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On optimality conditions and duality results for a new class of nonconvex nondifferentiable multicriteria optimization problems

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Abstract. In this paper, a new class of nonconvex nonsmooth multiobjective programming problems with both inequality and equality constraints is considered. Namely, the sufficient optimality conditions and Mond-Weir duality results are established for such nondifferentiable multicriteria optimization problems in which the involved functions are nondifferentiable $(b, \Psi, \Phi, \rho)^w$-univex functions (not necessarly with respect to the same functions b and Ψ and ρ). Then the aforesaid results developed here under $(b, \Psi, \Phi, \rho)^w$-uinvexity are applicable for a larger class of nonsmooth vector optimization problems than under other generalized convexity notions existing in the literature.

1. Introduction

Nonsmooth optimization provides analytical tools for studying optimization problems involving functions that are not differentiable in the usual sense. Several nonlinear nonsmooth analysis problems arise from areas of optimization theory, convex and nonconvex analysis, mathematical physics, game theory, differential equations, and nonlinear functional analysis.

The term vector optimization (or multiobjective programming) is used to denote a type of mathematical programming problems where two or more objectives are to be minimized subject to certain constraints. Investigation on sufficiency and duality has been one of the most attraction topics in the theory of multiobjective programming problems. It is well known that the concept of convexity and its various generalizations play an important role in deriving sufficient optimality conditions and various duality results for multicriteria optimization problems. In recent years, therefore, multiobjective programming has grown remarkably in different directions in the settings of optimality conditions and duality theory. It has been enriched by the applications of various types of nondifferentiable generalizations of convexity theory (see, for example, [1], [4], [5], [7], [12], [19], [23], [25], [27], [28], [39], [40], [42], and others).

The concepts univexity and generalized univexity have been used in many works in proving the fundamental results in optimization theory for new classes of nonconvex optimization problems. The univexity notion has been introduced for the first time by Bector et al. [11] for differentiable scalar optimization problems. Mishra [32] obtained Kuhn-Tucker type sufficient optimality conditions for a feasible point to be

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an efficient or properly efficient solution in the analyzed nonsmooth multiobjective programming problem under generalized type I univex functions. Jayswal [20] introduced new classes of generalized α-univex type I vector valued functions and he used these functions in proving a number of Kuhn-Tucker type sufficient optimality conditions and Mond-Weir duality results for a multiobjective programming problem with inequality constraints. Long [30] defined nonsmooth univex, quasiunivex and pseudounivex functions, and, by utilizing these concepts, he proved sufficient optimality conditions for a weakly efficient solution and duality results for the considered nonsmooth multiobjective programming problem. Recently, Dubey et al. [16] introduced the definition of a (strongly) K-G_f-pseudoinvex function which enable to study a pair of nondifferentiable K-G-Mond-Weir type symmetric multiobjective programming model with such nonconvex functions.

The concept of differentiable (Φ, ρ)-invexity was introduced by Caristi et al. [13] for smooth scalar optimization problems. It was generalized to the concept of $(\Phi, \rho)^w$-invexity by Stefanescu and Stefanescu [43] for proving optimality results for nonconvex semi-infinite minmax programming problems with $(\Phi, \rho)^w$-invex functions. Antczak [5] introduced the concept of nondifferentiable (Φ, ρ)-invexity and he established optimality conditions and duality results in the sense of Mond-Weir for nonsmooth vector optimization problems with such nonconvex functions. In [37], Ojha used the so-called concepts of generalized type I (Φ, ρ)-univex functions for proving sufficient optimality results and several Mond-Weir duality theorems for the considered nondifferentiable vector optimization problem with inequality constraints. In [6], Antczak and Verma introduced the concept of $(b, \Psi, \Phi, \rho)^w$-univexity in the scalar case and they established parametric optimality conditions and several duality theorems in the sense of Schaible for the class of considered multiobjective fractional programming problems with $(b, \Psi, \Phi, \rho)^w$-univex functions. Recently, under nondifferentiable vectorial $(\Phi, \rho)^w$-invexity assumptions, Antczak and Verma [7] proved optimality conditions and duality results for the nonsmooth considered multiobjective programming problem defined in a Banach space.

The main purpose of this paper is to investigate optimality conditions and duality results for a new class of nondifferentiable vector optimization problems in which every component of the involved functions is a locally Lipschitz function. Namely, we consider a nonsmooth multicriteria optimization problem in which all the invloved functions are $(b, \Psi, \Phi, \rho)^w$-univex (with respect to, not necessarily, the same b, Ψ and ρ). Then, the central purpose of this work is to discuss application of the vectorial nondifferentiable $(b, \Psi, \Phi, \rho)^w$-univexity notion in proving the fundamental results in optimization theory for the aforesaid class of nonconvex nondifferentiable vector optimization problems. Namely, we establish the sufficiency of the generalized Karush-Kuhn-Tucker necessary optimality conditions in the considered nondifferentiable multiobjective programming problem in which both objective and constraint functions are $(b, \Psi, \Phi, \rho)^w$-univex. In other words, we prove the sufficient optimality conditions for a feasible solution to be a weak Pareto solution and also for a Pareto solution in the aforesaid nonconvex multicriteria optimization problem. The sufficient optimality conditions established in the paper are illustrated by an example of a nonconvex nondifferentiable vector optimization problem with locally Lipschitz $(b, \Psi, \Phi, \rho)^w$-univex functions. Further, for the considered nondifferentiable multicriteria optimization problem, we define its vector dual problem in the sense of Mond-Weir. Then, also under $(b, \Psi, \Phi, \rho)^w$-univexity assumptions, we prove several duality theorems between both aforesaid vector optimization problems.

2. Preliminaries

Throughout this paper, we use the following conventions for vectors $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$ in the Euclidean space R^n:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;
(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, ..., n$;
(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, ..., n$;
(iv) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

In this section, we provide some definitions and some results from nonsmooth analysis that we shall use in the sequel.

Throughout this section, X is a nonempty subset of R^n.

