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Estimates for the zeros of a polynomial using matrix inequalities

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Abstract. In this paper, we apply several matrix inequalities to the generalized companion matrix of monic-polynomial and thereby obtain some new estimates for the moduli of their zeros.

1. Introduction

In the realm of applied mathematics, polynomial is a widely used mathematical entity. It finds its application in almost every domain of Science. Finding circular disks in the complex plane $\mathbb{C}$ containing all the zeros of a polynomial is a classical problem and have attracted the attention of numerous mathematicians. One can refer to [13] for a comprehensive account on classical results on this topic.

It is well-known that matrix methods can be used to obtain classical zero-bounds for polynomials (see [6, p. 316]). The roots of a given polynomial could be found as the eigenvalues of a companion matrix.

Linden [10] used (generalized) companion matrices, which are based on special multiplicative decompositions of the coefficients of the polynomial, to obtain estimates for the zeros of the polynomial $p(z)$ mainly by the application of Gersgorin's theorem to the companion matrices or by computing the singular values of the companion matrices.

Let

$$p(z) = z^n - a_1 z^{n-1} - a_2 z^{n-2} - \ldots - a_n \tag{1}$$

be a monic polynomial of degree n with complex coefficients, then one of the (generalized) companion matrix due to Linden can be illustrated in the following theorem.

Theorem 1.1. *[10] Let $p(z)$ be a monic polynomial of degree $n \geq 1$ given by* (1). *Let there exist complex numbers $c_1, c_2, \ldots, c_n$ and $b_1(\neq 0), b_2, \ldots, b_{n-1}$ such that*

$$\begin{cases} a_1 = & c_1, \\ a_2 = & c_2 b_1, \\ a_3 = & c_3 b_2 b_1, \\ \vdots & \\ a_n = & c_n b_{n-1} b_{n-2} \cdots b_2 b_1, \end{cases} \tag{2}$$

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and

$$C(p) = \begin{bmatrix} 0 & b_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & b_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & b_1 \\ c_n & c_{n-1} & \cdots & c_2 & c_1 \end{bmatrix}. \tag{3}$$

Then $\det(zI_n - C(p)) = p(z)$ *where* I_n *denotes the* $n \times n$ *identity matrix.*

In ([1], [3], [4]), using results on the numerical range and the numerical radius of the Frobenius companion matrix of $f(z)$, bounds for the zeros of $f(z)$ were given: In ([1], [4]) an estimate of the numerical radius of a matrix is applied to the Frobenius companion matrix, and in [3] a formula for the numerical radius of a matrix of rank one is applied to the Frobenius companion matrix which was decomposed in the sum of a matrix of rank one and a right shift matrix. Linden [12] extended these methods and gave further estimates for the zeros of $f(z)$ using properties of the numerical ranges and the numerical radii of some other types of companion matrices of $f(z)$. One such type of companion matrix is given by (3).
Let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices. For $A \in M_n(\mathbb{C})$, let $r(A)$, $w(A)$, and $\|A\|$ denote the spectral radius, the numerical radius and the spectral norm of A respectively. It is known that [6]

$$|\lambda_j(A)| \leq r(A) \leq w(A) \leq \|A\| = s_1(A),$$

where $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$ and $s_1(A), s_2(A), \ldots, s_n(A)$ are respectively the eigenvalues and singular values of A and are arranged so that $|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)|$ and $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. Recall that $s_j^2(A) = \lambda_j(A^*A) = \lambda_j(AA^*)$ for $j = 1, 2, \ldots, n$.

In this paper, we employ certain matrix inequalities concerning spectral radius, numerical radius and spectral norm to the generalized companion matrix of Linden to estimate new bounds for the zeros of a polynomial. The technique is similar to that of [8], [9], [14] and [15].

2. Lemmas

To achieve our goal, we need the following two lemmas due to Hou and Du [7].

Lemma 2.1. *Let* $A \in M_n(\mathbb{C})$ *be partitioned as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{ij} *is an* $n_i \times n_j$ *matrix for* $i, j = 1, 2$ *with* $n_1 + n_2 = n$. *If*

$$\widetilde{A} = \begin{bmatrix} \|A_{11}\| & \|A_{12}\| \\ \|A_{21}\| & \|A_{22}\| \end{bmatrix},$$

then $r(A) \leq r(\tilde{A})$.

