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Explicit solutions of the Yang-Baxter-like matrix equation for diagonalizable coefficient matrix with two distinct nonzero eigenvalues

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Abstract. Let A be a complex diagonalizable matrix with two distinct nonzero eigenvalues λ and μ, the Yang-Baxter-like matrix equation $AXA = XAX$ is reconsidered. We correct and improve the results in Shen et al. (2020) when $\lambda^2 - \lambda\mu + \mu^2 = 0$. We also improve the results in Shen et al. (2020) when $\lambda^2 - \lambda\mu + \mu^2 \neq 0$. We obtain the explicit structure of the solutions X for the Yang-Baxter-like matrix equation $AXA = XAX$, which are diagonalizable. Finally, we improve other existing relevant conclusions.

1. Introduction

Let A be an $n \times n$ diagonalizable complex matrix with two distinct nonzero eigenvalues. The quadratic matrix equation

$$AXA = XAX, \tag{1}$$

is often called the *Yang-Baxter-like matrix equation* (also called the star-triangle-like equation in statistical mechanics; see, e.g., in Part III of [9]) because of its connections with the classical Yang-Baxter equation arising in statistical mechanics [4, 16, 17]. The Yang-Baxter equation first appeared in theoretical physics, in a paper by the Nobel laureate C. N. Yang [16], and in statistical mechanics, in R. J. Baxter's work [4]. Later, it became one of the important equations of mathematical physics. It plays a crucial role in: Knot theory, braided categories, non-commutative descent theory, quantum computing, integrable systems, non-commutative geometry, and so on [1–3, 7, 10, 17].

Solving Yang-Baxter-like matrix equation (1) is equivalent to solving a polynomial system of n^2 quadratic equations with n^2 unknowns, which solving this system is a very challenging topic. This compelled many researchers to find solutions for particular A. All solutions have been constructed for rank-1 matrices A in [15], rank-2 matrices A in [18, 19], non-diagonalizable elementary matrices A in [20], idempotent matrices

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A ($A^2 = A$) in [11], $A^2 = I$ in [8, 12], $A^3 = A$ in [13], and diagonalizable matrices A with two different eigenvalues in [14]. However, some results in [14] is false.

In this paper, we solve the Yang-Baxter-like matrix equation (1) to derive all explicit solutions X when the given diagonalizable matrix A has two distinct nonzero eigenvalues λ and μ. We correct and improve the results in [14] when $\lambda^2-\lambda\mu+\mu^2 = 0$. We also improve the results in [14] when $\lambda^2-\lambda\mu+\mu^2 \neq 0$. We obtain the explicit structure of the solutions. We prove that the solutions are diagonalizable and the spectrum contained in the set $\{\lambda, \mu, 0\}$. We extend the research for which A is a Householder matrix $A = I - 2uu^H$ [6], A is a class of elementary matrices $A = I - uv^T$ ($v^T u \neq 0$) [5] and $A \neq \pm I$ is an $n\times n$ complex matrix satisfying $A^2 = I$ [8, 12], respectively. This is an important step to solve more general matrices.

2. Main results

In this section, we give main results. At first, we give an assumption as follows.

Assumption 2.1. *Let A be an $n \times n$ complex diagonalizable matrix with two distinct nonzero eigenvalues λ and μ, that is, $A = SJS^{-1}$ in which, S is a nonsingular matrix and*

$$J = diag(\lambda I_m, \mu I_{n-m}).$$

Recently, in [14], the authors gave the following results.

Theorem 2.1. *[14, Theorem 4.4, Theorem 4.5 and Theorem 4.6] Suppose that A satisfies Assumption 2.1.*

If $\lambda^2 - \lambda\mu + \mu^2 = 0$, then all solutions of the Yang-Baxter-like matrix equation (1) have the form

$$X = S\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}\left[\begin{array}{cc|cc} \lambda I_t & 0 & F & 0 \\ 0 & 0_{m-t} & 0 & 0 \\ \hline G & 0 & \mu I_k & 0 \\ 0 & 0 & 0 & 0_{n-m-k} \end{array}\right]\begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}S^{-1}, \tag{2}$$

in which, $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{(n-m)\times(n-m)}$ are any invertible matrices, $0 \leq t \leq m$, $0 \leq k \leq n-m$, F is an arbitrary $t \times k$ matrix, $G = (I - F^\dagger F)M(I - FF^\dagger)$, M is an arbitrary $k \times t$ matrix.

If $\lambda^2 - \lambda\mu + \mu^2 \neq 0$, then all solutions of the Yang-Baxter-like matrix equation (1) are

$$X = S\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\left[\begin{array}{ccc|ccc} \widehat{\lambda} I_r & 0 & 0 & C & 0 & 0 \\ 0 & \lambda I_v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0_{m-r-v} & 0 & 0 & 0 \\ \hline D & 0 & 0 & \widehat{\mu} I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu I_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{n-m-r-\tau} \end{array}\right]\begin{bmatrix} U^{-1} & 0 \\ 0 & V^{-1} \end{bmatrix}S^{-1}, \tag{3}$$

in which, $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{(n-m)\times(n-m)}$ are any invertible matrices, $0 \leq r \leq \min\{m, n-m\}$, $0 \leq v \leq m-r$, $0 \leq \tau \leq n-m-r$, $\widehat{\lambda} = \frac{\mu^2}{\mu-\lambda}$, $\widehat{\mu} = \frac{\lambda^2}{\lambda-\mu}$, C is an arbitrary $r \times r$ invertible matrix, and $D = \frac{-\lambda\mu(\lambda^2-\lambda\mu+\mu^2)}{(\lambda-\mu)^2}C^{-1}$.

The result in (2) is false. There is a counterexample as follows.