Definition 2.1. *[14] The Clarke generalized directional derivative of a locally Lipschitz function $f : X \to R$ at $x \in X$ in the direction $v \in R^n$, denoted $f^0(x; v)$, is given by*

$$f^0(x;v) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

Definition 2.2. *[14] The Clarke generalized subgradient of a locally Lipschitz function $f : X \to R$ at $x \in X$, denoted $\partial f(x)$, is defined as follows*

$$\partial f(x) = \left\{ \xi \in R^n : f^0(x; v) \geq \langle \xi, v \rangle \text{ for all } v \in X \right\}.$$

Definition 2.3. *The function $\varphi : X \to R$ is said to be quasi-convex if, for each $\alpha \in R$, the level set $\{x \in X : \varphi(x) \leqq \alpha\}$ is convex, or equivalently, if $\varphi(\lambda y + (1 - \lambda)x) \leqq \max\{\varphi(y), \varphi(x)\}$ for every $y, x \in X$ and $\lambda \in [0, 1]$.*

A stronger property is also considered as follows:

Definition 2.4. *The function $\varphi : X \to R$ is said to be strictly quasi-convex if it is quasi-convex and $\varphi(\lambda y + (1 - \lambda)x) < 0$, whenever $\varphi(y) < 0$, $\varphi(x) \leqq 0$ and $\lambda \in (0, 1)$.*

Proposition 2.5. *[43] If $\varphi : X \to R$ is a strictly quasi-convex function and there are $x^1, ..., x^k \in X$ such that $\varphi\left(x^i\right) \leqq 0$, $i = 1, ..., k$ and $\varphi\left(x^{i^*}\right) < 0$ for at least one $i^* \in \{1, ..., k\}$, then $\varphi\left(\sum_{i=1}^k \lambda_i x_i\right) < 0$ for every $\lambda = (\lambda_1, ..., \lambda_k) \geq 0$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\lambda_{i^*} > 0$.*

Remark 2.6. *There are various definitions of strict quasi-convexity appearing in the literature. The most frequently used seems to be originated in [9], and is defined by the inequality $\Phi(\lambda y + (1 - \lambda)x) < \min\{\Phi(y), \Phi(x)\}$, whenever $\Phi(y) \neq \Phi(x)$ and $\lambda \in (0, 1)$. But the property defined by the above inequality is known in the earlier literature as pseudo-convexity. Moreover, it is not stronger than the quasi-convexity (besides the topological framework of lower semi-continuity), so that the term "strict" is inadequately used. The next definition is stronger, asking the above strict inequality whenever $x \neq y$ and $\lambda \in (0, 1)$. Obviously, Definition 2.3 is a weaker version of this one.*

In [43], Stefanescu and Stefanescu introduced the definition of a differentiable $(\Phi, \rho)^w$-invex vector-valued function. Further, Antczak [4] generalized the concept of (Φ, ρ)-invexity to the case of nondifferentiable vector optimization problems with locally Lipschitz functions. In [11], Bector et al. introduced the concept of univexity for differentiable scalar optimization problems.

Based on the aforesaid concepts of generalized convexity, we now give the definition of $(b, \Psi, \Phi, \rho)^w$-univexity in a nondifferentiable vectorial case.

Let X be a nonempty subset of R^n and, moreover, let $b := (b_1, ..., b_k) : X \times X \to R^k$ with $b_i : X \times X \to R_+$ and $\Psi := (\Psi_1, ..., \Psi_k) : R \to R^k$ be given.

Definition 2.7. *Let a function $f : X \to R^k$ be defined on X, every f_i, $i = 1, ..., k$, be a locally Lipschitz function on X and $u \in X$. If there exist a function $\Phi : X \times X \times R^n \times R^k \to R$, where $\Phi(x, u, (\cdot, \cdot))$ is strictly quasi-convex on R^{n+1}, $\Phi(x, u, (0, a)) \geqq 0$ for all $x \in X$ and each $a \in R_+$ and $\rho = (\rho_1, ..., \rho_k) \in R^k$, where ρ_i, $i = 1, ..., k$, are real numbers, such that the inequalities*

$$b_i(x, \overline{x})\Psi_i(f_i(x) - f_i(\overline{x})) \geqq \Phi(x, \overline{x}; (\xi_i, \rho_i)), \ i = 1, .., k, \tag{1}$$

hold for any $\xi_i \in \partial f_i(\overline{x})$ and all $x \in X$, then f is said to be $(b, \Psi, \Phi, \rho)^w$-univex at $\overline{x}$ on X. If each inequality (1) is satisfied at each $\overline{x} \in X$, then f is said to be $(b, \Psi, \Phi, \rho)^w$-univex on X. Each function f_i, $i = 1, .., k$, satisfying (1) is said to be locally Lipschitz $(b_i, \Psi_i, \Phi, \rho_i)^w$-univex at $\overline{x}$ on X.

Definition 2.8. *Let a function $f : X \to R^k$ be defined on X, every f_i, $i = 1, ..., k$, be a locally Lipschitz function on X and $u \in X$. If there exist a function $\Phi : X \times X \times R^n \times R^k \to R$, where $\Phi(x, u, (\cdot, \cdot))$ is strictly quasi-convex on R^{n+1},*

$\Phi(x, u, (0, a)) \geqq 0$ *for all* $x \in X$ *and each* $a \in R_+$ *and* $\rho = (\rho_1, ..., \rho_k) \in R^k$, *where* ρ_i, $i = 1, ..., k$, *are real numbers, such that the inequalities*

$$b_i(x, \overline{x})\Psi_i(f_i(x) - f_i(\overline{x})) > \Phi(x, \overline{x}; (\xi_i, \rho_i)), \; i = 1, .., k, \tag{2}$$

hold for any $\xi_i \in \partial f_i(\overline{x})$ *and all* $x \in X$, $(x \neq \overline{x})$, *then* f *is said to be strictly* $(b, \Psi, \Phi, \rho)^w$*-univex at* $\overline{x}$ *on* X. *If each inequality (2) is satisfied at each* $\overline{x} \in X$, *then* f *is said to be strictly* $(b, \Psi, \Phi, \rho)^w$*-univex on* X. *Each function* f_i, $i = 1, .., k$, *satisfying (2) is said to be locally Lipschitz strictly* $(b_i, \Psi_i, \Phi, \rho_i)^w$*-univex at* $\overline{x}$ *on* X.

In order to define an analogous class of nondifferentiable vector (strictly) $(b, \Psi, \Phi, \rho)^w$-unincave functions, the direction of the inequality in the definition of these functions should be changed to the opposite one.

