ &< \frac{1}{2}\mathcal{D}_p(x_{n_{k_1}}, Tx_{n_{k_1}}) + \frac{1}{2}\mathcal{D}_p(x_{n_{k_1}+1}, Tx_{n_{k_1}+1}) \\ &< \frac{1}{2}\mathcal{D}_p(x_{n_{k_1}}, Tx_{n_{k_1}}) + \frac{1}{2}\mathcal{D}_p(x_{n_{k_1}}, Tx_{n_{k_1}}) \\ &< \frac{1}{2}\{\mathcal{D}_p(x_{n_{k_1}}, Tx_{n_{k_1}}) + \mathcal{D}_p(x_{n_{k_1}}, Tx_{n_{k_1}})\} = \mathcal{D}_p(x_{n_{k_1}}, Tx_{n_{k_1}}), \end{aligned}$$

which is a contradiction. Therefore, (10) holds for all $k \in \mathbb{N}_0$ immediately.
On using assumption (d), (10), $[x_{n_k}, z] \in \mathcal{R}$ and Proposition 3.16, we have

$$\mathcal{D}_p(x_{n_{k+1}}, Tz) \leq \mathcal{H}_p(Tx_{n_k}, Tz) \leq \phi\left(\mathcal{M}(x_{n_k}, z)\right). \tag{11}$$

If $\mathcal{M}(x_{n_k}, z) = \mathcal{D}_p(Tz, z) = \alpha$, then we have

$$\mathcal{D}_p(x_{n_{k+1}}, Tz) = \mathcal{H}_p(Tx_{n_k}, Tz) \leq \phi\left(\mathcal{M}(x_{n_k}, z)\right) \leq \phi\left(\mathcal{D}_p(z, Tz)\right) \leq \phi(\alpha).$$

Taking $k \to \infty$, we get

$$\begin{aligned} \mathcal{D}_p(Tz, z) &\leq \phi(\alpha) \\ \alpha &\leq \phi(\alpha) < \alpha, \end{aligned}$$

which is a contradiction. Otherwise, if

$$\mathcal{M}(x_{n_k}, z) = max\left\{p(x_{n_k}, z), p(x_{n_k}, x_{n_{k+1}}), \tfrac{1}{2}\left[\mathcal{D}_p(x_{n_k}, Tz) + p(z, x_{n_k+1})\right]\right\},$$

then due to the fact $x_n \xrightarrow{p} z$, there always exist $N = N(\alpha)$ such that

$$\mathcal{M}(x_{n_k}, z) \leq \tfrac{3}{4}\alpha \qquad \text{for all } k \geq N.$$

As ϕ is increasing, we have

$$\phi\left(\mathcal{M}(x_{n_k}, z)\right) \leq \phi\left(\frac{3}{4}\alpha\right) \qquad \forall k \geq N. \tag{12}$$

Employing (11) and (12) we have,

$$\mathcal{D}_p(x_{n_{k+1}}, Tz) \leq \mathcal{H}_p(Tx_{n_k}, Tz) \leq \phi\left(\mathcal{M}(x_{n_k}, z)\right) \leq \phi\left(\tfrac{3}{4}\alpha\right) \qquad \forall k \geq N.$$

On taking $k \to \infty$ and using Lemma 3.10, we get

$$\alpha = \phi\left(\tfrac{3}{4}\alpha\right) < \tfrac{3}{4}\alpha < \alpha,$$

which is a contradiction. Hence, $\alpha = 0$, so that

$$\mathcal{D}_p(z, Tz) = \alpha = 0.$$

Since $Tz \in C(X)$ is compact then Tz is closed, hence $z \in Tz$. Hence, z is a fixed point of T. □

Example 4.3. *Consider the metric space $X = [0, 1]$ with the partial metric p defined as $p(x, y) = max\{x, y\}$ and a binary relation $\mathcal{R} = \{(x, y) \in X \times X : x > y > \frac{1}{2}\}$ together with a multivalued mapping $T : X \to C(X)$ defined by*

$$T(x) = \begin{cases} \{\frac{x}{2}\} & \text{if } 0 \le x < \frac{1}{2}, \\ \left[\frac{1}{2}, \frac{3}{4}\right] & \text{if } \quad x = \frac{1}{2}, \\ \{\frac{x^3}{1+x^2}\} & \text{if } \ \frac{1}{2} < x \le 1. \end{cases}$$