Lemma 2.2. *Let* $A \in M_k(\mathbb{C})$, $B \in M_{k\times m}(\mathbb{C})$, $C \in M_{m\times k}(\mathbb{C})$ *and* $D \in M_m(\mathbb{C})$ *and let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. *Then*

$$\begin{aligned} w(T) &\leq w\left(\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix}\right) \\ &= \frac{1}{2}\left(\|A\| + \|D\| + \sqrt{(\|A\| - \|D\|)^2 + (\|B\| + \|C\|)^2}\right). \end{aligned}$$

3. Main Results

Through out this paper $c_1, c_2, \ldots, c_n$ and $b_1, b_2, \ldots, b_{n-1}$ denote the complex numbers satisfying (2). Our first result is obtained by using a property of spectral radius.

Theorem 3.1. *All the zeros of polynomial given by* (1) *of degree* $n \geq 3$ *lie in*

$$|z| \leq \frac{1}{2}\left\{b + |c_1| + \sqrt{(b - |c_1|)^2 + 4|b_1|\left(\sum_{j=2}^{n} |c_j|^2\right)^{1/2}}\right\},$$

where $b = \max\{|b_2|, |b_3|, \ldots, |b_{n-1}|\}$.

Proof. Partition the companion matrix $C(p)$ of $p(z)$ defined by (3) as:

$$C(p) = \left[\begin{array}{cccc|c} 0 & b_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & b_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & b_1 \\ \hline c_n & c_{n-1} & \cdots & c_2 & c_1 \end{array}\right] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

$$\text{where} \quad C_{11} = \begin{bmatrix} 0 & b_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & b_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & b_2 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad C_{12} = [0 \quad 0 \quad \ldots \quad 0 \quad b_1]^t,$$

$$C_{21} = [c_n \quad c_{n-1} \quad \ldots \quad c_2] \quad \text{and} \quad C_{22} = [c_1].$$

Let $\widetilde{C}(p) = \begin{bmatrix} ||C_{11}|| & ||C_{12}|| \\ ||C_{21}|| & ||C_{22}|| \end{bmatrix}$, therefore,

$$\widetilde{C}(p) = \begin{bmatrix} b & |b_1| \\ \left(\sum_{j=2}^{n} |c_j|^2\right)^{1/2} & |c_1| \end{bmatrix}, \qquad \text{where} \quad b = \max\{|b_2|, |b_3|, \ldots, |b_{n-1}|\}.$$

Now, by Lemma 2.1, we have

$$\begin{aligned} r(C(p)) &\leq r(\widetilde{C}(p)) \\ &= \frac{1}{2}\left\{b + |c_1| + \sqrt{(b - |c_1|)^2 + 4|b_1|\left(\sum_{j=2}^{n} |c_j|^2\right)^{1/2}}\right\}. \end{aligned}$$

Consequently all the zeros of $p(z)$ lie in

$$|z| \leq \frac{1}{2}\left\{b + |c_1| + \sqrt{(b - |c_1|)^2 + 4|b_1|\left(\sum_{j=2}^{n} |c_j|^2\right)^{1/2}}\right\}.$$

This completes the proof of Theorem 3.1. □

By suitably choosing the parameters c_j's and b_j's satisfying (2), new zero-bounds can be obtained, here we make some special choices.

Let $b_j = a_1 \neq 0$ for $j = 2, 3, \dots, n-1$, then $c_j = \dfrac{a_j}{a_1^{j-2} b_1}$ for $j = 2, 3, \dots, n-1$ and $c_n = \dfrac{a_n}{a_1^{n-2} b_1}$. By using these values in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Let $p(z)$ be a polynomial as given by* (1) *of degree $n \geq 3$ such that $a_1 \neq 0$, then all its zeros lie in*

$$|z| \leq |a_1| + \left(\sum_{j=2}^{n} \left| \frac{a_j}{a_1^{j-2}} \right|^2 \right)^{1/4}.$$

By choosing $b_1 = b_2 = \cdots = b_{n-1} = 1$ and $c_j = a_j$, $j = 1, 2, \dots, n$ in Theorem 3.1, the following result due to Kittaneh [8] can be obtained.

