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Szász-Baskakov-Stancu operators based on Boas-Buck-type polynomials

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Abstract. In this article, Stancu type generalization of the Szász-Baskakov operators including Boas-Buck type polynomials is given. The convergence properties are investigated in weighted space and the rate of convergence is found with the help of the weighted modulus of continuity. Theoretical results are shown with the help of Appell polynomials, Hermite polynomials and Gould-Hopper polynomials, respectively. Finally, we present some numerical examples to validate our theoretical results and compare the approximation errors of the newly constructed operators with the approximation errors of the Szász-Baskakov operators based on Boas-Buck-type polynomials.

1. Introduction

Approximation theory is concerned with how to approach uncomplicated functions and quantitatively characterize errors. The theory of approximation of functions start with the proof of Weierstrass theorem [1] and this theorem make a great contribution to the advancement of mathematical analysis.
In 1950, Szász [2] introduce the following operators known as Szász-Mirakjan operators

$$S_\eta(f;x) = e^{-\eta x}\sum_{\nu=0}^{\infty}\frac{(\eta x)^\nu}{\nu!}f\left(\frac{\nu}{\eta}\right) \tag{1}$$

where $\eta \in \mathbb{N}$, $x \geq 0$ and $f \in C[0,\infty)$.
In 1983, Prasad et al. [3] construct the following operators known as Szász-Mirakjan-Baskakov operators

$$D_\eta(f;x) = (\eta-1)\sum_{\nu=0}^{\infty}e^{-\eta x}\frac{(\eta x)^\nu}{\nu!}\int_0^\infty\begin{pmatrix}\eta+\nu-1\\ \nu\end{pmatrix}\frac{\zeta^\nu}{(1+\zeta)^{\eta+\nu}}f(\zeta)d\zeta. \tag{2}$$

Torun et al. [4] study some generalizations of the operators.
In 2012, Sucu et al. [5] investigate the following Boas-Buck-type linear positive operators

$$B_\eta(f;x) = \frac{1}{A(1)B(\eta x H(1))}\sum_{\nu=0}^{\infty}\rho_\nu(\eta x)f\left(\frac{\nu}{\eta}\right), \tag{3}$$

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where $\eta \in \mathbb{N}$, $x \geq 0$. Çekim et al. [6] study Kantorovich-Stancu operators involving Boas-Buck-type polynomials. The generating function of Boas-Buck type polynomials is as follows:

$$A(\tau) B(xH(\tau)) = \sum_{\nu=0}^{\infty} \rho_{\nu}(x) \tau^{\nu} \tag{4}$$

where

$$A(\tau) = \sum_{\iota=0}^{\infty} a_{\iota}\tau^{\iota},\ a_0 \neq 0, \tag{5}$$

$$B(\tau) = \sum_{\iota=0}^{\infty} b_{\iota}\tau^{\iota},\ b_{\iota} \neq 0 \quad (\iota \geq 0), \tag{6}$$

$$H(\tau) = \sum_{\iota=1}^{\infty} h_{\iota}\tau^{\iota},\ h_1 \neq 0 \tag{7}$$

are analytic functions.
We believe the following constraints to be true

i) The power series (4) - (7) converge in the disk $|\tau| < R, (R > 1)$,

ii) $A(1) \neq 0,\ H'(1) = 1, \rho_{\nu}(x) \geq 0,\ \nu = 0, 1, 2, ...,$

iii) $B : \mathbb{R} \to (0, \infty)$.

In 2022, the Szász-Baskakov operators containing polynomials of the Boas-Buck type are introduced by Sofyalıoğlu et al. [7] as follows

$$D_{\eta}^{*}(f;x) = \frac{\eta - 1}{A(1) B(\eta x H(1))} \sum_{\nu=0}^{\infty} \rho_{\nu}(\eta x) \int_{0}^{\infty} \binom{\eta + \nu - 1}{\nu} \frac{\zeta^{\nu}}{(1+\zeta)^{\eta+\nu}} f(\zeta) d\zeta, \tag{8}$$

where $f \in C[0, \infty)$, $x \geq 0$ and $\eta \in \mathbb{N}$.

The Szász-Baskakov-Stancu operators, which are based on polynomials of the Boas-Buck type, are presented in this work. Then, we demonstrate that the Appell, Hermite, and Gould-Hopper polynomials with particular options are among the polynomials of the Boas-Buck type.

Now, we define the Stancu-type Szász-Baskakov operators based on Boas-Buck-type polynomials

$$D_{\eta}^{(\theta,\gamma)}(f;x) = \frac{\eta - 1}{A(1) B(\eta x H(1))} \sum_{\nu=0}^{\infty} \rho_{\nu}(\eta x) \int_{0}^{\infty} \binom{\eta + \nu - 1}{\nu} \frac{\zeta^{\nu}}{(1+\zeta)^{\eta+\nu}} f\left(\frac{\eta\zeta + \theta}{\eta + \gamma}\right) d\zeta \tag{9}$$

where $\eta \in \mathbb{N}$, $x \geq 0$ and $f \in C[0, \infty)$.

2. Approximation properties of $D_{\eta}^{(\theta,\gamma)}(f;x)$

Lemma 2.1. *We have*

$$D_{\eta}^{(\theta,\gamma)}(1;x) = 1,$$

$$D_{\eta}^{(\theta,\gamma)}(\zeta;x) = \frac{\eta^2 B'(\eta x H(1))}{(\eta-2)(\eta+\gamma) B(\eta x H(1))} x + \frac{\eta(A'(1) + A(1)) + (\eta-2) A(1)\theta}{(\eta-2)(\eta+\gamma) A(1)}, \quad \eta > 2,$$

$$D_\eta^{(\theta,\gamma)}\left(\zeta^2;x\right) = \frac{\eta^4 B''\left(\eta x H(1)\right)}{(\eta-2)(\eta-3)(\eta+\gamma)^2 B\left(\eta x H(1)\right)}x^2 + \frac{\eta^3\left(A(1)(4+H''(1)+2\theta)+2A'(1)\right)-\eta^2\left(6\theta A(1)\right)B'\left(\eta x H(1)\right)}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)B\left(\eta x H(1)\right)}x$$
$$+\frac{\eta^2\left(A''(1)+(4+2\theta)A'(1)+(2+2\theta+\theta^2)A(1)\right)-\eta\left(6\theta A'(1)+(6\theta+5\theta^2)A(1)\right)+6\theta^2 A(1)}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)}, \quad \eta>3,$$

$$D_\eta^{(\theta,\gamma)}\left(\zeta^3;x\right) = \frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3 A(1)B\left(\eta x H(1)\right)}\left\{\eta^6 A(1)B'''\left(\eta x H(1)\right)x^3\right.$$
$$+\left(\eta^5\left(3A'(1)+A(1)(9+3\theta+3H''(1))\right)-\eta^4\left(12\theta A(1)\right)\right)B''\left(\eta x H(1)\right)x^2$$
$$+\left(\eta^4\left(3A''(1)+A'(1)(18+6\theta+3H''(1))+A(1)\left(18+12\theta+3\theta^2+(9+3\theta)H''(1)+H'''(1)\right)\right)\right.$$
$$\left.-\eta^3\left(24\theta A'(1)+A(1)\left(48\theta+21\theta^2+12\theta H''(1)\right)\right)+\eta^2\left(36\theta^2 A(1)\right)\right)B'\left(\eta x H(1)\right)x$$
$$+\left(\eta^3\left(A'''(1)+A''(1)(9+3\theta)+A'(1)\left(18+12\theta+3\theta^2\right)+A(1)\left(6+6\theta+3\theta^2+\theta^3\right)\right)\right.$$
$$+\eta^2\left(-12\theta A''(1)-A'(1)\left(48\theta+21\theta^2\right)-A(1)\left(24\theta+21\theta^2+9\theta^3\right)\right)$$
$$\left.\left.+\eta\left(36\theta^2 A'(1)+A(1)\left(36\theta^2+26\theta^3\right)\right)-24\theta^3 A(1)\right)B\left(\eta x H(1)\right)\right\}, \quad \eta>4,$$

$$D_\eta^{(\theta,\gamma)}\left(\zeta^4;x\right) = \frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 A(1)B\left(\eta x H(1)\right)}\left\{\eta^8 A(1)B^{(iv)}\left(\eta x H(1)\right)x^4\right.$$
$$+\left(\eta^7\left(A(1)(16+4\theta+6H''(1))+4A'(1)\right)-20\eta^6\theta A(1)\right)B'''\left(\eta x H(1)\right)x^3$$
$$+\left(\eta^6(A(1)\left(72+36\theta+6\theta^2+(48+12\theta)H''(1)+3\left(H''(1)\right)^2+4H'''(1)\right)\right.$$
$$+A'(1)(48+12\theta+12H''(1))+6A''(1))$$
$$\left.-\eta^5\left(A(1)\left(180\theta+54\theta^2+60\theta H''(1)\right)+60\theta A'(1)\right)+\eta^4\left(120\theta^2 A(1)\right)\right)B''\left(\eta x H(1)\right)x^2$$
$$+\left(\eta^5\left(A(1)\left(96+72\theta+24\theta^2+4\theta^3+\left(72+36\theta+6\theta^2\right)H''(1)+(16+4\theta)H'''(1)+H^{(iv)}(1)\right)\right)\right.$$
$$-\eta^4(A(1)\left(360\theta+216\theta^2+48\theta^3+\left(180\theta+54\theta^2\right)H''(1)+20\theta H'''(1)\right)$$
$$+A'(1)\left(360\theta+108\theta^2+60\theta H''(1)\right)+60\theta A''(1))$$
$$\left.+\eta^3\left(A(1)\left(480\theta^2+188\theta^3+120\theta^2 H''(1)\right)+A'(1)\left(240\theta^2\right)\right)-\eta^2\left(A(1)\left(240\theta^3\right)\right)\right)B'\left(\eta x H(1)\right)x$$
$$+\left(\eta^4(A(1)\left(24+24\theta+12\theta^2+4\theta^3+\theta^4\right)+A'(1)\left(96+74\theta+24\theta^2+4\theta^3\right)+A''(1)\left(72+36\theta+6\theta^2\right)\right.$$
$$+A'''(1)(16+4\theta)+A^{(iv)}(1))$$
$$-\eta^3(A(1)\left(120\theta+108\theta^2+48\theta^3+14\theta^4\right)+A'(1)\left(360\theta+216\theta^2+48\theta^3\right)+A''(1)\left(180\theta+54\theta^2\right)$$
$$+A'''(1)(20\theta))$$
$$+\eta^2\left(A(1)\left(240\theta^2+188\theta^3+71\theta^4\right)+A'(1)\left(480\theta^2+188\theta^3\right)+A''(1)\left(120\theta^2\right)\right)$$
$$\left.\left.-\eta\left(A(1)\left(240\theta^3+154\theta^4\right)+A'(1)\left(240\theta^3\right)\right)+A(1)\left(120\theta^4\right)\right)B\left(\eta x H(1)\right)\right\}, \quad \eta>5$$

for each $x\in[0,\infty)$.

