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Some extensions of Ostrowski type inequalities for q-symmetric integrals involving h-convex functions

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Abstract. This article presents the Ostrowski type inequalities for h-convex functions in the context of quantum variational calculus using the Montogmery identity involving q-symmetric integrals. Additionally, Hölder's and Power mean inequalities involving q-symmetric integral are powerful tools to prove the results. Certain novel Ostrowski type inequalities for P-convex function, s-convex function, Godunova levin function, and s-Godunova Levin function are established, which are special instances of inequalities found for h-convex functions. Some examples are also provided along with graphical illusions to demonstrate the validity of the new discoveries. Our findings are regarded as generalizations of some known inequities from the literature.

1. Introduction

Quantum calculus, which is also known as limitless calculus, is a fascinating field of mathematics. It replaces the classical derivative with a quantum difference operator, allowing it to handle sets of non-differentiable functions. Kac and Cheung's book covers numerous basics of quantum calculus [17]. Furthermore, it is important in many fields of physics, such as cosmic strings and black holes [23], conformal quantum mechanics [31], nuclear and high energy physics. For more detailed discussion of quantum calculus, see [1, 7, 12, 18]. Tariboon and Ntouyas [27] proposed the left quantum difference operator, and Noor et al. [19] utilised it to explore numerous Ostrowski type inequalities. Bermudo et al. presented the notion of right quantum derivative and integral [4] as well as numerous q-Hermite-Hadamard inequalities for convex functions. Later on Budak et al. used the left and right quantum integrals to show several Ostrowski type inequalities [6] for q-differentiable mappings.

In 2012, Artur and da Cruz, B proposed a notion known as the q-symmetric derivative in [2].
The q-symmetric quantum calculus [18] has proven beneficial in a variety of domains, most notably in quantum mechanics. [18] demonstrates that the q-symmetric derivative possesses key properties for the

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q-exponential function that the standard derivative lacks. The idea of symmetric differentiation is noteworthy because if a function is differentiable at a point, it is also symmetrically differentiable, but not vice versa. The absolute value function, $S_1(\ell_1) = |\ell_1|$, for instance, is not differentiable when $\ell_1 = 0$, but it is symmetrically differentiable with a symmetric derivative of zero [28]. Researchers have recently been interested in symmetric quantum variational calculus due to its significance; relevant work is appeared in [5, 10, 20, 32].

In the recent years, generalizations and extensions were made rapidly for convex set and convex functions. Sanja Varošanec [29] introduced the class of h-convex function which generalizes P-convex function [8], convex function, s-convex function [15], Godunova-Levin function [14], and s-Godunova-Levin function [13]. Ostrowski type inequalities are widely recognised for calculating the function's average value from its integral mean. Quadrature rule, probability and optimization theory, statistics, information theory, and integral operator theory all made use of the Ostrowski type inequalities. Generalizations of Ostrowski type inequalities are obtained for different functions including, h-convex function [26], generalized h-convex function [24], strongly h-convex function [22], Harmonically h-convex function [3] and Logarithmically h-convex function [16].

This paper aims to reproduce certain Ostrowski type inequalities for h-convex functions [29] in context of quantum variational calculus by using left and right q-symmetric integrals. Newly generated Ostrowski type inequalities via quantum symmetric integrals are thought to present more refined and generalized forms of previously established inequalities in [20].

The paper is organized as follows: In Section 2, some definitions of convex function, s-convex function, h-convex function and left and right q-symmetric derivatives and respective integrals are recalled. Montogmery identity, Hölder and Power mean inequalities are addressed for q-symmetric integrals. In Section 3, some Ostrowski type inequalities are obtained for h-convex function with the help of Montogmery identity together with inequalities of Section 2. Section 4 deals with examples of Ostrowski type inequalities, proved in Section 3. Summery of the findings is discussed in Section 5.

2. Preliminaries

This section goes through the essential concepts used throughout the paper, which includes some classes of convex functions, fundamentals of q-symmetric calculus and some preliminary inequalities in the context of q-symmetric variational calculus.

The function $S_1 : J_1 \subset \mathbb{R} \to \mathbb{R}$ is said to be P-convex function [8], If S_1 is nonnegative and for all $\ell_1, \ell_2 \in J_1$ and $u_1 \in [0, 1]$, we have

$$S_1(u_1(\ell_1) + (1 - u_1)\ell_2) \leq S_1(\ell_1) + S_1(\ell_2).$$

A function S_1 is said to be a convex function, if for $S_1 : J_1 \subset \mathbb{R} \to \mathbb{R}$, $\ell_1, \ell_2 \in J_1$ and for all $u_1 \in [0, 1]$, we have

$$S_1(u_1(\ell_1) + (1 - u_1)\ell_2) \leq u_1 S_1(\ell_1) + (1 - u_1)S_1(\ell_2).$$

A function $S_1 : [0, \infty) \to [0, \infty)$, is said to be s-convex function in second sense [15], if $s \in (0, 1]$, $\ell_1, \ell_2 \in [0, \infty)$ and for all $u_1 \in [0, 1]$, we have

$$S_1(u_1(\ell_1) + (1 - u_1)\ell_2) \leq u_1^s S_1(\ell_1) + (1 - u_1)^s S_1(\ell_2).$$

A function $S_1 : J_1 \to \mathbb{R}$ is said to be Godunova-Levin function [14], if S_1 is non-negative and for all $\ell_1, \ell_2 \in J_1$ and $u_1 \in (0, 1)$, we have

$$S_1(u_1(\ell_1) + (1 - u_1)\ell_2) \leq \frac{S_1(\ell_1)}{u_1} + \frac{S_1(\ell_2)}{(1 - u_1)}.$$

A non-negative function $S_1 : J_1 \to \mathbb{R}$ is called s-Godunova-Levin function [13], if for all $\ell_1, \ell_2 \in J_1$ and $u_1, s \in (0,1)$, the following holds:

$$S_1(u_1(\ell_1) + (1-u_1)\ell_2) \leq \frac{S_1(\ell_1)}{u_1^s} + \frac{S_1(\ell_2)}{(1-u_1)^s}.$$

Varošanec [29] introduced and studied a class of convex functions, which is named by h-convex function in the following way:

Let $h : [0,1] \to [0,\infty)$ and $S_1 : [\ell_1, \ell_2] \subset \mathbb{R} \to \mathbb{R}$. A function S_1 is h-convex function, if for all $u_1 \in (0,1)$, we have

$$S_1(u_1(\ell_1) + (1-u_1)\ell_2) \leq h(u_1)S_1(\ell_1) + h(1-u_1)S_1(\ell_2). \tag{1}$$

If inequality (1) is reversed, then S_1 is said to be h-concave.

A function $h : S_1 \to \mathbb{R}$ is called supper-multiplicative function [9], if

$$h(\ell_1\ell_2) \geq h(\ell_1)h(\ell_2), \tag{2}$$

holds for all $\ell_1, \ell_2 \in S_1$.

If inequality (2) is reversed, then h is said to be a sub-multiplicative function. If equality holds in (2), then h is said to be a multiplicative function.

The function $h : S_1 \to \mathbb{R}$ is called supper-additive function [11], if

$$h(\ell_1 + \ell_2) \geq h(\ell_1) + h(\ell_2),$$

holds for all $\ell_1, \ell_2 \in S_1$.

Quantum symmetric derivative respective integral including left and right parts are recalled from [20] and [30] respectively.

Let a function $S_1 : [\vartheta_1, \varphi_1] \in J_1 \subset \mathbb{R} \to \mathbb{R}$ be continuous on $[\vartheta_1, \varphi_1]$, then left ${}_{\vartheta_1}q$-symmetric derivative [20] at $\ell_1 \in [\vartheta_1, \varphi_1]$ is defined as:

$${}_{\vartheta_1}\tilde{D}_q S_1(\ell_1) = \frac{S_1(q^{-1}\ell_1 + (1-q^{-1})\vartheta_1) - S_1(q\ell_1 + (1-q)\vartheta_1)}{(q^{-1}-q)(\ell_1 - \vartheta_1)}, \ell_1 \neq \vartheta_1, \tag{3}$$

if it exists and finite, where ${}_{\vartheta_1}\tilde{D}_q S_1(\vartheta_1) = \lim_{\ell_1 \to \vartheta_1} {}_{\vartheta_1}\tilde{D}_q S_1(\ell_1)$.