So X with the partial metric p and the given relation $\mathcal{R}$ is a $\mathcal{R}$-complete partial metric space. And we see that the map T is not continuous because it is upper semi-continuous but not lower semi-continuous. For any $x \in (\frac{1}{2}, 1]$ we always have a point $y \in Tx = \{\frac{x^3}{1+x^2}\}$ with $x > y$ such that $(x, y) \in \mathcal{R}$. Therefore $X(T; \mathcal{R})$ is non-empty. And we can easily proved that the relation $\mathcal{R}$ is T-closed. Now we see that

$$\begin{aligned} \mathcal{H}_p(Tx, Ty) &= max\,\{\{Tx\}, \{Ty\}\} \\ &= max\left\{\frac{x^3}{1+x^2}, \frac{y^3}{1+y^2}\right\} \\ &= \frac{x^3}{1+x^2}. \end{aligned}$$

And we also have $p(x, y) = x, \mathcal{D}_p(x, Tx) = x, \mathcal{D}_p(y, Ty) = y, \mathcal{D}_p(x, Ty) = x, \mathcal{D}_p(y, Tx) = x$ and $\frac{1}{2}\{\mathcal{D}_p(x, Ty) + \mathcal{D}_p(y, Tx)\} = \frac{1}{2}(x + y)$. Then we conclude that

$$\begin{aligned} \mathcal{H}_p(Tx, Ty) &= \frac{x^3}{1+x^2} \\ &\le \phi\left\{max\left(p(x, y), \mathcal{D}_p(x, Tx), \mathcal{D}_p(y, Ty), \frac{1}{2}\{\mathcal{D}_p(x, Ty) + \mathcal{D}_p(y, Tx)\}\right)\right\} \\ &\le \phi\left\{max\left(x, x, y, \frac{1}{2}(x + y)\right)\right\} \\ &\le \phi(x) \\ &\le \frac{2}{3}x^2 \qquad\qquad \text{for } \ \phi(t) = \frac{2}{3}t^2. \end{aligned}$$

So all the hypotheses of the Theorem 4.2 are satisfied. Hence the mapping T has fixed points, namely 0 and $\frac{1}{2}$.

For better visualization of the contraction above, see below the 3D graph (On MATLAB R2015a) where dark surface represent the right hand side of the contraction whereas light surface represent the left hand side of the contraction.

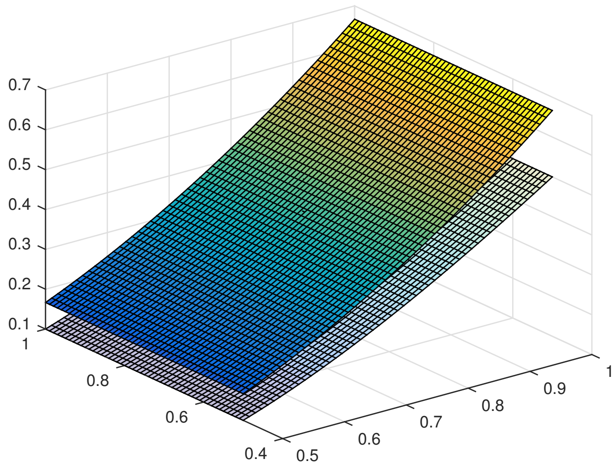

Figure 1: Visualization of the contraction condition

Under the universal relation (i.e., $\mathcal{R} = X \times X$), Theorems 4.2 remain sharpened versions of the following results (in the form of Remarks) in the context of the multivalued mapping, PMS, auxillary function $\phi \in \Phi'$ and Suzuki condition.

