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Convergence rate of precise asymptotics in the Baum-Katz laws of large numbers

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Abstract. Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables and $S_n = X_1 + X_2 + \cdots + X_n$. In the present paper, we study the precise asymptotics for the following series

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|S_n| \geq \varepsilon n^{1/p}\right) \quad \text{for all} \ \ \varepsilon > 0,$$

where $1 \leq p < 2$, and consider the convergence rate of the series, which extends the works in He and Xie [9].

1. Introduction

Hsu and Robbins [11] introduced the following concept of complete convergence. A sequence $\{X_n, n \geq 1\}$ of random variables is said to converge completely to a constant C if

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n - C| \geq \varepsilon) < \infty \ \ \text{for all} \ \ \varepsilon > 0.$$

By the Borel-Cantelli lemma, $X_n \to C$ completely implies $X_n \xrightarrow{a.s.} C$. Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables and $S_n = X_1 + X_2 + \cdots + X_n$. Hsu and Robbins [11] proved that if $\mathbb{E}X = 0$ and $\mathbb{E}X^2 < \infty$, then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|S_n| \geq \varepsilon n\right) < \infty. \tag{1.1}$$

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The converse was proved by Erdős [4]. The result of Hsu-Robbins-Erdős is a fundamental theorem in probability theory and has been generalized and extended in several directions. Spitzer [15] proved that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\left(|S_n| \geq \varepsilon n\right) < \infty \tag{1.2}$$

if and only if $\mathbb{E}X = 0$ and $\mathbb{E}|X| < \infty$. Katz [12] and Baum and Katz [1] generalized the work of Spitzer [15] and obtained that for $0 < p < 2$ and $r \geq p$,

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2} \mathbb{P}\left(|S_n| \geq \varepsilon n^{1/p}\right) < \infty, \ \text{ for } \ \varepsilon > 0 \tag{1.3}$$

if and only if $\mathbb{E}|X|^r < \infty$ and when $r \geq 1$, $\mathbb{E}X = 0$.

Heyde [10] studied the limit behavior of the series in (1.1) as $\varepsilon \to 0$. If $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = \sigma^2$, then

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} \mathbb{P}\left(|S_n| \geq \varepsilon n\right) = \sigma^2. \tag{1.4}$$

Spătaru [14] considered (1.2) and proved that if $\mathbb{E}X = 0$ and $\mathbb{E}|X| < \infty$, and the distribution of the random variable X belongs to the domain of attraction of a nondegenerate stable distribution G with characteristic exponent $1 < \alpha \leq 2$, i.e.,

$$\frac{S_n}{b_n} - a_n \xrightarrow{weak} G \ \text{ as } \ n \to \infty,$$

for suitable a_n and $b_n > 0$, then

$$\lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}(|S_n| \geq \varepsilon n) = \frac{\alpha}{\alpha - 1}.$$

Gut and Spătaru [6] studied (1.3) and obtained that if $\mathbb{E}X = 0$ and $\mathbb{E}|X| < \infty$, and the distribution of the random variable X belongs to the normal domain of attraction of a nondegenerate stable distribution G with characteristic exponent $1 < \alpha \leq 2$, then for $1 \leq p < r < \alpha$, we have

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{\alpha p}{\alpha - p} \frac{r-p}{p}} \sum_{n=1}^{\infty} n^{\frac{r}{p}-2} \mathbb{P}(|S_n| \geq \varepsilon n^{1/p}) = \frac{p}{r-p} \mathbb{E}|Z|^{\frac{\alpha p}{\alpha - p} \frac{r-p}{p}},$$

where Z is a random variable having the distribution G. In particular, if $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = \sigma^2 < \infty$, then

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n=1}^{\infty} n^{\frac{r}{p}-2} \mathbb{P}(|S_n| \geq \varepsilon n^{1/p}) = \frac{p}{r-p} \mathbb{E}|Z|^{\frac{2(r-p)}{2-p}}, \tag{1.5}$$

where Z has a normal distribution with mean 0 and variance σ^2.

Furthermore, (1.4) means that σ^2 can be approximated by $\varepsilon^2 \sum_{n=1}^{\infty} \mathbb{P}\left(|S_n| \geq \varepsilon n\right)$ as $\varepsilon \to 0$. Klesov [13] proved that if $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2 < \infty$ and $\mathbb{E}|X|^3 < \infty$, then

$$\varepsilon^2 \sum_{n=1}^{\infty} \mathbb{P}\left(|S_n| \geq \varepsilon n\right) - \sigma^2 = o(\varepsilon^{1/2}), \ \text{ as } \ \varepsilon \to 0.$$

He and Xie [9] improved the result of Klesov [13] and obtained that if $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2 < \infty$ and $\mathbb{E}|X|^3 < \infty$, then

$$\varepsilon^2 \sum_{n=1}^{\infty} \mathbb{P}\left(|S_n| \geq \varepsilon n\right) - \sigma^2 = o(\varepsilon), \ \text{ as } \ \varepsilon \to 0.$$

In the equation (1.5), p is assumed to satisfy the condition $1 \le p < r < \alpha$, which exclude the case $r = 2p$. Hence, in the present paper, one aim is to study the precise asymptotics for the following series

$$\varepsilon^{\frac{2p}{2-p}} \sum_{n=1}^{\infty} \mathbb{P}\left(|S_n| \ge \varepsilon n^{1/p}\right), \text{ as } \varepsilon \to 0, \tag{1.6}$$

where $1 \le p < 2$. The other aim is to consider the convergence rate of the series (1.6), which extends the works in He and Xie [9]. Throughout this paper, let C be a constant not depending on n, which may be different in different places.