Remark 2.1. *If* $\lim q \to 1$ *and* $\vartheta_1 = 0$, *then* (3) *reduces to classical derivative.*

Suppose that the function $S_1 : [\vartheta_1, \varphi_1] \in J_1 \subset R \to \mathbb{R}$ is continuous on $[\vartheta_1, \varphi_1]$, then the right ${}^{\varphi_1}q$-symmetric derivative [30] at $\ell_1 \in [\vartheta_1, \varphi_1]$ is

$${}^{\varphi_1}\tilde{D}_q S_1(\ell_1) = \frac{S_1(q\ell_1 + (1-q)\varphi_1) - S_1(q^{-1}\ell_1 + (1-q^{-1})\varphi_1)}{(q^{-1}-q)(\varphi_1 - \ell_1)}, \ell_1 \neq \varphi_1, \tag{4}$$

if it exists and finite, where ${}^{\varphi_1}\tilde{D}_q S_1(\varphi_1) = \lim_{\ell_1 \to \varphi_1} {}^{\varphi_1}\tilde{D}_q S_1(\ell_1)$.

Remark 2.2. *If* $\lim q \to 1$ *and* $\varphi_1 = 0$, *then* (4) *becomes the definition of classical derivative.*

For $\vartheta_1 = 0$ and $\varphi_1 = 0$ in (3) and (4) respectively, we obtained an interpretation of quantum symmetric derivative defined in [10].

The q-symmetric analogue of Cauchy's formula of power $(m_1 - m_2)^{r_1}$ is defined in [25] by

$$(m_1 - m_2)^{r_1} = m_1^{r_1} \frac{\prod_{i=0}^{\infty}(m_1 - m_2 q^{2i+1})}{\prod_{i=0}^{\infty}(m_1 - m_2 q^{2(i+r_1)+1})}, m_1 \neq 0, k_1 \in \mathbb{R}.$$

Also q-real number $[n]$, is defined by

$$[n] = \frac{q^{2n} - 1}{q^2 - 1}, n \in \mathbb{R}.$$

For any positive integer n, the following hold:

$$(1 - \ell_1)_{\tilde{q}}^{n} = \prod_{i=0}^{n-1}(1 - q^{2i+1}\ell_1), \tag{5}$$

and

$$(1 - \ell_1)_{\tilde{q}}^{-n} = \frac{1}{\prod_{i=0}^{n-1}(1 - q^{-(2i+1)}\ell_1)}. \tag{6}$$

We have the following assessments for $n \geq 1$:

$$\begin{aligned}
&(\vartheta_1 - \ell_1)_{\tilde{q}}^{n} = (\vartheta_1 - q\ell_1)(\vartheta_1 - q^3\ell_1)(\vartheta_1 - q^5\ell_1)\cdots(\vartheta_1 - q^{2n-1}\ell_1);\\
&{}_{\vartheta_1}\tilde{D}_q(\vartheta_1 - \ell_1)_{\tilde{q}}^{n} = -[n]q(\vartheta_1 - q\ell_1)_{\tilde{q}}^{n-1};\\
&(\vartheta_1 - q\ell_1)_{\tilde{q}}^{n} = -\frac{1}{q[n+1]}\,{}_{\vartheta_1}\tilde{D}_q(\vartheta_1 - \ell_1)_{\tilde{q}}^{n+1};\\
&\int(\vartheta_1 - \ell_1)_{\tilde{q}}^{n}\,{}_{\vartheta_1}\tilde{d}_q\ell_1 = -\frac{(\vartheta_1 - q^{-1}\ell_1)_{\tilde{q}}^{n+1}}{[n+1]}, (\vartheta_1 \neq -1)
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
&(\vartheta_1 - \ell_1)_{\tilde{q}}^{-n} = \frac{1}{(\vartheta_1 - q^{-1}\ell_1)(\vartheta_1 - q^{-3}\ell_1)(\vartheta_1 - q^{-5}\ell_1)\cdots(\vartheta_1 - q^{-(2n-1)}\ell_1)};\\
&{}_{\vartheta_1}\tilde{D}_q(\vartheta_1 - \ell_1)_{\tilde{q}}^{-n} = [-n]q^{-2}(\vartheta_1 - q\ell_1)_{\tilde{q}}^{-n+1};\\
&(\vartheta_1 - q\ell_1)_{\tilde{q}}^{-n} = \frac{1}{q^{-2}[-(n+1)]}\,{}_{\vartheta_1}\tilde{D}_q(\vartheta_1 - \ell_1)_{\tilde{q}}^{-(n+1)};\\
&\int(\vartheta_1 - \ell_1)_{\tilde{q}}^{-n}\,{}_{\vartheta_1}\tilde{d}_q\ell_1 = \frac{q(\vartheta_1 - q^{-1}\ell_1)_{\tilde{q}}^{-(n+1)}}{[-(n+1)]}, (\vartheta_1 \neq -1).
\end{aligned} \tag{8}$$

q-Symmetric Antiderivative : Suppose that $S_1 : [\vartheta_1, \varphi_1] \to \mathbb{R}$ is continuous function. Then left q-symmetric definite integral [20] on $[\vartheta_1, \varphi_1]$ is defined as:

$$\begin{aligned}
\int_{\vartheta_1}^{\ell_1} S_1(u_1)\,{}_{\vartheta_1}\tilde{d}_q u_1 &= (q^{-1} - q)(\ell_1 - \vartheta_1)\sum_{n=0}^{\infty} q^{2n+1}S_1(q^{2n+1}\ell_1 + (1 - q^{2n+1})\vartheta_1)\\
&= (1 - q^2)(\ell_1 - \vartheta_1)\sum_{n=0}^{\infty} q^{2n}S_1(q^{2n+1}\ell_1 + (1 - q^{2n+1})\vartheta_1),
\end{aligned} \tag{9}$$

for $\ell_1 \in [\vartheta_1, \varphi_1]$.

Let $\vartheta_1, \varphi_1 \in J_1$ and $\vartheta_1 < \varphi_1$. For $S_1 : J_1 \to \mathbb{R}$ and $q \in (0,1)$, the right q-symmetric integral of S_1 [30] from ϑ_1 to φ_1 is

$$\begin{aligned}\int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1 &= (q^{-1}-q)(\varphi_1-\ell_1)\sum_{n=0}^{\infty} q^{2n+1} S_1(q^{2n+1}\ell_1 + (1-q^{2n+1})\varphi_1)\\ &= (1-q^2)(\varphi_1-\ell_1)\sum_{n=0}^{\infty} q^{2n} S_1(q^{2n+1}\ell_1 + (1-q^{2n+1})\varphi_1),\end{aligned} \tag{10}$$

where $\ell_1 \in [\vartheta_1, \varphi_1]$.

To accomplish the goal, we are in need of the following lemma from [21].

Lemma 2.3. *Let $S_1 : J_1 \subset \mathbb{R} \to \mathbb{R}$ be q-symmetric differentiable function on J_1 and $\vartheta_1, \varphi_1 \in J_1$ for $\vartheta_1 < \varphi_1$. If ${}_{\vartheta_1}\tilde{D}_q S_1, {}^{\varphi_1}\tilde{D}_q S_1 \in L_1[\vartheta_1, \varphi_1]$, then the following q-symmetric integral equality is valid:*

$$\begin{aligned}&S_1(\ell_1) - \frac{1}{\varphi_1-\vartheta_1}\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\\ &= \frac{q^2(\ell_1-\vartheta_1)^2}{\varphi_1-\vartheta_1}\int_0^1 u_1\,{}_{\vartheta_1}\tilde{D}_q S_1(qu_1\ell_1 + (1-qu_1)\vartheta_1)\tilde{d}_q u_1\\ &\qquad - \frac{q^2(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\int_0^1 u_1\,{}^{\varphi_1}\tilde{D}_q S_1(qu_1\ell_1 + (1-qu_1)\varphi_1)\tilde{d}_q u_1.\end{aligned} \tag{11}$$

q-Symmetric Hölder's Inequalities:

