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Coordinated convex mapping approach to trapezoid type inequalities with generalized conformable integrals

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Abstract. In this study, some new trapezoid type inequalities are generalized for convex functions in coordinates by means of generalized conformable fractional integrals. For functions with convex absolute values of their partial derivatives, some new trapezoid type inequalities are obtained using the well-known Holder and Power mean inequalities. In addition, some findings of this study include some results based on Riemann Liouville fractional integrals and Riemann integral.

1. Introduction

Convexity theory is a special field for mathematics. Especially, it is preferred by many researchers as it has had a wide application area lately. For instance, engineering applications, optimization theory, energy systems and physics. The definition known for convex functions is as follows:

Definition 1. *[26]Let I be convex set on $\mathbb{R}$. The function $\chi : I \to \mathbb{R}$ is said to be convex on I, if it satisfies the following inequality:*

$$\chi(t\delta + (1-t)\,\rho) \leq t\chi(\delta) + (1-t)\chi(\rho) \tag{1.1}$$

for all $(\delta, \rho) \in I$ and $t \in [0,1]$. The mapping χ is a concave on I if the inequality (1.1) holds in reversed direction for all $t \in [0,1]$ and $\delta, \rho \in I$.

Firstly let us now consider a bidemensional interval $\Delta := [\lambda_1, \lambda_2] \times [\mu_1, \mu_2]$ in $\mathbb{R}^2$ with $\lambda_1 < \lambda_2$ and $\mu_1 < \mu_2$. A mapping $\chi : \Delta \to \mathbb{R}$ the definition for a mapping in co-ordinated convex is as follows:

Definition 2. *[25]A function $\chi : \Delta \to \mathbb{R}$ is called co-ordinated convex on Δ, for all $(\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \Delta$ and $t, s \in [0,1]$, if it satifies the following inequality:*

$$\begin{aligned} & \chi(t\lambda_1 + (1-t)\,\lambda_2, s\mu_1 + (1-s)\,\mu_2) \\ \leq\; & ts\,\chi(\lambda_1, \mu_1) + t(1-s)\chi(\lambda_1, \mu_2) + s(1-t)\chi(\lambda_2, \mu_1) + (1-t)(1-s)\chi(\lambda_2, \mu_2). \end{aligned} \tag{1.2}$$

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It is clear that all convex functions are convex on co-ordinates. However, not every function that is a convex function in coordinates has to be convex (see, [25]).

The most frequently used field in the analysis of mathematics in convex mappings is integral inequalities. These inequalities those which were composed by C. Hermite and J. Hadamard are prominent in literature (see, e.g., [26], [11, p.137],[16]). These inequalities state that if $\chi : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\lambda_1, \lambda_2 \in I$ with $\lambda_1 < \lambda_2$ then,

$$\chi\left(\frac{\lambda_1+\lambda_2}{2}\right) \leq \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2} \chi(\delta)d\delta \leq \frac{\chi(\lambda_1)+\chi(\lambda_2)}{2}. \tag{1.3}$$

If χ is concave, the inequality that is stated above is provided reversely. The references may be seen for the examples of Hermite-Hadamard's inequality for some convex function on the co-ordinates in mathematics literature ([25]-[18]). Recently, this inequality has been expanded by many researchers. The right side of the Hermite-Hadamard inequality, namely the trapezoid type inequality, has been the focus of many studies. Trapezoid type inequalities for convex functions were first derived by Dragomir and Agarwal in[27]. In [20], Sarikaya et al. generalized the inequalities (1.3) for fractional integrals and the authors also proved some corresponding trapezoid type inequalities.

In ([25]), Dragomir proved the Hermite-Hadamard inequality, which formed the basis of this article and is valid for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 1. *Suppose that $\chi : \Delta \to \mathbb{R}$ is co-ordinated convex, then we have the following inequalities:*

$$\begin{aligned}
\chi\left(\frac{\lambda_1+\lambda_2}{2}, \frac{\mu_1+\mu_2}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2} \chi\left(\delta, \frac{\mu_1+\mu_2}{2}\right)d\delta + \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \chi\left(\frac{\lambda_1+\lambda_2}{2}, \rho\right)d\rho\right] \\
&\leq \frac{1}{(\lambda_2-\lambda_1)(\mu_2-\mu_1)}\int_{\lambda_1}^{\lambda_2}\int_{\mu_1}^{\mu_2} \chi(\delta,\rho)d\rho d\delta \\
&\leq \frac{1}{4}\left[\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2} \chi(\delta,\mu_1)d\delta + \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2} \chi(\delta,\mu_2)d\delta\right. \\
&+ \left.\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \chi(\lambda_1,\rho)d\rho + \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \chi(\lambda_2,\rho)d\rho\right] \\
&\leq \frac{\chi(\lambda_1,\mu_1)+\chi(\lambda_1,\mu_2)+\chi(\lambda_2,\mu_1)+\chi(\lambda_2,\mu_2)}{4}.
\end{aligned} \tag{1.4}$$

The above inequalities are sharp. The inequalities in (1.4) hold in reverse direction if the mapping χ is a co-ordinated concave mapping.