2. Main results

Let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$. From (1.3), for $1 \le p < 2$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|S_n| \ge \varepsilon n^{1/p}\right) < \infty \text{ for } \varepsilon > 0 \tag{2.1}$$

if and only if $\mathbb{E}|X|^{2p} < \infty$. The following Theorem 2.1 gives the precise asymptotics for the series in (2.1) as $\varepsilon \to 0$.

Theorem 2.1. *Let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2$ and $\mathbb{E}|X|^{\alpha} < \infty$ for some $\alpha > \frac{2p}{2-p}$, where $1 < p < 2$. Then we have*

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{2p}{2-p}} \sum_{n=1}^{\infty} \mathbb{P}\left(|S_n| \ge \varepsilon n^{1/p}\right) = \left(\frac{\sigma}{\sigma_1}\right)^{\frac{2p}{2-p}} \mathbb{E}|X|^{\frac{2p}{2-p}}, \tag{2.2}$$

where σ_1^2 is the variance of some normal random variable ξ satisfying $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = \sigma_1^2$ and $\mathbb{E}|\xi|^{\frac{2p}{2-p}} = \mathbb{E}|X|^{\frac{2p}{2-p}}$

The following results give the convergence rate of the limit in (2.2).

Theorem 2.2. *Let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^{\alpha} < \infty$ for some $\alpha > \frac{2p}{2-p}$, where $6/5 \le p < 3/2$. Then we have*

$$\varepsilon^{\frac{2p}{2-p}} \sum_{n=1}^{\infty} \mathbb{P}\left(|S_n| \ge \varepsilon n^{1/p}\right) - \left(\frac{\sigma}{\sigma_1}\right)^{\frac{2p}{2-p}} \mathbb{E}|X|^{\frac{2p}{2-p}} = O\left(\varepsilon^{\frac{p}{2-p}}\right), \tag{2.3}$$

where σ_1^2 is the variance of some normal random variable ξ satisfying $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = \sigma_1^2$ and $\mathbb{E}|\xi|^{\frac{2p}{2-p}} = \mathbb{E}|X|^{\frac{2p}{2-p}}$.

Theorem 2.3. *Let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^{\alpha} < \infty$ for some $\frac{2p}{2-p} < \alpha < 3$, where $1 < p < 6/5$. Then we have*

$$\varepsilon^{\frac{2p}{2-p}} \sum_{n=1}^{\infty} \mathbb{P}\left(|S_n| \ge \varepsilon n^{1/p}\right) - \left(\frac{\sigma}{\sigma_1}\right)^{\frac{2p}{2-p}} \mathbb{E}|X|^{\frac{2p}{2-p}} = o\left(\varepsilon^{\frac{p(\alpha-2)}{2-p}}\right), \tag{2.4}$$

where σ_1^2 is the variance of some normal random variable ξ satisfying $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = \sigma_1^2$ and $\mathbb{E}|\xi|^{\frac{2p}{2-p}} = \mathbb{E}|X|^{\frac{2p}{2-p}}$.

Remark 2.1. *Intuitively, as $p \to 1$, from (2.2) and (2.4), we have*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \sum_{n=1}^{\infty} \mathbb{P}\left(|S_n| \geq \varepsilon n\right) = \sigma^2$$

and

$$\varepsilon^2 \sum_{n=1}^{\infty} \mathbb{P}\left(|S_n| \geq \varepsilon n\right) - \sigma^2 = o\left(\varepsilon^{\alpha-2}\right).$$

Hence Theorem 2.2 and Theorem 2.3 extend the results in He and Xie [9].

Remark 2.2. *In Theorem 2.2 and Theorem 2.3, we have discussed the cases $6/5 \leq p < 3/2$ and $1 < p < 6/5$ respectively. However, it is still an open problem for the case $3/2 \leq p < 2$.*

3. Proof of main results

Lemma 3.1. *[2] Let $X_1, X_2, \cdots, X_n$ be independent and not necessarily identically distributed random variables with zero means and finite variances. Define $W = \sum_{k=1}^{n} X_k$ and assume that $Var(W) = 1$. Let F be the distribution function of W and Φ the standard normal distribution function. Then there exists an absolute constant C such that for every real number x,*

$$|F(x) - \Phi(x)| \leq C \sum_{i=1}^{n} \left\{ \frac{\mathbb{E}X_i^2 I(|X_i| > 1 + |x|)}{(1 + |x|)^2} + \frac{\mathbb{E}|X_i|^3 I(|X_i| \leq 1 + |x|)}{(1 + |x|)^3} \right\}.$$

Furthermore, we have

$$\sup_x |F(x) - \Phi(x)| \leq 4.1 \sum_{i=1}^{n} \left\{ \mathbb{E}X_i^2 I(|X_i| > 1) + \mathbb{E}|X_i|^3 I(|X_i| \leq 1) \right\}.$$

Lemma 3.2. *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2$ and $\mathbb{E}|X|^{2+\delta} < \infty$ for some $0 \leq \delta \leq 1$. Then there exists an absolute constant C such that*

$$\sup_x \left| \mathbb{P}\left(\frac{S_n}{\sqrt{n}\sigma} \leq x \right) - \Phi(x) \right| \leq \frac{C}{n^{\delta/2}},$$

where Φ is the standard normal distribution function.

Proof. From Lemma 3.1, the lemma is easy to be obtained. □

Lemma 3.3. *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = \sigma^2$.*

(1) $\mathbb{E}|X|^3 < \infty$, then there exists an absolute constant C such that for every real number x,

$$\left| \mathbb{P}\left(\frac{S_n}{\sqrt{n}\sigma} \leq x \right) - \Phi(x) \right| \leq \frac{C\mathbb{E}|X|^3}{\sqrt{n}\sigma^3 (1 + |x|)^3},$$

where Φ is the standard normal distribution function.