Theorem 2.4. *Suppose that S_1 and R_1 are q-symmetric integrable functions on $[\vartheta_1, \varphi_1] \subset \mathbb{R}^+$, $0 < q < 1$ and $\frac{1}{n} + \frac{1}{m} = 1$ with $m > 1$, where $n, m \in \mathbb{R}$, then right q-symmetric Hölder inequality [21] is,*

$$\int_{\ell_1}^{\varphi_1} |S_1(u_1)R_1(u_1)|^{\varphi_1}\tilde{d}_q u_1 \le \left\{\int_{\ell_1}^{\varphi_1} (|S_1(u_1)|^n)\,{}^{\varphi_1}\tilde{d}_q u_1\right\}^{\frac{1}{n}}\left\{\int_{\ell_1}^{\varphi_1} (|R_1(u_1)|^m)\,{}^{\varphi_1}\tilde{d}_q u_1\right\}^{\frac{1}{m}}. \tag{12}$$

Theorem 2.5. *Assume that S_1 and R_1 are q-symmetric integrable functions on $[\varpi_1, \omega_1] \subset \mathbb{R}^+$, $0 < q < 1$ and $\frac{1}{n} + \frac{1}{m} = 1$ with $m > 1$; then left q-symmetric Hölder inequality [20], is*

$$\int_{\vartheta_1}^{\ell_1} |S_1(u_1)R_1(u_1)|_{\vartheta_1}\tilde{d}_q u_1 \le \left\{\int_{\vartheta_1}^{\ell_1} |S_1(u_1)|^n\,{}_{\vartheta_1}\tilde{d}_q u_1\right\}^{\frac{1}{n}}\left\{\int_{\vartheta_1}^{\ell_1} |R_1(u_1)|^m\,{}_{\vartheta_1}\tilde{d}_q u_1\right\}^{\frac{1}{m}}. \tag{13}$$

q-Symmetric Power Mean Inequalities:

Theorem 2.6. *Suppose that $\frac{1}{n} + \frac{1}{m} = 1$ where $n, m > 1$ are real numbers. If $\vartheta_1, \varphi_1 \in \mathbb{R}$ and $S_1, R_1 : [\vartheta_1, \varphi_1] \to \mathbb{R}$ are continuous functions, then from [20], we have*

$$\int_{\vartheta_1}^{\varphi_1} |S_1(u_1)R_1(u_1)|_{\vartheta_1}\tilde{d}_q u_1 \le \left\{\int_{\vartheta_1}^{\varphi_1} |S_1(u_1)|_{\vartheta_1}\tilde{d}_q u_1\right\}^{1-\frac{1}{m}}\left\{\int_{\vartheta_1}^{\varphi_1} |S_1(u_1)||G_1(u_1)|^m\,{}_{\vartheta_1}\tilde{d}_q u_1\right\}^{\frac{1}{m}}. \tag{14}$$

Theorem 2.7. *Let $\vartheta_1, \varphi_1 \in \mathbb{R}$ and $S_1, R_1 : [\vartheta_1, \varphi_1] \to \mathbb{R}$ are continuous functions, for $\frac{1}{n} + \frac{1}{m} = 1$ where $n, m > 1$ are real numbers, then from [21], one has*

$$\int_{\vartheta_1}^{\varphi_1} |S_1(u_1)G_1(u_1)|^{\varphi_1}\tilde{d}_q u_1 \le \left\{\int_{\vartheta_1}^{\varphi_1} |S_1(u_1)|^{\varphi_1}\tilde{d}_q u_1\right\}^{1-\frac{1}{m}}\left\{\int_{\vartheta_1}^{\varphi_1} |S_1(u_1)||G_1(u_1)|^{m\varphi_1}\tilde{d}_q u_1\right\}^{\frac{1}{m}}. \tag{15}$$

Throughout this paper, Assume $h : [0,1] \to [0,\infty)$ is a function, satisfying $h(u_1) \ge u_1$, $\forall u_1 \in [0,1]$.

3. Main Results

In this section, we generalise the results obtained in [26] and established some new Ostrowski-type inequalities for those functions for which absolute values of derivatives are h-convex functions.

3.1. q-Symmetric Ostrowski Type Inequalities:

Theorem 3.1. *Let $S_1 : J_1 \subset \mathbb{R} \to \mathbb{R}$ be q-symmetric differentiable on (ϑ_1, φ_1), for $\vartheta_1, \varphi_1 \in J_1$ and $\vartheta_1 < \varphi_1$ and ${}_{\vartheta_1}\tilde{D}_q S_1$, ${}^{\varphi_1}\tilde{D}_q S_1$ are two continuous and differentiable function on $[\vartheta_1, \varphi_1]$. Furthermore, assume $|{}_{\vartheta_1}\tilde{D}_q S_1(\ell_1)|$ and $|{}^{\varphi_1}\tilde{D}_q S_1(\ell_1)|$ are h-convex functions for $\ell_1 \in [\vartheta_1, \varphi_1]$, $q \in (0,1)$ and $|{}_{\vartheta_1}\tilde{D}_q S_1(\ell_1)| \leq \hat{M}_1$, $|{}^{\varphi_1}\tilde{D}_q S_1(\ell_1)| \leq \tilde{M}_2$, where $M_1 = \max\left\{\hat{M}_1, \tilde{M}_2\right\}$, then the following q-symmetric integral inequality is valid:*

$$\left|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)\,{}_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)\,{}^{\varphi_1}\tilde{d}_q u_1\Big)\right| \leq M_1 q^2\Big(\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\Big)\Big[\int_0^1 \Big(u_1 h(qu_1) + u_1 h(1 - qu_1)\Big)\tilde{d}_q u_1\Big]. \quad (16)$$

Proof. *Apply modulus on both sides of* (11), *it yeilds*

$$\begin{aligned}
&\left|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)\,{}_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)\,{}^{\varphi_1}\tilde{d}_q u_1\Big)\right| \\
&\leq \frac{q^2(\ell_1 - \vartheta_1)^2}{\varphi_1 - \vartheta_1}\int_0^1 \left|u_1\,{}_{\vartheta_1}\tilde{D}_q S_1(qu_1\ell_1 + (1 - qu_1)\vartheta_1)\tilde{d}_q u_1\right| \\
&+ \frac{{}^2(\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\int_0^1 \left|u_1\,{}^{\varphi_1}\tilde{D}_q S_1(qu_1\ell_1 + (1 - qu_1)\varphi_1)\tilde{d}_q u_1\right| \\
&\leq \frac{q^2(\ell_1 - \vartheta_1)^2}{\varphi_1 - \vartheta_1}\int_0^1 u_1|{}_{\vartheta_1}\tilde{D}_q S_1(qu_1(\ell_1) + (1 - qu_1)\vartheta_1)|\tilde{d}_q u_1 \\
&\qquad + \frac{q^2(\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\int_0^1 u_1|{}^{\varphi_1}\tilde{D}_q S_1(qu_1\ell_1 + (1 - qu_1)\varphi_1)|\tilde{d}_q u_1. \quad (17)
\end{aligned}$$

Using the h-convexity of q-symmetric derivative,

$$|{}_{\vartheta_1}\tilde{D}_q S_1(qu_1\ell_1 + (1 - qu_1)\vartheta_1)| \leq h(qu_1)|{}_{\vartheta_1}\tilde{D}_q S_1(\ell_1)| + h(1 - qu_1)|{}_{\vartheta_1}\tilde{D}_q S_1(\vartheta_1)|, \quad (18)$$

and

$$|{}^{\varphi_1}\tilde{D}_q S_1(qu_1\ell_1 + (1 - qu_1)\varphi_1)| \leq h(qu_1)|{}^{\varphi_1}\tilde{D}_q S_1(\ell_1)| + h(1 - qu_1)|{}^{\varphi_1}\tilde{D}_q S_1(\varphi_1)|. \quad (19)$$

