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On compact (limited) operators between Hilbert and Banach spaces

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Abstract. We give orthonormal characterizations of collectively compact (limited) sets of linear operators from a Hilbert space to a Banach space.

1. Introduction and preliminaries

A bounded linear operator between Hilbert spaces is compact if and only if it maps every orthonormal sequence onto a norm-null sequence (cf. [7, p. 95 and p. 293]). In Theorem 2.3, we extend this fact to linear operators from a Hilbert space to a Banach space. More generally, in Theorem 2.2 we prove that a sets of linear operators from a Hilbert space to a Banach space is collectively compact (resp., limited) if and only if the union of images of each orthonormal sequence is relatively compact (resp., limited). In the sequel, $\mathcal{H}$ denotes a real or complex Hilbert space.

We need the following two elementary and certainly well known lemmas. For convenience, we include the proof of the second one, whereas the proof of the first lemma is left as an exercise.

Lemma 1.1. Every w-null sequence (x_n) in $\mathcal{H}$ contains a subsequence (x_{n_k}) such that $|(x_{n_{k_1}}, x_{n_{k_2}})| \le 2^{-2(n_{k_1}+n_{k_2})}$ whenever $k_1 \neq k_2$.

Lemma 1.2. A linear operator $T : \mathcal{H} \to Y$ from a Hilbert space $\mathcal{H}$ to a Banach space Y is bounded if and only if T maps orthonormal sequences onto bounded sequences.

Proof. Only the sufficiency needs a proof. Let a linear operator $T : \mathcal{H} \to Y$ be bounded on each orthonormal sequence of $\mathcal{H}$. Suppose T is not bounded. Then, there exists a linearly independent normalized sequence (x_n) in $\mathcal{H}$ such that $\|Tx_n\| \ge 2^n$ for all n. Pick (by the Gram–Schmidt orthonormalization) an orthonormal sequence (u_n) in $\mathcal{H}$ such that $x_n \in \operatorname{span}\{u_i\}_{i=1}^n$ for all n. By the assumption, there exists an $M \in \mathbb{R}$ satisfying $\|Tu_n\| \le M$ for all $n \in \mathbb{N}$. Then

$$2^n \le \|Tx_n\| = \left\| \sum_{k=1}^{n} (x_n, u_k) T(u_k) \right\| \le nM \quad (\forall n \in \mathbb{N}),$$

which is absurd. The obtained contradiction completes the proof. □

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A subset A of a normed space X is limited [5] whenever $f_n \rightrightarrows 0(A)$ for every w*-null sequence (f_n) of X'. Each limited set is necessarily bounded. Indeed, otherwise let A be a limited subset of a normed space X and $\|a_n\| \geq n$ for a sequence (a_n) in A. Take a sequence (f_n) in X' such that $f_n(a_n) = 1$ and $\|f_n\| = \|a_n\|^{-1}$ for all n. Then $\|f_n\| \to 0$, and hence $f_n \xrightarrow{w^*} 0$. As A is limited, $f_n \rightrightarrows 0(A)$ in contrary with $f_n(a_n) = 1$ for all n. Therefore, the next corollary follows from Lemma 1.2.

Corollary 1.3. A linear operator $T : \mathcal{H} \to Y$ from a Hilbert space $\mathcal{H}$ to a Banach space Y is bounded if it maps orthonormal sequences onto limited sets.

For further unexplained terminology and notation, we refer to [1, 2, 4, 5, 7].

2. Main results

We begin with the following definition [1, 2].

Definition 2.1. A subset $\mathcal{T}$ of the space $\mathrm{L}(X, Y)$ of linear operators between normed spaces X and Y is

a) collectively (weakly) compact if the set $\mathcal{T}B_X = \bigcup\limits_{T\in\mathcal{T}} T(B_X)$ is relatively (weakly) compact.

b) collectively limited if the set $\mathcal{T}B_X$ is limited.

Thus, an operator T is (weakly) compact (resp., limited) if and only if the set $\{T\}$ is collectively (weakly) compact (resp., collectively limited). The following theorem is the main result of the paper.