Example 2.2. *Let $A = diag(1, 1, \frac{1+\sqrt{3}i}{2})$. The eigenvalues of A are 1, 1, and $\frac{1+\sqrt{3}i}{2}$. It is easy to verify that $1^2 - 1 \times \frac{1+\sqrt{3}i}{2} + (\frac{1+\sqrt{3}i}{2})^2 = 0$ and*

$$X = \left[\begin{array}{cc|c} 1 & -\frac{1+\sqrt{3}i}{2} & 1 \\ 0 & 1 & 0 \\ \hline 0 & 1 & \frac{1+\sqrt{3}i}{2} \end{array}\right]$$

is a solution of the Yang-Baxter-like matrix equation (1). In this case, we see that its upper left 2×2 corner is not diagonalizable. So the result in (2) does not hold.

The false in proving Theorem 4.3 in [14] is that

$$\widetilde{C^{(1)}}\widetilde{D^{(1)}} = \begin{bmatrix} -\frac{\lambda_1^2}{\lambda_2} I_{s_1} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \widetilde{D^{(1)}}\widetilde{C^{(1)}} = \begin{bmatrix} -\frac{\lambda_2^2}{\lambda_1} I_{s_2} & 0 \\ 0 & 0 \end{bmatrix}.$$

If $C^{(1)} \in \mathbb{C}^{t_1\times t_2}$, $D^{(1)} \in \mathbb{C}^{t_2\times t_1}$, $C^{(1)}([2:2:2s_1],:) = 0$, $C^{(1)}(:,[1:2:2s_2-1]) = 0$, $D^{(1)}([2:2:2s_2],:) = 0$, $D^{(1)}(:,[1:2:2s_1-1]) = 0$, $2s_1 \le t_1$, $2s_2 \le t_2$, Suppose that $\widetilde{C^{(1)}}$ is a matrix obtained from $C^{(1)}$ by deleting the rows $2j$ for $j = 1:s_1$ and columns $2j-1$ for $j = 1:s_2$. $\widetilde{D^{(1)}}$ is a matrix obtained from $D^{(1)}$ by deleting the rows $2j$ for $j = 1:s_2$ and columns $2j-1$ for $j = 1:s_1$. Suppose that $s_1 = 1$, $s_2 = 1$, $t_1 = 3$, $t_2 = 3$,

$$C^{(1)} = \begin{bmatrix} 0 & c_{12} & c_{13} \\ 0 & 0 & 0 \\ 0 & c_{32} & c_{33} \end{bmatrix}, \quad D^{(1)} = \begin{bmatrix} 0 & d_{12} & d_{13} \\ 0 & 0 & 0 \\ 0 & d_{32} & d_{33} \end{bmatrix},$$

then

$$\widetilde{C^{(1)}} = \begin{bmatrix} c_{12} & c_{13} \\ c_{32} & c_{33} \end{bmatrix}, \quad \widetilde{D^{(1)}} = \begin{bmatrix} d_{12} & d_{13} \\ d_{32} & d_{33} \end{bmatrix}.$$

It is obvious that $\widetilde{C^{(1)}}\widetilde{D^{(1)}}$ is not equal to the matrix

$$\begin{bmatrix} c_{13}d_{32} & c_{13}d_{33} \\ c_{32}d_{32} & c_{33}d_{33} \end{bmatrix}$$

obtained from $C^{(1)}D^{(1)}$ by deleting the row 2 and column 1.

According to Lemma 4.1 and 4.2, and Theorem 4.3 in [14], we have the following lemma. Then we will prove that the solutions of the Yang-Baxter-like matrix equation (1) is diagonalizable and the eigenvalues are contained in the set $\{\lambda, \mu, 0\}$ whenever $\lambda^2 - \lambda\mu + \mu^2 = 0$ or $\lambda^2 - \lambda\mu + \mu^2 \neq 0$.

Lemma 2.3. *[14, Lemma 4.1 and 4.2, Theorem 4.4] Suppose that A satisfies Assumption 2.1 with $\lambda^2 - \lambda\mu + \mu^2 = 0$. Then all solutions of the Yang-Baxter-like matrix equation (1) have the form*

$$X = S\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}\left[\begin{array}{cc|cc} \lambda I_t + N_1 & 0 & F & 0 \\ 0 & 0_{m-t} & 0 & 0 \\ \hline G & 0 & \mu I_k + N_2 & 0 \\ 0 & 0 & 0 & 0_{n-m-k} \end{array}\right]\begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}S^{-1}, \tag{4}$$

in which, $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{(n-m)\times(n-m)}$ are any invertible matrices, $0 \le t \le m$, $0 \le k \le n-m$, $F \in \mathbb{C}^{t\times k}$, $G \in \mathbb{C}^{k\times t}$. The elements of N_1, N_2 are zeros except that the second upper diagonal elements may be one or zero. $N_1^2 = 0$, $N_2^2 = 0$, $FG = -\frac{\lambda^2}{\mu}N_1$, $GF = -\frac{\mu^2}{\lambda}N_2$, $N_1F = 0$, $GN_1 = 0$, $FN_2 = 0$, $N_2G = 0$.

The following lemma is the key to deriving our theorem.

Lemma 2.4. *Let $B = \lambda I + \alpha N$. λ and α are scalars ($\lambda \neq 0$). If $N^2 = 0$, then*

$$B^{-1} = \frac{1}{\lambda}I - \frac{\alpha}{\lambda^2}N.$$

Proof. Since $N^2 = 0$, it is easy to verify that

$$(\lambda I + \alpha N)(\frac{1}{\lambda}I - \frac{\alpha}{\lambda^2}N) = I.$$

Thus

$$B^{-1} = \frac{1}{\lambda}I - \frac{\alpha}{\lambda^2}N.$$

This completes the proof. □

Based on Theorem 2.1, and Lemma 2.3 and 2.4, we give our results in the following theorem.