Remark 4.4. *If we replace $\phi(t) = \alpha t$ for $\alpha \in [0, 1)$ and the setting $\mathcal{M}(x, y)$ to be $\mathcal{N}(x, y)$, we obtain a sharpened version of the Ćirić fixed point theorem [7].*

Remark 4.5. *Under the setting of $\phi(t) = \beta t$ ($\beta \in [0, 1)$) and $\mathcal{M}(x, y) = p(x, y)$ in Theorem 4.2, we obtain the sharpened version of the Banach contraction principle [1].*

Remark 4.6. *Under the setting of $\phi(t) = \beta(2t)$ ($\beta \in [0, \frac{1}{2})$) and $\mathcal{M}(x, y) = \frac{1}{2}\left(\mathcal{D}_p(x, Ty) + \mathcal{D}_p(y, Tx)\right)$ in Theorem 4.2, we obtain an improved version of the Chatterjea fixed point theorem (see [3]).*

Remark 4.7. *If we replace $\mathcal{M}(x, y)$ by the condition $\mathcal{M}(x, y) = \{p(x, y), \frac{1}{2}[\mathcal{D}_p(x, Tx) + \mathcal{D}_p(y, Ty)]$, $\frac{1}{2}\left[\mathcal{D}_p(x, Ty) + \mathcal{D}_p(y, Tx)\right]\}$, we obtain a consequences of Theorem 4.2, which remains an improved version of Theorem 1.17 contained in Ahmadullah et al. [31].*

5. Stability of fixed point sets

The concept of stability is related to a system's limiting behaviours. It has been researched in numerous discrete and continuous dynamical systems settings, see [32, 33]. The stability of fixed points, known as the relationship between the convergence of a sequence of mappings and their fixed points, has also been extensively researched in a wide range of contexts, see [34, 35]. If the fixed point sets of a sequence of mappings converge to the set of fixed points of the limit mapping in the Hausdorff metric, they are said to be stable. Compared to single-valued mappings, multivalued mappings frequently have more fixed points. As a result, the set of fixed points for multivalued mappings widens and becomes more intriguing for the study of stability. In this section, we investigate the stability of the fixed point sets of the multivalued contraction discussed in the preceding section.

Theorem 5.1. *Let (X, p) be a $\mathcal{R}$-complete partial metric space endowed with a locally T-transitive binary relation $\mathcal{R}$ and $T_1, T_2 : X \to C(X)$ be two continuous multivalued mappings. Suppose that each T_1, T_2 satisfies the contractive* *condition* (d) *of Theorem 4.2, i.e., for all* $x, y \in X$ *with* $(x, y) \in \mathcal{R}$ *and* $\phi \in \Phi'$,

$$\frac{1}{2}\mathcal{D}_p(x, Tx) \leq p(x, y) \Longrightarrow \mathcal{H}_p(Tx, Ty) \leq \phi(\mathcal{M}(x, y)) \ \ \forall\, x, y \in X \ \ with \ \ (x, y) \in \mathcal{R}.$$

Then $\mathcal{H}_p(F(T_1), F(T_2)) \leq \Psi(L)$ *where* $L = \sup_{x\in X} \mathcal{H}_p(T_1(x), T_2(x))$ *and* $\Psi(t) = \phi^n(t)$.

Proof. From Theorem 4.2 the set of fixed points of T_1, T_2 are non empty. Let $x_0 \in F(T_1)$ that is $x_0 \in T_1(x_0)$. Then by Lemma 2.5 there exists a $x_1 \in T_2(x_0)$ such that

$$p(x_0, x_1) \leq \mathcal{H}_p(T_1(x_0), T_2(x_0)) \tag{13}$$

since $x_1 \in T_2(x_0)$ then by Lemma 2.6 there exists $x_2 \in T_2(x_1)$ such that

$$p(x_1, x_2) \leq \mathcal{H}_p(T_1(x_0), T_2(x_1)). \tag{14}$$

Then same as the proof of Theorem 4.2 we have $x_{n+1} \in T_2 x_n$ and $p(x_{n+1}, x_{n+2}) \leq \phi(p(x_n, x_{n+1}))$ which by properties of ϕ we get

$$p(x_{n+1}, x_{n+2}) \leq \phi(p(x_n, x_{n+1})) \leq \phi^2(p(x_{n-1}, x_n)) \leq \cdots \leq \phi^{n+1}(p(x_0, x_1)). \tag{15}$$

So the sequence $\{x_n\}$ will be Cauchy sequence as same as in Theorem 4.2. Then there exists a $x \in X$ such that

$$x_n \to x \quad \text{as } n \to \infty \tag{16}$$

and x is fixed point of T_2 i.e., $x \in T_2 x$. Now using (13) and definition of L, we have

$$p(x_0, x_1) \leq \mathcal{H}_p(T_1 x_0, T_2(x_0)) \leq L = \sup_{x\in X} \mathcal{H}_p(T_1(x), T_2(x)) \tag{17}$$