Theorem 2.2. Let $\mathcal{T} \subseteq \mathrm{L}(\mathcal{H}, Y)$, where $\mathcal{H}$ is a Hilbert space and Y is a Banach space. Then the following holds.

i) $\mathcal{T}$ is collectively compact if and only if the set $\{Tx_n : T \in \mathcal{T}; n \in \mathbb{N}\}$ is relatively compact for every orthonormal sequence (x_n) in $\mathcal{H}$.

ii) $\mathcal{T}$ is collectively limited if and only if the set $\{Tx_n : T \in \mathcal{T}; n \in \mathbb{N}\}$ is limited for every orthonormal sequence (x_n) in $\mathcal{H}$.

Proof. We may assume $\dim(\mathcal{H}) = \infty$. Only the sufficiency requires proof. First, observe $\sup\limits_{T\in\mathcal{T}} \|T\| \leq M < \infty$. Otherwise $\sup\limits_{T\in\mathcal{T}} \|Tx\| = \infty$ for some $x \in \mathcal{H}$, $\|x\| = 1$ by the Uniform Boundedness Principle. This leads to contradiction via taking an orthonormal sequence (x_n) in $\mathcal{H}$ with $x_1 = x$ because the set $\{Tx_1\}_{T\in\mathcal{T}}$ is not bounded, and hence not limited.

i) Suppose, in contrary, $\bigcup\limits_{T\in\mathcal{T}} T(B_{\mathcal{H}})$ is not relatively compact. Then, for some $\alpha > 0$, there exist a normalized sequence (y_n) in $\mathcal{H}$ and a sequence (T_n) in $\mathcal{T}$ satisfying $\|T_n y_n - T_m y_m\| \geq 3\alpha$ for all $m \neq n$.

Consider separately the following two mutually exclusive cases. **(A)**: (y_n) has no norm-convergent subsequence; and **(B)**: (y_n) has a norm-convergent subsequence.

(A). Since $B_{\mathcal{H}}$ is relatively weakly compact, by passing to subsequence, we may assume $y_n \xrightarrow{w} y$ and $\|y_n - y_m\| \geq C > 0$ for all $m \neq n$ and some C. By the weak lower semicontinuity of the norm (cf. [3, Lemma 6.22]), $\|y_n - y\| \geq C$ for all n. Let $z_n = \frac{y_n - y}{\|y_n - y\|}$. Then $z_n \xrightarrow{w} 0$. By the assumption, $(T_{n_j} y)$ is norm-Cauchy for some subsequence (T_{n_j}). We may assume $\|T_{n_j} y - T_{n_i} y\| \leq \min(j, i)^{-1}$ for all $j, i \in \mathbb{N}$. Then, for all $j \neq i$,

$$2\|T_{n_j} z_{n_j} - T_{n_i} z_{n_i}\| = 2\left\|\frac{T_{n_j}(y_{n_j} - y)}{\|y_{n_j} - y\|} - \frac{T_{n_i}(y_{n_i} - y)}{\|y_{n_i} - y\|}\right\| \geq \|T_{n_j}(y_{n_j} - y) - T_{n_i}(y_{n_i} - y)\| \geq$$

$$\|T_{n_j} y_{n_j} - T_{n_i} y_{n_i}\| - \|T_{n_j} y - T_{n_i} y\| \geq 3\alpha - \|T_{n_j} y - T_{n_i} y\| \geq 3\alpha - \min(j, i)^{-1}.$$

By applying Lemma 1.1 and passing to a further subsequence, we may assume that $|(z_{n_j}, z_{n_i})| \le 2^{-2(n_j+n_i)}$ and $\|T_{n_j}z_{n_j} - T_{n_i}z_{n_i}\| \ge \alpha$ for all $j \neq i$. By the Gram–Schmidt orthonormalization, there exists an orthonormal (x_{n_j}) in $\mathcal{H}$ with $\|x_{n_j} - z_{n_j}\| \le 2^{-n_j}$ for all j. By the compactness assumption, there exists a subsequence $(x_{n_{j_l}})$ such that $T_{n_{j_l}}x_{n_{j_l}} \xrightarrow{\|\cdot\|} w \in Y$, and hence

$$\alpha \le \|T_{n_{j_l}}z_{n_{j_l}} - T_{n_{j_p}}z_{n_{j_p}}\| \le \|T_{n_{j_l}}x_{n_{j_l}} - T_{n_{j_p}}x_{n_{j_p}}\| + \|T_{n_{j_l}}(x_{n_{j_l}} - z_{n_{j_l}})\| + \|T_{n_{j_l}}(x_{n_{j_p}} - z_{n_{j_p}})\| \le$$

$$\|T_{n_{j_l}}x_{n_{j_l}} - w\| + \|T_{n_{j_p}}x_{n_{j_p}} - w\| + (2^{-n_{j_l}} + 2^{-n_{j_p}})M \to 0 \qquad (l, p \to \infty).$$

A contradiction.