Remark 2.9. *Note that the definition of a nondifferentiable* (b, Ψ, Φ, ρ)*-univex vector-valued function generalizes and extends many other generalized convexity notions, which have been earlier defined in the literature. Indeed, from Definition 2.7, there are the following special cases:*

i) *If* $\Phi(x, \overline{x}, (\xi_i, \rho_i)) = \xi_i^T (x - \overline{x})$, $\Psi_i(a) \equiv a$ *and* $b(x, \overline{x}) \equiv 1$ *for all* $x, \overline{x} \in R^n$, *then we obtain the definition of a (nondifferentiable) convex function.*

ii) *If* $\Phi(x, \overline{x}, (\xi_i, \rho_i)) = \xi_i^T (x - \overline{x})$ *and* $\Psi_i(a) \equiv a$, *then we obtain the definition of a (nondifferentiable) b-convex function.*

iii) *If* $\Phi(x, \overline{x}, (\xi_i, \rho_i)) = \xi_i^T \eta(x, \overline{x})$ *for a certain mapping* $\eta : R^n \times R^n \to R^n$, $\Psi_i(a) \equiv a$ *and* $b(x, \overline{x}) \equiv 1$ *for all* $x, \overline{x} \in R^n$, *then we obtain the definition of a (locally Lipschitz) invex function (with respect to the function* η*) (see, Kim and Schaible [23], Lee [27] in a nonsmooth vectorial case).*

iv) *If* $\Phi(x, \overline{x}, (\xi_i, \rho_i)) = \xi_i^T \eta(x, \overline{x})$ *for a certain mapping* $\eta : R^n \times R^n \to R^n$, *then the definition of* $(b, \Psi, \Phi, \rho)^w$*-univexity reduces to the definition of a (locally Lipschitz) univex function (with respect to the function* η*) (see, Bector et al. [11] in a differentiable scalar case).*

v) *If* $\Phi(x, \overline{x}, (\xi_i, \rho_i)) = \frac{1}{b(x, \overline{x})} \xi_i^T \eta(x, \overline{x})$, $\Psi_i(a) \equiv a$, *and* $\eta : R^n \times R^n \to R^n$, *then we obtain the definition of a nondifferentiable b-invex function (with respect to the function* η*) (see, Li et al. [26]).*

vi) *If* $\Phi(x, \overline{x}, (\xi_i, \rho_i)) = \xi_i^T (x - \overline{x}) + \rho_i \left\| x - \overline{x} \right\|^2$, $\Psi_i(a) \equiv a$ *and* $b(x, \overline{x}) \equiv 1$ *for all* $x, \overline{x} \in R^n$, *then* $(b, \Psi, \Phi, \rho)^w$*-univexity reduces to the definition of a nonsmooth* ρ*-convex function defined by Vial [45] in the scalar case (see also Zalmai [47] in a nondifferentiable case).*

vii) *If* $\Phi(x, \overline{x}, (\xi_i, \rho_i)) = \xi_i^T \eta(x, \overline{x}) + \rho_i \left\| \theta(x, \overline{x}) \right\|^2$, $\Psi_i(a) \equiv a$ *and* $b(x, \overline{x}) \equiv 1$ *for all* $x, \overline{x} \in R^n$, $\eta : R^n \times R^n \to R^n$, $\theta : R^n \times R^n \to R^n$, $\theta(x, \overline{x}) \neq 0$, *whenever* $x \neq \overline{x}$, *then* $(b, \Psi_i, \Phi, \rho)^w$*-univexity reduces to the definition of a nonsmooth* ρ*-invex function (with respect to* η *and* θ*), in the scalar case introduced by Jeyakumar [21] (see also, Craven [15], Ahmad [2] and Suneja and Lalitha [44] in the vectorial case).*

viii) *If* $\Phi(x, \overline{x}, (\xi_i, \rho_i)) = \xi_i^T \eta(x, \overline{x}) + \rho_i \left\| x - \overline{x} \right\|^2$ *for all* $x, \overline{x} \in R^n$, $\eta : R^n \times R^n \to R^n$, *then the definition of a* $(b, \Psi, \Phi, \rho)^w$*-univex function reduces to the definition of a nonsmooth* ρ*-univex function (with respect to* η *and* θ*) (see, for example, Mishra [31], [33]).*

ix) *If* $\Phi(x, \overline{x}, (\xi_i, \rho_i)) = F(x, \overline{x}, \xi_i)$, *where* $F(x, \overline{x}, \cdot)$ *is a sublinear functional on* R^n, $\Psi_i(a) \equiv a$ *and* $b(x, \overline{x}) \equiv 1$ *for all* $x, \overline{x} \in R^n$, *then the definition of* $(b, \Psi, \Phi, \rho)^w$*-univexity reduces to the definition of F-convexity introduced by Hanson and Mond [17] in the scalar case.*

x) *If* $\Phi(x, \overline{x}, (\xi_i, \rho_i)) = F(x, \overline{x}, \xi_i) + \rho_i d^2(x, \overline{x})$, *where* $F(x, \overline{x}, \cdot)$ *is a sublinear functional on* R^n, $\Psi_i(a) \equiv a$ *and* $b(x, \overline{x}) \equiv 1$ *for all* $x, \overline{x} \in R^n$, *then the definition of* $(b, \Psi, \Phi, \rho)^w$*-univexity reduces to the definition of* (F, ρ)*-convexity considered by Liu [29], Craven [15] in the vectorial case.*

xi) *If* $\Phi(x, \overline{x}, (\xi_i, \rho_i)) = \alpha(x, \overline{x}) \xi_i^T \eta(x, \overline{x})$, *where* $\eta : R^n \times R^n \to R^n$, $\alpha : R^n \times R^n \to R_+ \backslash \{0\}$, $\alpha(x, \overline{x}) = \frac{1}{b(x, \overline{x})}$, $\Psi_i(a) \equiv a$, *then* $(b, \Psi, \Phi, \rho)^w$*-univexity reduces to the definition of a nonsmooth* α*-invex function (with respect to* η*) introduced by Mishra et al. [35].*

xii) *If* $\Phi(x, \overline{x}, (\xi_i, \rho_i)) = \alpha(x, \overline{x}) \xi_i^T \eta(x, \overline{x})$, *where* $\eta : R^n \times R^n \to R^n$, $\alpha : R^n \times R^n \to R_+ \backslash \{0\}$, $\alpha(x, \overline{x}) = \frac{1}{b(x, \overline{x})}$, *then* $(b, \Psi, \Phi, \rho)^w$*-univexity reduces to the definition of a nonsmooth* α*-univex function (with respect to* η *and* θ*) introduced by Jayswal et al. [18].*

xiii) *If* $\Phi(x,\overline{x},(\xi_i,\rho_i)) = F(x,\overline{x},\xi_i) + \rho_i d^2(x,\overline{x})$, *where* $F(x,\overline{x},\cdot)$ *is a sublinear functional on* R^n, $\Psi_i(a) \equiv a$, *and* $d: X \times X \to R$, *then* $(b,\Psi,\Phi,\rho)^w$*-univexity notion reduces to* (b,F,ρ)*-convexity, in the smooth vectorial case introduced by Pandian [38].*

xiv) *If* $\Psi_i(a) \equiv a$ *and* $b(x,\overline{x}) \equiv 1$ *for all* $x,\overline{x} \in R^n$, *then we obtain the definition of a locally Lipschitz* (Φ,ρ)*-invex function (see Antczak and Stasiak [8] in a scalar case, and Antczak [4] in a nondifferentiable vectorial case).*