Theorem 2.5. *Suppose that A satisfies Assumption 2.1. If $\lambda^2-\lambda\mu+\mu^2=0$, then all solutions of the Yang-Baxter-like matrix equation (1) have the form*

$$X = S\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} R \left[\begin{array}{cc|cc} \lambda I_t & 0 & 0 & 0 \\ 0 & 0_{m-t} & 0 & 0 \\ \hline 0 & 0 & \mu I_k & 0 \\ 0 & 0 & 0 & 0_{n-m-k} \end{array}\right] R^{-1} \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} S^{-1}, \tag{5}$$

$P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{(n-m)\times(n-m)}$ are any invertible matrices,

$$R = \left[\begin{array}{cc|cc} I_t & 0 & \frac{F}{\mu-\lambda} & 0 \\ 0 & I_{m-t} & 0 & 0 \\ \hline \frac{G}{\lambda-\mu} & 0 & I_k & 0 \\ 0 & 0 & 0 & I_{n-m-k} \end{array}\right],$$

$0 \le t \le m$, $0 \le k \le n-m$, $F \in \mathbb{C}^{t\times k}$, $G \in \mathbb{C}^{k\times t}$, $FG = -\frac{\lambda^2}{\mu}N_1$, $GF = -\frac{\mu^2}{\lambda}N_2$, $N_1F = 0$, $GN_1 = 0$, $FN_2 = 0$, $N_2G = 0$. The elements of N_1, N_2 are zeros except that the second upper diagonal elements may be one or zero. $N_1^2 = 0$, $N_2^2 = 0$. Thus all solutions X of the Yang-Baxter-like matrix equation (1) are diagonalizable and the eigenvalues of X are contained in the set $\{\lambda, \mu, 0\}$.

Proof. According to (4) in Lemma 2.3, let

$$Y_1 = \left[\begin{array}{cc|cc} \lambda I_t + N_1 & 0 & F & 0 \\ 0 & 0_{m-t} & 0 & 0 \\ \hline G & 0 & \mu I_k + N_2 & 0 \\ 0 & 0 & 0 & 0_{n-m-k} \end{array}\right].$$

Apply Lemma 2.4 and $N_1^2 = 0$, $N_2^2 = 0$, $FG = -\frac{\lambda^2}{\mu}N_1$, $GF = -\frac{\mu^2}{\lambda}N_2$, $N_1F = 0$, $GN_1 = 0$, $FN_2 = 0$, $N_2G = 0$, we obtain that the characteristic polynomial of Y_1 is

$$P_{Y_1}(x) = det(xI - Y_1) = x^{n-t-k}(x-\lambda)^t(x-\mu)^k.$$

Thus the eigenvalues λ, μ, and 0 of Y_1 have algebraic multiplicity t, k, and $n-t-k$, respectively. It is easy to verify that $Rank(\lambda I - Y_1) = n-t$, $Rank(\mu I - Y_1) = n-k$, $Rank(0I - Y_1) = t+k$. So Y_1 is diagonalizable. In fact, let

$$R = \left[\begin{array}{cc|cc} I_t & 0 & \frac{F}{\mu-\lambda} & 0 \\ 0 & I_{m-t} & 0 & 0 \\ \hline \frac{G}{\lambda-\mu} & 0 & I_k & 0 \\ 0 & 0 & 0 & I_{n-m-k} \end{array}\right].$$

Since $\lambda^2 - \lambda\mu + \mu^2 = 0$ and $GF = -\frac{\mu^2}{\lambda}N_2$, we have

$$det(R) = \left|\begin{array}{cc|cc} I_t & 0 & \frac{F}{\mu-\lambda} & 0 \\ 0 & I_{m-t} & 0 & 0 \\ \hline \frac{G}{\lambda-\mu} & 0 & I_k & 0 \\ 0 & 0 & 0 & I_{n-m-k} \end{array}\right| = \left|\begin{array}{cc|cc} I_t & 0 & \frac{F}{\mu-\lambda} & 0 \\ 0 & I_{m-t} & 0 & 0 \\ \hline 0 & 0 & I_k - \frac{N_2}{\mu-\lambda} & 0 \\ 0 & 0 & 0 & I_{n-m-k} \end{array}\right| = 1 \ne 0.$$

Thus matrix R is invertible and

$$R^{-1} = \left[\begin{array}{cc|cc} I_t + \frac{N_1}{\lambda-\mu} & 0 & \frac{F}{\lambda-\mu} & 0 \\ 0 & I_{m-t} & 0 & 0 \\ \hline \frac{G}{\mu-\lambda} & 0 & I_k + \frac{N_2}{\mu-\lambda} & 0 \\ 0 & 0 & 0 & I_{n-m-k} \end{array}\right].$$

Then

$$Y_1 = R\left[\begin{array}{cc|cc} \lambda I_t & 0 & 0 & 0 \\ 0 & 0_{m-t} & 0 & 0 \\ \hline 0 & 0 & \mu I_k & 0 \\ 0 & 0 & 0 & 0_{n-m-k} \end{array}\right]R^{-1}.$$

Thus, we get (5). So all solutions X of the Yang-Baxter-like matrix equation (1) are diagonalizable and the eigenvalues of X are contained in the set $\{\lambda, \mu, 0\}$. This completes the proof.

□

Theorem 2.6. *Suppose that A satisfies Assumption 2.1. If $\lambda^2 - \lambda\mu + \mu^2 \neq 0$, then all solutions of the Yang-Baxter-like matrix equation (1) have the form*

$$X = S\left[\begin{array}{cc} U & 0 \\ 0 & V \end{array}\right]W\left[\begin{array}{ccc|ccc} \lambda I_r & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda I_v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0_{m-r-v} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mu I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu I_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{n-m-r-\tau} \end{array}\right]W^{-1}\left[\begin{array}{cc} U^{-1} & 0 \\ 0 & V^{-1} \end{array}\right]S^{-1},$$

in which, $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{(n-m)\times(n-m)}$ are any invertible matrices,

$$W = \left[\begin{array}{ccc|ccc} \frac{\lambda-\mu}{\lambda^2-\lambda\mu+\mu^2}C & 0 & 0 & \frac{\lambda-\mu}{\lambda\mu}C & 0 & 0 \\ 0 & I_v & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m-r-v} & 0 & 0 & 0 \\ \hline I_r & 0 & 0 & I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & I_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n-m-r-\tau} \end{array}\right],$$

$0 \leq r \leq \min\{m, n-m\}$, $0 \leq v \leq m-r$, $0 \leq \tau \leq n-m-r$, and C is an arbitrary $r\times r$ invertible matrix. Thus, all solutions X of the Yang-Baxter-like matrix equation (1) are diagonalizable and the eigenvalues of X are contained in the set $\{\lambda, \mu, 0\}$.