Proof. It can be easily seen that

$$\sum_{\nu=0}^{\infty}\rho_\nu(\eta x)=1,\ \int_0^\infty \rho_\nu(\eta x)\,dx=\frac{1}{\eta} \tag{10}$$

and

$$\sum_{\nu=0}^{\infty}\binom{\eta+\nu-1}{\nu}\frac{\zeta^\nu}{(1+\zeta)^{\eta+\nu}}=1,\ \int_0^\infty\binom{\eta+\nu-1}{\nu}\frac{\zeta^\nu}{(1+\zeta)^{\eta+\nu}}d\zeta=\frac{1}{\eta-1}. \tag{11}$$

Using the generating functions given in (4), we obtain

$$\begin{aligned}
\sum_{\nu=0}^{\infty}\rho_\nu(\eta x) &= A(1)B(\eta x H(1)),\\
\sum_{\nu=0}^{\infty}\nu\rho_\nu(\eta x) &= A'(1)B(\eta x H(1)) + \eta x A(1)B'(\eta x H(1)),\\
\sum_{\nu=0}^{\infty}\nu^2\rho_\nu(\eta x) &= (A'(1)+A''(1))B(\eta x H(1)) + \eta x(A(1)(1+H''(1))+2A'(1))B'(\eta x H(1))\\
&\quad +\eta^2x^2A(1)B''(\eta x H(1)),\\
\sum_{\nu=0}^{\infty}\nu^3\rho_\nu(\eta x) &= (A'(1)+3A''(1)+A'''(1))B(\eta x H(1))\\
&\quad +\eta x(A(1)(1+3H''(1)+H'''(1))+A'(1)(6+3H''(1))+3A''(1))B'(\eta x H(1))\\
&\quad +\eta^2x^2(A(1)(3+3H''(1))+3A'(1))B''(\eta x H(1))+\eta^3x^3A(1)B'''(\eta x H(1)),\\
\sum_{\nu=0}^{\infty}\nu^4\rho_\nu(\eta x) &= \left(A'(1)+7A''(1)+6A'''(1)+A^{(iv)}(1)\right)B(\eta x H(1))\\
&\quad +\eta x\left(A(1)\left(1+7H''(1)+6H'''(1)+H^{(iv)}(1)\right)+A'(1)(14+18H''(1)+4H'''(1))\right.\\
&\quad \left.+A''(1)(18+6H''(1))+4A'''(1)\right)B'(\eta x H(1))\\
&\quad +\eta^2x^2\left(A(1)\left(7+18H''(1)+4H'''(1)+3(H''(1))^2\right)\right.\\
&\quad \left.+A'(1)(18+12H''(1))+6A''(1)\right)B''(\eta x H(1))\\
&\quad +\eta^3x^3(A(1)(6+6H''(1))+4A'(1))B'''(\eta x H(1))+\eta^4x^4A(1)B^{(iv)}(\eta x H(1)).
\end{aligned}$$

Considering these equations, we obtain the lemma's equalities.

Let it be K

$$K := \left\{f : x\in[0,\infty), \left|f(x)\right| \le ae^{bx},\ a,b \textit{ positive and finite}\right\}. \tag{12}$$

Additionally, presume

$$\lim_{\eta\to\infty}\frac{B'(\eta)}{B(\eta)}=1,\ \lim_{\eta\to\infty}\frac{B''(\eta)}{B(\eta)}=1,\ \lim_{\eta\to\infty}\frac{B'''(\eta)}{B(\eta)}=1,\ \lim_{\eta\to\infty}\frac{B^{(iv)}(\eta)}{B(\eta)}=1. \tag{13}$$

Theorem 2.2. *Let* (13) *and* $f\in C[0,\infty)\cap K$ *be persuaded. Later*

$$\lim_{\eta\to\infty}D_\eta^{(\theta,\gamma)}(f;x)=f(x) \tag{14}$$

and all $D_\eta^{(\theta,\gamma)}$ *operators uniformly converge in every compact subset of* $[0,\infty)$.

Proof. From Lemma 1 and (13), we get

$$\lim_{\eta\to\infty}D_\eta^{(\theta,\gamma)}(e_i;x)=e_i(x),\ i=0,1,2. \tag{15}$$

Thus, the newly constructed operators $D_\eta^{(\theta,\gamma)}$ uniformly converge in every compact subset of $[0,\infty)$. Later, we use the global Korovkin type theorem to wrap up the proof.

Lemma 2.3. *For every $x \in [0,\infty)$, we obtain*

$$
\begin{aligned}
D_{\eta}^{(\theta,\gamma)}(t-x;x) &= \left(\frac{\eta^2 B'(\eta x H(1))}{(\eta-2)(\eta+\gamma)B(\eta x H(1))}-1\right)x+\frac{\eta A'(1)+A(1)(\eta+(\eta-2)\theta)}{(\eta-2)(\eta+\gamma)A(1)}, \quad \eta>2,\\
D_{\eta}^{(\theta,\gamma)}\left((t-x)^2;x\right) &= \left(\frac{\eta^4 B''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 B(\eta x H(1))}-\frac{2\eta^2 B'(\eta x H(1))}{(\eta-2)(\eta+\gamma)B(\eta x H(1))}+1\right)x^2\\
&\quad+\left(\frac{(\eta^3(A(1)(4+H''(1)+2\theta)+2A'(1))-6\theta\eta^2 A(1))B'(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)B(\eta x H(1))}\right.\\
&\quad\left.-\frac{2A(1)(\eta+(\eta-2)\theta)+2\eta A'(1)}{(\eta-2)(\eta+\gamma)A(1)}\right)x\\
&\quad+\left(\frac{\eta^2(A(1)(2+2\theta+\theta^2)+A'(1)(4+2\theta)+A''(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)}\right)\\
&\quad-\frac{\eta(A(1)(6\theta+5\theta^2)+6\theta A'(1))+6\theta^2 A(1)}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)}, \quad \eta>3,\\
D_{\eta}^{(\theta,\gamma)}\left((t-x)^4;x\right) &= \left(\frac{\eta^8 B^{(iv)}(\eta x H(1))}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 B(\eta x H(1))}\right.\\
&\quad-\frac{4\eta^6 B'''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3 B(\eta x H(1))}\\
&\quad\left.+\frac{6\eta^4 B''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 B(\eta x H(1))}-\frac{4\eta^2 B'(\eta x H(1))}{(\eta-2)(\eta+\gamma)B(\eta x H(1))}+1\right)x^4\\
&\quad+\left(\frac{(\eta^7(A(1)(16+4\theta+6H''(1))+4A'(1))-20\eta^6\theta A(1))B'''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 A(1)B(\eta x H(1))}\right.\\
&\quad-\frac{\left(4\eta^5(A(1)(9+3\theta+3H''(1))+3A'(1))-48\eta^4\theta A(1)\right)B''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3 A(1)B(\eta x H(1))}\\
&\quad+\frac{(6\eta^3(A(1)(4+2\theta+H''(1))+2A'(1))-36\theta\eta^2 A(1))B'(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)B(\eta x H(1))}\\
&\quad\left.-\frac{4A(1)(\eta+(\eta-2)\theta)+4\eta A'(1)}{(\eta-2)(\eta+\gamma)A(1)}\right)x^3\\
&\quad+\left(\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 A(1)B(\eta x H(1))}\right.\\
&\quad\times\left(\left(\eta^6\left(A(1)\left(72+36\theta+6\theta^2+(48+12\theta)H''(1)+4H'''(1)+3(H''(1))^2\right)\right.\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+A'(1)(48+12\theta+12H''(1))+6A''(1))-\eta^5\left(A(1)\left(180\theta+54\theta^2+60\theta H''(1)\right)+60\theta A'(1)\right)\\
&+\eta^4\left(120\theta^2 A(1)\right)\Big)B''(\eta x H(1))\Big)\\
&+\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3 A(1)B(\eta x H(1))}\\
&\times\left(\left(-4\eta^4\left(A(1)\left(18+12\theta+3\theta^2+(9+3\theta)H''(1)+H'''(1)\right)+A'(1)(18+6\theta+3H''(1))+3A''(1)\right)\right.\right.\\
&+4\eta^3\left(A(1)\left(48\theta+21\theta^2+12\theta H''(1)\right)+A'(1)(24\theta)\right)-4\eta^2\left(36\theta^2 A(1)\right)\Big)B'(\eta x H(1))\Big)\\
&+\frac{1}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)}\\
&\times\left(6\eta^2\left(A(1)\left(2+2\theta+\theta^2\right)+A'(1)(4+2\theta)+A''(1)\right)-6\eta\left(A(1)\left(6\theta+5\theta^2\right)+A'(1)(6\theta)\right)+36\theta^2 A(1)\right)\Big)\Big)x^2
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 A(1)B(\eta xH(1))}\\
&\times\Big(\Big(\eta^5\Big(A(1)\Big(96+72\theta+24\theta^2+4\theta^3+\Big(72+36\theta+6\theta^2\Big)H''(1)+(16+4\theta)H'''(1)+H^{(iv)}(1)\Big)\\
&+A'(1)\Big(144+72\theta+12\theta^2+(48+12\theta)H''(1)+4H'''(1)\Big)+A''(1)(48+12\theta+6H''(1))+4A'''(1)\Big)\Big)\\
&-\eta^4\Big(A(1)\Big(360\theta+216\theta^2+48\theta^3+\Big(180\theta+54\theta^2\Big)H''(1)+20\theta H'''(1)\Big)\\
&+A'(1)\Big(360\theta+108\theta^2+60\theta H''(1)\Big)+60\theta A''(1)\Big)\\
&+\eta^3\Big(A(1)\Big(480\theta^2+188\theta^3+120\theta^2H''(1)\Big)+240\theta^2A'(1)\Big)-\eta^2\Big(240\theta^3A(1)\Big)\Big)B'(\eta xH(1))\\
&+\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3 A(1)}\\
&\times\Big(-4\eta^3\Big(A(1)\Big(6+6\theta+3\theta^2+\theta^3\Big)+A'(1)\Big(18+12\theta+3\theta^2\Big)+A''(1)(9+3\theta)+A'''(1)\Big)\\
&+4\eta^2\Big(A(1)\Big(24\theta+21\theta^2+9\theta^3\Big)+A'(1)\Big(48\theta+21\theta^2\Big)+12\theta A''(1)\Big)\\
&-4\eta\Big(A(1)\Big(36\theta^2+26\theta^3\Big)+36\theta^2A'(1)\Big)+96\theta^3A(1)\Big)\Big)x\\
&+\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 A(1)}\\
&\times\Big(\eta^4\Big(A(1)\Big(24+24\theta+12\theta^2+4\theta^3+\theta^4\Big)+A'(1)\Big(96+74\theta+24\theta^2+4\theta^3\Big)+A''(1)\Big(72+36\theta+6\theta^2\Big)\\
&+A'''(1)(16+4\theta)+A^{(iv)}(1)\Big)\Big)\\
&-\eta^3\Big(A(1)\Big(120\theta+108\theta^2+48\theta^3+14\theta^4\Big)+A'(1)\Big(360\theta+216\theta^2+48\theta^3\Big)\\
&+A''(1)\Big(180\theta+54\theta^2\Big)+A'''(1)(20\theta)\Big)\\
&+\eta^2\Big(A(1)\Big(240\theta^2+188\theta^3+71\theta^4\Big)+A'(1)\Big(480\theta^2+188\theta^3\Big)+A''(1)\Big(120\theta^2\Big)\Big)\\
&-\eta\Big(A(1)\Big(240\theta^3+154\theta^4\Big)+A'(1)\Big(240\theta^3\Big)\Big)+A(1)\Big(120\theta^4\Big),\quad \eta>5.
\end{aligned}
$$