3. Optimality conditions

In the paper, we consider the following nonsmooth multiobjective programming problem with both inequality and equality constraints:

$$\begin{aligned} &\text{minimize } f(x) := (f_1(x),...,f_k(x)) \\ &\text{subject to } \quad g_j(x) \leqq 0, \ j \in J = \{1,...m\}, \\ &h_t(x) = 0, \ t \in T = \{1,...,q\}, \\ &x \in X, \end{aligned} \tag{VP}$$

where $f_i : X \to R$, $i \in I = \{1,...,k\}$, $g_j : X \to R$, $j \in J$, and $h_t : X \to R$, $t \in T$, are locally Lipschitz functions on X, where X is a nonempty convex subset of R^n.

For the purpose of simplifying our presentation, we will next introduce some notations which will be used frequently throughout this paper. Let

$$D := \left\{ x \in X : g_j(x) \leqq 0, \ j \in J, h_t(x) = 0, t \in T \right\}$$

be the set of all feasible solutions in (VP). Further, we denote the set of active inequality constraints at point $\overline{x} \in X$, that is, $J(\overline{x}) = \left\{ j \in J : g_j(\overline{x}) = 0 \right\}$.

It is well-known that a weak Pareto solution and a Pareto solution are natural optimal solutions in multicriteria optimization problems.

Definition 3.1. *A feasible point* $\overline{x}$ *is said to be a weak Pareto solution (a weakly efficient solution) of (VP) if and only if there exists no other* $x \in D$ *such that* $f(x) < f(\overline{x})$.

Definition 3.2. *A feasible point* $\overline{x}$ *is said to be a Pareto solution (an efficient solution) of (VP) if and only if there exists no other* $x \in D$ *such that* $f(x) \leq f(\overline{x})$.

Now, we present the generalized Karush-Kuhn-Tucker necessary optimality conditions for nondifferentiable vector optimization problems with both inequality and equality constraints (see, for example, [5], [23]).

Theorem 3.3. *(Generalized Karush-Kuhn-Tucker necessary optimality conditions). Let* $\overline{x} \in D$ *be a weakly efficient solution in the considered vector optimization problem (VP). Further, we assume that the suitable constraint qualification (for example, the no nonzero abnormal multiplier constraint qualification [46]) is satisfied at* $\overline{x}$ *for (VP). Then, the generalized Karush-Kuhn-Tucker necessary optimality conditions are fulfilled at* $\overline{x}$ *for (VP). In other words, there exist* $\overline{\lambda} \in R^k$, $\overline{\mu} \in R^m$ *and* $\overline{\vartheta} \in R^q$ *such that*

$$0 \in \sum_{i=1}^{k} \overline{\lambda}_i \partial f_i(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j \partial g_j(\overline{x}) + \sum_{t=1}^{q} \overline{\vartheta}_t \partial h_t(\overline{x}), \tag{3}$$

$$\overline{\mu}_j g_j(\overline{x}) = 0, \ j \in J, \tag{4}$$

$$\overline{\lambda} \geq 0, \overline{\mu} \geqq 0. \tag{5}$$

Definition 3.4. *The point* $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\vartheta}\right) \in D \times R^k \times R^m \times R^q$ *is said to be a Karush-Kuhn-Tucker point in the considered vector optimization problem (VP) if the necessary optimality conditions (3)-(5) are fulfilled at* $\overline{x}$ *with Lagrange multipliers* $\overline{\lambda}$, $\overline{\mu}$ *and* $\overline{\vartheta}$.

Now, we prove the sufficient optimality conditions for weak efficiency of a feasible solution in the considered nonsmooth multiobjective programming problem (VP) under nonsmooth $(\Phi, \rho)^w$-univexity.

Theorem 3.5. *Let* $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\vartheta}\right) \in D \times R^k \times R^m \times R^q$ *be a Karush-Kuhn-Tucker point in the considered nonsmooth multiobjective programming problem (VP). Further, assume that the following sets of hypotheses are satisfied:*

a) each objective function f_i, $i = 1, ..., k$, *is* $(b_{f_i}, \Psi_{f_i}, \Phi, \rho_{f_i})^w$*-univex at* $\overline{x}$ *on* D *and* $a < 0 \Longrightarrow \Psi_{f_i}(a) < 0$;
b) each inequality constraint function g_j, $j \in J(\overline{x})$, *is* $(b_{g_j}, \Psi_{g_j}, \Phi, \rho_{g_j})^w$*-univex at* $\overline{x}$ *on* D *and* $a \leqq 0 \Longrightarrow \Psi_{g_j}(a) \leqq 0$;
c) each equality constraint function h_s, $t \in T^+(\overline{x})$, *is* $(b_{h_t}, \Psi_{h_t}, \Phi, \rho^+_{h_t})^w$*-univex at* $\overline{x}$ *on* D *and* ($\Psi_{h_s}(0) = 0$ *or* $a \leqq 0 \Longrightarrow \Psi_{h_s}(a) \leqq 0$);
d) each function $-h_s$, $t \in T^-(\overline{x})$, *is* $(b_{h_t}, \Psi_{h_t}, \Phi, \rho^-_{h_t})^w$*-univex at* $\overline{x}$ *on* D *and* ($\Psi_{h_t}(0) = 0$ *or* $a \leqq 0 \Longrightarrow \Psi_{h_t}(a) \leqq 0$);
e) $\sum_{i=1}^k \overline{\lambda}_i \rho_{f_i} + \sum_{j \in J(\overline{x})} \overline{\mu}_j \rho_{g_j} + \sum_{t \in T^+(\overline{x})} \overline{\vartheta}_t \rho^+_{h_t} - \sum_{t \in T^-(\overline{x})} \overline{\vartheta}_t \rho^-_{h_t} \geqq 0$.