The fractional calculus [[4]-[9]-[10]-[24]-[8]] is defined as any random real number or derivative and integral calculus in complex order. As a result of having various uses in other branches besides mathematics it is an updated study area. These definitions are the most notable definitions of Caputo, Riemann- Liouville, Grünwald- Letnikov play an important role in many fields such as physics, biology, engineering. However, it is known that these definitions have some diffuculties in spite of the availability. For instance, unless derivative of order in Riemann-Liouville fractional derivative definition is a naturel number, derivative of fixed function is not 0. Likewise, the function f must be differentiable in Caputo fractional derivatives. Moreover, many definitions of fractional derivatives do not provide the quotient formula, the product of two functions, and the chain rule. In order to overcome these and similar difficulties, comformable fractional derivative was defined by Khalil et al. in [[23]]. Khalil et al. described the higher order ($\alpha > 1$) fractional derivative and the fractional integral of order ($0 < \alpha \leq 1$). They also proved important theorems such as the product rule, the fractional mean value theorem. They solved conformable fractional differential equations for fractional exponential functions (see, [23]- [28]-[2]-[7]). Thus, conformable fractional integrals became an important field of study for many researchers. As an example to the authors working in this field; Abdeljawad, Khalil, Abdelhakim, Khan, Akkurt, Sarikaya, Budak etc. For all this, please see [[28]-[23]-[1]-[29]-[3]-[6]-[21]-[22]).

The definitions and mathematical underpinnings of conformable fractional calculus principles that are used later in this study are provided below.

Definition 3. *[4]For $\xi \in L_1[\eta_1, \eta_2]$, the Riemann-Liouville integrals of order $\alpha > 0$ are given by*

$$J^{\alpha}_{\eta_1+}\xi(\delta) = \frac{1}{\Gamma(\alpha)} \int_{\eta_1}^{\delta} (\delta - t)^{\alpha-1} \xi(t)dt, \quad \delta > \eta_1 \tag{1.5}$$

and

$$J^{\alpha}_{\eta_2-}\xi(\delta) = \frac{1}{\Gamma(\alpha)} \int_{\delta}^{\eta_2} (t - \delta)^{\alpha-1} \xi(t)dt, \quad \delta < \eta_2, \tag{1.6}$$

respectively. The Riemann-Liouville integrals will be equal to their classical integrals for the condition $\alpha = 1$.

Definition 4. *[19]Let $\xi \in L_1([\eta_1, \eta_2] \times [\vartheta_1, \vartheta_2])$. The Riemann-Liouville integrals $J^{\alpha,\beta}_{\eta_1+,\vartheta_1^+}, J^{\alpha,\beta}_{\eta_1+,\vartheta_2^-}, J^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}$ and $J^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}$ of order $\alpha, \beta > 0$ with $\eta_1, \vartheta_1 \geq 0$ are defined by*

$$J^{\alpha,\beta}_{\eta_1+,\vartheta_1^+}\xi(\delta, \rho) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\eta_1}^{\delta} \int_{\vartheta_1}^{\rho} (\delta - t)^{\alpha-1} (\rho - s)^{\beta-1} \xi(t, s)dsdt, \quad \delta > \eta_1, \rho > \vartheta_1, \tag{1.7}$$

$$J^{\alpha,\beta}_{\eta_1+,\vartheta_2^-}\xi(\delta, \rho) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\eta_1}^{\delta} \int_{\rho}^{\vartheta_2} (\delta - t)^{\alpha-1} (s - \rho)^{\beta-1} \xi(t, s)dsdt, \quad \delta > \eta_1, \rho < \vartheta_2, \tag{1.8}$$

$$J^{\alpha,\beta}_{\eta_2-,\vartheta_1^+}\xi(\delta, \rho) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\delta}^{\eta_2} \int_{\vartheta_1}^{\rho} (t - \delta)^{\alpha-1} (\rho - s)^{\beta-1} \xi(t, s)dsdt, \quad \delta < \eta_2, \rho > \vartheta_1, \tag{1.9}$$

and

$$J^{\alpha,\beta}_{\eta_2-,\vartheta_2^-}\xi(\delta, \rho) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\delta}^{\eta_2} \int_{\rho}^{\vartheta_2} (t - \delta)^{\alpha-1} (s - \rho)^{\beta-1} \xi(t, s)dsdt, \quad \delta < \eta_2, \rho < \vartheta_2, \tag{1.10}$$

respectively. Here Γ is the Gama function.