Proof. By means of

$$
\begin{aligned}
D_\eta^{(\theta,\gamma)}(t-x;x) &= D_\eta^{(\theta,\gamma)}(e_1;x)-xD_\eta^{(\theta,\gamma)}(1;x),\quad \eta>2,\\
D_\eta^{(\theta,\gamma)}\left((t-x)^2;x\right) &= D_\eta^{(\theta,\gamma)}(e_2;x)-2xD_\eta^{(\theta,\gamma)}(e_1;x)+x^2D_\eta^{(\theta,\gamma)}(1;x),\quad \eta>3,\\
D_\eta^{(\theta,\gamma)}\left((t-x)^4;x\right) &= D_\eta^{(\theta,\gamma)}(e_4;x)-4xD_\eta^{(\theta,\gamma)}(e_3;x)+6x^2D_\eta^{(\theta,\gamma)}(e_2;x)-4x^3D_\eta^{(\theta,\gamma)}(e_1;x)+x^4D_\eta^{(\theta,\gamma)}(1;x),\quad \eta>5
\end{aligned}
$$

the desired result of the lemma is obtained.

Lemma 2.4. *For every $x\in[0,\infty)$, we have*

$$
\begin{aligned}
\lim_{\eta\to\infty}\eta D_\eta^{(\theta,\gamma)}(t-x;x) &= x\varphi_1(x)+\frac{A(1)(1+\theta)+A'(1)}{A(1)},\\
\lim_{\eta\to\infty}\eta D_\eta^{(\theta,\gamma)}\left((t-x)^2;x\right) &= x^2\varphi_2(x)+x(2+H''(1)),\\
\lim_{\eta\to\infty}\eta^2 D_\eta^{(\theta,\gamma)}\left((t-x)^4;x\right) &= x^4\varphi_3(x)+x^3\varphi_4(x)+x^2\left(12+12H''(1)+3(H''(1))^2\right)
\end{aligned}
$$

where

$$\begin{aligned}
\varphi_1(x) &= \lim_{\eta\to\infty} \eta\left(\frac{\eta^2 B'(\eta x H(1))}{(\eta-2)(\eta+\gamma)B(\eta x H(1))}-1\right),\\
\varphi_2(x) &= \lim_{\eta\to\infty} \eta\left(\frac{\eta^4 B''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 B(\eta x H(1))}-\frac{2\eta^2 B'(\eta x H(1))}{(\eta-2)(\eta+\gamma)B(\eta x H(1))}+1\right),\\
\varphi_3(x) &= \lim_{\eta\to\infty} \eta^2\left(\frac{\eta^8 B^{(iv)}(\eta x H(1))}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 B(\eta x H(1))}\right.\\
&\quad -\frac{4\eta^6 B'''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3 B(\eta x H(1))}\\
&\quad \left.+\frac{6\eta^4 B''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 B(\eta x H(1))}-\frac{4\eta^2 B'(\eta x H(1))}{(\eta-2)(\eta+\gamma)B(\eta x H(1))}+1\right),\\
\varphi_4(x) &= \lim_{\eta\to\infty} \eta^2\left(\frac{\left(\eta^7(A(1)(16+4\theta+6H''(1))+4A'(1))-20\eta^6\theta A(1)\right)B'''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 A(1)B(\eta x H(1))}\right.\\
&\quad -\frac{\left(4\eta^5(A(1)(9+3\theta+3H''(1))+3A'(1))-48\eta^4\theta A(1)\right)B''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3 A(1)B(\eta x H(1))}\\
&\quad +\frac{\left(6\eta^3(A(1)(4+2\theta+H''(1))+2A'(1))-36\theta\eta^2 A(1)\right)B'(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)B(\eta x H(1))}\\
&\quad \left.-\frac{4A(1)(\eta+(\eta-2)\theta)+4\eta A'(1)}{(\eta-2)(\eta+\gamma)A(1)}\right).
\end{aligned}$$

3. Weighted approximation

To determine the rate of convergence of the unbounded function described on $[0,\infty)$, weighted space is necessary. In this way, we explore the approximation property of recently created $D_\eta^{(\theta,\gamma)}$ operators in the weighted space of functions on $[0,\infty)$ that have exponential growth. We start by recalling the weighted space representations. Let $\rho(x)=1+x^2$ be the weighted function and let R_f be a positive fixed. A normed linear space with $\|f\|=\sup_{x\in[0,\infty)}\frac{|f(x)|}{\rho(x)}$ is called $B_\rho([0,\infty))=\left\{f:[0,\infty)\to\mathbb{R} \mid |f(x)|\le R_f\rho(x)\right\}$.

$C_\rho([0,\infty))=\left\{f\in B_\rho([0,\infty)) \mid f \textit{ is continuous}\right\}$,
$C_\rho^*([0,\infty))=\left\{f\in C_\rho([0,\infty)) \mid \lim_{x\to\infty}\frac{f(x)}{\rho(x)}<\infty\right\}$. The relation between these spaces can be represented as $C_\rho^*([0,\infty))\subset C_\rho([0,\infty))\subset B_\rho([0,\infty))$.

Lemma 3.1. *Let the weight function be $\rho(x)=1+x^2$ and* (13) *be satisfied. If and only if a positive fixed R and $f\in C_\rho([0,\infty))$ with R being a positive constant*

$$\left\|D_\eta^{(\theta,\gamma)}(\rho;x)\right\|_\rho\le R. \tag{16}$$

exists, the series of linear and positive operators $D_\eta^{(\theta,\gamma)}$, $\eta>3$ act from $C_\rho([0,\infty))$ to $B_\rho([0,\infty))$.

Proof. Using Lemma 1 and substituting $\rho(x)=1+x^2$ we obtain

$$D_\eta^{(\theta,\gamma)}(\rho;x)=D_\eta^{(\theta,\gamma)}(1;x)+D_\eta^{(\theta,\gamma)}\left(\zeta^2;x\right)$$

$$= 1 + \frac{\eta^4 B''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 B(\eta x H(1))} x^2$$
$$+ \frac{\eta^3 (A(1)(4 + H''(1) + 2\theta) + 2A'(1)) - \eta^2 (6\theta A(1)) B'(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1) B(\eta x H(1))} x$$
$$+ \frac{\eta^2 \left(A''(1) + (4+2\theta) A'(1) + \left(2 + 2\theta + \theta^2\right) A(1)\right) - \eta \left(6\theta A'(1) + \left(6\theta + 5\theta^2\right) A(1)\right) + 6\theta^2 A(1)}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)}, \eta > 3.$$

Therefore, we write

$$\left\| D_\eta^{(\theta,\gamma)}(\rho; x) \right\|_\rho = \sup_{x \geq 0} \left\{ \left| \frac{\eta^4 B''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 B(\eta x H(1))} \right| \frac{x^2}{1+x^2} \right.$$
$$+ \left| \frac{\eta^3 (A(1)(4 + H''(1) + 2\theta) + 2A'(1)) - \eta^2 (6\theta A(1)) B'(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1) B(\eta x H(1))} \right| \frac{x}{1+x^2}$$
$$\left. + \left| \frac{\eta^2 \left(A''(1) + (4+2\theta) A'(1) + \left(2 + 2\theta + \theta^2\right) A(1)\right) - \eta \left(6\theta A'(1) + \left(6\theta + 5\theta^2\right) A(1)\right) + 6\theta^2 A(1)}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)} + 1 \right| \frac{1}{1+x^2} \right\}$$

where $\eta > 3$.