Proof. According to (3) in Theorem 2.1, let

$$Y_2 = \left[\begin{array}{ccc|ccc} \widehat{\lambda I_r} & 0 & 0 & C & 0 & 0 \\ 0 & \lambda I_v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0_{m-r-v} & 0 & 0 & 0 \\ \hline D & 0 & 0 & \widehat{\mu I_r} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu I_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{n-m-r-\tau} \end{array}\right]$$

and

$$W = \left[\begin{array}{ccc|ccc} \frac{\lambda-\mu}{\lambda^2-\lambda\mu+\mu^2}C & 0 & 0 & \frac{\lambda-\mu}{\lambda\mu}C & 0 & 0 \\ 0 & I_v & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m-r-v} & 0 & 0 & 0 \\ \hline I_r & 0 & 0 & I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & I_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n-m-r-\tau} \end{array}\right].$$

Then $det(W) = det(-\frac{(\lambda-\mu)^3}{\lambda\mu(\lambda^2-\lambda\mu+\mu^2)}C) \neq 0$. Thus matrix W is invertible and

$$W^{-1} = \left[\begin{array}{ccc|ccc} -\frac{\lambda\mu(\lambda^2-\lambda\mu+\mu^2)}{(\lambda-\mu)^3}C^{-1} & 0 & 0 & \frac{\lambda^2-\lambda\mu+\mu^2}{(\lambda-\mu)^2}I_r & 0 & 0 \\ 0 & I_v & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m-r-v} & 0 & 0 & 0 \\ \hline \frac{\lambda\mu(\lambda^2-\lambda\mu+\mu^2)}{(\lambda-\mu)^3}C^{-1} & 0 & 0 & -\frac{\lambda\mu}{(\lambda-\mu)^2}I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & I_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n-m-r-\tau} \end{array}\right].$$

Compute

$$W^{-1}Y_2W = \left[\begin{array}{ccc|ccc} \lambda I_r & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda I_v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0_{m-r-v} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mu I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu I_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{n-m-r-\tau} \end{array}\right],$$

so all solutions X of the Yang-Baxter-like matrix equation (1) are diagonalizable and the eigenvalues of X are contained in the set $\{\lambda, \mu, 0\}$. This completes the proof.

□

3. Application

Next we apply Theorem 2.5 and 2.6 to the case that $A = I - uv^T$ ($v^Tu \neq 0$) in [5]. Since the matrix A is $I - uv^T$ with two given nonzero n-dimensional complex vectors u and v such that $v^Tu \neq 0$. Clearly A is diagonalizable. In fact, let $v_1, \ldots, v_{n-1}$ be linearly independent vectors such that $v^Tv_j = 0$ for $j = 1, \ldots, n-1$, the matrix $S = [v_1, \ldots, v_{n-1}, u]$ is nonsingular such that $A = SJS^{-1}$, where $J = diag(I_{n-1}, 1 - v^Tu)$. If $v^Tu \neq 0$ and $v^Tu \neq 1$, since A is diagonalizable with two distinct nonzero eigenvalues, applying Theorem 2.5 and 2.6 , we have the following results.

Theorem 3.1. *Suppose $A = I - uv^T$ with $v^Tu \neq 0$ and $v^Tu \neq 1$. Let $A = SJS^{-1}$, where $J = diag(I_{n-1}, 1 - v^Tu)$. If $v^Tu = \frac{1\pm\sqrt{3}i}{2}$, then all solutions of the Yang-Baxter-like matrix equation (1) have the form*

$$X = S\begin{bmatrix} P & 0 \\ 0 & q \end{bmatrix}\left[\begin{array}{cc|c} I_t & 0 & 0 \\ 0 & 0_{n-1-t} & 0 \\ \hline 0 & 0 & 0 \end{array}\right]\begin{bmatrix} P^{-1} & 0 \\ 0 & q^{-1} \end{bmatrix}S^{-1},$$

or

$$X = S\begin{bmatrix} P & 0 \\ 0 & q \end{bmatrix}R\left[\begin{array}{cc|c} I_t & 0 & 0 \\ 0 & 0_{n-1-t} & 0 \\ \hline 0 & 0 & \frac{1\mp\sqrt{3}i}{2} \end{array}\right]R^{-1}\begin{bmatrix} P^{-1} & 0 \\ 0 & q^{-1} \end{bmatrix}S^{-1},$$

$q \neq 0$, $P \in \mathbb{C}^{(n-1)\times(n-1)}$ is any invertible matrix,

$$R = \left[\begin{array}{cc|c} I_t & 0 & \frac{-1\pm\sqrt{3}i}{2}f \\ 0 & I_{n-1-t} & 0 \\ \hline \frac{1\mp\sqrt{3}i}{2}g & 0 & 1 \end{array}\right],$$

$0 \leq t \leq n-1$, $f \in \mathbb{C}^{t\times 1}$, $g \in \mathbb{C}^{1\times t}$. The elements of N_1 is zeros except that the second upper diagonal elements may be one or zero. $N_1^2 = 0$, $fg = \frac{-1\mp\sqrt{3}i}{2}N_1$, $N_1f = 0$, $gN_1 = 0$.