Definition 5. *[8] For $\xi \in L_1[\eta_1, \eta_2]$, the fractional conformable integral operator ${}^{\beta}I^{\alpha}_{\eta_1+}f$ and ${}^{\beta}I^{\alpha}_{\eta_2-}f$ of order $\beta > 0$ and $\alpha \in (0, 1]$ are presented by*

$${}^{\beta}\mathcal{I}^{\alpha}_{\eta_1+}\xi(\delta) = \frac{1}{\Gamma(\beta)} \int_{\eta_1}^{\delta} \left(\frac{(\delta - \eta_1)^{\alpha} - (t - \eta_1)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\xi(t)}{(t - \eta_1)^{1-\alpha}} dt, \quad t > \eta_1 \tag{1.11}$$

and

$${}^{\beta}\mathcal{I}^{\alpha}_{\eta_2-}\xi(\delta) = \frac{1}{\Gamma(\beta)} \int_{\delta}^{\eta_2} \left(\frac{(\eta_2 - \delta)^{\alpha} - (\eta_2 - t)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\xi(t)}{(\eta_2 - t)^{1-\alpha}} dt, \quad t < \eta_2, \tag{1.12}$$

respectively.

Definition 6. *[14]Let $\xi \in L_1([\eta_1,\eta_2]\times[\vartheta_1,\vartheta_2])$ and let $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, α, $\beta \in \mathbf{C}$, $Re(\alpha) > 0$ and $Re(\beta) > 0$. The generalized conformable integral of order α, β of $\xi(\delta,\rho)$ is defined by;*

$$\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\right)(\delta,\rho) = \left[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\eta_1}^{\delta}\int_{\vartheta_1}^{\rho}\left(\frac{(\delta-\eta_1)^{\gamma_1}-(t-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \times\left(\frac{(\rho-\vartheta_1)^{\gamma_2}-(s-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(t,s)}{(t-\eta_1)^{1-\gamma_1}(s-\vartheta_1)^{1-\gamma_2}}dsdt\right], \tag{1.13}$$

$$\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\right)(\delta,\rho) = \left[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\delta}^{\eta_2}\int_{\vartheta_1}^{\rho}\left(\frac{(\eta_2-\delta)^{\gamma_1}-(\eta_2-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \times\left(\frac{(\rho-\vartheta_1)^{\gamma_2}-(s-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(t,s)}{(\eta_2-t)^{1-\gamma_1}(s-\vartheta_1)^{1-\gamma_2}}dsdt\right], \tag{1.14}$$

$$\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\right)(\delta,\rho) = \left[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\eta_1}^{\delta}\int_{\rho}^{\vartheta_2}\left(\frac{(\delta-\eta_1)^{\gamma_1}-(t-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \times\left(\frac{(\vartheta_2-\rho)^{\gamma_2}-(\vartheta_2-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(t,s)}{(t-\eta_1)^{1-\gamma_1}(\vartheta_2-s)^{1-\gamma_2}}dsdt\right], \tag{1.15}$$

and

$$\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\right)(\delta,\rho) = \left[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\delta}^{\eta_2}\int_{\rho}^{\vartheta_2}\left(\frac{(\eta_2-\delta)^{\gamma_1}-(\eta_2-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \times\left(\frac{(\vartheta_2-\rho)^{\gamma_2}-(\vartheta_2-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(t,s)}{(\eta_2-t)^{1-\gamma_1}(\vartheta_2-s)^{1-\gamma_2}}dsdt\right]. \tag{1.16}$$

Remark 1. *[14] If $\gamma_1 = \gamma_2 = 1$ in (1.13), (1.14),(1.15) and (1.16), we have (1.7)-(1.10) the Fractional integrals of the functions of two variables.*

Remark 2. *[14] If we consider $\alpha = 1$ and $\beta = 1$ in (1.13), (1.14),(1.15) and (1.16), we have*

$$\left(I^{1,1}_{\eta_1^+,\vartheta_1^+}\xi\right)(\delta,\rho) = \int_{\eta_1}^{\delta}\int_{\vartheta_1}^{\rho}\frac{\xi(t,s)}{(t-\eta_1)^{1-\gamma_1}(s-\vartheta_1)^{1-\gamma_2}}dsdt, \tag{1.17}$$

$$\left(I^{1,1}_{\eta_2^-,\vartheta_1^+}\xi\right)(\delta,\rho) = \int_{\delta}^{\eta_2}\int_{\vartheta_1}^{\rho}\frac{\xi(t,s)}{(\eta_2-t)^{1-\gamma_1}(s-\vartheta_1)^{1-\gamma_2}}dsdt, \tag{1.18}$$

$$\left(I^{1,1}_{\eta_1^+,\vartheta_2^-}\xi\right)(\delta,\rho) = \int_{\eta_1}^{\delta}\int_{\rho}^{\vartheta_2}\frac{\xi(t,s)}{(t-\eta_1)^{1-\gamma_1}(\vartheta_2-s)^{1-\gamma_2}}dsdt, \tag{1.19}$$

and

$$\left(I^{1,1}_{\eta_2^-,\vartheta_2^-}\xi\right)(\delta,\rho) = \int_{\delta}^{\eta_2}\int_{\rho}^{\vartheta_2}\frac{\xi(t,s)}{(\eta_2-t)^{1-\gamma_1}(\vartheta_2-s)^{1-\gamma_2}}dsdt. \tag{1.20}$$