Then using triangle inequality, locally T-transitivity and using (15), we have

$$\begin{aligned}
p(x_0, x) &\leq p(x_0, x_{n+1}) + p(x_{n+1}, x) - p(x_{n+1}, x_{n+1}) \\
&< p(x_0, x_{n+1}) + p(x_{n+1}, x) \\
&\leq \phi(p(x_0, x_n)) + \phi(p(x_n, x)) \\
&\leq \phi^n(p(x_0, x_1)) + \phi^n(p(x_1, x)) \\
&\leq \Psi(L) + \Psi(L) \\
&= 2\Psi(L).
\end{aligned}$$

Thus for given arbitrary $x_0 \in F(T_1)$ we always have $x \in F(T_2)$ with $p(x_0, x) \leq \Psi(L)$.
Similary, we can prove that for arbitrary $y_0 \in F(T_2)$, there exists a $y \in F(T_1)$ such that $p(y_0, y) \leq \Psi(L)$. Hence we can conclude that $\mathcal{H}_p(F(T_1), F(T_2)) \leq 2\Psi(L)$. □

Lemma 5.2. *Let* (X, p) *be a* $\mathcal{R}$*-complete partial metric space endowed with a locally* T*-transitive binary relation* $\mathcal{R}$ *and* $\{T_n : X \to C(X) \ \text{for all} \ n \in \mathbb{N}\}$ *be a sequence of multivalued mappings uniformly convergent to multivalued mapping* $T : X \to C(X)$. *If* $\{T_n\}$ *satisfies the contractive condition of Theorem 4.2 for every* $n \in \mathbb{N}$ *and* $\phi \in \Phi$ *then* T *also satisfy the contractive condition of Theorem 4.2.*

Proof. Since $\{T_n\}$ satisfy the contractive condition of Theorem 4.2 for every $n \in \mathbb{N}$, we have

$$\frac{1}{2}\mathcal{D}_p(x, T_n x) \leq p(x, y) \Longrightarrow \mathcal{H}_p(T_n x, T_n y) \leq \phi(\mathcal{M}(x, y)) \ \ \forall\, x, y \in X \ \ with \ \ (x, y) \in \mathcal{R}.$$

Since $\{T_n\}$ is uniformly convergent to T. Then using the properties of ϕ and taking limit $n \to \infty$ of the above contractive condition we get

$$\frac{1}{2}\mathcal{D}_p(x, Tx) \leq p(x, y) \Longrightarrow \mathcal{H}_p(Tx, Ty) \leq \phi(\mathcal{M}(x, y)) \ \ \forall\, x, y \in X \ \ with \ \ (x, y) \in \mathcal{R}.$$

which conclude that T also satisfy the contractive condition of Theorem 4.2. □

Next we are going to prove the stability result for our setting.

Theorem 5.3. *Let (X, p) be a $\mathcal{R}$-complete partial metric space endowed with a locally T-transitive binary relation $\mathcal{R}$ and $\{T_n : X \to C(X)$ for all $n \in \mathbb{N}\}$ be a sequence of multivalued mappings uniformly convergent to multivalued mapping $T : X \to C(X)$. Consider $\{T_n\}$ satisfies the contractive condition of Theorem 4.2 for every $n \in \mathbb{N}$ with $\phi \in \Phi$. Then*

$$\lim_{n\to\infty} \mathcal{H}_p(F(T_n), F(T)) = 0.$$

Then we conclude that fixed points sets of $\{T_n\}$ are stable.

Proof. By Lemma 5.2 we say that T satisfies the contractivity condition of Theorem 4.2. Assume that $L_n = \sup_{x\in X} \mathcal{H}_p(T_n x, Tx)$. Since T_n is uniformly convergent to T on X. Then

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \mathcal{H}_p(T_n x, Tx) = 0. \tag{18}$$

Then by Theorem 5.1, we get

$$\mathcal{H}_p(F(T_n), F(T)) \leq \Psi(L_n) \qquad \text{for every } n \in \mathbb{N}. \tag{19}$$

Then using (19) we get

$$\lim_{n\to\infty} \mathcal{H}_p(F(T_n), F(T)) \leq \lim_{n\to\infty} \Psi(L_n) = 0.$$