From

$$\sup_{x \geq 0} \frac{1}{1+x^2} = 1, \qquad \sup_{x \geq 0} \frac{x}{1+x^2} = \frac{1}{2}, \qquad \sup_{x \geq 0} \frac{x^2}{1+x^2} = 1, \tag{17}$$

we get

$$\left\| D_\eta^{(\theta,\gamma)}(\rho; x) \right\|_\rho \leq \frac{\eta^4 B''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 B(\eta x H(1))}$$
$$+ \frac{\eta^3 (A(1)(4 + H''(1) + 2\theta) + 2A'(1)) - \eta^2 (6\theta A(1)) B'(\eta x H(1))}{2(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1) B(\eta x H(1))}$$
$$+ \frac{\eta^2 \left(A''(1) + (4+2\theta) A'(1) + \left(2 + 2\theta + \theta^2\right) A(1)\right) - \eta \left(6\theta A'(1) + \left(6\theta + 5\theta^2\right) A(1)\right) + 6\theta^2 A(1)}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)}$$
$$+1, \quad \eta > 3.$$

By using the limits in (13), a positive fixed R is present

$$\left\| D_\eta^{(\theta,\gamma)}(\rho; x) \right\|_\rho \leq R.$$

From Lemma 5, it understandable that the $D_\eta^{(\theta,\gamma)}$ operators act from $C_\rho([0,\infty))$ to $B_\rho([0,\infty))$.

Theorem 3.2. *$D_\eta^{(\theta,\gamma)}$, $\eta > 3$ go from $C_\rho([0,\infty))$ to $B_\rho([0,\infty))$ in a series of positive, linear operators such that*

$$\lim_{\eta \to \infty} \left\| D_\eta^{(\theta,\gamma)}\left(\zeta^i; x\right) - x^i \right\|_\rho = 0, \quad i = 0, 1, 2. \tag{18}$$

For every $f \in C_\rho^R([0,\infty))$ later,

$$\lim_{\eta \to \infty} \left\| D_\eta^{(\theta,\gamma)} f - f \right\|_\rho = 0. \tag{19}$$

Theorem 3.3. *Assume that $\rho(x) = 1 + x^2$ is the weight function, and verify that* (13)*. Later, for each $f \in C_\rho^R([0,\infty))$,*

$$\lim_{\eta\to\infty}\left\|D_\eta^{(\theta,\gamma)}f - f\right\|_\rho = 0.$$

Proof. To illustrate this theorem, one only needs to provide the weighted Korovkin theorem examples for $i = 0, 1, 2$

$$\lim_{\eta\to\infty}\left\|D_\eta^{(\theta,\gamma)}\left(\zeta^i; x\right) - x^i\right\|_\rho = 0.$$

By considering the test functions given in Lemma 1, we have

$$\lim_{\eta\to\infty}\left\|D_\eta^{(\theta,\gamma)}(1;x) - 1\right\|_\rho = 0. \tag{20}$$

After that, by using Lemma 1 and (17), we get

$$\begin{aligned}\left\|D_\eta^{(\theta,\gamma)}(e_1;x) - e_1(x)\right\|_\rho &= \sup_{x\geq 0}\left\{\left|\frac{\eta^2 B'(\eta x H(1))}{(\eta-2)(\eta+\gamma)B(\eta x H(1))} - 1\right|\frac{x}{1+x^2} + \left|\frac{\eta(A'(1)+A(1)) + (\eta-2)A(1)\theta}{(\eta-2)(\eta+\gamma)A(1)}\right|\frac{1}{1+x^2}\right\} \\ &\leq \frac{1}{2}\left|\frac{\eta^2 B'(\eta x H(1))}{(\eta-2)(\eta+\gamma)B(\eta x H(1))} - 1\right| + \left|\frac{\eta(A'(1)+A(1)) + (\eta-2)A(1)\theta}{(\eta-2)(\eta+\gamma)A(1)}\right|, \ \eta > 2\end{aligned}$$

which implies that

$$\lim_{\eta\to\infty}\left\|D_\eta^{(\theta,\gamma)}(e_1;x) - e_1(x)\right\|_\rho = 0. \tag{21}$$

Likewise,

$$\begin{aligned}\left\|D_\eta^{(\theta,\gamma)}(e_2;x) - e_2(x)\right\|_\rho &= \sup_{x\geq 0}\left\{\left|\frac{\eta^4 B''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 B(\eta x H(1))} - 1\right|\frac{x^2}{1+x^2}\right. \\ &\quad + \left|\frac{\eta^3(A(1)(4+H''(1)+2\theta)+2A'(1)) - \eta^2(6\theta A(1))B'(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)B(\eta x H(1))}\right|\frac{x}{1+x^2} \\ &\quad + \left|\frac{\eta^2\left(A''(1) + (4+2\theta)A'(1) + \left(2+2\theta+\theta^2\right)A(1)\right)}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)}\right. \\ &\quad \left.\left. - \frac{\eta\left(6\theta A'(1) + \left(6\theta+5\theta^2\right)A(1)\right) - 6\theta^2 A(1)}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)}\right|\frac{1}{1+x^2}\right\} \\ &\leq \left|\frac{\eta^4 B''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 B(\eta x H(1))} - 1\right| \\ &\quad + \frac{1}{2}\left|\frac{\eta^3(A(1)(4+H''(1)+2\theta)+2A'(1)) - \eta^2(6\theta A(1))B'(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)B(\eta x H(1))}\right| \\ &\quad + \left|\frac{\eta^2\left(A''(1) + (4+2\theta)A'(1) + \left(2+2\theta+\theta^2\right)A(1)\right)}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)}\right. \\ &\quad \left. - \frac{\eta\left(6\theta A'(1) + \left(6\theta+5\theta^2\right)A(1)\right) - 6\theta^2 A(1)}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)}\right|, \ \eta > 3\end{aligned}$$

which implies that

$$\lim_{\eta\to\infty}\left\|D_{\eta}^{(\theta,\gamma)}(e_2;x)-e_2(x)\right\|_{\rho}=0. \tag{22}$$

From (20), (21) and (22) , we have

$$\lim_{\eta\to\infty}\left\|D_{\eta}^{(\theta,\gamma)}\left(\zeta^i;x\right)-x^i\right\|_{\rho}=0,\quad i=0,1,2. \tag{23}$$

We eventually derive the desired outcome

$$\lim_{\eta\to\infty}\left\|D_{\eta}^{(\theta,\gamma)}f-f\right\|_{\rho}=0. \tag{24}$$

from Theorem 6.

4. Weighted modulus of continuity

The convergence of $\omega(f,\delta)$ to 0 as $\delta\to 0$ is not achieved if f is not uniformly continuous in $[0,\infty)$. Therefore, Gadjieva and Doğru [8] defined the weighted modulus of continuity in 1998 as follows:

$$\Omega(f,\delta)=\sup_{x\geq 0,\ |h|\leq\delta}\frac{\left|f(x+h)-f(x)\right|}{(1+x^2)(1+h^2)}.$$

Following that, in 2006, Yüksel and Ispir [9] provided the following definition of weighted modulus of continuity

$$\Omega(f,\delta)=\sup_{x\geq 0}\sup_{0<h\leq\delta}\frac{\left|f(x+h)-f(x)\right|}{1+(x+h)^2}$$

where $f\in C_{\rho}^{*}[0,\infty)$.

Lemma 4.1. *[9] If $f\in C_{\rho}^{*}[0,\infty)$ then*

i) *As a function of δ, $\Omega(f,x)$ increases monotonously,*

ii) $\lim_{\delta\to 0^+}\Omega(f,x)=0$,

iii) $\Omega(f,\lambda x)\leq(1+\lambda)\Omega(f,x)$, *for any $\lambda\in[0,\infty)$.*

Now, using the weighted modulus of continuity, we will determine the rate of convergence for $f\in C_{\rho}^{*}[0,\infty)$.

Theorem 4.2. *If $f\in C_{\rho}^{*}[0,\infty)$, later*

$$\sup_{x\in[0,\infty)}\frac{\left|D_{\eta}^{(\theta,\gamma)}(f;x)-f(x)\right|}{(1+x^2)^{\frac{5}{2}}}\leq 2\left(2+\Lambda_0^{*}(x)+\left(\Lambda_1^{*}(x)\right)^{\frac{1}{2}}\right)\Omega\left(f,\left(\Lambda_0^{*}(x)\right)^{\frac{1}{2}}\right)$$