Theorem 2. *[15] Assume ξ is a co-ordinated convex function that goes from $[\eta_1,\eta_2]\times[\vartheta_1,\vartheta_2]$ into $\mathbb{R}$ and let $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, α, $\beta \in (0,1]$, $Re(\alpha) > 0$ and $Re(\beta) > 0$. The following inequality holds for generalized conformable fractional integrals.*

$$\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \tag{1.21}$$

$$
\begin{aligned}
&\leq \frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&\quad+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+\left.{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\
&\leq \frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}.
\end{aligned}
$$

2. Trapezoid Type Inequalities for Co-Ordinated Convex Functions

Let's start with the following lemma, which will form the basic structure of our article to obtain our main results.

Lemma 1. *Let $\xi:\triangle\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[\eta_1,\eta_2]\times[\vartheta_1,\vartheta_2]$ in $\mathbb{R}^2$ with $0\leq\eta_1<\eta_2$, $0\leq\vartheta_1<\vartheta_2$. If $\frac{\partial^2\xi(t,s)}{\partial t\partial s}\in L_1(\Delta)$, then the following identity:*

$$
\begin{aligned}
&\frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}} \qquad (2.1)\\
&\times\left[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&\left.+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]-A\\
&=\frac{\gamma_1^{\alpha}\gamma_2^{\beta}(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}\\
&\times\left[\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)dsdt\right.\\
&-\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)dsdt\\
&-\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)dsdt\\
&\left.+\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)dsdt\right]
\end{aligned}
$$

where

$$
\begin{aligned}
A&=\frac{2^{\gamma_2\beta-2}\,\gamma_2^{\beta}\,\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}} \qquad (2.2)\\
&\left[{}^{\gamma_2}I^{\beta}_{\vartheta_1^+}\xi\left(\eta_1,\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_2}I^{\beta}_{\vartheta_2^-}\xi\left(\eta_1,\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_2}I^{\beta}_{\vartheta_1^+}\xi\left(\eta_2,\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_2}I^{\beta}_{\vartheta_2^-}\xi\left(\eta_2,\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\
&+\frac{2^{\gamma_1\alpha-2}\,\gamma_1^{\alpha}\,\Gamma(\alpha+1)}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\\
&\left[{}^{\gamma_1}I^{\alpha}_{\eta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_1\right)+{}^{\gamma_1}I^{\alpha}_{\eta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_2\right)^{\gamma_1}I^{\alpha}_{\eta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_1\right)+{}^{\gamma_1}I^{\alpha}_{\eta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_2\right)\right].
\end{aligned}
$$

Proof. By integration by parts, we get

$$\begin{aligned}
I_1 &= \int_0^1\int_0^1 \left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)dsdt \\
&= \int_0^1 \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\left\{\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\frac{-2}{(\eta_2-\eta_1)}\frac{\partial\xi}{\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\Bigg|_0^1\right. \\
&\quad \left.+\int_0^1 \frac{2\alpha}{(\eta_2-\eta_1)}\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\frac{\partial\xi}{\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)dt\right\}ds \\
&= \int_0^1 \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\left\{\left(\frac{1}{\gamma_1}\right)^{\alpha}\left(\frac{-2}{\eta_2-\eta_1}\right)\frac{\partial\xi}{\partial s}\left(\eta_1,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right. \\
&\quad \left.+\frac{2\alpha}{(\eta_2-\eta_1)}\int_0^1 \left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\frac{\partial\xi}{\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)dt\right\}ds \\
&= \frac{-2}{(\eta_2-\eta_1)\gamma_1^{\alpha}}\int_0^1 \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial\xi}{\partial s}\left(\eta_1,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)ds \\
&\quad +\frac{2\alpha}{(\eta_2-\eta_1)}\left[\int_0^1 \left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\right. \\
&\quad \left.\times\left\{\int_0^1 \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial\xi}{\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)ds\right\}dt\right] \\
&= \frac{-2}{(\eta_2-\eta_1)}\left(\frac{1}{\gamma_1}\right)^{\alpha}\left[\left(\frac{1}{\gamma_2}\right)^{\beta}\frac{-2}{(\vartheta_2-\vartheta_1)}\xi(\eta_1,\vartheta_1)\right. \\
&\quad \left..+\frac{2\beta}{(\vartheta_2-\vartheta_1)}\int_0^1 \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}\xi\left(\eta_1,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)ds\right] \\
&\quad +\frac{2\alpha}{(\eta_2-\eta_1)}\int_0^1 \left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left\{\left(\frac{1}{\gamma_2}\right)^{\beta}\frac{-2}{(\vartheta_2-\vartheta_1)}\xi\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\vartheta_1\right)\right. \\
&\quad \left.+\frac{2\beta}{(\vartheta_2-\vartheta_1)}\int_0^1 \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}\xi\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)ds\right\}dt \\
&= \frac{4}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_1^{\alpha}\gamma_2^{\beta}}\xi(\eta_1,\vartheta_1) \\
&\quad -\frac{4\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\left(\frac{1}{\gamma_1}\right)^{\alpha}\int_0^1 \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}\xi\left(\eta_1,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)ds
\end{aligned}$$