(2) If $\mathbb{E}|X|^{2+\delta} < \infty$ for some $0 < \delta < 1$, then there exists a bounded and decreasing function $\phi(x)$ defined on the interval $(0, \infty)$ such that $\lim_{x\to\infty} \phi(x) = 0$ and

$$\left| \mathbb{P}\left(\frac{S_n}{\sqrt{n}\sigma} \leq x \right) - \Phi(x) \right| \leq \frac{\phi(\sqrt{n}(1 + |x|))}{n^{\delta/2}(1 + |x|)^{2+\delta}}.$$

Proof. From Lemma 3.1, we have

$$\begin{aligned}
&\left|\mathbb{P}\left(\frac{S_n}{\sqrt{n}\sigma} \le x\right) - \Phi(x)\right| \\
\le& C\left\{\frac{\mathbb{E}X^2 I(|X| > (1+|x|)\sqrt{n}\sigma)}{\sigma^2(1+|x|)^2} + \frac{\mathbb{E}|X|^3 I(|X| \le (1+|x|)\sqrt{n}\sigma)}{\sqrt{n}\sigma^3(1+|x|)^3}\right\} \\
\le& \frac{C\mathbb{E}|X|^3}{\sqrt{n}\sigma^3(1+|x|)^3}
\end{aligned}$$

and

$$\sup_x \left|\mathbb{P}\left(\frac{S_n}{\sqrt{n}\sigma} \le x\right) - \Phi(x)\right| \le \frac{C}{\sqrt{n}}.$$

Furthermore, it is easy to see

$$\frac{\mathbb{E}X^2 I(|X| > (1+|x|)\sqrt{n}\sigma)}{\sigma^2(1+|x|)^2} \le \frac{\mathbb{E}|X|^{2+\delta} I(|X| > (1+|x|)\sqrt{n}\sigma)}{n^{\delta/2}\sigma^{2+\delta}(1+|x|)^{2+\delta}}$$

and

$$\begin{aligned}
&\frac{\mathbb{E}|X|^3 I(|X| \le (1+|x|)\sqrt{n}\sigma)}{\sqrt{n}\sigma^3(1+|x|)^3} \\
=& \frac{\mathbb{E}|X|^3 I\left(\sqrt{(1+|x|)\sqrt{n}\sigma} < |X| \le (1+|x|)\sqrt{n}\sigma\right)}{\sqrt{n}\sigma^3(1+|x|)^3} \\
&+ \frac{\mathbb{E}|X|^3 I\left(|X| \le \sqrt{(1+|x|)\sqrt{n}\sigma}\right)}{\sqrt{n}\sigma^3(1+|x|)^3} \\
\le& \frac{\mathbb{E}|X|^{2+\delta} I\left(\sqrt{(1+|x|)\sqrt{n}\sigma} < |X| \le (1+|x|)\sqrt{n}\sigma\right)}{n^{\delta/2}\sigma^{2+\delta}(1+|x|)^{2+\delta}} \\
&+ \frac{\mathbb{E}|X|^{2+\delta} I\left(|X| \le \sqrt{(1+|x|)\sqrt{n}\sigma}\right)}{n^{(1+\delta)/4}\sigma^{(5+\delta)/2}(1+|x|)^{(5+\delta)/2}} \\
\le& \frac{\mathbb{E}|X|^{2+\delta} I\left(|X| > \sqrt{(1+|x|)\sqrt{n}\sigma}\right)}{n^{\delta/2}\sigma^{2+\delta}(1+|x|)^{2+\delta}} \\
&+ \frac{\mathbb{E}|X|^{2+\delta} I\left(|X| \le \sqrt{(1+|x|)\sqrt{n}\sigma}\right)}{n^{\delta/2}\sigma^{2+\delta}(1+|x|)^{2+\delta}} \frac{1}{n^{(1-\delta)/4}\sigma^{(1-\delta)/2}}.
\end{aligned}$$

Hence if we take

$$\phi(\sqrt{n}(1+|x|)) = \frac{\mathbb{E}|X|^{2+\delta} I(|X| > \sqrt{(1+|x|)\sqrt{n}\sigma})}{\sigma^{2+\delta}} + \frac{\mathbb{E}|X|^{2+\delta}}{n^{(1-\delta)/4}\sigma^{(5-\delta)/2}},$$

then the desired result can be obtained. □

Proof. **[Proof of Theorem 2.1]** From the following elementary inequalities: for any random variable Y with $\mathbb{E}|Y| < \infty$,

$$\sum_{n=1}^{\infty} \mathbb{P}(|Y| \ge n) \le \mathbb{E}|Y| \le \sum_{n=0}^{\infty} \mathbb{P}(|Y| \ge n),$$

then we have

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\frac{|\xi|}{\sigma_1}\geq\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)\leq\left(\frac{\sigma}{\sigma_1}\right)^{\frac{2p}{2-p}}\frac{\mathbb{E}|\xi|^{\frac{2p}{2-p}}}{\varepsilon^{\frac{2p}{2-p}}}\leq\sum_{n=0}^{\infty}\mathbb{P}\left(\frac{|\xi|}{\sigma_1}\geq\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right),$$

which implies that

$$\lim_{\varepsilon\to0}\varepsilon^{\frac{2p}{2-p}}\sum_{n=1}^{\infty}\mathbb{P}\left(\frac{|\xi|}{\sigma_1}\geq\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)=\left(\frac{\sigma}{\sigma_1}\right)^{\frac{2p}{2-p}}\mathbb{E}|\xi|^{\frac{2p}{2-p}}. \tag{3.1}$$