Then $\overline{x}$ *is a weak Pareto solution in problem (VP).*

Proof. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not a weak Pareto solution in (VP). Then, by Definition 3.1, there exists an other feasible solution $\widetilde{x}$ such that

$$f_i(\widetilde{x}) < f_i(\overline{x}), \; i = 1, ..., k. \tag{6}$$

By assumption, $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\vartheta}\right) \in D \times R^k \times R^m \times R^q$ is a Karush-Kuhn-Tucker point in the considered nonsmooth multiobjective programming problem (VP). By (3), there exist $\xi_i \in \partial f_i(\overline{x})$, $i \in I$, $\zeta_j \in \partial g_j(\overline{x})$, $j \in J$, $\varsigma_t \in \partial h_t(\overline{x})$, $t \in T$, such that

$$\sum_{i=1}^k \overline{\lambda}_i \xi_i + \sum_{j \in J(\overline{x})} \overline{\mu}_j \zeta_j + \sum_{t \in T^+(\overline{x})} \overline{\vartheta}_t \varsigma_t + \sum_{t \in T^-(\overline{x})} \overline{\vartheta}_t \varsigma_t = 0. \tag{7}$$

From Definition 2.7, it follows that the following inequality $\Phi(\widetilde{x}, \overline{x}, (0, a)) \geqq 0$ is satisfied for any $a \geqq 0$. Hence, by (7), the assumption $\sum_{i=1}^k \overline{\lambda}_i \rho_{f_i} + \sum_{j \in J(\overline{x})} \overline{\mu}_j \rho_{g_j} + \sum_{t \in T^+(\overline{x})} \overline{\vartheta}_t \rho^+_{h_t} - \sum_{t \in T^-(\overline{x})} \overline{\vartheta}_t \rho^-_{h_t} \geqq 0$ implies

$$\Phi\left(\widetilde{x}, \overline{x}, \frac{1}{\overline{A}}\left(\sum_{i=1}^k \overline{\lambda}_i \xi_i + \sum_{j \in J(\overline{x})} \overline{\mu}_j \zeta_j + \sum_{t \in T^+(\overline{x})} \overline{\vartheta}_t \varsigma_t + \sum_{t \in T^-(\overline{x})} \overline{\vartheta}_t \varsigma_t, \right.\right. \tag{8}$$

$$\left.\left. \sum_{i=1}^k \overline{\lambda}_i \rho_{f_i} + \sum_{j \in J(\overline{x})} \overline{\mu}_j \rho_{g_j} + \sum_{t \in T^+(\overline{x})} \overline{\vartheta}_t \rho^+_{h_t} - \sum_{t \in T^-(\overline{x})} \overline{\vartheta}_t \rho^-_{h_t}\right)\right) \geqq 0,$$

where

$$\overline{A} = \sum_{i=1}^k \overline{\lambda}_i + \sum_{j=1}^m \overline{\mu}_j + \sum_{t \in T^+(\overline{x})} \overline{\vartheta}_j - \sum_{t \in T^-(\overline{x})} \overline{\vartheta}_j > 0. \tag{9}$$

Let us denote

$$\overline{\alpha}_i = \frac{\overline{\lambda}_i}{\overline{A}}, \; i \in I(\overline{x}), \tag{10}$$

$$\overline{\beta}_j = \frac{\overline{\mu}_j}{\overline{A}}, \; j \in J(\overline{x}), \tag{11}$$

$$\overline{\gamma}_t^+ = \frac{\overline{\vartheta}_t}{\overline{A}}, \; t \in T^+(\overline{x}), \tag{12}$$

$$\overline{\gamma}_t^- = \frac{-\overline{\vartheta}_t}{\overline{A}}, \; t \in T^-(\overline{x}). \tag{13}$$

Then, by $\overline{\lambda} \geq 0$, $\sum_{i=1}^k \overline{\lambda}_i = 1$, it follows that $\overline{\alpha} := (\overline{\alpha}_1, ..., \overline{\alpha}_k) \geq 0$, $0 \leqq \overline{\alpha}_i \leqq 1$, $i \in I$, $\overline{\beta}_j = \left(\overline{\beta}_1, ..., \overline{\beta}_m\right) \geqq 0$, $0 \leqq \overline{\beta}_j \leqq 1$, $j \in J$, $0 \leqq \overline{\gamma}_t^+ \leqq 1$, $t \in T^+(\overline{x})$, $0 \leqq \overline{\gamma}_t^- \leqq 1$, $t \in T^-(\overline{x})$, and, moreover,

$$\sum_{i=1}^k \overline{\alpha}_i + \sum_{j\in J(\overline{x})} \overline{\beta}_j + \sum_{t\in T^+(\overline{x})} \overline{\gamma}_t^+ + \sum_{t\in T^-(\overline{x})} \overline{\gamma}_t^- = 1. \tag{14}$$

Using assumptions a)-d), by Definition 2.7, the following inequalities

$$b_{f_i}(\widetilde{x}, \overline{x}) \Psi_{f_i}(f_i(\widetilde{x}) - f_i(\overline{x})) \geqq \Phi\left(\widetilde{x}, \overline{x}, \left(\xi_i, \rho_{f_i}\right)\right), \quad i \in I, \tag{15}$$

$$b_{g_j}(\widetilde{x}, \overline{x}) \Psi_{g_j}\left(g_j(\widetilde{x}) - g_j(\overline{x})\right) \geqq \Phi\left(\widetilde{x}, \overline{x}, \left(\zeta_j, \rho_{g_j}\right)\right), \quad j \in J(\overline{x}), \tag{16}$$

$$b_{h_t}(\widetilde{x}, \overline{x}) \Psi_{h_t}(h_t(\widetilde{x}) - h_t(\overline{x})) \geqq \Phi\left(\widetilde{x}, \overline{x}, \left(\varsigma_t, \rho_{h_t}^+\right)\right), \quad t \in T^+(\overline{x}), \tag{17}$$

$$b_{h_t}(\widetilde{x}, \overline{x}) \Psi_{h_t}(-h_t(\widetilde{x}) + h_t(\overline{x})) \geqq \Phi\left(\widetilde{x}, \overline{x}, \left(-\varsigma_t, \rho_{h_t}^-\right)\right), \quad t \in T^-(\overline{x}) \tag{18}$$

hold for each $\xi_i \in \partial f_i(\overline{x})$, $i \in I$, $\zeta_j \in \partial g_j(\overline{x})$, $j \in J(\overline{x})$, $\varsigma_t \in \partial h_t(\overline{x})$, $t \in T^+(\overline{x}) \cup T^-(\overline{x})$. By (6) and the properties of b_{f_i} and Ψ_{f_i}, $i = 1, ..., k$, it follows that

$$b_{f_i}(\widetilde{x}, \overline{x}) \Psi_{f_i}(f_i(\widetilde{x}) - f_i(\overline{x})) < 0, \; i = 1, ..., k. \tag{19}$$