If $v^Tu \neq \frac{1\pm\sqrt{3}i}{2}$, then all solutions of the Yang-Baxter-like matrix equation (1) have the form

$$X = S\begin{bmatrix} \widetilde{P} & 0 \\ 0 & \widetilde{q} \end{bmatrix}\left[\begin{array}{cc|c} I_r & 0 & 0 \\ 0 & 0_{n-r-1} & 0 \\ \hline 0 & 0 & l \end{array}\right]\begin{bmatrix} \widetilde{P}^{-1} & 0 \\ 0 & \widetilde{q}^{-1} \end{bmatrix}S^{-1},$$

or

$$X = S\begin{bmatrix} \widetilde{P} & 0 \\ 0 & \widetilde{q} \end{bmatrix}W\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & 0_{n-2-k} & 0 \\ \hline 0 & 0 & 0 & 1-v^Tu \end{array}\right]W^{-1}\begin{bmatrix} \widetilde{P}^{-1} & 0 \\ 0 & \widetilde{q}^{-1} \end{bmatrix}S^{-1},$$

in which, $\widetilde{q} \neq 0$, $\widetilde{P} \in \mathbb{C}^{(n-1)\times(n-1)}$ are any invertible matrices, $0 \leq r \leq n-1$, l is either 0 or $1 - v^Tu$, $0 \leq k \leq n-2$,

$$W = \left[\begin{array}{ccc|c} \frac{v^Tu}{1-v^Tu+(v^Tu)^2}c & 0 & 0 & \frac{v^Tu}{1-v^Tu}c \\ 0 & I_k & 0 & 0 \\ 0 & 0 & I_{n-2-k} & 0 \\ \hline 1 & 0 & 0 & 1 \end{array}\right], \quad \forall\, c \neq 0.$$

When $v^Tu = \frac{1\pm\sqrt{3}i}{2}$, since $N_1^2 = 0$, $fg = \frac{-1\mp\sqrt{3}i}{2}N_1$, $N_1f = 0$, $gN_1 = 0$, if $f = 0$ or $g = 0$, then $N_1 = 0$. If $fg \neq 0$, then $rank(N_1) = 1$. If $v^Tu \neq 0$ and $v^Tu \neq 1$, the formula X in Theorem 3.1 are more general than the formula in Theorem 2.1 by J. Ding and H. Tian [5]. We obtain the explicit structure of the solutions X for the Yang-Baxter-like matrix equation $AXA = XAX$ when $A = I - uv^T$ with $v^Tu \neq 0$ and $v^Tu \neq 1$. Also, we prove that the solutions X are diagonalizable and the eigenvalues are contained in the set $\{1, 1 - v^Tu, 0\}$.

As a direct consequence of Theorem 2.6 and 3.1, we look for all solutions of the Yang-Baxter-like matrix equation (1) with a Householder matrix $A = I - 2uu^H$, where u is a unit vector in $\mathbb{C}^n$ and u^H is the conjugate transpose of u. Householder transformations are widely used in numerical linear algebra, for example, to annihilate the entries below the main diagonal of a matrix to perform QR decompositions and in the first step of the QR algorithm. They are also widely used for transforming to a Hessenberg form. 1 is an eigenvalue of A with multiplicity $n-1$ and -1 is the other eigenvalue of A with multiplicity 1. Let $u_1, \ldots, u_{n-1}$ be orthonormal vectors such that $u^Hu_j = 0$ ($u_j^Hu_j = 1$) for $j = 1, \ldots, n-1$, the matrix $S = [u_1, \ldots, u_{n-1}, u]$ is a unitary matrix such that $A = SJS^{-1}$, where $J = diag(I_{n-1}, -1)$. Thus we have the following results.

Theorem 3.2. *Suppose $A = I - 2uu^H$ with $u^Hu = 1$. Let $A = SJS^{-1}$, where $J = diag(I_{n-1}, -1)$. Then all solutions of the Yang-Baxter-like matrix equation (1) have the form*

$$X = S\begin{bmatrix} \widetilde{P} & 0 \\ 0 & \widetilde{q} \end{bmatrix}\left[\begin{array}{cc|c} I_r & 0 & 0 \\ 0 & 0_{n-r-1} & 0 \\ \hline 0 & 0 & l \end{array}\right]\begin{bmatrix} \widetilde{P}^{-1} & 0 \\ 0 & \widetilde{q}^{-1} \end{bmatrix}S^{-1},$$

or

$$X = S\begin{bmatrix} \widetilde{P} & 0 \\ 0 & \widetilde{q} \end{bmatrix}W\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & 0_{n-2-k} & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right]W^{-1}\begin{bmatrix} \widetilde{P}^{-1} & 0 \\ 0 & \widetilde{q}^{-1} \end{bmatrix}S^{-1},$$

in which, $\widetilde{q} \neq 0$, $\widetilde{P} \in \mathbb{C}^{(n-1)\times(n-1)}$ are any invertible matrices, $0 \leq r \leq n-1$, l is either 0 or -1, $0 \leq k \leq n-2$,

$$W = \left[\begin{array}{ccc|c} \frac{2}{3}c & 0 & 0 & -2c \\ 0 & I_k & 0 & 0 \\ 0 & 0 & I_{n-2-k} & 0 \\ \hline 1 & 0 & 0 & 1 \end{array}\right], \quad \forall\, c \neq 0.$$

We obtain the explicit structure of the solutions X for the Yang-Baxter-like matrix equation (1) when A is a Householder matrix. Also, we prove that the solutions X are diagonalizable and the eigenvalues are contained in the set $\{1,-1,0\}$. Thus, the formula X in Theorem 3.2 are better than the formula in Corollary 3.1 and Theorem 4.1 by Q. Dong and J. Ding [6].

Let $A \neq \pm I$ be an $n\times n$ complex matrix satisfying $A^2 = I$. The matrix A is diagonalizable with eigenvalues 1 and -1. Let m be the multiplicity of 1. Then $A = SJS^{-1}$ for a nonsingular matrix S, where $J = diag(I_m, -I_{n-m})$.