In order to obtain (2.2), it is enough to show

$$\lim_{\varepsilon\to0}\varepsilon^{\frac{2p}{2-p}}\left(\sum_{n=1}^{\infty}\mathbb{P}\left(\frac{|S_n|}{\sqrt{n}\sigma}\geq\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)-\sum_{n=1}^{\infty}\mathbb{P}\left(\frac{|\xi|}{\sigma_1}\geq\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)\right)=0. \tag{3.2}$$

For every n, define

$$\Delta_n=\sup_{-\infty<x<\infty}\left|\mathbb{P}\left(\frac{|S_n|}{\sqrt{n}\sigma}\geq x\right)-\mathbb{P}\left(\frac{|\xi|}{\sigma_1}\geq x\right)\right|,$$

then from Lemma 3.2, we have

$$\lim_{n\to\infty}\Delta_n=0.$$

Hence, for any $M>0$, we have

$$\lim_{\varepsilon\to0}\varepsilon^{\frac{2p}{2-p}}\left(\sum_{n=1}^{[M\varepsilon^{-\frac{2p}{2-p}}]}\mathbb{P}\left(\frac{|S_n|}{\sqrt{n}\sigma}\geq\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)-\sum_{n=1}^{[M\varepsilon^{-\frac{2p}{2-p}}]}\mathbb{P}\left(\frac{|\xi|}{\sigma_1}\geq\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)\right)=0. \tag{3.3}$$

By taking $t>0$ such that $t>\frac{2p}{2-p}$, we get

$$\begin{aligned}\varepsilon^{\frac{2p}{2-p}}\sum_{n=[M\varepsilon^{-\frac{2p}{2-p}}]+1}^{\infty}\mathbb{P}\left(\frac{|\xi|}{\sigma_1}\geq\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)&\leq C\varepsilon^{\frac{2p}{2-p}}\sum_{n=[M\varepsilon^{-\frac{2p}{2-p}}]+1}^{\infty}\frac{\mathbb{E}|\xi|^t}{\varepsilon^t n^{\frac{2-p}{2p}t}}\\&\leq C\varepsilon^{\frac{2p}{2-p}-t}\frac{1}{(M\varepsilon^{-\frac{2p}{2-p}})^{\frac{2-p}{2p}t-1}}\leq\frac{C}{M^{\frac{2-p}{2p}t-1}},\end{aligned}$$

which implies

$$\lim_{M\to\infty}\lim_{\varepsilon\to0}\varepsilon^{\frac{2p}{2-p}}\sum_{n=[M\varepsilon^{-\frac{2p}{2-p}}]+1}^{\infty}\mathbb{P}\left(\frac{|\xi|}{\sigma_1}\geq\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)=0. \tag{3.4}$$

By using Rosenthal's inequality (noting $\alpha>2$), we have

$$\begin{aligned}\varepsilon^{\frac{2p}{2-p}}\sum_{n=[M\varepsilon^{-\frac{2p}{2-p}}]+1}^{\infty}\mathbb{P}\left(|S_n|\geq\varepsilon n^{1/p}\right)&\leq\varepsilon^{\frac{2p}{2-p}}\sum_{n=[M\varepsilon^{-\frac{2p}{2-p}}]+1}^{\infty}\frac{\mathbb{E}|S_n|^\alpha}{\varepsilon^\alpha n^{\alpha/p}}\\&\leq C\varepsilon^{\frac{2p}{2-p}-\alpha}\sum_{n=[M\varepsilon^{-\frac{2p}{2-p}}]+1}^{\infty}\frac{1}{n^{\alpha/p}}\left(\left(\sum_{i=1}^{n}\mathbb{E}X_i^2\right)^{\alpha/2}+\sum_{i=1}^{n}\mathbb{E}|X_i|^\alpha\right)\\&\leq C\varepsilon^{\frac{2p}{2-p}-\alpha}\sum_{n=[M\varepsilon^{-\frac{2p}{2-p}}]+1}^{\infty}\frac{1}{n^{\frac{\alpha}{p}-\frac{\alpha}{2}}}\mathbb{E}|X|^\alpha\\&\leq C\varepsilon^{\frac{2p}{2-p}-\alpha}\frac{1}{\left[M\varepsilon^{-\frac{2p}{2-p}}\right]^{\frac{\alpha}{p}-\frac{\alpha}{2}-1}}\leq\frac{C}{M^{\frac{\alpha}{p}-\frac{\alpha}{2}-1}}\end{aligned}$$

which yields

$$\lim_{M\to\infty}\lim_{\varepsilon\to 0}\varepsilon^{\frac{2p}{2-p}}\sum_{n=[M\varepsilon^{-\frac{2p}{2-p}}]+1}^{\infty}\mathbb{P}\left(|S_n|\geq \varepsilon n^{1/p}\right)=0. \tag{3.5}$$