Corollary 3.3. *All the zeros of polynomial given by* (1) *of degree $n \geq 3$ lie in*

$$|z| \leq \frac{1}{2} \left\{ |a_1| + 1 + \sqrt{(|a_1| - 1)^2 + 4 \left(\sum_{j=2}^{n} |a_j|^2 \right)^{1/2}} \right\}.$$

Example 1. Let $p(z) = z^3 - z^2 - 2z - 0.5$ be a polynomial. Here, $a_1 = 1$, $a_2 = 2$ and $a_3 = 0.5$.
Let $b_1 = 1$, $b_2 = \frac{1}{2}$ and $c_1 = 1$, $c_2 = 2$, $c_3 = 1$, such that

$$\begin{aligned} a_1 &= c_1, \\ a_2 &= c_2 b_1, \\ a_3 &= c_3 b_2 b_1. \end{aligned}$$

Then by applying Theorem 3.1 to $p(z)$, it follows that all the zeros of $p(z)$ lie in the disc $|z| \leq 2.2660$. Whereas if we apply the result of Kittaneh [8] (Theorem 1) to $p(z)$, it follows that all the zeros of $p(z)$ lie in the disc $|z| \leq 2.4357$.

If $a_j \neq 0$, $j = 1, 2, \dots, n$, then by taking $b_1 = c_1 = a_1$, $b_j = c_j = \dfrac{a_j}{a_{j-1}}$, $j = 2, 3, \dots, n-1$, $c_n = \dfrac{a_n}{a_{n-1}}$, in Theorem 3.1, we obtain the following result:

Corollary 3.4. *All the zeros of polynomial given by* (1) *of degree $n \geq 3$ lie in*

$$|z| \leq \frac{1}{2} \left\{ b + |a_1| + \sqrt{(b - |a_1|)^2 + 4|a_1| \left(\sum_{j=2}^{n} \left| \frac{a_j}{a_{j-1}} \right|^2 \right)^{1/2}} \right\},$$

where $b = \max \left\{ \left| \frac{a_2}{a_1} \right|, \left| \frac{a_3}{a_2} \right|, \dots, \left| \frac{a_{n-1}}{a_{n-2}} \right| \right\}$.

Next, if we choose $b_1 = b_2 = \cdots = b_{n-1} = b = \max\limits_{2 \leq j \leq n} |a_j|^{1/j}$, $c_j = \dfrac{a_j}{b^{j-1}}$, $j = 1, 2, \dots, n$, in Theorem 3.1, we get the following result:

Corollary 3.5. *All the zeros of polynomial given by* (1) *of degree $n \geq 3$ lie in*

$$|z| \leq \frac{1}{2} \left\{ b + |a_1| + \sqrt{(b - |a_1|)^2 + 4b \left(\sum_{j=2}^{n} \left| \frac{a_j}{b^{j-1}} \right|^2 \right)^{1/2}} \right\},$$

where $b = \max\limits_{2 \leq j \leq n} |a_j|^{1/j}$.

Our next result is obtained by using some properties of the numerical radius of a matrix.

Theorem 3.6. *All the zeros of polynomial given by* (1) *of degree* $n \geq 3$ *lie in*

$$|z| \leq \frac{1}{2}\left\{|c_1| + \sqrt{\sum_{j=1}^{n} |c_j|^2} + b + \sqrt{b^2 + |b_1|^2}\right\},$$

where $b = \max\{|b_2|, |b_3|, \ldots, |b_{n-1}|\}$.

Proof. The companion matrix $C(p)$ of $p(z)$ given by (3) can be expressed as

$$C(p) = Q + R,$$

$$\text{where} \quad Q = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \\ c_n & c_{n-1} & \cdots & c_2 & c_1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0 & b_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & b_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & b_1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Now partition Q as

$$Q = \left[\begin{array}{cccc|c} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \\ \hline c_n & c_{n-1} & \cdots & c_2 & c_1 \end{array}\right] = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix},$$

$$\text{where} \quad D_{11} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad D_{12} = [0 \quad 0 \quad \ldots \quad 0 \quad 0]^t,$$

$$D_{21} = [c_n \quad c_{n-1} \quad \ldots \quad c_3 \quad c_2] \quad \text{and} \quad D_{22} = [c_1].$$

Invoking Lemma 2.2, we get

$$\begin{aligned} w(Q) \leq & w\left(\begin{bmatrix} \|D_{11}\| & \|D_{12}\| \\ \|D_{21}\| & \|D_{22}\| \end{bmatrix}\right) \\ &= \frac{1}{2}\left\{|c_1| + \sqrt{|c_1|^2 + \left(\sum_{j=2}^{n} |c_j|^2\right)}\right\} \\ &= \frac{1}{2}\left\{|c_1| + \sqrt{\sum_{j=1}^{n} |c_j|^2}\right\}. \end{aligned}$$