where

$$
\begin{aligned}
\Lambda_0^*(x) &= \left(\frac{\eta^4 B''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 B(\eta x H(1))} - 1\right) \\
&+\frac{1}{2}\left(\frac{\eta^3(A(1)(4+H''(1)+2\theta)+2A'(1)) - \eta^2(6\theta A(1))B'(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1) B(\eta x H(1))}\right) \\
&+\left(\frac{\eta^2(A''(1)+(4+2\theta)A'(1)+(2+2\theta+\theta^2)A(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)}\right. \\
&\left.-\frac{\eta(6\theta A'(1)+(6\theta+5\theta^2)A(1)) - 6\theta^2 A(1)}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)}\right), \ \eta > 3, \\
\Lambda_1^*(x) &= \left(\frac{\eta^8 B^{(iv)}(\eta x H(1))}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 B(\eta x H(1))} - \frac{4\eta^6 B'''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3 B(\eta x H(1))}\right. \\
&\left.+\frac{6\eta^4 B''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 B(\eta x H(1))} - \frac{4\eta^2 B'(\eta x H(1))}{(\eta-2)(\eta+\gamma) B(\eta x H(1))} + 1\right) \\
&+\frac{3\sqrt{3}}{16}\left(\frac{(\eta^7(A(1)(16+4\theta+6H''(1))+4A'(1)) - 20\eta^6\theta A(1))B'''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 A(1) B(\eta x H(1))}\right. \\
&-\frac{\left(4\eta^5(A(1)(9+3\theta+3H''(1))+3A'(1)) - 48\eta^4\theta A(1)\right)B''(\eta x H(1))}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3 A(1) B(\eta x H(1))} \\
&+\frac{(6\eta^3(A(1)(4+2\theta+H''(1))+2A'(1)) - 36\theta\eta^2 A(1))B'(\eta x H(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1) B(\eta x H(1))} \\
&\left.-\frac{4A(1)(\eta+(\eta-2)\theta)+4\eta A'(1)}{(\eta-2)(\eta+\gamma)A(1)}\right) \\
&+\frac{1}{4}\left\{\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 A(1) B(\eta x H(1))}\right. \\
&\times\left(\left(\eta^6\left(A(1)\left(72+36\theta+6\theta^2+(48+12\theta)H''(1)+4H'''(1)+3(H''(1))^2\right)+A'(1)(48+12\theta+12H''(1))\right)\right.\right. \\
&\left.\left.+6A''(1) - \eta^5\left(A(1)\left(180\theta+54\theta^2+60\theta H''(1)\right)+60\theta A'(1)\right)+\eta^4\left(120\theta^2 A(1)\right)\right)B''(\eta x H(1))\right) \\
&+\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3 A(1) B(\eta x H(1))} \\
&\times\left(\left(-4\eta^4\left(A(1)\left(18+12\theta+3\theta^2+(9+3\theta)H''(1)+H'''(1)\right)+A'(1)(18+6\theta+3H''(1))+3A''(1)\right)\right.\right. \\
&\left.\left.+4\eta^3\left(A(1)\left(48\theta+21\theta^2+12\theta H''(1)\right)+A'(1)(24\theta)\right) - 4\eta^2\left(36\theta^2 A(1)\right)\right)B'(\eta x H(1))\right) \\
&+\frac{1}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)} \\
&\left.\times\left(6\eta^2\left(A(1)\left(2+2\theta+\theta^2\right)+A'(1)(4+2\theta)+A''(1)\right) - 6\eta\left(A(1)\left(6\theta+5\theta^2\right)+A'(1)(6\theta)\right)+36\theta^2 A(1)\right)\right\} \\
&+\frac{3\sqrt{3}}{16}\left\{\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 A(1) B(\eta x H(1))}\right. \\
&\times\left(\left(\eta^5\left(A(1)\left(96+72\theta+24\theta^2+4\theta^3+\left(72+36\theta+6\theta^2\right)H''(1)+(16+4\theta)H'''(1)+H^{(iv)}(1)\right)\right.\right.\right. \\
&\left.\left.+A'(1)\left(144+72\theta+12\theta^2+(48+12\theta)H''(1)+4H'''(1)\right)+A''(1)(48+12\theta+6H''(1))+4A'''(1)\right)\right) \\
&-\eta^4\left(A(1)\left(360\theta+216\theta^2+48\theta^3+\left(180\theta+54\theta^2\right)H''(1)+20\theta H'''(1)\right)\right. \\
&\left.+A'(1)\left(360\theta+108\theta^2+60\theta H''(1)\right)+60\theta A''(1)\right) \\
&+\eta^3\left(A(1)\left(480\theta^2+188\theta^3+120\theta^2 H''(1)\right)+240\theta^2 A'(1)\right) - \eta^2\left(240\theta^3 A(1)\right)\Big)B'(\eta x H(1))
\end{aligned}
$$

$$+\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3 A(1)}$$
$$+\Big(-4\eta^3\Big(A(1)\Big(6+6\theta+3\theta^2+\theta^3\Big)+A'(1)\Big(18+12\theta+3\theta^2\Big)+A''(1)(9+3\theta)+A'''(1)\Big)$$
$$+4\eta^2\Big(A(1)\Big(24\theta+21\theta^2+9\theta^3\Big)+A'(1)\Big(48\theta+21\theta^2\Big)+12\theta A''(1)\Big)$$
$$-4\eta\Big(A(1)\Big(36\theta^2+26\theta^3\Big)+36\theta^2 A'(1)\Big)+96\theta^3 A(1)\Big)\Big\}$$
$$+\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 A(1)}$$
$$\times\Big(\eta^4\Big(A(1)\Big(24+24\theta+12\theta^2+4\theta^3+\theta^4\Big)+A'(1)\Big(96+74\theta+24\theta^2+4\theta^3\Big)+A''(1)\Big(72+36\theta+6\theta^2\Big)$$
$$+A'''(1)(16+4\theta)+A^{(iv)}(1)\Big)\Big)$$
$$-\eta^3\Big(A(1)\Big(120\theta+108\theta^2+48\theta^3+14\theta^4\Big)+A'(1)\Big(360\theta+216\theta^2+48\theta^3\Big)+A''(1)\Big(180\theta+54\theta^2\Big)+A'''(1)(20\theta)\Big)$$
$$+\eta^2\Big(A(1)\Big(240\theta^2+188\theta^3+71\theta^4\Big)+A'(1)\Big(480\theta^2+188\theta^3\Big)+A''(1)\Big(120\theta^2\Big)\Big)$$
$$-\eta\Big(A(1)\Big(240\theta^3+154\theta^4\Big)+A'(1)\Big(240\theta^3\Big)\Big)+A(1)\Big(120\theta^4\Big),\quad \eta>5.$$

Proof. We obtain

$$\begin{aligned}\left|f(t)-f(x)\right| &\leq \Big(1+(x+|t-x|)^2\Big)\Big(1+\frac{|t-x|}{\delta}\Big)\Omega(f,\delta)\\ &\leq 2\Big(1+x^2\Big)\Big(1+(t-x)^2\Big)\Big(1+\frac{|t-x|}{\delta}\Big)\Omega(f,\delta).\end{aligned}$$

from the weighted modulus of continuity and Lemma 8. Then by applying $D_\eta^{(\theta,\gamma)}$ for both sides of the inequality, we have

$$\left|D_\eta^{(\theta,\gamma)}(f;x)-f(x)\right|\leq 2\Big(1+x^2\Big)\Big(1+D_\eta^{(\theta,\gamma)}\Big((t-x)^2;x\Big)+D_\eta^{(\theta,\gamma)}\Big(\Big(1+(t-x)^2\Big)\frac{|t-x|}{\delta};x\Big)\Big)\Omega(f,\delta).$$

We obtain

$$\begin{aligned}\left|D_\eta^{(\theta,\gamma)}(f;x)-f(x)\right| &\leq 2\Big(1+x^2\Big)\Big\{1+D_\eta^{(\theta,\gamma)}\Big((t-x)^2;x\Big)\\ &\quad+\frac{1}{\delta}\Big(\Big(D_\eta^{(\theta,\gamma)}\Big((t-x)^2;x\Big)\Big)^{\frac12}+\Big(D_\eta^{(\theta,\gamma)}\Big((t-x)^4;x\Big)\Big)^{\frac12}\Big(D_\eta^{(\theta,\gamma)}\Big((t-x)^2;x\Big)\Big)^{\frac12}\Big)\Big\}\Omega(f,\delta).\end{aligned}$$

by applying the Cauchy-Schwarz inequality. From central moments of operator, we can write

$D_\eta^{(\theta,\gamma)}\Big((t-x)^2;x\Big)\leq\Lambda_0^*(x)\Big(1+x^2\Big)$ and $D_\eta^{(\theta,\gamma)}\Big((t-x)^4;x\Big)\leq\Lambda_1^*(x)\Big(1+x^2\Big)^2$.

So, when we choose $\delta=\Lambda_0^*(x)$, we give

$$\begin{aligned}\left|D_\eta^{(\theta,\gamma)}(f;x)-f(x)\right| &\leq 2\Big(1+x^2\Big)\Big\{1+\Lambda_0^*(x)\Big(1+x^2\Big)+\Big(1+x^2\Big)^{\frac12}+\Big(\Lambda_1^*(x)\Big)^{\frac12}\Big(1+x^2\Big)^{\frac32}\Big\}\Omega(f,\delta)\\ &\leq 2\Big(1+x^2\Big)^{\frac52}\Big\{2+\Lambda_0^*(x)+\Big(\Lambda_1^*(x)\Big)^{\frac12}\Big\}\Omega(f,\delta).\end{aligned}$$

Eventually,

$$\sup_{x\in[0,\infty)}\frac{\left|D_\eta^{(\theta,\gamma)}(f;x)-f(x)\right|}{(1+x^2)^{\frac52}}\leq 2\Big(2+\Lambda_0^*(x)+\Big(\Lambda_1^*(x)\Big)^{\frac12}\Big)\Omega\Big(f,\Big(\Lambda_0^*(x)\Big)^{\frac12}\Big)$$

is desired result.

5. Voronovskaya-type theorem

Theorem 5.1. *We have*

$$\lim_{\eta\to\infty}\eta\left(D_{\eta}^{(\theta,\gamma)}(f;x)-f(x)\right)=f'(x)\left(x\varphi_1(x)+\frac{A(1)(1+\theta)+A'(1)}{A(1)}\right)+\frac{1}{2}f''(x)\left(x^2\varphi_2(x)+(2+H''(1))x\right)$$

uniformly for $f,f',f''\in C[0,\infty)\cap K$ *and* $x\in[0,\infty)$, *in every compact subset of* $[0,\infty)$, *where* $\varphi_1(x)$ *and* $\varphi_2(x)$ *are determined by Lemma 4.*

Proof. This theorem is demonstrated using Taylor's expansion

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2}f''(x)+g(t,x)(t-x)^2 \tag{25}$$

where $g(t,x)\in C[0,\infty)\cap K$ and $\lim_{t\to x}g(t,x)=0$. By applying the operator $D_{\eta}^{(\theta,\gamma)}$ to the above relation we get

$$D_{\eta}^{(\theta,\gamma)}(f;x)-f(x)=f'(x)D_{\eta}^{(\theta,\gamma)}(t-x;x)+\frac{1}{2}f''(x)D_{\eta}^{(\theta,\gamma)}\left((t-x)^2;x\right)+D_{\eta}^{(\theta,\gamma)}\left(g(t,x)(t-x)^2;x\right).$$

By using the Cauchy-Schwarz inequality for $D_{\eta}^{(\theta,\gamma)}\left(g(t,x)(t-x)^2;x\right)$, we get

$$\left|D_{\eta}^{(\theta,\gamma)}(f;x)-f(x)\right|\le\left(D_{\eta}^{(\theta,\gamma)}\left(g^2(t,x);x\right)\right)^{\frac{1}{2}}\left(D_{\eta}^{(\theta,\gamma)}\left((t-x)^4;x\right)\right)^{\frac{1}{2}}.$$

Since $g(t,x)\to0$ as $t\to x$,

$$\lim_{\eta\to\infty}D_{\eta}^{(\theta,\gamma)}\left(g^2(t,x);x\right)=g^2(x,x)=0. \tag{26}$$

It is properly validated in every compact subset of $[0,\infty)$. Then

$$\lim_{\eta\to\infty}\eta\left(D_{\eta}^{(\theta,\gamma)}(f;x)-f(x)\right)=f'(x)\lim_{\eta\to\infty}\eta\left(D_{\eta}^{(\theta,\gamma)}(t-x;x)\right)+\frac{1}{2}f''(x)\lim_{\eta\to\infty}\eta\left(D_{\eta}^{(\theta,\gamma)}\left((t-x)^2;x\right)\right).$$

Finally, from Lemma 4 and (26) we obtaine result.