Therefore by substituting (18) *and* (19) *in* (17), *we have*

$$\Big|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\Big)\Big|$$
$$\leq \frac{q^2(\ell_1 - \vartheta_1)^2}{\varphi_1 - \vartheta_1}\int_0^1 \Big[u_1 h(qu_1)|_{\vartheta_1}\tilde{D}_q S_1(\ell_1)| + u_1 h(1 - qu_1)|_{\vartheta_1}\tilde{D}_q S_1(\vartheta_1)|\Big]\tilde{d}_q u_1$$
$$+ \frac{q^2(\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\int_0^1 \Big[u_1 h(qu_1)|^{\varphi_1}\tilde{D}_q S_1(\ell_1)| + u_1 h(1 - qu_1)|^{\varphi_1}\tilde{D}_q S_1(\varphi_1)|\Big]\tilde{d}_q u_1$$
$$\leq \frac{\hat{M}_1 q^2(\ell_1 - \vartheta_1)^2}{\varphi_1 - \vartheta_1}\Big[\int_0^1 u_1 h(qu_1)_0\tilde{d}_q u_1 + \int_0^1 u_1 h(1 - qu_1)\tilde{d}_q u_1\Big]$$
$$+ \frac{\tilde{M}_2 q^2(\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\Big[\int_0^1 u_1 h(qu_1)\tilde{d}_q u_1 + \int_0^1 u_1 h(1 - qu_1)\tilde{d}_q u_1\Big]$$
$$\leq M_1 q^2\Big(\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\Big)\Big[\int_0^1 \Big(u_1 h(qu_1) + u_1 h(1 - qu_1)\Big)\tilde{d}_q u_1\Big].$$

□

Remark 3.2. ***(i)*** *If $q = 1$ and h is supper multiplicative then, the inequality* (16) *becomes Theorem 2 of [26];*
(ii) *If we choose $h(u_1) = u_1$, then the inequality* (16) *reduces to Theorem 4 of [21].*

Corollary 3.3. *If $h(u_1) = u_1^s$, $u_1 \in (0, 1)$ and $s \in (0, 1]$ then the inequality* (16), *reduces to the following inequality involving s-convex functions:*

$$\Big|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\Big)\Big|$$
$$\leq M_1 q^2\Big(\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\Big)\int_0^1 \Big(u_1^{s+1} + u_1(1 - u_1)^s\Big)\tilde{d}_q u_1$$
$$= M_1 q^2\Big(\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\Big)\Big[\frac{u_1^{s+2}}{[s+2]}\Big|_0^1 + \int_0^1 u_1\frac{q\tilde{D}_q(1 - q^{-1}u_1)_{\tilde{q}}^{s+1}}{[s+1]}\tilde{d}_q u_1\Big]$$
$$= M_1 q^2\Big(\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\Big)\Big[\frac{1}{[s+2]} + \frac{qu_1(1 - q^{-1}u_1)_{\tilde{q}}^{s+1}}{[s+1]}\Big|_0^1$$
$$- \frac{q}{[s+1]}\int_0^1 (1 - q^{-1}u_1)_{\tilde{q}}^{s+1}\tilde{d}_q u_1\Big]$$
$$= M_1 q^2\Big(\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\Big)\Big[\frac{1}{[s+2]} + \frac{q(1 - q^{-1})_{\tilde{q}}^{s+1}}{[s+1]} - \frac{q}{[s+1]}\Big(\frac{q(1 - q^{-1}u_1)_{\tilde{q}}^{s+2}}{[s+2]}\Big|_0^1\Big)\Big]$$
$$= M_1 q^2\Big(\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\Big)\Big[\frac{1}{[s+2]} + \frac{q^2(1 - q^{-1})_{\tilde{q}}^{-s+1}}{[s+1]}$$
$$- \frac{q^2}{[s+1]}\Big(\frac{(1 - q^{-1})_{\tilde{q}}^{s+2}}{[s+2]} - \frac{q^2}{[s+2]}\Big)\Big].$$

Corollary 3.4. *Choose $h(u_1) = 1$ in* (16), *then the inequality* (16) *becomes the following inequality for P-convex function:*

$$\Big|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\Big)\Big| \leq \frac{2M_1 q^2}{[2]}\Big(\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\Big).$$

Corollary 3.5. *If $h(u_1) = \frac{1}{u_1}$ in inequality* (16)*, then the inequality* (16) *reduces to the following inequality involving Godunova-Levin function:*

$$\begin{aligned}
&\left|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\Big)\right| \\
&\leq M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\int_0^1\Big(\frac{1}{q} + \frac{u_1}{1-qu_1}\Big)\tilde{d}_q u_1 \\
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big[\int_0^1 1 + u_1(1-u_1)^{-1}\tilde{d}_q u_1\Big] \\
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big[1 + \int_0^1 \frac{u_1 q\tilde{D}_q(1-q^{-1}u_1)^{-(1+1)}}{[-(1+1)]}\tilde{d}_q u_1\Big] \\
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big[1 + \frac{qu_1(1-q^{-1}u_1)_{\tilde{q}}^{-2}}{[-2]}\Big|_0^1 - \int_0^1 \frac{q(1-q^{-1}u_1)_{\tilde{q}}^{-2}}{[-2]}\Big] \\
&\qquad = M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big[1 + \frac{q(1-q^{-1})_{\tilde{q}}^{-2}}{[-2]} - \frac{q^2}{[-2]}\Big(\frac{(1-q^{-1})_{\tilde{q}}^{-3}}{[-3]} - \frac{1}{[-3]}\Big)\Big].
\end{aligned}$$

Corollary 3.6. *If $h(u_1) = \frac{1}{u_1^s}$, $u_1 \in (0,1)$ and $s \in (0,1)$, then the inequality* (16) *gives the next inequality involving s-Godunova-Levin function in second sense:*

$$\begin{aligned}
&\left|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\Big)\right| \\
&\leq M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\int_0^1\Big(\frac{u_1}{(qu_1)^s} + \frac{u_1}{(1-qu_1)^s}\Big)\tilde{d}_q u_1 \\
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big[\frac{1}{q^s}\int_0^1\Big(u_1^{1-s} + u_1(1-qu_1)^{-s}\Big)\tilde{d}_q u_1\Big] \\
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big[\frac{1}{q^s}\frac{u_1^{2-s}}{[2-s]}\Big|_0^1 + \int_0^1 u_1\frac{q^2\tilde{D}_q(1-u_1)_{\tilde{q}}^{-(s+1)}}{[-(s+1)]}\tilde{d}_q u_1\Big] \\
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big[\frac{1}{q^s[2-s]} + q^2\frac{u_1(1-u_1)_{\tilde{q}}^{-(s+1)}}{[-(s+1)]}\Big|_0^1 \\
&- \frac{q^2}{[-(s+1)]}\int_0^1 (1-u_1)_{\tilde{q}}^{-(s+1)}\tilde{d}_q u_1\Big] \\
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big[\frac{1}{q^s[2-s]} - \frac{q^2}{[-(s+1)]}\Big(\frac{q(1-q^{-1}u_1)_{\tilde{q}}^{-(s+2)}}{[-(s+2)]}\Big|_0^1\Big)\Big] \\
&\qquad = M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big[\frac{1}{q^s[2-s]} - \frac{q^3}{[-(s+1)]}\Big(\frac{(1-q^{-1})_{\tilde{q}}^{-(s+2)}}{[-(s+2)]} - \frac{1}{[-(s+2)]}\Big)\Big]. \qquad (20)
\end{aligned}$$

Theorem 3.7. *Let $S_1 : J_1 \subset \mathbb{R} \to \mathbb{R}$ be q-symmetric differentiable on (ϑ_1, φ_1), for $\vartheta_1, \varphi_1 \in J_1$ and $\vartheta_1 < \varphi_1$ and ${}_{\vartheta_1}\tilde{D}_q S_1$, ${}^{\varphi_1}\tilde{D}_q S_1$ are two continuous and differentiable function on $[\vartheta_1, \varphi_1]$. Furthermore, assume $|{}_{\vartheta_1}\tilde{D}_q S_1(\ell_1)|^m$ and $|{}^{\varphi_1}\tilde{D}_q S_1(\ell_1)|^m$ are h-convex functions for $\ell_1 \in [\vartheta_1, \varphi_1]$, $q \in (0,1)$ and $m > 1$, $n = \frac{m}{m-1}$. Also $|{}_{\vartheta_1}\tilde{D}_q S_1(\ell_1)| \leq \hat{M}_1$, $|{}^{\varphi_1}\tilde{D}_q S_1(\ell_1)| \leq \tilde{M}_2$. Define $M_1 = \max\{\hat{M}_1, \tilde{M}_2\}$, then the following q-symmetric integral inequality is valid:*

$$\left|S_1(\ell_1)-\frac{1}{\varphi_1-\vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1}S_1(u_1)_{\vartheta_1}\tilde{d}_qu_1+\int_{\ell_1}^{\varphi_1}S_1(u_1)^{\varphi_1}\tilde{d}_qu_1\Big)\right|$$
$$\leq\frac{M_1q^2}{[n+1]^{\frac{1}{n}}}\Big[\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big]\Big(\int_0^1\Big(h(qu_1)+h(1-qu_1)\Big)\tilde{d}_qu_1\Big)^{\frac{1}{m}},\quad(21)$$

for each $\ell_1\in[\vartheta_1,\varphi_1]$.