Partition R as

$$R = \left[\begin{array}{cccc|c} 0 & b_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & b_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & b_1 \\ \hline 0 & 0 & \cdots & 0 & 0 \end{array}\right] = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},$$

$$\text{where}\quad E_{11} = \begin{bmatrix} 0 & b_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & b_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & b_2 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad E_{12} = [0 \quad 0 \quad \ldots \quad 0 \quad b_1]^t,$$

$$E_{21} = [0 \quad 0 \quad \ldots \quad 0] \quad \text{and} \quad E_{22} = [0].$$

Employing Lemma 2.2 again , we get

$$\begin{aligned} w(R) &\le w\left(\begin{bmatrix} \|E_{11}\| & \|E_{12}\| \\ \|E_{21}\| & \|E_{22}\| \end{bmatrix}\right) \\ &= \frac{1}{2}\left\{b + \sqrt{b^2 + |b_1|^2}\right\}, \text{where} \quad b = \max\{|b_2|, |b_3|, \ldots, |b_{n-1}|\}. \end{aligned}$$

Now by using sub-additive property of numerical radius, we have

$$\begin{aligned} w(C(p)) &= w(Q + R) \\ &\le w(Q) + w(R) \\ &= \frac{1}{2}\left\{|c_1| + \sqrt{\sum_{j=1}^{n} |c_j|^2} + b + \sqrt{b^2 + |b_1|^2}\right\}. \end{aligned}$$

Accordingly, all the zeros of polynomial $p(z)$ lie in

$$|z| \le \frac{1}{2}\left\{|c_1| + \sqrt{\sum_{j=1}^{n} |c_j|^2} + b + \sqrt{b^2 + |b_1|^2}\right\}.$$

This completes the proof of Theorem 3.6. □

By suitably choosing the parameters c_j's and b_j's satisfying (2), new zero-bounds can be obtained, here we make some special choices.

Let $b_j = a_1 \neq 0$ for $j = 2, 3, \ldots, n-1$, then $c_j = \dfrac{a_j}{a_1^{j-2} b_1}$ for $j = 2, 3, \ldots, n-1$ and $c_n = \dfrac{a_n}{a_1^{n-2} b_1}$. By using these values in Theorem 3.6, we obtain the following corollary.

Corollary 3.7. *Let $p(z)$ be a polynomial as given by* (1) *of degree $n \ge 3$ such that $a_1 \neq 0$, then all its zeros lie in*

$$|z| \le |a_1| + \frac{1}{2}\left\{\sqrt{|a_1|^2 + \sum_{j=2}^{n} \left|\frac{a_j}{a_1^{j-2} b_1}\right|^2} + \sqrt{|a_1|^2 + |b_1|^2}\right\}.$$

By choosing $b_1 = b_2 = \cdots = b_{n-1} = 1$ and $c_j = a_j$, $j = 1, 2, \ldots, n$ in Theorem 3.6, the following result can be obtained.

Corollary 3.8. *All the zeros of polynomial given by* (1) *of degree $n \ge 3$ lie in*

$$|z| \le \frac{1}{2}\left\{|a_1| + \sqrt{\sum_{j=1}^{n} |a_j|^2} + \sqrt{2} + 1\right\}.$$

If $a_j \neq 0$, $j = 1, 2, \ldots, n$, then by taking $b_1 = c_1 = a_1$, $b_j = c_j = \dfrac{a_j}{a_{j-1}}$, $\quad j = 2, 3, \ldots, n-1$, $c_n = \dfrac{a_n}{a_{n-1}}$, in Theorem 3.6, we obtain the following result:

Corollary 3.9. *All the zeros of polynomial given by* (1) *of degree* $n \geq 3$ *lie in*

$$|z| \leq \frac{1}{2}\left\{|a_1| + \sqrt{|a_1|^2 + \left(\sum_{j=2}^{n}\left|\frac{a_j}{a_{j-1}}\right|^2\right)} + b + \sqrt{b^2 + |b_1|^2}\right\},$$

where $b = \max\left\{\left|\frac{a_2}{a_1}\right|, \left|\frac{a_3}{a_2}\right|, \ldots, \left|\frac{a_{n-1}}{a_{n-2}}\right|\right\}$.