Then we have $\lim_{n\to\infty} \mathcal{H}_p(F(T_n), F(T)) = 0$, which amount to say that fixed points set of T_n are stable. □

6. Application to Fractal Space

Let (X, p) be a Partial metric space and $C(X)$ be the collection of all non-empty compact subsets of X. Define,

$$r_p(U, V) := \inf \{r(u, v) : u \in U, v \in V\}.$$

$$\rho_p(U, V) := \sup \left\{r_p(u, V) : u \in U\right\} = \sup_{u\in U} \inf_{v\in V} p(u, v).$$

So, then

$$\rho_p(V, U) := \sup \left\{r_p(v, U) : v \in V\right\} = \sup_{v\in V} \inf_{u\in U} p(v, u).$$

Then the Hausdorff metric induced by p is defined by

$$\mathcal{H}_p(U, V) = max \left\{\sup_{u\in U} p(u, V), \sup_{v\in V} p(v, U)\right\} = max \left\{\rho_p(U, V), \rho_p(V, U)\right\}$$

for all $U, V \in C(X)$ where $p(u, V) = \inf_{v\in V} p(u, v)$.

Hutchinson [36] and Barnsley [37] initiated an ingenious way to define and construct fractals as compact invariant subsets of an abstract complete metric space with respect to the union of contractions $T_i (i = 1, 2, \ldots n)$. Hutchinsons established that the operator

$$F(U) = T_1(U) \cup T_2(U) \cup \ldots \cup T_n(U), \quad U \subset X,$$

is a contraction with respect to the Hausdorff distance. Thus, the contraction mapping principle can be applied to the iteration of Hutchison operator F. Consequently, whatever the initial image is chosen to start the iteration under the Iterated Function System (IFS), for example U_0, the generated sequence

$$U_{k+1} = F(U_k) \quad k = 0, 1, \ldots$$

will tend towards a distinguish image, the attractor U_∞ of the IFS. Moreover, this image is invariant, i.e., $F(U_\infty) = U_\infty$.

Now, we construct a results in the form of Lemma 3.2 contained in [38] for the partial metric space under the Suzuki-generalized Ćirić type Matkowski-contractive mapping.

Lemma 6.1. *Let (X,p) be a partial metric space and $T : X \to X$ a continuous Suzuki-generalized Ćirić type Matkowski contractive mapping w.r.t. operator F, i.e.,*
$\frac{1}{2}\mathcal{H}_p(U,T(U)) \leq \mathcal{H}_p(U,V) \implies \mathcal{H}_p(F_T(U),F_T(V)) \leq \phi(\mathcal{M}_T(U,V))$ *for all* $U,V \in C(X)$ *where,*
$\mathcal{M}_T(U,V) = max\left\{\mathcal{H}_p(U,V),\mathcal{H}_p(U,T(U)),\mathcal{H}_p(V,T(V)),\frac{1}{2}[\mathcal{H}_p(U,T(V))+\mathcal{H}_p(V,T(U))]\right\}$. *Then* $F_T : C(X) \to C(X)$ *is also a Suzuki-generalized Ćirić type Matkowski contractive mapping, where* $\forall\ Z \in C(X),\ \ F_T(Z) = T(Z)$.

Proof. Let $U,V \in C(X)$ and any point $u_0 \in U$. Using the compactness of U, there always exists $v_{u_0} \in V$ such that $\inf_{v\in V} p(u_0,v) = p(u_0,v_{u_0})$. Then we have

$$\inf_{v\in V}\phi(p(u_0,v)) \leq \phi(p(u_0,v_{u_0})) = \phi\left(\inf_{v\in V}p(u_0,v)\right).$$

Because $\phi : [0,\infty) \to [0,\infty)$ is increasing, it follows that

$$\phi\left(\inf_{v\in V}p(u_0,v)\right) \leq \phi\left(\sup_{u\in U}\inf_{v\in V}p(u,v)\right) \leq \phi\left(\mathcal{H}_p(U,V)\right).$$