Combining (15) and (19), we obtain

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\xi_i, \rho_{f_i}\right)\right) < 0, \quad i \in I. \tag{20}$$

Further, by $\widetilde{x}, \overline{x} \in D$ and the assumptions imposed on b_{g_j}, Ψ_{g_j}, $j \in J(\overline{x})$, b_{h_t}, Ψ_{h_t}, $t \in T^+(\overline{x}) \cup T^-(\overline{x})$, we have, respectively,

$$b_{g_j}(\widetilde{x}, \overline{x}) \Psi_{g_j}\left(g_j(\widetilde{x}) - g_j(\overline{x})\right) \leqq 0, \; j \in J(\overline{x}), \tag{21}$$

$$b_{h_t}(\widetilde{x}, \overline{x}) \Psi_{h_t}(h_t(\widetilde{x}) - h_t(\overline{x})) \leqq 0, \; t \in T^+(\overline{x}), \tag{22}$$

$$b_{h_t}(\widetilde{x}, \overline{x}) \Psi_{h_t}(-h_t(\widetilde{x}) + h_t(\overline{x})) \leqq 0, \; t \in T^-(\overline{x}). \tag{23}$$

Combining (16) and (21), (17) and (22), (18) and (23), we get, respectively,

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\zeta_j, \rho_{g_j}\right)\right) \leqq 0, \quad j \in J(\overline{x}), \tag{24}$$

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\varsigma_t, \rho_{h_t}^+\right)\right) \leqq 0, \quad t \in T^+(\overline{x}), \tag{25}$$

$$\Phi\left(\widetilde{x}, \overline{x}, \left(-\varsigma_t, \rho_{h_t}^-\right)\right) \leqq 0, \quad t \in T^-(\overline{x}). \tag{26}$$

By Definition 2.7, it follows that $\Phi(\widetilde{x}, \overline{x}, \cdot)$ is a strictly quasi-convex function on R^{n+1}. Since (20) and (24)-(26) are satisfied, Proposition 2.5 implies

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\sum_{i=1}^k \overline{\alpha}_i \xi_i + \sum_{j\in J(\overline{x})} \overline{\beta}_j \zeta_j + \sum_{t\in T^+(\overline{x})} \overline{\gamma}_t^+ \varsigma_t + \sum_{t\in T^-(\overline{x})} \overline{\gamma}_t^- (-\varsigma_t), \right.\right. \tag{27}$$

$$\sum_{i=1}^{k}\overline{\alpha}_i\rho_{f_i}+\sum_{j\in J(\overline{x})}\overline{\beta}_j\rho_{g_j}+\sum_{t\in T^{+}(\overline{x})}\overline{\gamma}_t^{+}\rho_{h_t}^{+}+\sum_{t\in T^{-}(\overline{x})}\overline{\gamma}_t^{-}\rho_{h_t}^{-}\Bigg)\Bigg)<0.$$

Using (10)-(13) in (27), we get that the inequality

$$\Phi\Bigg(\widetilde{x},\overline{x}\,,\frac{1}{A}\Bigg(\sum_{i=1}^{k}\overline{\lambda}_i\xi_i+\sum_{j\in J(\overline{x})}\overline{\mu}_j\zeta_j+\sum_{t\in T^{+}(\overline{x})}\overline{\vartheta}_t\varsigma_t+\sum_{t\in T^{-}(\overline{x})}\overline{\vartheta}_t\varsigma_t,$$

$$\sum_{i=1}^{k}\overline{\lambda}_i\rho_{f_i}+\sum_{j\in J(\overline{x})}\overline{\mu}_j\rho_{g_j}+\sum_{t\in T^{+}(\overline{x})}\overline{\vartheta}_t\rho_{h_t}^{+}-\sum_{t\in T^{-}(\overline{x})}\overline{\vartheta}_t\rho_{h_t}^{-}\Bigg)\Bigg)<0$$

holds, contradicts (8). Hence, this means that $\overline{x}$ is a weakly efficient solution in (VP) and completes the proof of this theorem. □

In order to prove the sufficient optimality conditions for a Pareto solution in the nonsmooth multiobjective programming problem (VP) with nonsmooth (Φ,ρ)-invex functions, some stronger hypotheses should be assumed.

Theorem 3.6. *Let $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\vartheta}\right)\in D\times R^k\times R^m\times R^q$ be a Karush-Kuhn-Tucker point in the considered nonsmooth multiobjective programming problem (VP). Further, assume that any one of the following hypotheses is satisfied:*
Further, assume that either one of the following two sets of hypotheses is satisfied:

A)
- *a) each objective function f_i, $i=1,...,k$, is strictly $(b_{f_i},\Psi_{f_i},\Phi,\rho_{f_i})^w$-univex at $\overline{x}$ on D and $a<0\Longrightarrow\Psi_{f_i}(a)<0$;*
- *b) each inequality constraint function g_j, $j\in J(\overline{x})$, is $(b_{g_j},\Psi_{g_j},\Phi,\rho_{g_j})^w$-univex at $\overline{x}$ on D and $a\leqq 0\Longrightarrow\Psi_{g_j}(a)\leqq 0$;*
- *c) each equality constraint function h_s, $t\in T^{+}(\overline{x})$, is $(b_{h_t},\Psi_{h_t},\Phi,\rho_{h_t}^{+})^w$-univex at $\overline{x}$ on D and $(\Psi_{h_s}(0)=0$ or $a\leqq 0\Longrightarrow\Psi_{h_s}(a)\leqq 0)$;*
- *d) each function $-h_s$, $t\in T^{-}(\overline{x})$, is $(b_{h_t},\Psi_{h_t},\Phi,\rho_{h_t}^{-})^w$-univex at $\overline{x}$ on D and $(\Psi_{h_t}(0)=0$ or $a\leqq 0\Longrightarrow\Psi_{h_t}(a)\leqq 0)$;*
- *e) $\sum_{i=1}^{k}\overline{\lambda}_i\rho_{f_i}+\sum_{j\in J(\overline{x})}\overline{\mu}_j\rho_{g_j}+\sum_{t\in T^{+}(\overline{x})}\overline{\vartheta}_t\rho_{h_t}^{+}-\sum_{t\in T^{-}(\overline{x})}\overline{\vartheta}_t\rho_{h_t}^{-}\geqq 0$.*