(B). Assume (y_n) has a norm-convergent subsequence, say $y_{n_j} \xrightarrow{\|\cdot\|} y$. Let $\|y_{n_j} - y\| \le M^{-1}\alpha$ for $j \ge j_0$. Then, for all $j, i \ge j_0$,

$$\|T_{n_j}y - T_{n_i}y\| = \|T_{n_j}y - T_{n_j}y_{n_j} + T_{n_j}y_{n_j} - T_{n_i}y_{n_i} + T_{n_i}y_{n_i} - T_{n_i}y\| \ge$$

$$-\|T_{n_j}y - T_{n_j}y_{n_j}\| + \|T_{n_j}y_{n_j} - T_{n_i}y_{n_i}\| - \|T_{n_i}y_{n_i} - T_{n_i}y\| \ge$$

$$3\alpha - M(\|y_{n_j} - y\| + \|y_{n_i} - y\|) \ge 3\alpha - M(M^{-1}\alpha + M^{-1}\alpha) = \alpha,$$

which leads to a contradiction by taking an orthonormal sequence (z_n) with $z_1 = y$. It follows that $\bigcup\limits_{T\in\mathcal{T}} T(B_{\mathcal{H}})$ is relatively compact.

ii) Suppose in contrary $\bigcup\limits_{T\in\mathcal{T}} T(B_{\mathcal{H}})$ is not limited. Then, for some w*-null sequence (f_n) in Y' there exist an $\alpha > 0$, a sequence (T_n) in $\mathcal{T}$, and a normalized sequence (x_n) in $\mathcal{H}$ satisfying $|f_n(T_nx_n)| \ge \alpha$ for all n.

Let $0 \neq z \in \mathcal{H}$. Take any orthonormal sequence (z_n) with $z_1 = \frac{z}{\|z\|}$. Since $|f_n(T_nz)| \le \sup\limits_{k\in\mathbb{N};T\in\mathcal{T}} |f_n(Tz_k)|$ for all n, and $\lim\limits_{n\to\infty} \sup\limits_{k\in\mathbb{N};T\in\mathcal{T}} |f_n(Tz_k)| = 0$ by the assumption, then $|(T'_nf_n)z| = |f_n(T_nz)| \to 0$. So, the sequence (T'_nf_n) is w*-null in $\mathcal{H}'$, and hence (by identifying $\mathcal{H}$ with $\mathcal{H}''$) the sequence (T'_nf_n) is w-null in the Hilbert space $\mathcal{H}'$. Since $|(T'_nf_n)x_n| = |f_n(T_nx_n)| \ge \alpha$ then $\|T'_nf_n\| \ge \alpha$ for all n. Thus, by scaling (f_n), we may assume $\|T'_nf_n\| = 1$ for all n.

Lemma 1.1 implies the existence of a subsequence $(T'_{n_k}f_{n_k})$ satisfying

$$|(T'_{n_{k_1}}f_{n_{k_1}}, T'_{n_{k_2}}f_{n_{k_2}})| \le 2^{-2(n_{k_1}+n_{k_2})} \qquad (k_1 \neq k_2).$$

Applying the Gram–Schmidt orthonormalization to $(T'_{n_k}f_{n_k})$, we obtain an orthonormal sequence (z_{n_k}) in $\mathcal{H}'$ such that $\|z_{n_k} - T'_{n_k}f_{n_k}\| \le 2^{-k}$ for all k. Pick a bi-orthogonal for (z_{n_k}) sequence (y_{n_k}) in $\mathcal{H}$, i.e, (y_{n_k}) is orthonormal and $z_{n_k}(y_{n_k}) = 1$ for every k. By the assumption, $\{Ty_{n_l}\}_{l\in\mathbb{N};T\in\mathcal{T}}$ is a limited subset of Y. Thus, $f_n \rightrightarrows 0\,(\{Ty_{n_l}\}_{l\in\mathbb{N};T\in\mathcal{T}})$, and hence $T'_nf_n \rightrightarrows 0\Big(\{y_{n_l}\}_{l=1}^{\infty}\Big)$ violating

$$\limsup_{n\to\infty}\left(\sup_{l\in\mathbb{N}}\left|(T'_nf_n)y_{n_l}\right|\right) \ge \limsup_{k\to\infty}\left(\sup_{l\in\mathbb{N}}\left|(T'_{n_k}f_{n_k})y_{n_l}\right|\right) \ge$$

$$\limsup_{k\to\infty}\left|(T'_{n_k}f_{n_k})y_{n_k}\right| = \limsup_{k\to\infty}\left|z_{n_k}(y_{n_k})\right| = 1.$$