From (3.3), (3.4) and (3.5), the claim (3.2) holds. □

Proof. [**Proof of Theorem 2.2**] Since $p \geq 6/5$, we have $\alpha > 3$. By using non-uniform estimate of the central limit theorem (see Lemma 3.3), for any $x \in \mathbb{R}$, we have

$$\left|\mathbb{P}\left(\frac{S_n}{\sqrt{n}\sigma}\leq x\right)-\Phi(x)\right|\leq \frac{C\mathbb{E}|X|^3}{\sqrt{n}\sigma^3(1+|x|)^3} \tag{3.6}$$

where C is an absolute positive constant, σ^2 is the variance of X and $\Phi(\cdot)$ is the distribution function of standard normal random variable. Let $\{\xi, \xi_n, n\geq 1\}$ be a sequence of i.i.d. normal random variables with $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2=\sigma_1^2$ and $\mathbb{E}|\xi|^{\frac{2p}{2-p}}=\mathbb{E}|X|^{\frac{2p}{2-p}}$, then we have

$$\begin{aligned}
&\left|\varepsilon^{\frac{2p}{2-p}}\sum_{n=1}^{\infty}\mathbb{P}\left(|S_n|>\varepsilon n^{1/p}\right)-\left(\frac{\sigma}{\sigma_1}\right)^{\frac{2p}{2-p}}\mathbb{E}|X|^{\frac{2p}{2-p}}\right|\\
\leq&\varepsilon^{\frac{2p}{2-p}}\left|\sum_{n=1}^{\infty}\mathbb{P}\left(\frac{1}{\sqrt{n}\sigma}|S_n|>\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)-\sum_{n=1}^{\infty}\mathbb{P}\left(\frac{1}{\sqrt{n}\sigma_1}\left|\sum_{k=1}^{n}\xi_k\right|>\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)\right|\\
&+\left|\varepsilon^{\frac{2p}{2-p}}\sum_{n=1}^{\infty}\mathbb{P}\left(\frac{1}{\sqrt{n}\sigma_1}\left|\sum_{k=1}^{n}\xi_k\right|>\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)-\left(\frac{\sigma}{\sigma_1}\right)^{\frac{2p}{2-p}}\mathbb{E}|X|^{\frac{2p}{2-p}}\right|\\
=:&I_n+II_n.
\end{aligned} \tag{3.7}$$

It is easy to see that $\frac{1}{\sqrt{n}\sigma_1}\sum_{k=1}^{n}\xi_k$ is a standard normal random variable. Let η be a standard normal random variable, then from (3.6), we have

$$\begin{aligned}
&\left|\mathbb{P}\left(\frac{1}{\sqrt{n}\sigma}|S_n|>\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)-\mathbb{P}\left(\frac{1}{\sqrt{n}\sigma_1}\left|\sum_{k=1}^{n}\xi_k\right|>\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)\right|\\
=&\left|\mathbb{P}\left(\frac{1}{\sqrt{n}\sigma}|S_n|>\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)-\mathbb{P}\left(|\eta|>\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)\right|\\
\leq&\frac{C\mathbb{E}|X|^3}{\sqrt{n}\sigma^3\left(1+\left|\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right|\right)^3}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
I_n\leq&\varepsilon^{\frac{2p}{2-p}}\sum_{n=1}^{\infty}\frac{C\mathbb{E}|X|^3}{\sqrt{n}\sigma^3\left(1+\left|\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right|\right)^3}\\
\leq&\varepsilon^{\frac{2p}{2-p}}\sum_{n=1}^{\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{2p}{2-p}}\right]}\frac{C}{\sqrt{n}}+\varepsilon^{\frac{2p}{2-p}}\sum_{n=\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{2p}{2-p}}\right]+1}^{\infty}\frac{C}{\sqrt{n}\left|\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right|^3}\leq C\varepsilon^{\frac{p}{2-p}}.
\end{aligned} \tag{3.8}$$

Now we estimate the term II_n. It is easy to check that

$$
\begin{aligned}
\varepsilon^{\frac{2p}{2-p}} \sum_{n=1}^{\infty} \mathbb{P}\left(\frac{1}{\sqrt{n}\sigma_1}\left|\sum_{k=1}^{n}\xi_k\right| > \frac{\varepsilon}{\sigma} n^{\frac{2-p}{2p}}\right) &= \varepsilon^{\frac{2p}{2-p}} \sum_{n=1}^{\infty} \mathbb{P}\left(|\eta| > \frac{\varepsilon}{\sigma} n^{\frac{2-p}{2p}}\right) \\
&= \varepsilon^{\frac{2p}{2-p}} \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \mathbb{P}\left(\frac{\varepsilon}{\sigma} k^{\frac{2-p}{2p}} < |\eta| \le \frac{\varepsilon}{\sigma}(k+1)^{\frac{2-p}{2p}}\right) \\
&= \varepsilon^{\frac{2p}{2-p}} \sum_{n=1}^{\infty} n\mathbb{P}\left(\frac{\varepsilon}{\sigma} n^{\frac{2-p}{2p}} < |\eta| \le \frac{\varepsilon}{\sigma}(n+1)^{\frac{2-p}{2p}}\right) \\
&= \varepsilon^{\frac{2p}{2-p}} \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} n \int_{\frac{\varepsilon}{\sigma} n^{\frac{2-p}{2p}}}^{\frac{\varepsilon}{\sigma}(n+1)^{\frac{2-p}{2p}}} e^{-\frac{t^2}{2}} dt
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}|X|^{\frac{2p}{2-p}} = \mathbb{E}|\xi|^{\frac{2p}{2-p}} &= \sqrt{\frac{2}{\pi}} \frac{1}{\sigma_1} \int_0^{\infty} x^{\frac{2p}{2-p}} e^{-\frac{x^2}{2\sigma_1^2}} dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^{\infty} (t\sigma_1)^{\frac{2p}{2-p}} e^{-\frac{t^2}{2}} dt \\
&= \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \int_{\frac{\varepsilon}{\sigma} n^{\frac{2-p}{2p}}}^{\frac{\varepsilon}{\sigma}(n+1)^{\frac{2-p}{2p}}} (t\sigma_1)^{\frac{2p}{2-p}} e^{-\frac{t^2}{2}} dt.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
II_n &= \left|\varepsilon^{\frac{2p}{2-p}} \sum_{n=1}^{\infty} \mathbb{P}\left(\frac{1}{\sqrt{n}\sigma_1}\left|\sum_{k=1}^{n}\xi_k\right| > \frac{\varepsilon}{\sigma} n^{\frac{2-p}{2p}}\right) - \left(\frac{\sigma}{\sigma_1}\right)^{\frac{2p}{2-p}} \mathbb{E}|X|^{\frac{2p}{2-p}}\right| \\
&\le \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \left| \int_{\frac{\varepsilon}{\sigma} n^{\frac{2-p}{2p}}}^{\frac{\varepsilon}{\sigma}(n+1)^{\frac{2-p}{2p}}} \left(n\varepsilon^{\frac{2p}{2-p}} - (t\sigma)^{\frac{2p}{2-p}}\right) e^{-\frac{t^2}{2}} dt \right| + \sqrt{\frac{2}{\pi}} \int_0^{\frac{\varepsilon}{\sigma}} (t\sigma)^{\frac{2p}{2-p}} e^{-\frac{t^2}{2}} dt.
\end{aligned}
$$