Proof. Taking modulus in (11), yields

$$\left|S_1(\ell_1)-\frac{1}{\varphi_1-\vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1}S_1(u_1)_{\vartheta_1}\tilde{d}_qu_1+\int_{\ell_1}^{\varphi_1}S_1(u_1)^{\varphi_1}\tilde{d}_qu_1\Big)\right|$$
$$\leq\frac{q^2(\ell_1-\vartheta_1)^2}{\varphi_1-\vartheta_1}\int_0^1u_1|_{\vartheta_1}\tilde{D}_qS_1(qu_1\ell_1+(1-qu_1)\vartheta_1)|\tilde{d}_qu_1$$
$$+\frac{q^2(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\int_0^1u_1|^{\varphi_1}\tilde{D}_qS_1(qu_1\ell_1+(1-qu_1)\varphi_1)|\tilde{d}_qu_1.\quad(22)$$

Using the inequalities (12) and (13) on right hand side of inequality (22),

$$\left|S_1(\ell_1)-\frac{1}{\varphi_1-\vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1}S_1(u_1)_{\vartheta_1}\tilde{d}_qu_1+\int_{\ell_1}^{\varphi_1}S_1(u_1)^{\varphi_1}\tilde{d}_qu_1\Big)\right|$$
$$\leq\frac{q^2(\ell_1-\vartheta_1)^2}{\varphi_1-\vartheta_1}\Big[\Big(\int_0^1u_1^n\tilde{d}_qu_1\Big)^{\frac{1}{n}}\Big(\int_0^1|_{\vartheta_1}\tilde{D}_qS_1(qu_1\ell_1+(1-qu_1)\vartheta_1)|^m\tilde{d}_qu_1\Big)^{\frac{1}{m}}\Big]$$
$$+\frac{q^2(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big[\Big(\int_0^1u_1^n\tilde{d}_qu_1\Big)^{\frac{1}{n}}\Big(\int_0^1|^{\varphi_1}\tilde{D}_qS_1(qu_1\ell_1+(1-qu_1)\varphi_1)|^m\tilde{d}_qu_1\Big)^{\frac{1}{m}}\Big].\quad(23)$$

From the definition of h-convexity, we have

$$|_{\vartheta_1}\tilde{D}_qS_1(qu_1\ell_1+(1-qu_1)\vartheta_1)|^m\leq h(qu_1)|_{\vartheta_1}\tilde{D}_qS_1(\ell_1)|^m+h(1-qu_1)|_{\vartheta_1}\tilde{D}_qS_1(\vartheta_1)|^m,\quad(24)$$

and

$$|^{\varphi_1}\tilde{D}_qS_1(qu_1\ell_1+(1-qu_1)\varphi_1)|^m\leq h(qu_1)|^{\varphi_1}\tilde{D}_qS_1(\ell_1)|^m+h(1-qu_1)|^{\varphi_1}\tilde{D}_qS_1(\varphi_1)|^m.\quad(25)$$

Using (24) and (25) in (23), we get

$$\left|S_1(\ell_1)-\frac{1}{\varphi_1-\vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1}S_1(u_1)_{\vartheta_1}\tilde{d}_qu_1+\int_{\ell_1}^{\varphi_1}S_1(u_1)^{\varphi_1}\tilde{d}_qu_1\Big)\right|$$
$$\leq\frac{q^2(\ell_1-\vartheta_1)^2}{\varphi_1-\vartheta_1}\Big(\int_0^1u_{10}^n\tilde{d}_qu_1\Big)^{\frac{1}{n}}\Big[\int_0^1\Big(h(qu_1)|_{\vartheta_1}\tilde{D}_qS_1(\ell_1)|^m+h(1-qu_1)|_{\vartheta_1}\tilde{D}_qS_1(\vartheta_1)|^m\Big)\tilde{d}_qu_1\Big]^{\frac{1}{m}}$$
$$+\frac{q^2(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big(\int_0^1u_1^n\tilde{d}_qu_1\Big)^{\frac{1}{n}}\Big[\int_0^1\Big(h(qu_1)|^{\varphi_1}\tilde{D}_qS_1(\ell_1)|^m+h(1-qu_1)|^{\varphi_1}\tilde{D}_qS_1(\varphi_1)|^m\Big)\tilde{d}_qu_1\Big]^{\frac{1}{m}}$$
$$\leq\frac{q^2(\ell_1-\vartheta_1)^2}{\varphi_1-\vartheta_1}\tilde{M}_1\Big[\Big(\int_0^1u_1^n\tilde{d}_qu_1\Big)^{\frac{1}{n}}\Big(\int_0^1\Big(h(qu_1)+h(1-qu_1)\Big)\tilde{d}_qu_1\Big)^{\frac{1}{m}}\Big]$$
$$+\frac{q^2(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\tilde{M}_2\Big[\Big(\int_0^1u_1^n\tilde{d}_qu_1\Big)^{\frac{1}{n}}\Big(\int_0^1\Big(h(qu_1)+h(1-qu_1)\Big)\tilde{d}_qu_1\Big)^{\frac{1}{m}}\Big]$$
$$\leq\frac{M_1q^2}{[n+1]^{\frac{1}{n}}}\Big[\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big]\Big(\int_0^1\Big(h(qu_1)+h(1-qu_1)\Big)\tilde{d}_qu_1\Big)^{\frac{1}{m}}.$$

□

Remark 3.8. ***(i)*** *If h is supper additive and $q = 1$ then, the inequality* (21) *becomes Theorem 3 of [26];*
(ii) *If $h(u_1) = u_1$, then the inequality* (21) *reduces to Theorem 5 of [21].*

Corollary 3.9. *If we take $h(u_1) = u_1^s$ and $s \in (0, 1]$ in inequality* (21)*, then it reduces to the following inequality involving s-convex function:*

$$\begin{aligned}
&\left|\left(S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\left(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\right)\right)\right| \\
&\leq \frac{M_1 q^2}{[n+1]^{\frac{1}{n}}}\left[\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\right]\left(\int_0^1 \left((qu_1)^s + (1 - qu_1)^s\right)\tilde{d}_q u_1\right)^{\frac{1}{m}} \\
&= \frac{M_1 q^2}{[n+1]^{\frac{1}{n}}}\left[\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\right]\left(q^s \frac{u_1^{s+1}}{[s+1]}\Big|_0^1 + \frac{q^2(1-u_1)_{\tilde{q}}^{s+1}}{[s+1]}\Big|_0^1\right)^{\frac{1}{m}} \\
&\qquad = \frac{M_1 q^2}{[n+1]^{\frac{1}{n}}}\left[\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\right]\left(\frac{q^s}{[s+1]} - \frac{q^2}{[s+1]}\right)^{\frac{1}{m}}.
\end{aligned}$$

Corollary 3.10. *If we choose $h(u_1) = 1$, then* (21) *becomes the following inequality involving P-convex function:*

$$\begin{aligned}
&\left|\left(S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\left(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\right)\right)\right| \\
&\qquad \leq \frac{(2)^{1/m} M_1 q^2}{[n+1]^{1/n}}\left(\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\right).
\end{aligned}$$

Corollary 3.11. *If we choose $h(u_1) = \frac{1}{u_1}$ and $u_1 \in (0, 1)$, then the inequality* (21) *becomes the following inequality involving Godunova-Levin function:*

$$\begin{aligned}
&\left|\left(S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\left(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\right)\right)\right| \\
&\leq \frac{M_1 q^2}{[n+1]^{\frac{1}{n}}}\left[\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\right]\left(\int_0^1 \left(\frac{1}{qu_1} + \frac{1}{1 - qu_1}\right)\tilde{d}_q u_1\right)^{\frac{1}{m}} \\
&= \frac{M_1 q^2}{[n+1]^{\frac{1}{n}}}\left[\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\right]\left(\frac{u_1^{-(1+1)}}{q[-(1+1)]}\Big|_0^1 + \frac{q^2(1-u_1)_{\tilde{q}}^{-(1+1)}}{[-(1+1)]}\Big|_0^1\right)^{\frac{1}{m}} \\
&\qquad = \frac{M_1 q^2}{[n+1]^{\frac{1}{n}}}\left[\frac{(\ell_1 - \vartheta_1)^2 + (\varphi_1 - \ell_1)^2}{\varphi_1 - \vartheta_1}\right]\left(\frac{1}{q[-2]} - \frac{q^2}{[-2]}\right)^{\frac{1}{m}}.
\end{aligned}$$