The obtained contradiction completes the proof. □

The author does not know whether it is possible to replace compactness by weak compactness in Theorem 2.2 *i*).

Next, we give an orthonormal characterization of compact operators.

Theorem 2.3. Let $T : \mathcal{H} \to Y$ be a linear operator from a Hilbert space $\mathcal{H}$ to a Banach space Y. The following conditions are equivalent.

i) T is compact.

ii) T maps every orthonormal subset of $\mathcal{H}$ into a relatively compact set.

iii) T maps every orthonormal sequence of $\mathcal{H}$ to a norm-null sequence.

iv) A set $A_\varepsilon = \{a \in A : \|Ta\| \geq \varepsilon\}$ is finite for every orthonormal basis A of $\mathcal{H}$ and every $\varepsilon > 0$.

v) T maps every orthonormal basis of $\mathcal{H}$ into a relatively compact set.

Proof. We may assume $\dim(\mathcal{H}) = \infty$.

i) $\Longrightarrow$ *iii*): Let (x_n) be an orthonormal sequence in $\mathcal{H}$. Then $x_n \xrightarrow{\mathrm{w}} 0$, and hence $\|Tx_n\| \to 0$ because T is compact.

iii) $\Longrightarrow$ *i*): It follows from Theorem 2.2 *i*).

ii) $\Longrightarrow$ *iii*): The operator T is bounded by Lemma 1.2. Let (u_n) be an orthonormal sequence of $\mathcal{H}$. On the way to a contradiction, suppose $\|Tu_n\| \not\to 0$. By passing to a subsequence, we may suppose $\|Tu_n\| \geq M$ for all n and some $M > 0$. By *i*), the set $\{Tu_n : n \in \mathbb{N}\}$ is relatively compact, and hence (Tu_n) has a norm-convergent subsequence, say $\lim\limits_{j\to\infty} \|Tu_{n_j} - y\| = 0$ for some $y \in Y$. As (u_{n_j}) is orthonormal, $u_{n_j} \xrightarrow{\mathrm{w}} 0$. Since T is bounded, $Tu_{n_j} \xrightarrow{\mathrm{w}} 0$. As $Tu_{n_j} \xrightarrow{\|\cdot\|} y$, we obtain $y = 0$, and hence (Tu_{n_j}) is norm-null, which is absurd since $\|Tu_n\| \geq M > 0$ for all n.

iii) $\Longrightarrow$ *ii*): Let A be an orthonormal subset of $\mathcal{H}$. Pick a sequence (y_n) in $T(A)$. We need to show that $y_{n_j} \xrightarrow{\|\cdot\|} y$ for some subsequence (y_{n_j}) and some $y \in Y$. Clearly, (y_n) has a norm-convergent subsequence when some term y_n of (y_n) occurs infinitely many times. So, suppose each term of the sequence (y_n) occurs at most finitely many times. Keeping the first of occurrences of each y_n in (y_n) and removing others, we obtain a subsequence (y_{n_j}) whose terms are distinct. Pick an $x_n \in A$ with $y_n = Tx_n$ for each $n \in \mathbb{N}$. Then (x_{n_j}) is an orthonormal sequence of $\mathcal{H}$, and hence $y_{n_j} = Tx_{n_j} \xrightarrow{\|\cdot\|} 0$ by *iii*).

Implications *iii*) $\Longrightarrow$ *iv*) $\Longrightarrow$ *v*) are trivial, while *v*) $\Longrightarrow$ *ii*) is a consequence of the fact that each orthonormal $S \subseteq \mathcal{H}$ has an extension to an orthonormal basis. □

The condition *v*) of Theorem 2.3 cannot be replaced by the condition that T maps some orthonormal basis of $\mathcal{H}$ into a relatively compact set (see, for example [7, p.292]).
The following weakly compact version of Theorem 2.3 follows immediately from Lemma 1.2 due to the fact that every bounded linear operator $T : \mathcal{H} \to Y$ is weakly compact.