$$
\begin{aligned}
&-\frac{4\alpha}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\left(\frac{1}{\gamma_2}\right)^{\beta}\int\limits_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\xi\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\vartheta_1\right)dt\\
&+\frac{4\alpha\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\left[\int\limits_0^1\int\limits_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}\right.\\
&\left.\times\,\xi\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)dsdt\right]. \qquad (2.3)
\end{aligned}
$$

In (2.3), using the change of the variables $u=\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2$ and $v=\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2$, we can write

$$
\begin{aligned}
=\;&\frac{4}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_1^{\alpha}\gamma_2^{\beta}}\xi(\eta_1,\vartheta_1)-\left(\frac{2}{\vartheta_2-\vartheta_1}\right)^{\gamma_2\beta}\Gamma(\beta)\left({}^{\gamma_2}I^{\beta}_{\vartheta_1^+}\xi\right)\left(\eta_1,\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&-\left(\frac{2}{\eta_2-\eta_1}\right)^{\gamma_1\alpha}\Gamma(\alpha)^{\gamma_1}I^{\alpha}_{\eta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\vartheta_1\right)\\
&+\frac{4\alpha\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \qquad (2.4)
\end{aligned}
$$

Thus, similarly, by integration by parts it follows that

$$
\begin{aligned}
I_2\;=\;&\int\limits_0^1\int\limits_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)dsdt\\
=\;&\frac{-4}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_1^{\alpha}\gamma_2^{\beta}}\xi(\eta_1,\vartheta_2)+\left(\frac{2}{\vartheta_2-\vartheta_1}\right)^{\gamma_2\beta}\Gamma(\beta)\left({}^{\gamma_2}I^{\beta}_{\vartheta_2^-}\xi\right)\left(\eta_1,\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&+\left(\frac{2}{\eta_2-\eta_1}\right)^{\gamma_1\alpha}\Gamma(\alpha)\left({}^{\gamma_1}I^{\alpha}_{\eta_1^+}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\vartheta_2\right)\\
&-\frac{4\alpha\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right), \qquad (2.5)
\end{aligned}
$$

$$
\begin{aligned}
I_3\;=\;&\int\limits_0^1\int\limits_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)dsdt\\
=\;&\frac{-4}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_1^{\alpha}\gamma_2^{\beta}}\xi(\eta_2,\vartheta_1)+\left(\frac{2}{\vartheta_2-\vartheta_1}\right)^{\gamma_2\beta}\Gamma(\beta)\left({}^{\gamma_2}I^{\beta}_{\vartheta_1^+}\xi\right)\left(\eta_2,\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&+\left(\frac{2}{\eta_2-\eta_1}\right)^{\gamma_1\alpha}\Gamma(\alpha)\left({}^{\gamma_1}I^{\alpha}_{\eta_2^-}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\vartheta_1\right)\\
&-\frac{4\alpha\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right), \qquad (2.6)
\end{aligned}
$$

and

$$
\begin{aligned}
I_4\;=\;&\int\limits_0^1\int\limits_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)dsdt\\
=\;&\frac{4}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_1^{\alpha}\gamma_2^{\beta}}\xi(\eta_2,\vartheta_2)-\left(\frac{2}{\vartheta_2-\vartheta_1}\right)^{\gamma_2\beta}\Gamma(\beta)\left({}^{\gamma_2}I^{\beta}_{\vartheta_2^-}\xi\right)\left(\eta_2,\frac{\vartheta_1+\vartheta_2}{2}\right)
\end{aligned}
$$

$$-\left(\frac{2}{\eta_2-\eta_1}\right)^{\gamma_1\alpha}\Gamma(\alpha)\left({}^{\gamma_1}I^{\alpha}_{\eta_2^-}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\vartheta_2\right)$$
$$+\frac{4\alpha\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \tag{2.7}$$

By equalities from (2.4)-(2.7), we obtain

$$\begin{aligned}
&\frac{\gamma_1^{\alpha}\gamma_2^{\beta}(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}\left[I_1-I_2-I_3+I_4\right]\\
=\;&\frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\\
&\times\left[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&\left.+\;{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]-A.
\end{aligned}$$

This completes the proof. □

Next, we start to state the first theorem containing the Hermite-Hadamard type inequality for generalized conformable fractional integrals.