Theorem 3.3. *Suppose $A^2 = I$ and $A \neq \pm I$. Let $A = SJS^{-1}$, where $J = diag(I_m, -I_{n-m})$. Then all solutions of the Yang-Baxter-like matrix equation (1) have the form*

$$X = S\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} W \left[\begin{array}{ccc|ccc} I_r & 0 & 0 & 0 & 0 & 0 \\ 0 & I_v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0_{m-r-v} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{n-m-r-\tau} \end{array}\right] W^{-1} \begin{bmatrix} U^{-1} & 0 \\ 0 & V^{-1} \end{bmatrix} S^{-1},$$

in which, $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{(n-m)\times(n-m)}$ are any invertible matrices,

$$W = \left[\begin{array}{ccc|ccc} \frac{2}{3}C & 0 & 0 & -2C & 0 & 0 \\ 0 & I_v & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m-r-v} & 0 & 0 & 0 \\ \hline I_r & 0 & 0 & I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & I_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n-m-r-\tau} \end{array}\right],$$

$0 \le r \le \min\{m, n-m\}$, $0 \le v \le m-r$, $0 \le \tau \le n-m-r$, and C is an arbitrary $r\times r$ invertible matrix.

We have constructed the structure of the solutions of the Yang-Baxter-like matrix equation (1). The solutions X are diagonalizable. The eigenvalues of X constitute a subset of $\{0,1,-1\}$. The formula X in Theorem 3.3 are better than the results in [8, 12].

4. Numerical examples

This section contains two examples to illustrate our theoretical results.

Example 4.1. *Let $u = (1,0,-i)^H$, $v = (\frac{1}{2}, 1, \frac{\sqrt{3}}{2})^H$, and*

$$A = I - uv^H = \begin{bmatrix} \frac{1}{2} & -1 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2}i & -i & 1-\frac{\sqrt{3}}{2}i \end{bmatrix}.$$

We choose $v_1 = (2,-1,0)^T$ and $v_2 = (\sqrt{3},0,1)^T$ so that $A = SJS^{-1}$, where

$$S = \begin{bmatrix} 2 & \sqrt{3} & 1 \\ -1 & 0 & 0 \\ 0 & 1 & i \end{bmatrix} \quad and \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1-\sqrt{3}}{2} \end{bmatrix}.$$

Since $v^H u = \frac{1+\sqrt{3}i}{2}$, by Theorem 3.1, we get the follow solutions.

Case I: $t = 0$

$$X = 0, \quad X = \begin{bmatrix} \frac{1}{2} & 1 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & i & -\frac{\sqrt{3}i}{2} \end{bmatrix}.$$

Case II: $t = 1$

$$X = \begin{bmatrix} -\frac{(2\sqrt{3}-2i)p_1p_2+(3-\sqrt{3}i)p_2p_3}{4(p_1p_4-p_2p_3)} & -\frac{4p_1p_4+(2\sqrt{3}-2i)p_1p_2+2\sqrt{3}p_3p_4+(3-\sqrt{3}i)p_2p_3}{2(p_1p_4-p_2p_3)} & -\frac{(2+2\sqrt{3}i)p_1p_2+(\sqrt{3}+3i)p_2p_3}{4(p_1p_4-p_2p_3)} \\ \frac{(\sqrt{3}-i)p_1p_2}{4(p_1p_4-p_2p_3)} & \frac{2p_1p_4+(\sqrt{3}-i)p_1p_2}{2(p_1p_4-p_2p_3)} & \frac{(1+\sqrt{3}i)p_1p_2}{4(p_1p_4-p_2p_3)} \\ -\frac{(\sqrt{3}-i)p_2p_3}{4(p_1p_4-p_2p_3)} & -\frac{2p_3p_4+(\sqrt{3}-i)p_2p_3}{2(p_1p_4-p_2p_3)} & -\frac{(1+\sqrt{3}i)p_2p_3}{4(p_1p_4-p_2p_3)} \end{bmatrix},$$

for all $p_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, $p_1p_4 - p_2p_3 \neq 0$.

$$X = \begin{bmatrix} -\frac{(2\sqrt{3}-2i)p_1p_2+(3-\sqrt{3}i)p_2p_3+(\sqrt{3}-i)qp_2g}{4(p_1p_4-p_2p_3)} + \frac{1}{2} & -\frac{4p_1p_4+(2\sqrt{3}-2)p_1p_2+2\sqrt{3}p_3p_4+(3-\sqrt{3}i)p_2p_3+2qp_4g+(\sqrt{3}-i)qp_2g}{4(p_1p_4-p_2p_3)} + 1 & -\frac{(2+2\sqrt{3}i)p_1p_2+(\sqrt{3}+3i)p_2p_3+(1-\sqrt{3}i)qp_2g}{4(p_1p_4-p_2p_3)} - \frac{\sqrt{3}}{2} \\ \frac{(\sqrt{3}-i)p_1p_2}{4(p_1p_4-p_2p_3)} & \frac{2p_1p_4+(\sqrt{3}-1)p_1p_2}{2(p_1p_4-p_2p_3)} & \frac{(1+\sqrt{3}i)p_1p_2}{4(p_1p_4-p_2p_3)} \\ -\frac{(\sqrt{3}-i)p_2p_3+(1+\sqrt{3}i)qp_2g}{4(p_1p_4-p_2p_3)} + \frac{i}{2} & -\frac{2p_3p_4+(\sqrt{3}-i)p_2p_3+2iqp_4g+(1+\sqrt{3}i)qp_2g}{2(p_1p_4-p_2p_3)} + i & -\frac{(1+\sqrt{3}i)p_2p_3+(-\sqrt{3}+i)qp_2g}{4(p_1p_4-p_2p_3)} - \frac{\sqrt{3}i}{2} \end{bmatrix}$$

for all $q, g, p_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, $p_1p_4 - p_2p_3 \neq 0$.