Proposition 2.4. Let $T : \mathcal{H} \to Y$ be a linear operator from a Hilbert space $\mathcal{H}$ to a Banach space Y. The following conditions are equivalent.

i) T is weakly compact.

ii) T maps every orthonormal subset of $\mathcal{H}$ into a weakly relatively compact set.

iii) T maps every orthonormal sequence of $\mathcal{H}$ to a w-null sequence.

iv) T maps every orthonormal basis of $\mathcal{H}$ into a relatively weakly compact set.

Recall that a linear operator $T : X \to Y$ between normed spaces is limited if $T(B_X)$ is limited in Y, where B_X is the closed unit ball of X. It follows immediately that a bounded linear operator T is limited if T' carries w*-null sequences to norm-null ones. It is well known that compact operators from $\mathcal{H}$ to Y agree with limited operators whenever Y is reflexive or separable [5]. In general, they are different.

Example 2.5. Let $\mathcal{B}$ be an ortonormal basis in an infinite dimensional Hilbert space $\mathcal{H}$, and let $Y = \ell^\infty(\mathcal{B})$ be the space of bounded scalar functions on $\mathcal{B}$. The operator $T : \mathcal{H} \to Y$ defined by $(Tx)(u) = (x, u)$ for $x \in \mathcal{H}$ and $u \in \mathcal{B}$ is not compact (the u-th coordinate of Tu is one whereas all others are zeros). Since $\mathcal{B}$ is an orthonormal basis, $T(B_{\mathcal{H}}) \subseteq B_{c_0(\mathcal{B})}$. It follows from Phillip's lemma (cf. [4, Theorem 4.67]) that $B_{c_0(\mathcal{B})}$ is a limited subset of Y, and hence T is limited.

Theorem 2.6. Let $T : \mathcal{H} \to Y$ be a linear operator from an infinite dimensional Hilbert space $\mathcal{H}$ to a Banach space Y. The following conditions are equivalent.

i) T is limited.

ii) T maps every orthonormal subset of $\mathcal{H}$ onto a limited set.

iii) T maps every orthonormal basis of $\mathcal{H}$ onto a limited set.

iv) T maps every orthonormal sequence of $\mathcal{H}$ onto a limited set.

Proof. Implications *i*) $\Longrightarrow$ *ii*) $\Longrightarrow$ *iii*) $\Longrightarrow$ *iv*) are trivial, and *iv*) $\Longrightarrow$ *i*) follows from Theorem 2.2 *ii*). □

The condition *ii*) of Theorem 2.6 cannot be replaced by the condition that T carries some orthonormal basis of $\mathcal{H}$ onto a limited set. Indeed, limited sets in $\mathcal{H}$ agree with relatively compact sets [5], and hence limited operators from $\mathcal{H}$ to $\mathcal{H}$ coincide with compact ones. So, apply again [7, p.292]. The orthonormal sets, bases, and sequences in Lemma 1.2, in Theorems 2.2–2.6, and in Proposition 2.4 can be replaced by bounded orthogonal sequences. The Banach lattice setting, where the bounded disjointedness can be used instead of bounded orthogonality is much more rigid. It produces new classes of disjointedly (weakly) compact and disjointedly limited operators [6].

References

[1] P. M. Anselone, *Collectively Compact Operator Approximation Theory and Applications to Integral Equations*, Prentice-Hall, 1971.
[2] P. M. Anselone, T. W. Palmer, *Collectively compact sets of linear operators*, Pacific J. Math., **25**(3) (1968), 417–422.
[3] C. D. Aliprantis, K. Border, *Infinite dimensional analysis. A Hitchhiker's Guide*, (3rd edition), Springer, 2006.
[4] C. D. Aliprantis, O. Burkinshaw, *Positive operators*, Springer, 2006.
[5] J. Bourgain, J. Diestel, *Limited operators and strict cosingularity*, Math. Nachr., **119** (1984), 55–58.
[6] E. Emelyanov, N. Erkurşun-Özcan, S. Gorokhova, *d-Operators in Banach Lattices*, https://arxiv.org/abs/2401.08792v3 (2024).
[7] P. R. Halmos, *A Hilbert Space Problem Book*, Springer, 1982.