Corollary 3.12. *If $h(u_1) = \frac{1}{u_1^s}$, $s \in (0, 1)$ and $u_1 \in (0, 1)$ in inequality* (21)*, then it reduces to the following inequality* *involving s-Godunova-Levin function in second sense:*

$$\begin{aligned}
&\left|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\Big)\right| \\
&\le \frac{M_1 q^2}{[n+1]^{\frac{1}{n}}}\Big[\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big]\Big(\int_0^1 \Big(\frac{1}{(qu_1)^s} + \frac{1}{(1-qu_1)^s}\Big)\tilde{d}_q u_1\Big)^{\frac{1}{m}} \\
&= \frac{M_1 q^2}{[n+1]^{\frac{1}{n}}}\Big[\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big]\Big(\frac{1}{q^s}\int_0^1 u_1^{-s}\tilde{d}_q u_1 + \int_0^1 (1-qu_1)^{-s}\tilde{d}_q u_1\Big)^{\frac{1}{m}} \\
&= \frac{M_1 q^2}{[n+1]^{\frac{1}{n}}}\Big[\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big]\Big(\frac{1}{q^s}\frac{u_1^{-(s+1)}}{[-(s+1)]}\Big|_0^1 + \frac{q^2(1-u_1)_{\tilde{q}}^{-(s+1)}}{[-(s+1)]}\Big|_0^1\Big)^{\frac{1}{m}} \\
&\qquad = \frac{M_1 q^2}{[n+1]^{\frac{1}{n}}}\Big[\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big]\Big(\frac{1}{q[-(s+1)]} - \frac{q^2}{[-(s+1)]}\Big)^{\frac{1}{m}}.
\end{aligned}$$

Theorem 3.13. *If all the suppositions of Theorem 3.7 hold. Then we also have the following inequality:*

$$\begin{aligned}
&\left|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\Big)\right| \\
&\qquad \le M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-(1/m)}\Big[\int_0^1 \Big(u_1 h(qu_1) + u_1 h(1-qu_1)\Big)\tilde{d}_q u_1\Big]^{1/m}, \quad (26)
\end{aligned}$$

for each $\ell_1 \in [\vartheta_1, \varphi_1]$.

Proof. *By using* (11) *and the q-symmetric power-mean inequality, we get*

$$\begin{aligned}
&\left|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\Big)\right| \\
&\le \frac{q^2(\ell_1-\vartheta_1)^2}{\varphi_1-\vartheta_1}\int_0^1 u_1|_{\vartheta_1}\tilde{D}_q S_1(qu_1\ell_1 + (1-qu_1)\vartheta_1)|\tilde{d}_q u_1 \\
&+ \frac{q^2(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\int_0^1 u_1|^{\varphi_1}\tilde{D}_q S_1(qu_1\ell_1 + (1-qu_1)\varphi_1)|\tilde{d}_q u_1 \\
&\le \frac{q^2(\ell_1-\vartheta_1)^2}{\varphi_1-\vartheta_1}\Big(\int_0^1 u_1\tilde{d}_q u_1\Big)^{1-(1/m)}\Big(\int_0^1 u_1|_{\vartheta_1}\tilde{D}_q S_1(qu_1\ell_1 + (1-qu_1)\vartheta_1|^m\tilde{d}_q u_1\Big)^{1/m} \\
&\qquad + \frac{q^2(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big(\int_0^1 u_1\tilde{d}_q u_1\Big)^{1-(1/m)}\Big(\int_0^1 u_1|^{\varphi_1}\tilde{D}_q S_1(qu_1\ell_1 + (1-qu_1)\varphi_1)|^m\tilde{d}_q u_1\Big)^{1/m}. \quad (27)
\end{aligned}$$

Consider,

$$\begin{aligned}
&\int_0^1 u_1|_{\vartheta_1}\tilde{D}_q S_1(qu_1\ell_1 + (1-qu_1)\vartheta_1)|^m\tilde{d}_q u_1 \\
&\le \int_0^1 u_1 h(qu_1)|_{\vartheta_1}\tilde{D}_q S_1(\ell_1)|^m\tilde{d}_q u_1 + \int_0^1 u_1 h(1-qu_1)|_{\vartheta_1}\tilde{D}_q S_1(\vartheta_1)|^m\tilde{d}_q u_1 \\
&\qquad \le \hat{M}_1^m\int_0^1 \Big(u_1 h_1(qu_1) + u_1 h_1(1-qu_1)\Big)\tilde{d}_q u_1, \quad (28)
\end{aligned}$$

and

$$\int_0^1 u_1|^{\varphi_1}\tilde{D}_q S_1(qu_1(\ell_1) + (1-qu_1)\varphi_1|^m\tilde{d}_q u_1 \le \tilde{M}_2^m\int_0^1 \Big(u_1 h(qu_1) + u_1 h(1-qu_1)\Big)\tilde{d}_q u_1. \quad (29)$$

Use (28) *and* (29) *in* (27) *to get,*

$$\Big|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\Big)\Big|$$
$$\leq M_1 q^2\Big(\int_0^1 u_1 \tilde{d}_q u_1\Big)^{1-\frac{1}{m}}\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big[\int_0^1 [u_1 h(qu_1) + u_1 h(1-qu_1)]\tilde{d}_q u_1\Big]^{\frac{1}{m}}$$
$$\leq M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}}\Big[\int_0^1 \Big(u_1 h(qu_1) + u_1 h(1-qu_1)\Big)\tilde{d}_q u_1\Big]^{\frac{1}{m}}.$$

□

Remark 3.14. ***(i)*** *If h is supper multiplicative and* $q = 1$*, then the inequality* (26) *reduces to [26, Theorem 4];*
(ii) *If* $h(u_1) = u_1$*, then the inequality* (26)*, becomes the inequality [21, Theorem 8].*

Corollary 3.15. *If we choose* $h(u_1) = u_1^s$ *where* $s \in (0, 1]$ *and* $u_1 \in (0, 1)$ *in inequality* (26)*, then the inequality* (26) *reduces to the following form involving s-convex function:*

$$\Big|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\Big)\Big|$$
$$\leq M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-(1/m)}\Big[\int_0^1 [u_1(qu_1)^s + u_1(1-qu_1)^s\tilde{d}_q u_1]\Big]^{\frac{1}{m}}$$

$$= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}}\Big[\int_0^1 \Big(q^s u_1^{s+1} + \frac{q^2 u_1 \tilde{D}_q(1-qu_1)^{s+1}}{[s+1]}\Big)\tilde{d}_q u_1\Big]^{\frac{1}{m}}$$
$$= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}}\Big[\frac{q^s u_1^{s+2}}{[s+2]}\Big|_0^1 + \frac{u_1 q^2(1-u_1)_{\tilde{q}}^{s+1}}{[s+1]}\Big|_0^1$$
$$- \int_0^1 \frac{q^2(1-u_1)_{\tilde{q}}^{s+1}}{[s+1]}\tilde{d}_q u_1\Big]^{\frac{1}{m}}$$
$$= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}}\Big[\frac{q^s}{[s+2]} - \frac{q^2}{[s+1]}\Big(\frac{qu_1\tilde{D}_q(1-q^{-1}u_1)^{s+2}}{[s+2]}\Big|_0^1\Big)\Big]^{\frac{1}{m}}$$
$$= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}}\Big[\frac{q^s}{[s+2]} - \frac{q^3}{[s+1]}\Big(\frac{(1-q^{-1})^{s+2}}{[s+2]} + \frac{1}{[s+2]}\Big)\Big]^{\frac{1}{m}}.$$

Corollary 3.16. *For* $h(u_1) = 1$ *in* (26)*, then the inequality* (26) *becomes the next inequality for P-convex functions:*

$$\Big|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\Big)\Big|$$
$$\leq (2)^{1/m} M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2+(\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-(1/m)}.$$