$$X = \begin{bmatrix} -\frac{(2\sqrt{3}-2i)p_1p_2+(3-\sqrt{3}i)p_2p_3}{4(p_1p_4-p_2p_3)} + \frac{(2+2\sqrt{3}i)p_1f+(\sqrt{3}+3i)p_3f}{4q} + \frac{1}{2} & -\frac{4p_1p_4+(2\sqrt{3}-2i)p_1p_2+2\sqrt{3}p_3p_4+(3-\sqrt{3}i)p_2p_3}{2(p_1p_4-p_2p_3)} + \frac{(2+2\sqrt{3}i)p_1f+(\sqrt{3}+3i)p_3f}{2q} + 1 & -\frac{(2+2\sqrt{3}i)p_1p_2+(\sqrt{3}+3i)p_2p_3}{4(p_1p_4-p_2p_3)} - \frac{(2\sqrt{3}+6i)p_1f+(3+3\sqrt{3}i)p_3f}{4q} - \frac{\sqrt{3}}{2} \\ \frac{(\sqrt{3}-i)p_1p_2}{4(p_1p_4-p_2p_3)} - \frac{(1+\sqrt{3}i)p_1f}{4q} & \frac{2p_1p_4+(\sqrt{3}-i)p_1p_2}{2(p_1p_4-p_2p_3)} - \frac{(1+\sqrt{3}i)p_1f}{2q} & \frac{(1+\sqrt{3}i)p_1p_2}{4(p_1p_4-p_2p_3)} + \frac{(\sqrt{3}+3i)p_1f}{4q} \\ -\frac{(\sqrt{3}-i)p_2p_3}{4(p_1p_4-p_2p_3)} + \frac{(1+\sqrt{3}i)p_3f}{4q} + \frac{i}{2} & -\frac{2p_3p_4+(\sqrt{3}-i)p_2p_3}{2(p_1p_4-p_2p_3)} + \frac{(1+\sqrt{3}i)p_3f}{2q} + i & -\frac{(1+\sqrt{3}i)p_2p_3}{4(p_1p_4-p_2p_3)} - \frac{(\sqrt{3}+3i)p_3f}{4q} - \frac{\sqrt{3}i}{2} \end{bmatrix}$$

for all $q, f, p_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, $q \neq 0$, $p_1p_4 - p_2p_3 \neq 0$.

Case III: $t = 2$

$$X = \begin{bmatrix} \frac{3-\sqrt{3}i}{4} & -\frac{1+\sqrt{3}i}{2} & \frac{\sqrt{3}+3i}{4} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}-i}{4} & \frac{\sqrt{3}-i}{2} & \frac{1+\sqrt{3}i}{4} \end{bmatrix}, \quad X = \begin{bmatrix} \frac{5-\sqrt{3}i}{4} & \frac{1-\sqrt{3}i}{2} & \frac{-\sqrt{3}+3i}{4} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}+i}{4} & \frac{\sqrt{3}+i}{2} & \frac{1-\sqrt{3}i}{4} \end{bmatrix},$$

and

$$X = \begin{bmatrix} \frac{(\sqrt{3}-i)q(g_2p_1-g_1p_2)}{4(p_1p_4-p_2p_3)} + \frac{(2+2\sqrt{3}i)(p_1f_1+p_2f_2)+(\sqrt{3}+3i)(p_3f_1+p_4f_2)}{4q} + \frac{1}{2} & \frac{-2q(g_1p_4-g_2p_3)+(\sqrt{3}-i)q(g_2p_1-g_1p_2)}{2(p_1p_4-p_2p_3)} + \frac{(2+2\sqrt{3}i)(p_1f_1+p_2f_2)+(\sqrt{3}+3i)(p_3f_1+p_4f_2)}{2q} - 1 & \frac{(1+\sqrt{3}i)q(g_2p_1-g_1p_2)}{4(p_1p_4-p_2p_3)} - \frac{(2\sqrt{3}+6i)(p_1f_1+p_2f_2)+(3+3\sqrt{3}i)(p_3f_1+p_4f_2)}{4q} + \frac{\sqrt{3}-2+(3+2\sqrt{3})i}{4} \\ -\frac{(1+\sqrt{3}i)(p_1f_1+p_2f_2)}{4(p_1p_4-p_2p_3)} & 1 - \frac{(1+\sqrt{3}i)(p_1f_1+p_2f_2)}{2q} & \frac{(\sqrt{3}+3i)(p_1f_1+p_2f_2)}{4q} \\ \frac{(1+\sqrt{3}i)q(g_2p_1-g_1p_2)}{4(p_1p_4-p_2p_3)} + \frac{(1+\sqrt{3}i)(p_3f_1+p_4f_2)}{4q} - \frac{i}{2} & \frac{-2iq(g_1p_4-g_2p_3)+(1+\sqrt{3}i)q(g_2p_1-g_1p_2)}{2(p_1p_4-p_2p_3)} + \frac{(1+\sqrt{3}i)(p_3f_1+p_4f_2)}{2q} - i & \frac{(-\sqrt{3}+i)q(g_2p_1-g_1p_2)}{4(p_1p_4-p_2p_3)} - \frac{(\sqrt{3}+3i)(p_3f_1+p_4f_2)}{4q} + \frac{1-2\sqrt{3}+(\sqrt{3}-2)i}{4} \end{bmatrix},$$

for all $p_i, f_j \in \mathbb{C}$, $i = 1, 2, 3, 4$, $j = 1, 2$, $q \neq 0$, $f_1g_1 + f_2g_2 = 0$, $\begin{bmatrix} f_1g_1 & f_1g_2 \\ f_2g_1 & f_2g_2 \end{bmatrix} = 0$.

Example 4.2. *Let* $u = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})^T$, *and*

$$A = I - 2uu^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We choose $v_1 = (1, 0, 1)^T$ *and* $v_2 = (0, 1, 0)^T$ *so that* $A = SJS^{-1}$, *where*

$$S = \begin{bmatrix} 1 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ 1 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

By Theorem 3.2, we get the follow solutions.