Next, if we choose $b_1 = b_2 = \cdots = b_{n-1} = b = \max\limits_{2 \leq j \leq n} |a_j|^{1/j}$, $c_j = \dfrac{a_j}{b^{j-1}}$, $\quad j = 1, 2, \ldots, n$, in Theorem 3.6, we get the following result:

Corollary 3.10. *All the zeros of polynomial given by* (1) *of degree* $n \geq 3$ *lie in*

$$|z| \leq \frac{1}{2}\left\{|a_1| + \sqrt{\left(\sum_{j=1}^{n}\left|\frac{a_j}{b^{j-1}}\right|^2\right)} + \left(1 + \sqrt{2}\right)b\right\},$$

where $b = \max\limits_{2 \leq j \leq n} |a_j|^{1/j}$.

Example 2. Let $p(z) = z^3 - 3z^2 - 4z + 0.5$ be a polynomial. Here, $a_1 = 3$, $a_2 = 4$ and $a_3 = -0.5$. Let $b_1 = 2$, $b_2 = -0.5$ and $c_1 = 3$, $c_2 = 2$, $c_3 = \frac{1}{2}$, such that

$$\begin{aligned} a_1 &= c_1, \\ a_2 &= c_2 b_1, \\ a_3 &= c_3 b_2 b_1. \end{aligned}$$

Then by applying Theorem 3.6 to $p(z)$, it follows that all the zeros of $p(z)$ lie in the disc $|z| \leq 4.6007$. Whereas if we apply the result of Linden [11] to $p(z)$, it follows that all the zeros of $p(z)$ lie in the disc $|z| \leq 5.0207$.

Our third result is obtained by using again a property of numerical radius.

Theorem 3.11. *All the zeros of polynomial given by* (1) *of degree* $n \geq 3$ *lie in*

$$|z| \leq \frac{1}{2}\left\{b + |c_1| + \sqrt{(b - |c_1|)^2 + \left(|b_1| + \sqrt{\sum_{j=2}^{n}|c_j|^2}\right)^2}\right\},$$

where $b = \max\{|b_2|, |b_3|, \ldots, |b_{n-1}|\}$.

Proof. Let $u = [0 \quad 0 \quad \ldots \quad 0 \quad b_1]^t$ be the $(n-1)$ column vector and $v = [c_n \quad c_{n-1} \quad \ldots \quad c_3 \quad c_2]$ be the $(n-1)$ row vector. Then the companion matrix $C(p)$ of $p(z)$ given by (3) can be expressed as

$$C(p) = \begin{bmatrix} T & u \\ v & c_1 \end{bmatrix}, \quad \text{where} \quad T = \begin{bmatrix} 0 & b_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & b_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & b_2 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n-1 \times n-1}.$$

Now in the light of Lemma 2.2, we have

$$w(C(p)) \le w\left(\begin{bmatrix} ||T|| & ||u|| \\ ||v|| & ||c_1|| \end{bmatrix}\right)$$
$$= \frac{1}{2}\left\{b + |c_1| + \sqrt{(b - |c_1|)^2 + \left(|b_1| + \sqrt{\sum_{j=2}^{n} |c_j|^2}\right)^2}\right\},$$

where $b = \max\{|b_2|, |b_3|, \ldots, |b_{n-1}|\}$.
Hence all the zeros of $p(z)$ lie in

$$|z| \le \frac{1}{2}\left\{b + |c_1| + \sqrt{(b - |c_1|)^2 + \left(|b_1| + \sqrt{\sum_{j=2}^{n} |c_j|^2}\right)^2}\right\}.$$

This completes the proof of Theorem 3.11. □

By suitably choosing the parameters c_j's and b_j's satisfying (2), new zero-bounds can be obtained, here we make some special choices.

Let $b_j = a_1 \neq 0$ for $j = 2, 3, \ldots, n-1$, then $c_j = \dfrac{a_j}{a_1^{j-2} b_1}$ for $j = 2, 3, \ldots, n-1$ and $c_n = \dfrac{a_n}{a_1^{n-2} b_1}$. By using these values in Theorem 3.11, we obtain the following corollary.

Corollary 3.12. *Let $p(z)$ be a polynomial as given by* (1) *of degree $n \ge 3$ such that $a_1 \neq 0$, then all its zeros lie in*

$$|z| \le |a_1| + \frac{1}{2}\left\{\sqrt{\left(|b_1| + \sqrt{\sum_{j=2}^{n} \left|\frac{a_j}{a_1^{j-2} b_1}\right|^2}\right)^2}\right\}$$

By choosing $b_1 = b_2 = \cdots = b_{n-1} = 1$ and $c_j = a_j$, $j = 1, 2, \ldots, n$ in Theorem 3.11, the following result can be obtained.