Theorem 3. *Let $\xi:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on Δ with $0\le\eta_1<\eta_2$, $0\le\vartheta_1<\vartheta_2$. If $\frac{\partial^2\xi(t,s)}{\partial t\partial s}$ is a convex function on the co-ordinates on Δ, then the inequality below holds.*

$$\begin{aligned}
&\left|\frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\right. &(2.8)\\
&\times\left[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&\left.\left.+\;{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]-A\right|\\
\le\;&\frac{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16\,\gamma_1\gamma_2}B\left(\alpha+1,\frac{1}{\gamma_1}\right)B\left(\beta+1,\frac{1}{\gamma_2}\right)\\
&\times\left[\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\right],
\end{aligned}$$

where A is defined by (2.2) and $B(\cdot,\cdot)$ refers to the Beta function.

Proof. From Lemma 1, we acquire

$$\begin{aligned}
&\left|\frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\right. &(2.9)\\
&\times\left[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&\left.\left.+\;{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]-A\right|\\
\le\;&\frac{\gamma_1^{\alpha}\gamma_2^{\beta}(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}
\end{aligned}$$

$$\times\left[\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|dsdt\right.$$
$$+\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right|dsdt$$
$$+\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|dsdt$$
$$\left.+\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right|dsdt\right]$$

Since $\frac{\partial^2\xi}{\partial t\partial s}$ is convex function on the co-ordinates on Δ, then one has:

$$\left|\frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\right.$$
$$\times\left[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.$$
$$\left.\left.+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]-A\right|$$
$$\leq\frac{\gamma_1^{\alpha}\gamma_2^{\beta}(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}$$
$$\times\left\{\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right.$$
$$\left[\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|\right.$$
$$\left.+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\right]dsdt$$
$$+\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}$$
$$\left[\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|\right.$$
$$\left.+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\right]dsdt$$
$$+\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}$$
$$\left[\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|\right.$$

$$
\begin{aligned}
&+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\Bigg]dsdt\\
&+\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\\
&\Bigg[\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|\\
&+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\Bigg]dsdt\Bigg\}\\
=\;&\frac{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16\,\gamma_1\gamma_2}B\left(\alpha+1,\frac{1}{\gamma_1}\right)B\left(\beta+1,\frac{1}{\gamma_2}\right)\\
&\times\left[\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\right],
\end{aligned}
$$

which finishes the proof. □

Remark 3. *In Theorem 3, if we choose $\gamma_1 = 1$ and $\gamma_2 = 1$, the following inequalities are achieved*

$$
\begin{aligned}
&\Bigg|\frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\\
&\times\Bigg[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\Bigg]-A\Bigg|\\
\leq\;&\frac{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}\frac{1}{\alpha+1}\frac{1}{\beta+1}\left[\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\right].
\end{aligned}
$$

Theorem 4. *Let $\xi : \triangle \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on Δ with $0 \leq \eta_1 < \eta_2$, $0 \leq \vartheta_1 < \vartheta_2$. If $\left|\frac{\partial^2\xi}{\partial t\partial s}\right|^q$ $q > 1$, is a convex function on the co-ordinates on Δ, then the inequality below holds.*

$$
\begin{aligned}
&\Bigg|\frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}} \qquad (2.10)\\
&\times\Bigg[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\Bigg]-A\Bigg|\\
\leq\;&\frac{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}\left[\frac{16}{\gamma_1\gamma_2}B\left(\alpha p+1,\frac{1}{\gamma_1}\right)B\left(\beta p+1,\frac{1}{\gamma_2}\right)\right]^{\frac{1}{p}}\\
&\times\left[\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right]^{\frac{1}{q}}
\end{aligned}
$$

where A is defined by (2.2) and $B(\cdot,\cdot)$ refers to the Beta function and $\frac{1}{p} = 1 - \frac{1}{q}$.