Case I: $r = 0$

$$X = 0, \quad X = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix},$$

and

$$X = \begin{bmatrix} -\frac{4cu_1u_4+3\sqrt{2}vu_4}{16c(u_1u_4-u_2u_3)} - \frac{\sqrt{2}u_1c}{2v} + \frac{1}{4} & \frac{4cu_1u_2+3\sqrt{2}vu_2}{8c(u_1u_4-u_2u_3)} & -\frac{4cu_1u_4+3\sqrt{2}vu_4}{16c(u_1u_4-u_2u_3)} + \frac{\sqrt{2}u_1c}{2v} - \frac{1}{4} \\ -\frac{u_3u_4}{4(u_1u_4-u_2u_3)} - \frac{\sqrt{2}u_3c}{2v} & \frac{u_2u_3}{2(u_1u_4-u_2u_3)} & -\frac{u_3u_4}{4(u_1u_4-u_2u_3)} + \frac{\sqrt{2}u_3c}{2v} \\ \frac{-4cu_1u_4+3\sqrt{2}vu_4}{16c(u_1u_4-u_2u_3)} - \frac{\sqrt{2}u_1c}{2v} - \frac{1}{4} & \frac{4cu_1u_2-3\sqrt{2}vu_2}{8c(u_1u_4-u_2u_3)} & \frac{-4cu_1u_4+3\sqrt{2}vu_4}{16c(u_1u_4-u_2u_3)} + \frac{\sqrt{2}u_1c}{2v} + \frac{1}{4} \end{bmatrix},$$

for all $c, v, u_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, $v \neq 0$, $c \neq 0$, $u_1u_4 - u_2u_3 \neq 0$.

Case II: $r = 1$

$$X = \begin{bmatrix} \frac{u_1u_4}{2(u_1u_4-u_2u_3)} & -\frac{u_1u_2}{u_1u_4-u_2u_3} & \frac{u_1u_4}{2(u_1u_4-u_2u_3)} \\ \frac{u_3u_4}{2(u_1u_4-u_2u_3)} & -\frac{u_2u_3}{u_1u_4-u_2u_3} & \frac{u_3u_4}{2(u_1u_4-u_2u_3)} \\ \frac{u_1u_4}{2(u_1u_4-u_2u_3)} & -\frac{u_1u_2}{u_1u_4-u_2u_3} & -\frac{u_1u_4}{u_1u_4-u_2u_3} \end{bmatrix},$$

for all $u_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, $u_1u_4 - u_2u_3 \neq 0$.

$$X = \begin{bmatrix} \frac{u_1u_4}{2(u_1u_4-u_2u_3)} + \frac{1}{2} & -\frac{u_1u_2}{u_1u_4-u_2u_3} & \frac{u_1u_4}{2(u_1u_4-u_2u_3)} - \frac{1}{2} \\ \frac{u_3u_4}{2(u_1u_4-u_2u_3)} & -\frac{u_2u_3}{u_1u_4-u_2u_3} & \frac{u_3u_4}{2(u_1u_4-u_2u_3)} \\ \frac{u_1u_4}{2(u_1u_4-u_2u_3)} - \frac{1}{2} & -\frac{u_1u_2}{u_1u_4-u_2u_3} & -\frac{u_1u_4}{u_1u_4-u_2u_3} + \frac{1}{2} \end{bmatrix},$$

for all $u_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, $u_1u_4 - u_2u_3 \neq 0$.

$$X = \begin{bmatrix} -\frac{4cu_1u_4+8cu_2u_3+3\sqrt{2}vu_4}{16c(u_1u_4-u_2u_3)} - \frac{\sqrt{2}u_1c}{2v} + \frac{1}{4} & \frac{12cu_1u_2+3\sqrt{2}vu_2}{8c(u_1u_4-u_2u_3)} & -\frac{4cu_1u_4+8cu_2u_3+3\sqrt{2}vu_4}{16c(u_1u_4-u_2u_3)} + \frac{\sqrt{2}u_1c}{2v} - \frac{1}{4} \\ -\frac{3u_3u_4}{4(u_1u_4-u_2u_3)} - \frac{\sqrt{2}u_3c}{2v} & \frac{u_2u_3+2u_1u_4}{2(u_1u_4-u_2u_3)} & -\frac{3u_3u_4}{4(u_1u_4-u_2u_3)} + \frac{\sqrt{2}u_3c}{2v} \\ -\frac{4cu_1u_4+8cu_2u_3-3\sqrt{2}vu_4}{16c(u_1u_4-u_2u_3)} - \frac{\sqrt{2}u_1c}{2v} - \frac{1}{4} & \frac{12cu_1u_2-3\sqrt{2}vu_2}{8c(u_1u_4-u_2u_3)} & -\frac{4cu_1u_4+8cu_2u_3-3\sqrt{2}vu_4}{16c(u_1u_4-u_2u_3)} + \frac{\sqrt{2}u_1c}{2v} + \frac{1}{4} \end{bmatrix},$$

for all $c, v, u_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, $v \neq 0$, $c \neq 0$, $u_1u_4 - u_2u_3 \neq 0$.

Case III: $r = 2$

$$X = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

5. Conclusions

When the given matrix A is diagonalizable matrix with two distinct nonzero eigenvalues λ and μ, we have derived all explicit expression for the solutions X of the Yang-Baxter-like matrix equation (1) under the conditions that $\lambda^2 - \lambda\mu + \mu^2 = 0$ and $\lambda^2 - \lambda\mu + \mu^2 \neq 0$, respectively. We correct and improve the results in

Shen et al. [14] when $\lambda^2 - \lambda\mu + \mu^2 = 0$. We also improve the results in Shen et al. [14] when $\lambda^2 - \lambda\mu + \mu^2 \neq 0$. We prove that the solutions are diagonalizable and the spectrum contained in the set $\{\lambda, \mu, 0\}$. We improve the research for which A is a Householder matrix $A = I - 2uu^H$ [6], A is a class of elementary matrices $A = I - uv^T$ ($v^T u \neq 0$) [5] and $A \neq \pm I$ is an $n \times n$ complex matrix satisfying $A^2 = I$ [8, 12], respectively. This is an important step to solve more general matrices. Finding all the solutions of the Yang-Baxter-like matrix equation (1) for a general matrix A is a hard task, which is continuing research work in the future.

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