For every $n \ge 1$, by using the integral mean value theorem, there exists a constant $a_n \in \left(\frac{\varepsilon}{\sigma} n^{\frac{2-p}{2p}}, \frac{\varepsilon}{\sigma}(n+1)^{\frac{2-p}{2p}}\right)$, such that

$$
\begin{aligned}
&\int_{\frac{\varepsilon}{\sigma} n^{\frac{2-p}{2p}}}^{\frac{\varepsilon}{\sigma}(n+1)^{\frac{2-p}{2p}}} \left(n\varepsilon^{\frac{2p}{2-p}} - (t\sigma)^{\frac{2p}{2-p}}\right) e^{-\frac{t^2}{2}} dt \\
=& e^{-\frac{a_n^2}{2}} \int_{\frac{\varepsilon}{\sigma} n^{\frac{2-p}{2p}}}^{\frac{\varepsilon}{\sigma}(n+1)^{\frac{2-p}{2p}}} \left(n\varepsilon^{\frac{2p}{2-p}} - (t\sigma)^{\frac{2p}{2-p}}\right) dt \\
=& e^{-\frac{a_n^2}{2}} \left(\frac{1}{\sigma}\right) \varepsilon^{\frac{2+p}{2-p}} \left[n\left((n+1)^{\frac{2-p}{2p}} - n^{\frac{2-p}{2p}}\right)\right] \\
&- e^{-\frac{a_n^2}{2}} \left(\frac{1}{\sigma}\right) \varepsilon^{\frac{2+p}{2-p}} \left[\frac{2-p}{2+p}\left((n+1)^{\frac{2+p}{2p}} - n^{\frac{2+p}{2p}}\right)\right].
\end{aligned}
$$

By using Taylor formula, we have

$$
\begin{aligned}
n\left|(n+1)^{\frac{2-p}{2p}} - n^{\frac{2-p}{2p}}\right| &= n n^{\frac{2-p}{2p}} \left|\left(1+\frac{1}{n}\right)^{\frac{2-p}{2p}} - 1\right| \\
&= \frac{2-p}{2p} n^{\frac{2-p}{2p}} + \frac{(2-p)(2-3p)}{8p^2} n^{\frac{2-3p}{2p}} + O\left(n^{\frac{2-5p}{2p}}\right)
\end{aligned}
$$

and

$$\frac{2-p}{2+p}\left|(n+1)^{\frac{2+p}{2p}}-n^{\frac{2+p}{2p}}\right| = \frac{2-p}{2+p}n^{\frac{2+p}{2p}}\left|\left(1+\frac{1}{n}\right)^{\frac{2+p}{2p}}-1\right|$$
$$=\frac{2-p}{2p}n^{\frac{2-p}{2p}}+\frac{(2-p)^2}{8p^2}n^{\frac{2-3p}{2p}}+O\left(n^{\frac{2-5p}{2p}}\right)$$

which implies that

$$\left|\int_{\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}}^{\frac{\varepsilon}{\sigma}(n+1)^{\frac{2-p}{2p}}}\left(n\varepsilon^{\frac{2p}{2-p}}-(t\sigma)^{\frac{2p}{2-p}}\right)e^{-\frac{t^2}{2}}dt\right| \le Ce^{-\frac{a_n^2}{2}}\varepsilon^{\frac{2+p}{2-p}}\left(n^{\frac{2-3p}{2p}}+O\left(n^{\frac{2-5p}{2p}}\right)\right).$$

Since

$$\begin{aligned}
&\varepsilon^{\frac{2+p}{2-p}}\sum_{n=1}^{\infty}e^{-\frac{a_n^2}{2}}n^{\frac{2-3p}{2p}}\\
\le& C\varepsilon^{\frac{2+p}{2-p}}\int_1^{\infty}x^{\frac{2-3p}{2p}}e^{-\frac{1}{2}\frac{\varepsilon^2}{\sigma^2}x^{\frac{2-p}{p}}}dx\\
=& C\varepsilon^{\frac{2+p}{2-p}}\int_{\varepsilon^2}^{\infty}\left(\frac{t}{\varepsilon^2}\right)^{\frac{2-3p}{2(2-p)}}\left(\frac{1}{\varepsilon^2}\right)^{\frac{p}{2-p}}\frac{p}{2-p}t^{\frac{2p-2}{2-p}}e^{-\frac{1}{2}\frac{t}{\sigma^2}}dt\\
\le& C\varepsilon^{\frac{2p}{2-p}},
\end{aligned}$$

we get

$$\sum_{n=1}^{\infty}\left|\int_{\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}}^{\frac{\varepsilon}{\sigma}(n+1)^{\frac{2-p}{2p}}}\left(n\varepsilon^{\frac{2p}{2-p}}-(t\sigma)^{\frac{2p}{2-p}}\right)e^{-\frac{t^2}{2}}dt\right| \le C\varepsilon^{\frac{2p}{2-p}}. \tag{3.9}$$