Corollary 3.13. *All the zeros of polynomial given by* (1) *of degree $n \ge 3$ lie in*

$$|z| \le \frac{1}{2}\left\{|a_1| + 1 + \sqrt{(|a_1| - 1)^2 + \left(1 + \sqrt{\sum_{j=2}^{n} |a_j|^2}\right)^2}\right\}.$$

If $a_j \neq 0$, $j = 1, 2, \ldots, n$, then by taking $b_1 = c_1 = a_1$, $b_j = c_j = \dfrac{a_j}{a_{j-1}}$, $j = 2, 3, \ldots, n-1$, $c_n = \dfrac{a_n}{a_{n-1}}$, in Theorem 3.11, we obtain the following result:

Corollary 3.14. *All the zeros of polynomial given by* (1) *of degree $n \ge 3$ lie in*

$$|z| \le \frac{1}{2}\left\{b + |a_1| + \sqrt{(b - |a_1|)^2 + \left(|a_1| + \sqrt{\sum_{j=2}^{n} \left|\frac{a_j}{a_{j-1}}\right|^2}\right)^2}\right\},$$

where $b = \max\left\{\left|\frac{a_2}{a_1}\right|, \left|\frac{a_3}{a_2}\right|, \ldots, \left|\frac{a_{n-1}}{a_{n-2}}\right|\right\}$.

Next, if we choose $b_1 = b_2 = \cdots = b_{n-1} = b = \max_{2\le j\le n} |a_j|^{1/j}$, $c_j = \frac{a_j}{b^{j-1}}$, $j = 1, 2, \ldots, n$, in Theorem 3.11, we get the following result:

Corollary 3.15. *All the zeros of polynomial given by* (1) *of degree* $n \geq 3$ *lie in*

$$|z| \leq \frac{1}{2}\left\{b + |a_1| + \sqrt{(b - |a_1|)^2 + \left(b + \sqrt{\sum_{j=2}^{n}\left|\frac{a_j}{b^{j-1}}\right|^2}\right)^2}\right\}.$$

where $b = \max_{2\le j\le n} |a_j|^{1/j}$.

Example 3. Let $p(z) = z^3 - 3z^2 - 4z - 6$ be a polynomial. Here, $a_1 = 3$, $a_2 = 4$ and $a_3 = 6$.
Let $b_1 = 2$, $b_2 = 1$ and $c_1 = 3$, $c_2 = 2$, $c_3 = 3$, such that

$$\begin{aligned} a_1 &= c_1, \\ a_2 &= c_2 b_1, \\ a_3 &= c_3 b_2 b_1. \end{aligned}$$

Then by applying Theorem 3.11 to $p(z)$, it follows that all the zeros of $p(z)$ lie in the disc $|z| \leq 4.9758$. Whereas if we apply the result of Fujii and Kubo [3] to $p(z)$, it follows that all the zeros of $p(z)$ lie in the disc $|z| \leq 6.1122$.

Now we set up an example where we compare our results with the classical results concerning to the zero bounds of polynomials obtained by famous mathematicians.
Example 4. Consider the polynomial equation $p(z) = z^4 - z^2 - z + 3 = 0$. Here, $a_1 = 0$, $a_2 = 1$, $a_3 = 1$, and $a_4 = -3$.
Let $b_1 = 2$, $b_2 = 1$, $b_3 = \frac{1}{2}$ and $c_1 = 0$, $c_2 = \frac{1}{2}$, $c_3 = \frac{1}{2}$, $c_4 = -3$ such that

$$\begin{aligned} a_1 &= c_1, \\ a_2 &= c_2 b_1, \\ a_3 &= c_3 b_2 b_1, \\ a_4 &= c_4 b_3 b_2 b_1. \end{aligned}$$

Then the upper bounds for the zeros of this polynomial equation $z^4 - z^2 - z + 3 = 0$ estimated by our results are much better than the estimates obtained by different mathematicians as shown in table below:

Bound	Value	Bound	Value
Theorem 3.1	3.0326	Cauchy [5]	4
Theorem 3.2	3.1591	Montel [2]	5
Theorem 3.3	3.0898	Carmichael and Mason [5]	3.4641

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