Since u_0 was arbitrary, then $\sup_{u\in U}\phi\left(\inf_{v\in V}p(u,v)\right) \leq \phi\left(\mathcal{H}_p(U,V)\right)$.
Then for all $u \in U$ and $v \in V$, we have

$$\sup_{u\in U}\inf_{v\in V}\phi(p(u,v)) \leq \sup_{u\in U}\phi(\inf_{v\in V}p(u,v)) \leq \phi(\mathcal{M}_T(U,V)).$$
$$\sup_{v\in V}\inf_{u\in U}\phi(p(u,v)) \leq \sup_{v\in V}\phi(\inf_{u\in U}p(u,v)) \leq \phi(\mathcal{M}_T(U,V)).$$

Further for all $u \in U$ and $v \in V$,

$$\frac{1}{2}p(u,Tu) \leq p(u,v) \implies \frac{1}{2}\rho_p(U,T(U)) \leq \frac{1}{2}\mathcal{H}_p(U,T(U)) \leq \mathcal{H}_p(U,V).$$

and

$$\frac{1}{2}p(v,Tv) \leq p(u,v) \implies \frac{1}{2}\rho_p(V,T(V)) \leq \frac{1}{2}\mathcal{H}_p(V,T(V)) \leq \mathcal{H}_p(U,V).$$

Next, we have

$$\rho_p\left(F_T(U),F_T(V)\right) = \sup_{Tu\in F(U)}\inf_{Tv\in F(V)}p(Tu,Tv) = \sup_{u\in U}\inf_{v\in V}p(Tu,Tv).$$

Since T is Suzuki-generalized Ćirić type Matkowski mapping, we have $\frac{1}{2}\rho_p(U,T(U)) \leq \frac{1}{2}\mathcal{H}_p(U,T(U))$, then the above inequality reduces to

$$\rho_p\left(F_T(U),F_T(V)\right) = \sup_{u\in U}\inf_{v\in V}p(Tu,Tv) \leq \phi(\mathcal{M}_T(U,V)).$$

Similarly, $\frac{1}{2}\rho_p(V,T(V)) \leq \frac{1}{2}\mathcal{H}_p(V,T(V))$ we always have,

$$\rho_p\left(F_T(V),F_T(U)\right) = \sup_{v\in V}\inf_{u\in U}p(Tu,Tv) \leq \phi(\mathcal{M}_T(U,V)).$$

Since Hausdorff metric is symmetric $\left(i.e., \mathcal{H}_p(U,V) = \mathcal{H}_p(V,U)\right)$ we get

$$\mathcal{H}_p\left(F_T(U),F_T(V)\right) = max\left\{\rho_p\left(F_T(U),F_T(V)\right),\rho_p\left(F_T(V),F_T(U)\right)\right\},$$

hence we always have

$$\frac{1}{2}\mathcal{H}_p(U,T(U)) \leq \mathcal{H}_p(U,V) \implies \mathcal{H}_p\left(F_T(U),F_T(V)\right) \leq \phi(\mathcal{M}_T(U,V))$$

for all $U,V \in C(X)$. Therefore, F_T is a Suzuki-generalized Ćirić type Matkowski contractive mapping.

Lemma 6.2. *[39] Let (X, p) be a complete partial metric space. Then $(C(X), \mathcal{H}_p)$ is also a complete partial metric space.*

Lemma 6.3. *Let (X, p) be a partial metric space and $T_n : C(X) \to C(X)(n = 1, 2, \ldots, p)$ continuous Suzuki-generalized Ćirić type Matkowski contractive mapping, i.e., for all $U, V \in C(X)$,*

$$\tfrac{1}{2}\mathcal{H}_p(U, T_n(U)) \le \mathcal{H}_p(U, V) \implies \mathcal{H}_p(T_n(U), T_n(V)) \le \phi_n(\mathcal{M}_{T_n}(U, V)).$$

Define $T : C(X) \to C(X)$ by $T(U) = T_1(U) \cup T_2(U) \cup \cdots \cup T_p(U) = \cup_{n=1}^{p} T_n(U)$ for each $U \in C(X)$. Then T also satisfies

$$\tfrac{1}{2}\mathcal{H}_p(U, T(U)) \le \mathcal{H}_p(U, V) \implies \mathcal{H}_p\left(T(U), T(V)\right) \le \lambda(\mathcal{M}_T(U, V))$$

for all $U, V \in C(X)$, where $\lambda = max\{\phi_n : n = 1, 2, \ldots, p\}$.