B)
- *a) the Lagrange multipliers are positive, that is, $\overline{\lambda}_i>0$, $i\in I$, and each objective function f_i, $i=1,...,k$, is $(b_{f_i},\Psi_{f_i},\Phi,\rho_{f_i})^w$-univex at $\overline{x}$ on D and $a<0\Longrightarrow\Psi_{f_i}(a)<0$;*
- *b) each inequality constraint function g_j, $j\in J(\overline{x})$, is $(b_{g_j},\Psi_{g_j},\Phi,\rho_{g_j})^w$-univex at $\overline{x}$ on D and $a\leqq 0\Longrightarrow\Psi_{g_j}(a)\leqq 0$;*
- *c) each equality constraint function h_s, $t\in T^{+}(\overline{x})$, is $(b_{h_t},\Psi_{h_t},\Phi,\rho_{h_t}^{+})^w$-univex at $\overline{x}$ on D and $(\Psi_{h_s}(0)=0$ or $a\leqq 0\Longrightarrow\Psi_{h_s}(a)\leqq 0)$;*
- *d) each function $-h_s$, $t\in T^{-}(\overline{x})$, is $(b_{h_t},\Psi_{h_t},\Phi,\rho_{h_t}^{-})^w$-univex at $\overline{x}$ on D and $(\Psi_{h_t}(0)=0$ or $a\leqq 0\Longrightarrow\Psi_{h_t}(a)\leqq 0)$;*
- *e) $\sum_{i=1}^{k}\overline{\lambda}_i\rho_{f_i}+\sum_{j\in J(\overline{x})}\overline{\mu}_j\rho_{g_j}+\sum_{t\in T^{+}(\overline{x})}\overline{\vartheta}_t\rho_{h_t}^{+}-\sum_{t\in T^{-}(\overline{x})}\overline{\vartheta}_t\rho_{h_t}^{-}\geqq 0$.*

Then $\overline{x}$ is a Pareto solution in (VP).

Proof. Proof of theorem is similar to the proof of Theorem 3.5. □

Now, we illustrate the optimality results established in this section. Namely, we consider the example of a nondifferentiable vector optimization problem with nondifferentiable $(b,\Psi,\Phi,\rho)^w$-univex functions, not necessarily, with respect to the same b, Ψ and ρ.

Example 3.7. *Consider the following nondifferentiable multiobjective programming problem*

$$f(x) = \left(\arctan\left(\frac{|x_1|}{x_1^2+x_2^2+1}\right),\ \arctan\left(\left(x_1^2+x_2^2+1\right)|x_1x_2|\right)\right) \to \min \qquad (VP1)$$
$$g(x) = -\arctan\left(|x_1x_2|\right) \leqq 0.$$

It is not difficult to see that $D = \left\{(x_1, x_2) \in R^2 : \arctan\left(|x_1x_2|\right) \geqq 0\right\}$ *and* $\overline{x} = (0, 0)$ *is such a feasible solution at which the Generalized Karush-Kuhn-Tucker necessary optimality conditions (3)-(5) are satisfied. It can be shown, by Definition 2.7, that each objective function* f_i*,* $i = 1, 2$*, is locally Lipschitz* $\left(b_{f_i}, \Psi_{f_i}, \Phi, \rho_{f_i}\right)^w$*-univex at* $\overline{x}$ *on* D *and the constraint function* g *is* $\left(b_g, \Psi_g, \Phi, \rho_g\right)^w$*-univex at* $\overline{x}$ *on* D*, where*

$$b_{f_1}(x,\overline{x}) = x_1^2 + x_2^2 + 1,\ b_{f_2}(x,\overline{x}) = \frac{1}{x_1^2 + x_2^2 + 1},\ b_g(x,\overline{x}) = 1,$$

$$\Psi_{f_1}(a) = \tan(a),\ \Psi_{f_2}(a) = \tan(a),\ \Psi_g(a) = \tan(a),$$

$$\Phi\left(x, \overline{x}, (\varsigma, \rho)\right) = \arctan\left(\varsigma_1 |x_1|\right) + \arctan\left(\varsigma_2 |x_2|\right) + \arctan(\rho)\left(\arctan|x_1x_2| - \arctan\left|\overline{x}_1\overline{x}_2\right|\right)$$

$$\rho_{f_1} = 0,\ \rho_{f_2} = 1,\ \rho_g = -\tan(1),$$

where $\varsigma \in \partial\varphi(\overline{x})$*, where* φ *denotes* f_1 *or* f_2 *or* g*, respectively, and* ρ *is equal to* ρ_{f_1}*,* ρ_{f_2} *or* ρ_g*, respectively. Since all the assumptions of Theorem 3.5 are fulfilled,* $\overline{x}$ *is a weak Pareto solution in the considered nonsmooth multiobjective programming problem. Note that we are not able to prove that* $\overline{x}$ *is a weak Pareto solution in the considered nonconvex nonsmooth vector optimization problem (VP1) under other generalized convexity notions existing in the literature, for example, under invexity [27], [23], b-invexity [10], r-invexity [3], F-convexity [17], G-invexity [4], V-r-invexity [5], univexity [32]. This is a consequence of the fact that not every stationary point of functions constituting problem (VP1) is a global minimum of such functions. Whereas one of the main property of the concepts generalized convexity notions mentioned above is that a stationary point of every function belonging to these classes of generalized convex functions is its global minimizer. Further, also the sufficient optimality conditions under* (Φ, ρ)*-invexity are not applicable because the functional* $\Phi(x, \overline{x}, \cdot)$ *is not convex for all* $x \in D$ *as it is requirement in the definition of this generalized convexity notion (see [5]). Taking into account even this example, the concept of nondifferentiable* $(b, \Psi, \Phi, \rho)^w$*-univexity is useful in order to prove the (weakly) efficiency of a feasible solution (or in other words, the sufficiency of generalized Karush-Kuhn-Tucker necessary optimality conditions) for a larger class of nonconvex nondifferentaible multicriteria optimization problems in comparison to other generalized convexity notions, earlier defined in the literature.*

4. Mond-Weir duality

In this section, we define a vector dual problem in the Mond-Weir sense for the considered nonsmooth multicriteria optimization problem (VP). Then, under $(b, \Psi, \Phi, \rho)^w$-univexity hypotheses, we prove several families of duality results between the primal vector optimization problem and its Mond-Weir dual problem.