Proof. From Lemma, we have inequality (2.9). By using the well known Hölder's inequality for double integrals in I_5 and since $\left|\frac{\partial^2\xi}{\partial t\partial s}\right|^q$ is convex functions on the co-ordinates on $\triangle$, we get

$$I_5 = \left\{\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right. \tag{2.11}$$

$$\left.\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|dsdt\right\}$$

$$\leq \left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha p}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta p}dsdt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|^q dsdt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\gamma_1^{\alpha}}\frac{1}{\gamma_2^{\beta}}\left(\int_0^1\int_0^1(1-(1-t)^{\gamma_1})^{\alpha p}(1-(1-s)^{\gamma_2})^{\beta p}dsdt\right)^{\frac{1}{p}}$$

$$\times\left\{\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q\right.$$

$$\left.+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q dsdt\right\}^{\frac{1}{q}}$$

$$\leq \frac{1}{\gamma_1^{\alpha}}\frac{1}{\gamma_2^{\beta}}\left(\frac{1}{\gamma_1}\frac{1}{\gamma_2}B\left(\alpha p+1,\frac{1}{\gamma_1}\right)B\left(\beta p+1,\frac{1}{\gamma_2}\right)\right)^{\frac{1}{p}}$$

$$\times\left(\frac{9}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q+\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q+\frac{1}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}}$$

Where we take advantage of the fact:

$$(\varpi-\sigma)^j \leq \varpi^j-\sigma^j,$$

for any $\varpi > \sigma \geq 0$ and $j \geq 1$.

And similarly,

$$I_6 = \left\{\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right. \tag{2.12}$$

$$\left.\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right|dsdt\right\}$$

$$\leq \left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha p}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta p}dsdt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right|^q dsdt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\gamma_1^{\alpha}}\frac{1}{\gamma_2^{\beta}}\left(\frac{1}{\gamma_1}\frac{1}{\gamma_2}B\left(\alpha p+1,\frac{1}{\gamma_1}\right)B\left(\beta p+1,\frac{1}{\gamma_2}\right)\right)^{\frac{1}{p}}$$
$$\times\left(\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\frac{9}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q+\frac{1}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q+\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}}$$

$$I_7=\left\{\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right. \tag{2.13}$$
$$\left.\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|dsdt\right\}$$
$$\leq \left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha p}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta p}dsdt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|^q dsdt\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{\gamma_1^{\alpha}}\frac{1}{\gamma_2^{\beta}}\left(\frac{1}{\gamma_1}\frac{1}{\gamma_2}B\left(\alpha p+1,\frac{1}{\gamma_1}\right)B\left(\beta p+1,\frac{1}{\gamma_2}\right)\right)^{\frac{1}{p}}$$
$$\times\left(\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\frac{1}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q+\frac{9}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q+\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}}$$

and

$$I_8=\left\{\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right. \tag{2.14}$$
$$\left.\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right|dsdt\right\}$$
$$\leq \left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha p}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta p}dsdt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right|^q dsdt\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{\gamma_1^{\alpha}}\frac{1}{\gamma_2^{\beta}}\left(\frac{1}{\gamma_1}\frac{1}{\gamma_2}B\left(\alpha p+1,\frac{1}{\gamma_1}\right)B\left(\beta p+1,\frac{1}{\gamma_2}\right)\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q+\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q+\frac{9}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}}.$$

If we substitute from (2.11)-(2.14) in (2.9), we obtain the first inequality of (2.10) is achieved. □

Remark 4. *If we take $\gamma_1 = 1$ and $\gamma_2 = 1$ in Teorem 4, the following inequalities are achieved*

$$\begin{aligned}
&\left|\frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\right. \qquad (2.15)\\
&\times\left[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&\left.\left.+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]-A\right|\\
\le\ &\frac{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}\left[\frac{16}{(\alpha p+1)(\beta p+1)}\right]^{\frac{1}{p}}\\
&\times\left[\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 5. *Assume $\xi : \triangle \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on Δ with $0 \le \eta_1 < \eta_2$, $0 \le \vartheta_1 < \vartheta_2$. If $\left|\frac{\partial^2\xi}{\partial t\partial s}\right|^q$ $q \ge 1$, is a convex function on the co-ordinates on Δ, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\right. \qquad (2.16)\\
&\times\left[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&\left.\left.+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]-A\right|\\
\le\ &\frac{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}\left(\frac{1}{4}\right)^{\frac{1}{q}}\left(B\left(\alpha+1,\frac{1}{\gamma_1}\right)B\left(\beta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}}\\
&\times\left\{\left(\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\right.\right.\\
&\quad+\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q\\
&\quad+\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q\\
&\quad\left.+\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}}\\
&\quad+\left(\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\right.\\
&\quad+\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q\\
&\quad+\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q\\
&\quad\left.+\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\right.$$
$$+\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q$$
$$+\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q$$
$$\left.+\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\right.$$
$$+\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q$$
$$+\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q$$
$$\left.+\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}}.$$

Here A is defined as in (2.2).