Corollary 3.17. *If we take* $h(u_1) = \frac{1}{u_1}$ *for* $u_1 \in (0, 1)$*, then the inequality* (26) *becomes the following inequality* *involving Godunova-Levin function:*

$$\begin{aligned}
&\left|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\Big)\right| \\
&\leq M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}}\Big[\int_0^1 \frac{1}{q}\tilde{d}_q u_1 + \int_0^1 \frac{u_1}{1-qu_1}\tilde{d}_q u_1\Big]^{\frac{1}{m}} \\
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}}\Big[\frac{1}{q} + \int_0^1 u_1(1-qu_1)^{-1}\tilde{d}_q u_1\Big]^{\frac{1}{m}} \\
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}}\Big[\frac{1}{q} + \frac{qu_1\tilde{D}_q(1-q^{-1}u_1)_{\tilde{q}}^{-(1+1)}}{[-(1+1)]}\Big|_0^1 \\
&- \int_0^1 \frac{q(1-q^{-1}u_1)_{\tilde{q}}^{-(1+1)}}{[-(1+1)]}\tilde{d}_q u_1\Big]^{\frac{1}{m}} \\
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}}\Big[\frac{1}{q} + \frac{q(1-q^{-1})_{\tilde{q}}^{-2}}{[-2]} - \frac{q^2}{[-2]}\frac{u_1(1-q^{-1}u_1)_{\tilde{q}}^{-3}}{[-3]}\Big|_0^1\Big]^{\frac{1}{m}} \\
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}}\Big[\frac{1}{q} + \frac{q(1-q^{-1})_{\tilde{q}}^{-2}}{[-2]} \\
&\qquad - \frac{q^2}{[-2]}\Big(\frac{(1-q^{-1})_{\tilde{q}}^{-3}}{[-3]} - \frac{1}{[-3]}\Big)\Big]^{\frac{1}{m}}.
\end{aligned}$$

Corollary 3.18. *For $h(u_1) = \frac{1}{u_1^s}$, $s \in (0,1)$ and $u_1 \in (0,1)$. The inequality (26) takes the form of the following inequality involving s-Godunova-Levin function in second sense:*

$$\begin{aligned}
&\left|S_1(\ell_1) - \frac{1}{\varphi_1 - \vartheta_1}\Big(\int_{\vartheta_1}^{\ell_1} S_1(u_1)_{\vartheta_1}\tilde{d}_q u_1 + \int_{\ell_1}^{\varphi_1} S_1(u_1)^{\varphi_1}\tilde{d}_q u_1\Big)\right| \\
&\qquad \leq M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-(1/m)}\Big[\int_0^1 \frac{u_1}{qu_1^s}\tilde{d}_q u_1 + \int_0^1 \frac{u_1}{(1-qu_1)^s}\tilde{d}_q u_1\Big]^{1/m}
\end{aligned}$$

$$\begin{aligned}
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}}\Big[\frac{1}{q}\int_0^1 u_1^{1-s}\tilde{d}_q u_1 + \int_0^1 u_1(1-qu_1)^{-s}\tilde{d}_q u_1\Big]^{\frac{1}{m}} \\
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}}\Big[\frac{u_1^{2-s}}{[2-s]}\Big|_0^1 + \int_0^1 \frac{u_1 q^2\tilde{D}_q(1-u_1)_{\tilde{q}}^{-(s+1)}}{[-(s+1)]}\tilde{d}_q u_1\Big]^{\frac{1}{m}} \\
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}}\Big[\frac{1}{[2-s]} - \frac{q^2}{[-(s+1)]}\Big(u_1(1-u_1)_{\tilde{q}}^{-(s+1)}\Big|_0^1 \\
&- \frac{q^2}{[-(s+1)]}\int_0^1 (1-u_1)_{\tilde{q}}^{-(s+1)}\tilde{d}_q u_1\Big)\Big]^{\frac{1}{m}} \\
&= M_1 q^2\Big(\frac{(\ell_1-\vartheta_1)^2 + (\varphi_1-\ell_1)^2}{\varphi_1-\vartheta_1}\Big)\Big(\frac{1}{[2]}\Big)^{1-\frac{1}{m}} \\
&\qquad \times\Big[\frac{1}{[2-s]} - \frac{q^2}{[-(s+1)]}\Big(\frac{q(1-q^{-1})_{\tilde{q}}^{-(s+2)}}{[-(s+2)]} - \frac{q}{[-(s+2)]}\Big)\Big]^{1/m}.
\end{aligned}$$

4. Examples

In this section, we present some examples to show the validity of Ostrowski type inequalities obtained in Section 3. Graphical presentation of these inequalities are also included.
Let us first calculate the value of M_1. Choose $q = 1/2$, $S_1(\ell_1) = {\ell_1}^2$, $\vartheta_1 = -1$ and $\varphi_1 = 1$, to get:

$$_{\vartheta_1}\tilde{D}_q S_1(\ell_1) = \frac{5\ell_1}{2} + \frac{1}{2},$$

and

$$^{\varphi_1}\tilde{D}_q S_1(\ell_1) = \frac{5\ell_1}{2} - \frac{1}{2}.$$

So, in the following examples we choose the value of $M_1 = 3$ because $M_1 = \max\left\{\hat{M}_1, \tilde{M}_2\right\}$, where $|_{\vartheta_1}\tilde{D}_q S_1(\ell_1)| \leq \hat{M}_1$ and $|^{\varphi_1}\tilde{D}_q S_1(\ell_1)| \leq \tilde{M}_2$.

Example 4.1. *Let $S_1(\ell_1) = \ell_1^2$ and $h(u_1) = u_1^2$, then for fixed $q = 1/2$; $\vartheta_1 = -1$; $\varphi_1 = 1$; and $M_1 = 3$. Theorem 3.1 has the following evaluation:*

$$\left|S_1(\ell_1) - \frac{1}{2}\Big(\int_{-1}^{\ell_1} S_1(u_1)_{-1}\tilde{d}_q u_1 + \int_{\ell_1}^{1} S_1(u_1)^{1}\tilde{d}_q u_1\Big)\right|$$

$$\leq \frac{3}{4}\Big(\frac{(\ell_1+1)^2 + (1-\ell_1)^2}{2}\Big)\Big[\int_0^1 \Big(u_1(qu_1)^2 + u_1(1-qu_1)^2\Big)\tilde{d}_q u_1\Big]$$

$$\Rightarrow \Big|\ell_1^2 - \frac{1}{2}(1-q^2)\Big((\ell_1+1)\sum_{n=0}^{\infty} q^{2n}(q^{2n+1}\ell_1 - (1-q^{2n+1}))^2$$

$$+ (1-\ell_1)\sum_{n=0}^{\infty} q^{2n}(q^{2n+1}\ell_1 + (1-q^{2n+1}))^2\Big)\Big| \leq \frac{3}{4}(\ell_1^2+1)\Big[\int_0^1 (2q^2u_1^3 - 2qu_1^2 + u_1)\tilde{d}_q u_1\Big]$$

$$\Rightarrow \Big|\ell_1^2 - \frac{1}{2}(\Big(\frac{4}{21}(6\ell_1^2+2) + 2 - \frac{4}{5}(2\ell_1^2+2)\Big)\Big| \leq \frac{3}{4}(\ell_1^2+1)\Big(\frac{2q^2}{[4]} - \frac{2q}{[3]} + \frac{1}{[2]}\Big).$$

Hence

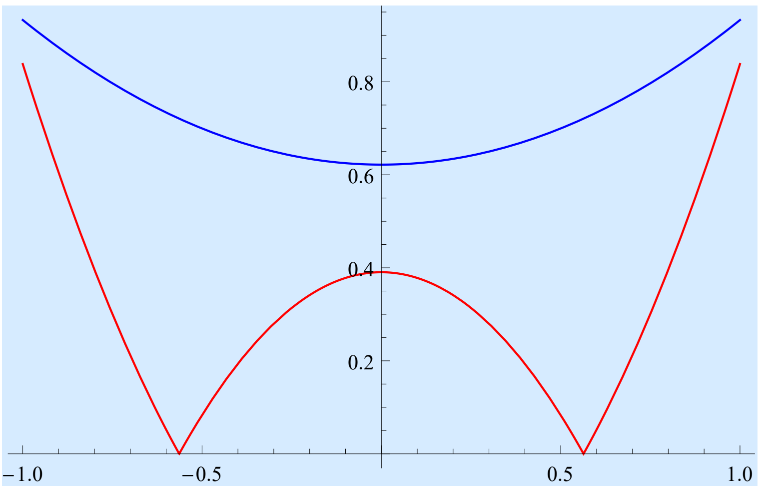

Figure 1: Blue and Red lines show right and left sides of inequality (30) respectively.