Furthermore, we have

$$\int_0^{\frac{\varepsilon}{\sigma}}(t\sigma)^{\frac{2p}{2-p}}e^{-\frac{t^2}{2}}dt \le C\varepsilon^{\frac{2+p}{2-p}} \le C\varepsilon^{\frac{2p}{2-p}}. \tag{3.10}$$

By (3.9) and (3.10), we get

$$II_n \le C\varepsilon^{\frac{2p}{2-p}}. \tag{3.11}$$

At last, from (3.7), (3.8) and (3.11), we have

$$\left|\varepsilon^{\frac{2p}{2-p}}\sum_{n=1}^{\infty}\mathbb{P}\left(|S_n-n\mu|>\varepsilon n^{1/p}\right)-\left(\frac{\sigma}{\sigma_1}\right)^{\frac{2p}{2-p}}\mathbb{E}|X|^{\frac{2p}{2-p}}\right| \le C\varepsilon^{\frac{p}{2-p}}.$$

□

Proof. [**Proof of Theorem 2.3**] For the case $1 < p < 6/5$, we have $2 < \alpha < 3$. By using non-uniform estimate of the central limit theorem (see Lemma 3.3), there exists a bounded and decreasing function $\phi(x)$ defined on the interval $(0, \infty)$ such that $\lim_{x\to\infty}\phi(x) = 0$ and

$$\left|\mathbb{P}\left(\frac{S_n}{\sqrt{n}\sigma}\le x\right)-\Phi(x)\right| \le \frac{\phi(\sqrt{n}(1+|x|))}{n^{\delta/2}(1+|x|)^{2+\delta}} \quad \text{for any } x\in\mathbb{R}, \tag{3.12}$$

where $0 < \delta = \alpha - 2 < 1$, σ^2 is the variance of X and $\Phi(\cdot)$ is the distribution function of standard normal random variable. As the similar proof as Theorem 2.2, let $\{\xi, \xi_n, n \geq 1\}$ be a sequence of i.i.d. normal random variables with $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = \sigma_1^2$ and $\mathbb{E}|\xi|^{\frac{2p}{2-p}} = \mathbb{E}|X|^{\frac{2p}{2-p}}$, then we have

$$\begin{aligned}
&\left|\varepsilon^{\frac{2p}{2-p}}\sum_{n=1}^{\infty}\mathbb{P}\left(|S_n| > \varepsilon n^{1/p}\right) - \left(\frac{\sigma}{\sigma_1}\right)^{\frac{2p}{2-p}}\mathbb{E}|X|^{\frac{2p}{2-p}}\right| \\
\leq&\varepsilon^{\frac{2p}{2-p}}\left|\sum_{n=1}^{\infty}\mathbb{P}\left(\frac{1}{\sqrt{n}\sigma}|S_n| > \frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right) - \sum_{n=1}^{\infty}\mathbb{P}\left(\frac{1}{\sqrt{n}\sigma_1}\left|\sum_{k=1}^{n}\xi_k\right| > \frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)\right| \\
&+\left|\varepsilon^{\frac{2p}{2-p}}\sum_{n=1}^{\infty}\mathbb{P}\left(\frac{1}{\sqrt{n}\sigma_1}\left|\sum_{k=1}^{n}\xi_k\right| > \frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right) - \left(\frac{\sigma}{\sigma_1}\right)^{\frac{2p}{2-p}}\mathbb{E}|X|^{\frac{2p}{2-p}}\right| \\
=:&I_n + II_n.
\end{aligned} \tag{3.13}$$

It is easy to see that $\frac{1}{\sqrt{n}\sigma_1}\sum_{k=1}^{n}\xi_k$ is a standard normal random variable. Let η be a standard normal random variable, then from (3.12) we have

$$\begin{aligned}
&\left|\mathbb{P}\left(\frac{1}{\sqrt{n}\sigma}|S_n| > \frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right) - \mathbb{P}\left(\frac{1}{\sqrt{n}\sigma_1}\left|\sum_{k=1}^{n}\xi_k\right| > \frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)\right| \\
=&\left|\mathbb{P}\left(\frac{1}{\sqrt{n}\sigma}|S_n| > \frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right) - \mathbb{P}\left(|\eta| > \frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right)\right| \\
\leq&\frac{\phi\left(\sqrt{n}\left(1 + \left|\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right|\right)\right)}{n^{\delta/2}\left(1 + \left|\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right|\right)^{2+\delta}}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
I_n \leq&\varepsilon^{\frac{2p}{2-p}}\sum_{n=1}^{\infty}\frac{\phi\left(\sqrt{n}\left(1 + \left|\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right|\right)\right)}{n^{\delta/2}\left(1 + \left|\frac{\varepsilon}{\sigma}n^{\frac{2-p}{2p}}\right|\right)^{2+\delta}} \\
\leq&C\varepsilon^{\frac{2p}{2-p}}\sum_{n=1}^{\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{2p}{2-p}}\right]}\frac{\phi\left(\sqrt{n}\right)}{n^{\delta/2}} + C\varepsilon^{\frac{2p}{2-p}-2-\delta}\sum_{n=\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{2p}{2-p}}\right]+1}^{\infty}\frac{\phi\left((\sigma/\varepsilon)^{\frac{p}{2-p}}\right)}{n^{\frac{\delta}{2}+\frac{(2-p)(2+\delta)}{2p}}}.
\end{aligned} \tag{3.14}$$