6. Special cases of the operators $D_{\eta}^{(\theta,\gamma)}$

In this section, under specific choices of the analytical functions A, B, and H, we find a few exceptional polynomials.

6.1. Appell polynomials

In order to demonstrate the recurrence connection,

$$R_{\eta}'(x)=\eta R_{\eta-1}(x),\eta=1,2,... \tag{27}$$

Appell [10] introduces the sequences of $\eta-$degree polynomials R_η, where $\eta=1,2,....$
A power series with the formula $A(\tau)=\sum_{\iota=0}^{\infty}a_\iota\tau^\iota,\ a_0\neq0$ such that

$$A(\tau)e^{\tau x}=\sum_{\eta=0}^{\infty}R_\eta(x)\tau^\eta \tag{28}$$

exists.
Choosing $B(t) = e^t$ and $H(t) = t$ in (4), then we get Appell polynomials.
Now, we give moments, central moments and important theorems for our operator including Appell polynomials.

Lemma 6.1. *If $x \in [0, \infty)$ exists, then*

$$\begin{aligned}
D_\eta^A(1;x) &= 1,\\
D_\eta^A(\zeta;x) &= \frac{\eta^2}{(\eta-2)(\eta+\gamma)}x + \frac{\eta(A'(1)+A(1)) + (\eta-2)A(1)\theta}{(\eta-2)(\eta+\gamma)A(1)}, \quad \eta > 2,\\
D_\eta^A(\zeta^2;x) &= \frac{\eta^4}{(\eta-2)(\eta-3)(\eta+\gamma)^2}x^2 + \frac{\eta^3(A(1)(4+H''(1)+2\theta)+2A'(1)) - \eta^2(6\theta A(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2A(1)}x\\
&+ \frac{\eta^2(A''(1)+(4+2\theta)A'(1)+(2+2\theta+\theta^2)A(1)) - \eta(6\theta A'(1)+(6\theta+5\theta^2)A(1)) + 6\theta^2A(1)}{(\eta-2)(\eta-3)(\eta+\gamma)^2A(1)}, \quad \eta > 3,\\
D_\eta^A(\zeta^3;x) &= \frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3A(1)}\left\{\eta^6A(1)x^3 + \left[\eta^5(3A'(1)+A(1)(9+3\theta+3H''(1))) - \eta^4(12\theta A(1))\right]x^2\right.\\
&+[\eta^4\left(3A''(1)+A'(1)(18+6\theta+3H''(1)) + A(1)\left(18+12\theta+3\theta^2+(9+3\theta)H''(1)+H'''(1)\right)\right)\\
&-\eta^3\left(24\theta A'(1)+A(1)\left(48\theta+21\theta^2+12\theta H''(1)\right)\right) + \eta^2\left(36\theta^2A(1)\right)]x\\
&+[\eta^3\left(A'''(1)+A''(1)(9+3\theta)+A'(1)\left(18+12\theta+3\theta^2\right)+A(1)\left(6+6\theta+3\theta^2+\theta^3\right)\right)\\
&+\eta^2\left(-12\theta A''(1) - A'(1)\left(48\theta+21\theta^2\right) - A(1)\left(24\theta+21\theta^2+9\theta^3\right)\right)\\
&\left.+\eta\left(36\theta^2A'(1)+A(1)\left(36\theta^2+26\theta^3\right)\right) - 24\theta^3A(1)]\right\}, \quad \eta > 4,\\
D_\eta^A(\zeta^4;x) &= \frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4A(1)}\left\{\eta^8A(1)x^4\right.\\
&+\left[\eta^7(A(1)(16+4\theta+6H''(1))+4A'(1)) - 20\eta^6\theta A(1)\right]x^3\\
&+[\eta^6(A(1)\left(72+36\theta+6\theta^2+(48+12\theta)H''(1)+3(H''(1))^2+4H'''(1)\right)+A'(1)(48+12\theta+12H''(1))\\
&+6A''(1)) - \eta^5\left(A(1)\left(180\theta+54\theta^2+60\theta H''(1)\right)+60\theta A'(1)\right)+\eta^4\left(120\theta^2A(1)\right)]x^2\\
&+[\eta^5\left(A(1)\left(96+72\theta+24\theta^2+4\theta^3+\left(72+36\theta+6\theta^2\right)H''(1)+(16+4\theta)H'''(1)+H^{(iv)}(1)\right)\right)\\
&-\eta^4(A(1)\left(360\theta+216\theta^2+48\theta^3+\left(180\theta+54\theta^2\right)H''(1)+20\theta H'''(1)\right)+A'(1)\left(360\theta+108\theta^2+60\theta H''(1)\right)\\
&+60\theta A''(1)) + \eta^3\left(A(1)\left(480\theta^2+188\theta^3+120\theta^2H''(1)\right)+A'(1)\left(240\theta^2\right)\right) - \eta^2\left(A(1)\left(240\theta^3\right)\right)]x\\
&+[\eta^4(A(1)\left(24+24\theta+12\theta^2+4\theta^3+\theta^4\right)+A'(1)\left(96+74\theta+24\theta^2+4\theta^3\right)+A''(1)\left(72+36\theta+6\theta^2\right)\\
&+A'''(1)(16+4\theta)+A^{(iv)}(1))\\
&-\eta^3(A(1)\left(120\theta+108\theta^2+48\theta^3+14\theta^4\right)+A'(1)\left(360\theta+216\theta^2+48\theta^3\right)+A''(1)\left(180\theta+54\theta^2\right)\\
&+A'''(1)(20\theta)) + \eta^2\left(A(1)\left(240\theta^2+188\theta^3+71\theta^4\right)+A'(1)\left(480\theta^2+188\theta^3\right)+A''(1)\left(120\theta^2\right)\right)\\
&\left.-\eta\left(A(1)\left(240\theta^3+154\theta^4\right)+A'(1)\left(240\theta^3\right)\right)+A(1)\left(120\theta^4\right)]\right\}, \quad \eta > 5.
\end{aligned}$$

Lemma 6.2. *For every $x \in [0, \infty)$, the following identities verify*

$$\begin{aligned}
D_\eta^A(t-x;x) &= \left(\frac{\eta^2}{(\eta-2)(\eta+\gamma)} - 1\right)x + \frac{\eta A'(1)+A(1)(\eta+(\eta-2)\theta)}{(\eta-2)(\eta+\gamma)A(1)}, \quad \eta > 2,\\
D_\eta^A\left((t-x)^2;x\right) &= \left(\frac{\eta^4}{(\eta-2)(\eta-3)(\eta+\gamma)^2} - \frac{2\eta^2}{(\eta-2)(\eta+\gamma)} + 1\right)x^2\\
&+\left(\frac{(\eta^3(A(1)(4+2\theta)+2A'(1)) - 6\theta\eta^2A(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2A(1)} - \frac{2A(1)(\eta+(\eta-2)\theta)+2\eta A'(1)}{(\eta-2)(\eta+\gamma)A(1)}\right)x
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\eta^2\left(A(1)\left(2+2\theta+\theta^2\right)+A'(1)(4+2\theta)+A''(1)\right)}{(\eta-2)(\eta-3)(\eta+\gamma)^2A(1)}\right)\\
&-\frac{\eta\left(A(1)\left(6\theta+5\theta^2\right)+6\theta A'(1)\right)-6\theta^2A(1)}{(\eta-2)(\eta-3)(\eta+\gamma)^2A(1)},\quad \eta>3,
\end{aligned}
$$

$$
\begin{aligned}
D_\eta^A\left((t-x)^4;x\right) =\; &\left(\frac{\eta^8}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4}-\frac{4\eta^6}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3}\right.\\
&\left.+\frac{6\eta^4}{(\eta-2)(\eta-3)(\eta+\gamma)^2}-\frac{4\eta^2}{(\eta-2)(\eta+\gamma)}+1\right)x^4\\
&+\left(\frac{\left(\eta^7\left(A(1)(16+4\theta)+4A'(1)\right)-20\eta^6\theta A(1)\right)}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4A(1)}-\frac{\left(4\eta^5\left(A(1)(9+3\theta)+3A'(1)\right)-48\eta^4\theta A(1)\right)}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3A(1)}\right.\\
&\left.+\frac{\left(6\eta^3\left(A(1)(4+2\theta)+2A'(1)\right)-36\theta\eta^2A(1)\right)}{(\eta-2)(\eta-3)(\eta+\gamma)^2A(1)}-\frac{4A(1)(\eta+(\eta-2)\theta)+4\eta A'(1)}{(\eta-2)(\eta+\gamma)A(1)}\right)x^3\\
&+\left(\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4A(1)}\right.\\
&\times\left(\eta^6\left(A(1)\left(72+36\theta+6\theta^2\right)+A'(1)(48+12\theta)+6A''(1)\right)\right.\\
&\left.-\eta^5\left(A(1)\left(180\theta+54\theta^2\right)+60\theta A'(1)\right)+\eta^4\left(120\theta^2A(1)\right)\right)\\
&+\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3A(1)}\\
&\times\left(-4\eta^4\left(A(1)\left(18+12\theta+3\theta^2\right)+A'(1)(18+6\theta)+3A''(1)\right)\right.\\
&\left.+4\eta^3\left(A(1)\left(48\theta+21\theta^2\right)+A'(1)(24\theta)\right)-4\eta^2\left(36\theta^2A(1)\right)\right)\\
&+\frac{1}{(\eta-2)(\eta-3)(\eta+\gamma)^2A(1)}\\
&\times\left(6\eta^2\left(A(1)\left(2+2\theta+\theta^2\right)+A'(1)(4+2\theta)+A''(1)\right)\right.\\
&\left.\left.-6\eta\left(A(1)\left(6\theta+5\theta^2\right)+A'(1)(6\theta)\right)+36\theta^2A(1)\right)\right)x^2\\
&+\left(\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4A(1)}\right.\\
&\times\left(\eta^5\left(A(1)\left(96+72\theta+24\theta^2+4\theta^3\right)+A'(1)\left(144+72\theta+12\theta^2\right)+A''(1)(48+12\theta)+4A'''(1)\right)\right.\\
&-\eta^4\left(A(1)\left(360\theta+216\theta^2+48\theta^3\right)+A'(1)\left(360\theta+108\theta^2\right)+60\theta A''(1)\right)\\
&\left.+\eta^3\left(A(1)\left(480\theta^2+188\theta^3\right)+240\theta^2A'(1)\right)-\eta^2\left(240\theta^3A(1)\right)\right)\\
&+\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3A(1)}\\
&\times\left(-4\eta^3\left(A(1)\left(6+6\theta+3\theta^2+\theta^3\right)+A'(1)\left(18+12\theta+3\theta^2\right)+A''(1)(9+3\theta)+A'''(1)\right)\right.\\
&+4\eta^2\left(A(1)\left(24\theta+21\theta^2+9\theta^3\right)+A'(1)\left(48\theta+21\theta^2\right)+12\theta A''(1)\right)\\
&\left.\left.-4\eta\left(A(1)\left(36\theta^2+26\theta^3\right)+36\theta^2A'(1)\right)+96\theta^3A(1)\right)\right)x\\
&+\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4A(1)}\\
&\times\left(\eta^4\left(A(1)\left(24+24\theta+12\theta^2+4\theta^3+\theta^4\right)+A'(1)\left(96+74\theta+24\theta^2+4\theta^3\right)+A''(1)\left(72+36\theta+6\theta^2\right)\right.\right.\\
&\left.\left.+A'''(1)(16+4\theta)+A^{(iv)}(1)\right)\right)\\
&-\eta^3\left(A(1)\left(120\theta+108\theta^2+48\theta^3+14\theta^4\right)+A'(1)\left(360\theta+216\theta^2+48\theta^3\right)\right.
\end{aligned}
$$