$$\left|\frac{129\ell_1^2 - 41}{105}\right| \leq \frac{3}{4}\Big(\frac{740}{1785}\Big)(\ell_1^2+1). \tag{30}$$

Example 4.2. *Take* $S_1(\ell_1) = \ell_1^2$ *and* $h(u_1) = u_1^2$*;* $\vartheta_1 = -1$*;* $\varphi_1 = 1$*;* $n = 2$ *;* $m = 2$ *and* $M_1 = 3$ *in Theorem 3.7, then Theorem 3.7 becomes:*

$$\left|S_1(\ell_1) - \frac{1}{2}\Big(\int_{-1}^{\ell_1} S_1(u_1)_{-1}\tilde{d}_q u_1 + \int_{\ell_1}^{1} S_1(u_1)^{1}\tilde{d}_q u_1\Big)\right| \leq \frac{3}{4}\Big(\frac{16}{21}\Big)^{\frac{1}{2}}\Big(\frac{(\ell_1+1)^2+(1-\ell_1)^2}{2}\Big)$$
$$\Big[\int_0^1 \Big((qu_1)^2 + (1-qu_1)^2\Big)\tilde{d}_q u_1\Big]^{\frac{1}{2}}$$
$$\Rightarrow \Big|\ell_1^2 - \frac{1}{2}(1-q^2)\Big((\ell_1+1)\sum_{n=0}^{\infty} q^{2n}(q^{2n+1}\ell_1 - (1-q^{2n+1}))^2 + (1-\ell_1)$$
$$\sum_{n=0}^{\infty} q^{2n}(q^{2n+1}\ell_1 + (1-q^{2n+1}))^2\Big)\Big| \leq \frac{3}{(21)^{\frac{1}{2}}}(\ell_1^2+1)\Big[\int_0^1 (2q^2u_1^2 - 2qu_1 + 1)\tilde{d}_q u_1\Big]^{\frac{1}{2}}$$
$$\Rightarrow \Big|\ell_1^2 - \frac{1}{2}(\Big(\frac{4}{21}(6\ell_1^2+2) + 2 - \frac{4}{5}(2\ell_1^2+2)\Big)\Big| \leq (\ell_1^2+1)\Big(\frac{2q^2}{[3]} - \frac{2q}{[2]} + 1\Big).$$

Therefore,

$$\left|\frac{129\ell_1^2 - 41}{105}\right| \leq \frac{3}{(21)^{\frac{1}{2}}}\Big(\frac{61}{105}\Big)^{\frac{1}{2}}(\ell_1^2+1). \tag{31}$$

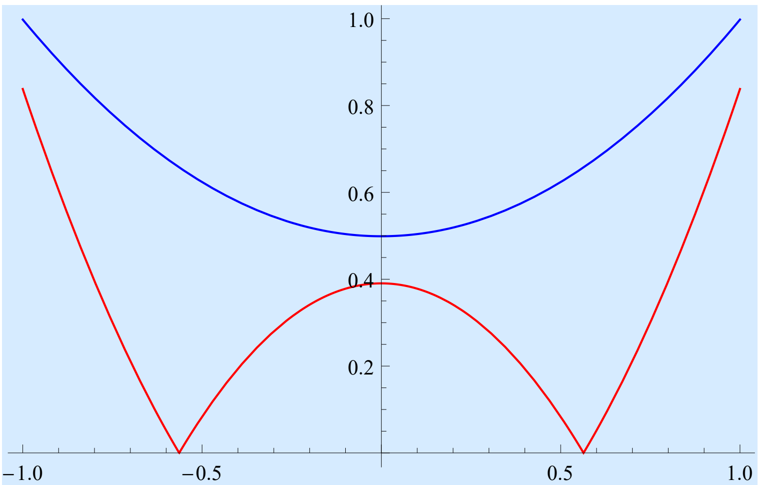

Figure 2: Blue and Red lines show the right and left sides of inequality (31).

Example 4.3. *Let* $S_1(\ell_1) = {\ell_1}^2$ *and* $h(u_1) = u_1^2$*;* $\vartheta_1 = -1$*;* $\varphi_1 = 1$*;* $q = 1/2$*;* $M_1 = 3$*; and* $m = 2$ *in Theorem 3.13.*

Then Theorem 3.13 yields the following result:

$$\left|S_1(\ell_1)-\frac{1}{2}\Big(\int_{-1}^{\ell_1} S_1(u_1)\,{}_{-1}\tilde{d}_q u_1+\int_{\ell_1}^{1} S_1(u_1)\,{}^{1}\tilde{d}_q u_1\Big)\right|\le\frac{3}{4}\Big(\frac{4}{5}\Big)^{\frac{1}{2}}\Big(\frac{(\ell_1+1)^2+(1-\ell_1)^2}{2}\Big)$$

$$\Big[\int_0^1\Big(u_1(qu_1)^2+u_1(1-qu_1)^2\Big)\tilde{d}_q u_1\Big]$$

$$\Rightarrow\left|\ell_1^2-\frac{1}{2}(1-q^2)\Big((\ell_1+1)\sum_{n=0}^{\infty}q^{2n}(q^{2n+1}\ell_1-(1-q^{2n+1}))^2+(1-\ell_1)\right.$$

$$\left.\sum_{n=0}^{\infty}q^{2n}(q^{2n+1}\ell_1+(1-q^{2n+1}))^2\Big)\right|$$

$$\le\frac{3}{4}\Big(\frac{4}{5}\Big)^{\frac{1}{2}}(\ell_1^2+1)\Big[\int_0^1(2q^2u_1^3-2qu_1^2+u_1)\tilde{d}_q u_1\Big]^{\frac{1}{2}}.$$

$$\Rightarrow\left|\frac{129\ell_1^2-41}{105}\right|\le\frac{3}{4}\Big(\frac{4}{5}\Big)^{\frac{1}{2}}\Big(\frac{740}{1785}\Big)^{\frac{1}{2}}(\ell_1^2+1). \tag{32}$$

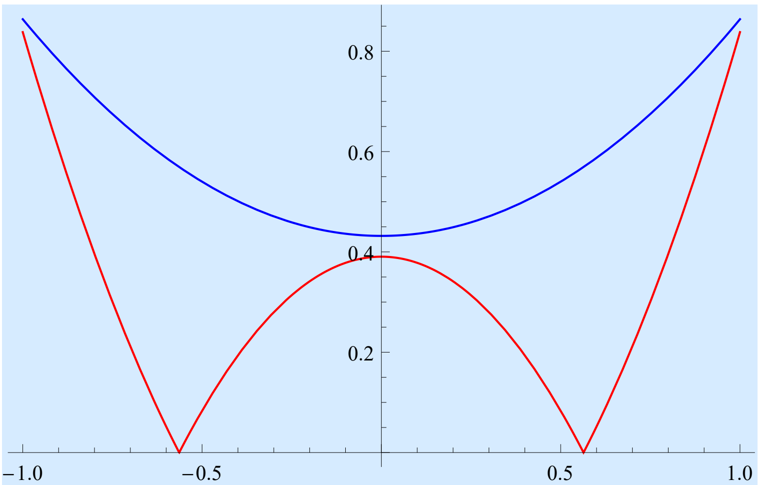

Figure 3: Blue and Red lines show the right and left sides of inequality (32).

5. Conclusion

This article utilises the left and right q-symmetric integrals to establish novel generalisations of Ostrowski type inequalities for h-convex functions. Additionally, several special novel findings include the cases of $h(s)=s^{-m}$, $m\in(0,1]$ and some existing inequalities of [6, 20, 21, 26]. It would be interesting to establish identical inequalities using some other types of convexities.

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7. Availability of data and materials

No data were used to support this study.

8. Competing interests

The authors declare that they have no competing interests.

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