Proof. With help of the equality (2.9) and by using Power-Mean inequality in I_9, we get

$$I_9=\left[\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right.$$
$$\left.\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|dsdt\right]$$
$$\leq\left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right.$$
$$\left.\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|^q dsdt\right)^{\frac{1}{q}}$$

Taking into account convexity on the co-ordinates on $\triangle$ of $\left|\frac{\partial^2\xi}{\partial t\partial s}\right|^q$, then we acquire

$$\leq\left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}dsdt\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right.$$

$$\times\left\{\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q\right.$$
$$\left.+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q dsdt\right\}^{\frac{1}{q}}$$

In this inequality, using the change of variables, we can write

$$\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta \tag{2.17}$$
$$\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|dsdt$$
$$\leq\left(\frac{1}{\gamma_1^{\alpha+1}}\frac{1}{\gamma_2^{\beta+1}}B\left(\alpha+1,\frac{1}{\gamma_1}\right)B\left(\beta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}}\left(\frac{1}{4}\frac{1}{\gamma_1^{\alpha+1}}\frac{1}{\gamma_2^{\beta+1}}\right.$$
$$\times\left\{\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\right.$$
$$+\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q$$
$$+\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q$$
$$\left.\left.+\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Similarly we have,

$$I_{10}=\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta \tag{2.18}$$
$$\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right|dsdt$$
$$\leq\left(\frac{1}{\gamma_1^{\alpha+1}}\frac{1}{\gamma_2^{\beta+1}}B\left(\alpha+1,\frac{1}{\gamma_1}\right)B\left(\beta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}}\left(\frac{1}{4}\frac{1}{\gamma_1^{\alpha+1}}\frac{1}{\gamma_2^{\beta+1}}\right.$$
$$\times\left\{\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\right.$$
$$+\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q$$
$$+\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q$$
$$\left.\left.+\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)\right\}^{\frac{1}{q}}$$

and

$$I_{11}=\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta} \tag{2.19}$$

$$\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|dsdt$$

$$\leq\left(\frac{1}{\gamma_1^{\alpha+1}}\frac{1}{\gamma_2^{\beta+1}}B\left(\alpha+1,\frac{1}{\gamma_1}\right)B\left(\beta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}}\left(\frac{1}{4}\frac{1}{\gamma_1^{\alpha+1}}\frac{1}{\gamma_2^{\beta+1}}\right.$$

$$\times\left\{\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\right.$$

$$+\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q$$

$$+\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q$$

$$\left.\left.+\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)\right\}^{\frac{1}{q}}$$

finally

$$I_{12}=\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta} \tag{2.20}$$

$$\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right|dsdt$$

$$\leq\left(\frac{1}{\gamma_1^{\alpha+1}}\frac{1}{\gamma_2^{\beta+1}}B\left(\alpha+1,\frac{1}{\gamma_1}\right)B\left(\beta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}}\left(\frac{1}{4}\frac{1}{\gamma_1^{\alpha+1}}\frac{1}{\gamma_2^{\beta+1}}\right.$$

$$\times\left\{\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\right.$$

$$+\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q$$

$$+\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q$$

$$\left.\left.+\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)\right\}^{\frac{1}{q}}.$$

By considering (2.17)-(2.20) in (2.9), we obtain the desired inequality (2.16). □

Remark 5. *If we assign $\gamma_1=1$ and $\gamma_2=1$ in Teorem 5, then we have following inequality:*

$$\left|\frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\right.$$

$$\times\left[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.$$

$$
\begin{aligned}
& + \; {}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\Bigg] - A\Bigg| \\
\leq \; & \frac{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}\left(\frac{1}{4}\right)^{\frac{1}{q}}\left(\frac{1}{\alpha+1}\frac{1}{\beta+1}\right)^{1-\frac{1}{q}} \\
& \times\Bigg\{\Bigg(\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q \\
& \quad + \left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q \\
& \quad + \left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q \\
& \quad + \left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\Bigg)^{\frac{1}{q}} \\
& + \Bigg(\left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q \\
& \quad + \left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q \\
& \quad + \left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q \\
& \quad + \left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\Bigg)^{\frac{1}{q}} \\
& + \Bigg(\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q \\
& \quad + \left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q \\
& \quad + \left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q \\
& \quad + \left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\Bigg)^{\frac{1}{q}} \\
& + \Bigg(\left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q \\
& \quad + \left[B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q \\
& \quad + \left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q \\
& \quad + \left[2B\left(\alpha+1,\frac{1}{\gamma_1}\right)-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\beta+1,\frac{1}{\gamma_2}\right)-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\Bigg)^{\frac{1}{q}}\;.
\end{aligned}
$$

3. Conclusion

In this research, we acquired some inequality of trapezoid type for co-ordinated convex functions by means of conformable fractional integrals. In the future studies, researchers can obtain some new inequalities with the aid of the different kinds of co-ordinated convex mappings or other types of fractional integral operators.

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