Proof. We prove the above lemma by the principle of mathematical induction. For $n = 1$ the result is obvious. For $n = 2$, we have

$$\begin{aligned}\mathcal{H}_p(T(U), T(V)) &= \mathcal{H}_p(T_1(U) \cup T_2(U), T_1(V) \cup T_2(V)) \\ &\le max\left\{\mathcal{H}_p(T_1(U), T_1(V)), \mathcal{H}_p(T_1(V), T_2(V))\right\}.\end{aligned}$$

Since each T_1 and T_2 are Suzuki-generalized Ćirić type Matkowski contractive, that is

$$\begin{aligned}\tfrac{1}{2}\mathcal{H}_p(U, T_1(U)) \le \mathcal{H}_p(U, V) &\implies \mathcal{H}_p(T_1(U), T_1(V)) \le \phi_1(\mathcal{M}_{T_1}(U, V)) \\ \tfrac{1}{2}\mathcal{H}_p(U, T_2(U)) \le \mathcal{H}_p(U, V) &\implies \mathcal{H}_p(T_2(U), T_2(V)) \le \phi_2(\mathcal{M}_{T_2}(U, V)),\end{aligned}$$

then we have

$$\begin{aligned}\mathcal{H}_p(T(U), T(V)) &\le max\left\{\phi_1(\mathcal{M}_{T_1}(U, V)), \phi_2(\mathcal{M}_{T_2}(U, V))\right\} \\ &= \lambda(max\{\mathcal{H}_p(U, V), \mathcal{H}_p(U, T_1(U) \cup T_2(U)), \mathcal{H}_p(V, T_1(V) \cup T_2(V)), \\ &\qquad \frac{1}{2}\left[\mathcal{H}_p(U, T_1(V) \cup T_2(V)) + \mathcal{H}_p(V, T_1(U) \cup T_2(U))\right]\}) \\ &= \lambda\left(max\left\{\mathcal{H}_p(U, V), \mathcal{H}_p(U, TU), \mathcal{H}_p(V, TV)\right\}\right) \\ &= \lambda(\mathcal{M}_T(U, V)),\end{aligned}$$

where $\lambda = max\{\phi_1, \phi_2\}$.
Now as a consequences of Theorem 4.2 and Lemmas 6.1 and 6.3, we get the following result in fractal spaces.

Theorem 6.4. *Let (X, p) be a complete partial metric space and $T_n : C(X) \to C(X)$ continuous Suzuki-generalized Ćirić type Matkowski contractive mapping. Then the transformation $T : C(X) \to C(X)$ defined by $T(U) = \cup_{n=1}^{p} T_n(U)$ for each $U \in C(X)$ satisfying the following condition*

$$\frac{1}{2}\mathcal{H}_p(U, T(U)) \le \mathcal{H}_p(U, V) \implies \mathcal{H}_p\left(T(U), T(V)\right) \le \lambda(\mathcal{M}(U, V)),$$

for all $U, V \in C(X)$, where $\lambda = max\{\phi_n : n = 1, 2, \ldots, p\}$.
Moreover,
(1): T has a unique fixed point U in $C(X)$; and
(2): $\lim_{n\to\infty} T^n(V) = U$ for all $V \in C(X)$.

Proof. Define the binary relation as

$$\mathcal{R} = \Big\{(U, V) \in C(X) \times C(X) \quad \text{such that} \quad U \subseteq V\Big\}.$$

Then T is well-defined and $\mathcal{R}$ on $C(X)$ is T-closed. As given that one of $T_n(U) \subseteq U$ then $T(U) = \cup_{n=1}^{p} T_n(U) \subseteq U$ implies that $(U, T(U)) \in \mathcal{R}$, which amount to say that $X(T, \mathcal{R})$ is non-empty. By Lemma 6.3 we can say that the mapping T satisfies Suzuki-generalized Ćirić type Matkowski contractive mapping for any $(U, V) \in \mathcal{R}$. Also T is $\mathcal{R}$-continuous being union of continuous map. Then by Theorem 4.2 we can say that T has fixed point. Then by the help of Theorem 3.1 of [38] we can say that T has a fixed point U in $C(X)$ and $\lim_{n\to\infty} T^n(V) = U$ for all $V \in C(X)$. □

Remark 6.5. *If we see the results of Rhoades [40] in setting of partial metric space, Theorem 4.2 and 6.4 generalizes certain results of [36, 38].*

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