$$+A''(1)\left(180\theta+54\theta^2\right)+A'''(1)(20\theta)\right)$$
$$+\eta^2\left(A(1)\left(240\theta^2+188\theta^3+71\theta^4\right)+A'(1)\left(480\theta^2+188\theta^3\right)+A''(1)\left(120\theta^2\right)\right)$$
$$-\eta\left(A(1)\left(240\theta^3+154\theta^4\right)+A'(1)\left(240\theta^3\right)\right)+A(1)\left(120\theta^4\right),\quad \eta>5.$$

Lemma 6.3. *We obtain following results*

$$\begin{aligned}\lim_{\eta\to\infty}\eta D_\eta^A(t-x;x) &= (2-\beta)x+\frac{A(1)(1+\theta)+A'(1)}{A(1)},\\ \lim_{\eta\to\infty}\eta D_\eta^A\left((t-x)^2;x\right) &= x^2+2x.\end{aligned}$$

Theorem 6.4. *We obtain*

$$\lim_{\eta\to\infty}\eta\left(D_\eta^A(f;x)-f(x)\right)=\left((2-\beta)x+\frac{A(1)(1+\theta)+A'(1)}{A(1)}\right)f'(x)+\frac{1}{2}\left(x^2+2x\right)f''(x)$$

uniformly on every compact subset of $[0,\infty)$ *for* $f,f',f''\in C[0,\infty)\cap K$ *and* $x\in[0,\infty)$.

Theorem 6.5. *If* $f\in C_\rho^*[0,\infty)$,

$$\sup_{x\in[0,\infty)}\frac{\left|D_\eta^A(f;x)-f(x)\right|}{(1+x^2)^{\frac{5}{2}}}\le 2\left(2+\Lambda_0^A(x)+\left(\Lambda_1^A(x)\right)^{\frac{1}{2}}\right)\Omega\left(f;\left(\Lambda_0^A(x)\right)^{\frac{1}{2}}\right)$$

where

$$\begin{aligned}\Lambda_0^A(x) &= \left(\frac{\eta^4}{(\eta-2)(\eta-3)(\eta+\gamma)^2}-\frac{2\eta^2}{(\eta-2)(\eta+\gamma)}+1\right)\\ &\quad+\frac{1}{2}\left(\frac{(\eta^3(A(1)(4+2\theta)+2A'(1))-6\theta\eta^2A(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2A(1)}-\frac{2A(1)(\eta+(\eta-2)\theta)+2\eta A'(1)}{(\eta-2)(\eta+\gamma)A(1)}\right)\\ &\quad+\left(\frac{\eta^2(A(1)(2+2\theta+\theta^2)+A'(1)(4+2\theta)+A''(1))-\eta(A(1)(6\theta+5\theta^2)+6\theta A'(1))+6\theta^2A(1)}{(\eta-2)(\eta-3)(\eta+\gamma)^2A(1)}\right),\quad \eta>3,\\ \Lambda_1^A(x) &= \left(\frac{\eta^8}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4}-\frac{4\eta^6}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3}\right.\\ &\quad\left.+\frac{6\eta^4}{(\eta-2)(\eta-3)(\eta+\gamma)^2}-\frac{4\eta^2}{(\eta-2)(\eta+\gamma)}+1\right)\\ &\quad+\frac{3\sqrt{3}}{16}\left(\frac{(\eta^7(A(1)(16+4\theta)+4A'(1))-20\eta^6\theta A(1))}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4A(1)}-\frac{\left(4\eta^5(A(1)(9+3\theta)+3A'(1))-48\eta^4\theta A(1)\right)}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3A(1)}\right.\\ &\quad\left.+\frac{(6\eta^3(A(1)(4+2\theta)+2A'(1))-36\theta\eta^2A(1))}{(\eta-2)(\eta-3)(\eta+\gamma)^2A(1)}-\frac{4A(1)(\eta+(\eta-2)\theta)+4\eta A'(1)}{(\eta-2)(\eta+\gamma)A(1)}\right)\\ &\quad+\frac{1}{4}\left(\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4A(1)}\right.\\ &\quad\times\left(\eta^6\left(A(1)\left(72+36\theta+6\theta^2\right)+A'(1)(48+12\theta)+6A''(1)\right)\right.\\ &\quad-\eta^5\left(A(1)\left(180\theta+54\theta^2\right)+60\theta A'(1)\right)+\eta^4\left(120\theta^2A(1)\right)\Big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3 A(1)}\\
&\times\Big(-4\eta^4\Big(A(1)\big(18+12\theta+3\theta^2\big)+A'(1)(18+6\theta)+3A''(1)\Big)\\
&+4\eta^3\Big(A(1)\big(48\theta+21\theta^2\big)+A'(1)(24\theta)\Big)-4\eta^2\big(36\theta^2 A(1)\big)\\
&+\frac{1}{(\eta-2)(\eta-3)(\eta+\gamma)^2 A(1)}\\
&\times\Big(6\eta^2\Big(A(1)\big(2+2\theta+\theta^2\big)+A'(1)(4+2\theta)+A''(1)\Big)\\
&-6\eta\Big(A(1)\big(6\theta+5\theta^2\big)+A'(1)(6\theta)\Big)+36\theta^2 A(1)\Big)\Big)\\
&+\frac{3\sqrt{3}}{16}\Bigg(\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 A(1)}\\
&\times\Big(\eta^5\Big(A(1)\big(96+72\theta+24\theta^2+4\theta^3\big)+A'(1)\big(144+72\theta+12\theta^2\big)+A''(1)(48+12\theta)+4A'''(1)\Big)\\
&-\eta^4\Big(A(1)\big(360\theta+216\theta^2+48\theta^3\big)+A'(1)\big(360\theta+108\theta^2\big)+60\theta A''(1)\Big)\\
&+\eta^3\Big(A(1)\big(480\theta^2+188\theta^3\big)+240\theta^2 A'(1)\Big)-\eta^2\big(240\theta^3 A(1)\big)\Big)\\
&+\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta+\gamma)^3 A(1)}\\
&\times\Big(-4\eta^3\Big(A(1)\big(6+6\theta+3\theta^2+\theta^3\big)+A'(1)\big(18+12\theta+3\theta^2\big)+A''(1)(9+3\theta)+A'''(1)\Big)\\
&+4\eta^2\Big(A(1)\big(24\theta+21\theta^2+9\theta^3\big)+A'(1)\big(48\theta+21\theta^2\big)+12\theta A''(1)\Big)\\
&-4\eta\Big(A(1)\big(36\theta^2+26\theta^3\big)+36\theta^2 A'(1)\Big)+96\theta^3 A(1)\Big)\Big)\\
&+\frac{1}{(\eta-2)(\eta-3)(\eta-4)(\eta-5)(\eta+\gamma)^4 A(1)}\\
&\times\Big(\eta^4\Big(A(1)\big(24+24\theta+12\theta^2+4\theta^3+\theta^4\big)+A'(1)\big(96+74\theta+24\theta^2+4\theta^3\big)+A''(1)\big(72+36\theta+6\theta^2\big)\\
&+A'''(1)(16+4\theta)+A^{(iv)}(1)\Big)\Big)\\
&-\eta^3\Big(A(1)\big(120\theta+108\theta^2+48\theta^3+14\theta^4\big)+A'(1)\big(360\theta+216\theta^2+48\theta^3\big)\\
&+A''(1)\big(180\theta+54\theta^2\big)+A'''(1)(20\theta)\Big)\\
&+\eta^2\Big(A(1)\big(240\theta^2+188\theta^3+71\theta^4\big)+A'(1)\big(480\theta^2+188\theta^3\big)+A''(1)\big(120\theta^2\big)\Big)\\
&-\eta\Big(A(1)\big(240\theta^3+154\theta^4\big)+A'(1)\big(240\theta^3\big)\Big)+A(1)\big(120\theta^4\big),\quad \eta>5.
\end{aligned}
$$

Under specific choices of A, we thereby obtain Hermite and Gould-Hopper polynomials.