It is easy to see that $\frac{\delta}{2} + \frac{(2-p)(2+\delta)}{2p} > 1$, then we have

$$\begin{aligned}
&\varepsilon^{\frac{2p}{2-p}-2-\delta}\sum_{n=\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{2p}{2-p}}\right]+1}^{\infty}\frac{\phi\left((\sigma/\varepsilon)^{\frac{p}{2-p}}\right)}{n^{\frac{\delta}{2}+\frac{(2-p)(2+\delta)}{2p}}} \\
\leq&C\varepsilon^{\frac{2p}{2-p}-2-\delta}\phi\left(\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2-p}}\right)\left(\frac{\sigma}{\varepsilon}\right)^{\frac{2p}{2-p}\left(1-\frac{\delta}{2}-\frac{(2-p)(2+\delta)}{2p}\right)} \\
\leq&C\phi\left(\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2-p}}\right)\varepsilon^{\frac{p\delta}{2-p}} = o\left(\varepsilon^{\frac{p\delta}{2-p}}\right).
\end{aligned} \tag{3.15}$$

Since $0 < \delta < 1$, then we have

$$\begin{aligned}\varepsilon^{\frac{2p}{2-p}} \sum_{n=1}^{\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2-p}}\right]} \frac{\phi\left(\sqrt{n}\right)}{n^{\delta/2}} \leq & C\varepsilon^{\frac{2p}{2-p}} \sum_{n=1}^{\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2-p}}\right]} \frac{1}{n^{\delta/2}} \\ \leq & C\varepsilon^{\frac{2p}{2-p}} \left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2-p}}\right]^{1-\frac{\delta}{2}} \leq C\varepsilon^{\frac{2p}{2-p}-\frac{p}{2-p}(1-\frac{\delta}{2})} = C\varepsilon^{\frac{p}{2-p}\left(1+\frac{\delta}{2}\right)}\end{aligned}$$

and

$$\begin{aligned}&\varepsilon^{\frac{2p}{2-p}} \sum_{n=\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2-p}}\right]+1}^{\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{2p}{2-p}}\right]} \frac{\phi\left(\sqrt{n}\right)}{n^{\delta/2}} \leq C\varepsilon^{\frac{2p}{2-p}} \phi\left(\sqrt{\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2-p}}\right]}\right) \sum_{n=\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2-p}}\right]+1}^{\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{2p}{2-p}}\right]} \frac{1}{n^{\delta/2}} \\ \leq & C\varepsilon^{\frac{2p}{2-p}} \phi\left(\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2(2-p)}}\right)\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{2p}{2-p}}\right]^{1-\frac{\delta}{2}} \leq C\phi\left(\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2(2-p)}}\right)\varepsilon^{\frac{2p}{2-p}-\frac{2p}{2-p}(1-\frac{\delta}{2})} \\ = & C\phi\left(\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2(2-p)}}\right)\varepsilon^{\frac{p\delta}{2-p}},\end{aligned}$$

which implies

$$\begin{aligned}&\varepsilon^{\frac{2p}{2-p}} \sum_{n=1}^{\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{2p}{2-p}}\right]} \frac{\phi\left(\sqrt{n}\right)}{n^{\delta/2}} \\ = & \varepsilon^{\frac{2p}{2-p}} \sum_{n=1}^{\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2-p}}\right]} \frac{\phi\left(\sqrt{n}\right)}{n^{\delta/2}} + \varepsilon^{\frac{2p}{2-p}} \sum_{n=\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2-p}}\right]+1}^{\left[\left(\frac{\sigma}{\varepsilon}\right)^{\frac{2p}{2-p}}\right]} \frac{\phi\left(\sqrt{n}\right)}{n^{\delta/2}} \\ \leq & C\varepsilon^{\frac{p}{2-p}\left(1+\frac{\delta}{2}\right)} + C\phi\left(\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2(2-p)}}\right)\varepsilon^{\frac{p\delta}{2-p}} \\ = & C\varepsilon^{\frac{p\delta}{2-p}}\left(\varepsilon^{\frac{p(1-\frac{\delta}{2})}{2-p}} + \phi\left(\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2(2-p)}}\right)\right) \\ = & o\left(\varepsilon^{\frac{p\delta}{2-p}}\right).\end{aligned} \tag{3.16}$$

Here we used the fact

$$\varepsilon^{\frac{p(1-\frac{\delta}{2})}{2-p}} \to 0 \quad \text{and} \quad \phi\left(\left(\frac{\sigma}{\varepsilon}\right)^{\frac{p}{2(2-p)}}\right) \to 0$$

as $\varepsilon \to 0$. From (3.14), (3.15) and (3.16), we have

$$I_n = o\left(\varepsilon^{\frac{p\delta}{2-p}}\right). \tag{3.17}$$

Now we estimate the term II_n. By using the same discussions as Theorem 2.2, we get

$$II_n \leq C\varepsilon^{\frac{2p}{2-p}} = o\left(\varepsilon^{\frac{p\delta}{2-p}}\right). \tag{3.18}$$

At last, from (3.13), (3.17) and (3.18), we have

$$\left|\varepsilon^{\frac{2p}{2-p}}\sum_{n=1}^{\infty}\mathbb{P}\left(|S_n-n\mu|>\varepsilon n^{1/p}\right)-\left(\frac{\sigma}{\sigma_1}\right)^{\frac{2p}{2-p}}\mathbb{E}|X|^{\frac{2p}{2-p}}\right|=o\left(\varepsilon^{\frac{p\delta}{2-p}}\right).$$

□

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