6.1.1. Hermite polynomials

If $A(\tau)=e^{\frac{-k\tau^2}{2}}$ later $R_\eta(x)=H_\eta^{(k)}(x)$ is the Hermite polynomials of variance k , which is

$$H_\nu^{(k)}(x)=\sum_{\mu=0}^{\left[\frac{\nu}{2}\right]}\left(\frac{-k}{2}\right)^{\mu}\frac{\nu!}{\mu!(\nu-2\mu)!}x^{\nu-2\mu}. \tag{29}$$

Here [.] indicates the integer moiety. Hermite polynomials have the following generating function

$$\sum_{\nu=0}^{\infty}\frac{\tau^\nu}{\nu!}H_\nu^{(k)}(x)=e^{\tau x-\frac{k\tau^2}{2}}. \tag{30}$$

Following are presented as

$$D_{\eta}^{H}(x)=(\eta-1)\,e^{\frac{k}{2-\eta x}}\sum_{\nu=0}^{\infty}\frac{H_{\nu}^{(k)}(\eta x)}{\nu!}\int_{0}^{\infty}\begin{pmatrix}\eta+\nu-1\\ \nu\end{pmatrix}\frac{\zeta^{\nu}}{(1+\zeta)^{\eta+\nu}}f\left(\frac{\eta\zeta+\theta}{\eta+\gamma}\right)d\zeta,\ \ k\leq 0. \tag{31}$$

the Stancu-type Szász-Baskakov operators containing Hermite polynomials of variance k.

Lemma 6.6. *We get following results*

$$\lim_{\eta\to\infty}\eta D_{\eta}^{H}(t-x;x) = (2-\beta)\,x+(1+\theta-k)\,,$$
$$\lim_{\eta\to\infty}\eta D_{\eta}^{H}\left((t-x)^{2};x\right) = x^{2}+2x.$$

Theorem 6.7. *On every compact subset of* $[0,\infty)$, *we have*

$$\lim_{\eta\to\infty}\eta\left(D_{\eta}^{H}(f;x)-f(x)\right)=((2-\beta)\,x+(1+\theta-k))\,f'(x)+\frac{1}{2}\left(x^{2}+2x\right)f''(x)$$

uniformly for $f,f',f''\in C[0,\infty)\cap K$ *and* $x\in[0,\infty)$.

Theorem 6.8. *If* $f\in C_{\rho}^{*}[0,\infty)$,

$$\sup_{x\in[0,\infty)}\frac{\left|D_{\eta}^{H}(f;x)-f(x)\right|}{(1+x^{2})^{\frac{5}{2}}}\leq 2\left(2+\Lambda_{0}^{H}(x)+\left(\Lambda_{1}^{H}(x)\right)^{\frac{1}{2}}\right)\Omega\left(f;\left(\Lambda_{0}^{H}(x)\right)^{\frac{1}{2}}\right).$$

Here, $\Lambda_{0}^{H}(x)$ *and* $\Lambda_{1}^{H}(x)$ *are found by substituting the* $A(t)$ *function and its derivatives at the 2nd and 4th central moments, respectively.*

By choosing $f(x)=\frac{\sin x}{(x^{2}+2)^{2}}$ and $f(x)=xe^{-x}$ for $k=-0.5$, $\theta=1$, $\gamma=2$ we compare the error estimation of Stancu-type Szász-Baskakov operators based on Boas-Buck-type polynomials $D_{\eta}^{(\theta,\gamma)}(f;x)$ with Szász-Baskakov operators based on Boas-Buck-type polynomials $D_{\eta}^{*}(f;x)$ [7], including Hermite polynomials by means of the weighted modulus of continuity in Table 1 and Table 2.
For $f(x)=\frac{\sin x}{(x^{2}+2)^{2}}$, the operator given in (8) and our operators error approximation are compared in Table 1.

η	$D_{\eta}^{(\theta,\gamma)}(f;x)$	$D_{\eta}^{*}(f;x)$
10^{2}	0.0739258642	0.1477314273
10^{3}	0.0198623687	0.0449245858
10^{4}	0.0061629816	0.0141485722
10^{5}	0.0019451463	0.0044723395
10^{6}	0.0006149903	0.0014142200
10^{7}	0.0001944732	0.0004472137

Table 1. Error estimation for $D_{\eta}^{(\theta,\gamma)}(f;x)$ and $D_{\eta}^{*}(f;x)$ by using the weighted modulus of continuity

For $f(x)=xe^{-x}$, the operator given in (8) and our operators error approximation are given in Table 2.

η	$D_{\eta}^{(\theta,\gamma)}(f;x)$	$D_{\eta}^{*}(f;x)$
10^2	0.0099332385	0.0198540906
10^3	0.0027069976	0.0060729394
10^4	0.0008446046	0.0019325439
10^5	0.0002670689	0.0006133277
10^6	0.0000844892	0.0001942193
10^7	0.0000267225	0.0000614443

Table 2. $D_{\eta}^{(\theta,\gamma)}(f;x)$ and $D_{\eta}^{*}(f;x)$ error estimation using the weighted modulus of continuity

6.1.2. Gould-Hopper polynomials

If $A(\tau) = e^{h\tau^{\mu}}$ later $R_{\eta}(x) = G_{\eta}^{\mu}(x,h)$ is the Gould-Hopper polynomials [11]. These polynomials have the following clear form:

$$G_{\nu}^{\mu}(x,h) = \sum_{\psi=0}^{[\frac{\nu}{2}]} \frac{\nu!}{\psi!(\nu-\mu\psi)!} h^{\psi} x^{\nu-\mu\psi}. \tag{32}$$

Here [.] indicates the integer part. Gould-Hopper polynomials have the following generating function

$$\sum_{\nu=0}^{\infty} \frac{\tau^{\nu}}{\nu!} G_{\nu}^{\mu}(x,h) = e^{\tau x + h\tau^{\mu}}. \tag{33}$$

After the Stancu-type Szász-Baskakov operators based on Boas-Buck-type polynomials including Gould-Hopper polynomials as follows

$$D_{\eta}^{G}(x) = (\eta-1)e^{-h-\eta x} \sum_{\nu=0}^{\infty} \frac{G_{\nu}^{\mu}(\eta x,h)}{\nu!} \int_0^{\infty} \begin{pmatrix} \eta+\nu-1 \\ \nu \end{pmatrix} \frac{\zeta^{\nu}}{(1+\zeta)^{\eta+\nu}} f\left(\frac{\eta\zeta+\theta}{\eta+\gamma}\right) d\zeta, \ \ h \le 0. \tag{34}$$

Theorem 6.9. *We receive*

$$\lim_{\eta\to\infty} \eta\left(D_{\eta}^{G}(f;x) - f(x)\right) = ((2-\beta)x + (1+\theta+h\mu)) f'(x) + \frac{1}{2}\left(x^2+2x\right) f''(x)$$

uniformly for the variables $f, f', f'' \in C[0,\infty) \cap K$ *and* $x \in [0,\infty)$ *on all compact subsets of* $[0,\infty)$.

Theorem 6.10. *If* $f \in C_{\rho}^{*}[0,\infty)$,

$$\sup_{x\in[0,\infty)} \frac{\left|D_{\eta}^{G}(f;x) - f(x)\right|}{(1+x^2)^{\frac{5}{2}}} \le 2\left(2 + \Lambda_0^{G}(x) + \left(\Lambda_1^{G}(x)\right)^{\frac{1}{2}}\right)\Omega\left(f;\left(\Lambda_0^{G}(x)\right)^{\frac{1}{2}}\right).$$

Here, $\Lambda_0^{G}(x)$ *and* $\Lambda_1^{G}(x)$ *are found by substituting the* $A(t)$ *function and its derivatives at the 2nd and 4th central moments, respectively.*

By selecting $f(x) = xe^{-x}$ and $f(x) = \frac{\sin x}{(x^2+2)^2}$ for $\mu = 0.2$, $h = 0.005$, $\theta = 1$, $\gamma = 2$ we compare the error estimation of Stancu-type Szász-Baskakov operators based on Boas-Buck-type polynomials $D_{\eta}^{(\theta,\gamma)}(f;x)$ with Szász-Baskakov operators based on Boas-Buck-type polynomials $D_{\eta}^{*}(f;x)$ [7], including Gould-Hopper polynomials by means of the weighted modulus of continuity in Table 3 and Table 4.
For $f(x) = xe^{-x}$, the operator given in (8) and our operators error approximation are compared in Table 3.

η	$D_{\eta}^{(\theta,\gamma)}(f;x)$	$D_{\eta}^{*}(f;x)$
10^2	0.0190523910	0.0196608062
10^3	0.0060205105	0.0060666427
10^4	0.0019221424	0.0019323402
10^5	0.0006102189	0.0006133212
10^6	0.0001932403	0.0001942191
10^7	0.0000611347	0.0000614443

Table 3. Error approximation for $D_{\eta}^{(\theta,\gamma)}(f;x)$ and $D_{\eta}^{*}(f;x)$ by using the weighted modulus of continuity for $\mu = 0.2, h = 0.005$

For $f(x) = \frac{\sin x}{(x^2+2)^2}$, the operator given in (8) and our operators error approximation are given in Table 4.

η	$D_{\eta}^{(\theta,\gamma)}(f;x)$	$D_{\eta}^{*}(f;x)$
10^2	0.1418929662	0.1463270353
10^3	0.0445319428	0.0448774276
10^4	0.0140719885	0.0141470727
10^5	0.0044496308	0.0044722920
10^6	0.0014070866	0.0014142184
10^7	0.0004449595	0.0004472137

Table 4. Error approximation for $D_{\eta}^{(\theta,\gamma)}(f;x)$ and $D_{\eta}^{*}(f;x)$ by using the weighted modulus of continuity for $\mu = 0.2, h = 0.005$

7. Conclusion

This study provides Stancu type-Szász-Baskakov operators using polynomials of the Boas-Buck type. Every compact subset of $[0,\infty)$ exhibits uniform convergence $D_{\eta}^{(\theta,\gamma)}$, and the weighted modulus of continuity is used to estimate the rate of convergence of the unbounded function specified on this set. The Voronovskaya-type theorem for the quantitative asymptotic approximation is then demonstrated. Finally, given some specific choices of analytical functions A, B and H, specific polynomials are obtained. The weighted modulus of continuity is then used to illustrate error approximations of $D_{\eta}^{(\theta,\gamma)}$ containing particular polynomials.

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