Now, for the considered nonsmooth vector optimization problem (VP), we define the following vector Mond-Weir dual problems as follows:

$$\begin{aligned} & f(y) \to V\text{-max} \\ \text{s.t. } & 0 \in \textstyle\sum_{i=1}^{k} \lambda_i \partial f_i(y) + \sum_{j=1}^{m} \mu_j \partial g_j(y) + \sum_{t=1}^{q} \vartheta_t \partial h_t(y), \\ & \textstyle\sum_{j=1}^{m} \mu_j g_j(y) \geqq 0, \qquad \text{(VD)} \\ & \textstyle\sum_{t=1}^{q} \vartheta_t h_t(y) \geqq 0, \\ & \lambda \in R^k,\ \lambda \geq 0,\ \textstyle\sum_{i=1}^{k} \lambda_i = 1,\ \mu \in R^m,\ \mu \geqq 0,\ \vartheta \in R^q. \end{aligned}$$

We denote by S the set of all feasible solutions in the vectorial Mond-Weir dual problem (VD). Further, denote $Y = \{y \in X : (y, \lambda, \mu, \vartheta) \in S\}$.

Theorem 4.1. *(Weak duality): Let x and $(y, \lambda, \mu, \vartheta)$ be feasible solutions for the vector optimization problems (VP) and (VD), respectively. Further, assume that each function f_i, $i \in I$, is locally Lipschitz $\left(b_{f_i}, \Psi_{f_i}, \Phi, \rho_{f_i}\right)^w$-univex at y on $D \cup Y$, $\sum_{j=1}^m \mu_j g_j(\cdot)$ is locally Lipschitz $\left(b_g, \Psi_g, \Phi, \rho_g\right)^w$-univex at y on $D \cup Y$, $\sum_{t=1}^q \vartheta_t h_t(\cdot)$ is locally Lipschitz $(b_h, \Psi_h, \Phi, \rho_h)^w$-univex at y on $D \cup Y$. If $\sum_{i=1}^k \lambda_i \rho_{f_i} + \rho_g + \rho_h \geqq 0$, then*

$$f(x) \nless f(y).$$

Proof. We proceed by contradiction. Suppose, contrary to the result, that there exist $x \in D$ and $(y, \lambda, \mu, \vartheta) \in \Omega$ such that

$$f_i(x) < f_i(y), \quad i = 1, ..., k. \tag{28}$$

By assumption, f_i, $i \in I$, is locally Lipschitz $\left(b_{f_i}, \Psi_{f_i}, \Phi, \rho_{f_i}\right)^w$-univex at y on $D \cup Y$. Hence, by Definition 2.7, the inequalities

$$b_{f_i}(z, y)\, \Psi_{f_i}(f_i(z) - f_i(y)) \geqq \Phi\left(z, y, \left(\xi_i, \rho_{f_i}\right)\right), i \in I \tag{29}$$

hold for all $z \in D \cup Y$ and for any $\xi_i \in \partial f_i(y)$. Thus, they are also satisfied for $z = x \in D$. Thus, inequalities (29) give

$$b_{f_i}(x, y)\, \Psi_{f_i}(f_i(x) - f_i(y)) \geqq \Phi\left(x, y, \left(\xi_i, \rho_{f_i}\right)\right), i \in I. \tag{30}$$

By (28) and the assumptions imposed on functions b_{f_i} and Ψ_{f_i}, $i \in I$, one has

$$b_{f_i}(x, y)\, \Psi_{f_i}(f_i(x) - f_i(y)) < 0.$$

Hence, the inequalities above yield

$$\Phi\left(x, y, \left(\xi_i, \rho_{f_i}\right)\right) < 0, i \in I. \tag{31}$$

By assumptions, $\sum_{j=1}^m \mu_j g_j$ is locally Lipschitz $\left(b_g, \Psi_g, \Phi, \rho_g\right)^w$-univex at y on $D \cup Y$, $\sum_{t=1}^q \vartheta_t h_t$ is locally Lipschitz $(b_h, \Psi_h, \Phi, \rho_h)^w$-univex at y on $D \cup Y$. Hence, by Definition 2.7, the inequalities

$$b_g(x, y)\, \Psi_g\left(\sum_{j=1}^m \mu_j g_j(x) - \sum_{j=1}^m \mu_j g_j(y)\right) \geqq \Phi\left(x, y, \left(\sum_{j=1}^m \mu_j \zeta_j, \rho_g\right)\right), \tag{32}$$

$$b_h(x, y)\, \Psi_h\left(\sum_{t=1}^q \vartheta_t h_t(x) - \sum_{t=1}^q \vartheta_t h_t(y)\right) \geqq \Phi\left(x, y, \left(\sum_{t=1}^q \vartheta_t \varsigma_t, \rho_h\right)\right) \tag{33}$$

hold for each $\zeta_j \in \partial g_j(y)$, $j \in J$ and $\varsigma_t \in \partial h_t(y)$, $t \in T$, respectively. Using the assumptions imposed on the functions b_g, Ψ_g, b_h, Ψ_h, we have, respectively,

$$b_g(x, y)\, \Psi_g\left(\sum_{j=1}^m \mu_j g_j(x) - \sum_{j=1}^m \mu_j g_j(y)\right) \leqq 0, \tag{34}$$

$$b_h(x, y)\, \Psi_h\left(\sum_{t=1}^q \vartheta_t h_t(x) - \sum_{t=1}^q \vartheta_t h_t(y)\right) \leqq 0. \tag{35}$$

Combining (32) and (34), (33) and (35), we get, respectively,

$$\Phi\left(x, y, \left(\sum_{j=1}^m \mu_j \zeta_j, \rho_g\right)\right) \leqq 0, \tag{36}$$

$$\Phi\left(x,y,\left(\sum_{t=1}^{q}\vartheta_t\varsigma_t,\rho_h\right)\right)\leqq 0. \tag{37}$$

It is known that, by Definition 2.7, $\Phi(x,y,\cdot)$ is strictly quasi-convex on R^{n+1}. Hence, by Proposition 2.5, (31), (36) and (37) imply

$$\Phi\left(x,y,\left(\sum_{i=1}^{k}\frac{\lambda_i}{3}\xi_i+\frac{1}{3}\sum_{j=1}^{m}\mu_j\zeta_j+\frac{1}{3}\sum_{t=1}^{q}\vartheta_t\varsigma_t\,,\sum_{i=1}^{k}\frac{\lambda_i}{3}\rho_{f_i}+\frac{1}{3}\rho_g+\frac{1}{3}\rho_h\right)\right)<0. \tag{38}$$

Thus, (38) yields

$$\Phi\left(x,y,\frac{1}{3}\left(\sum_{i=1}^{k}\lambda_i\xi_i+\sum_{j=1}^{m}\mu_j\zeta_j+\sum_{t=1}^{q}\vartheta_t\varsigma_t\,,\sum_{i=1}^{k}\lambda_i\rho_{f_i}+\rho_g+\rho_h\right)\right)<0. \tag{39}$$

Using $(y,\lambda,\mu,\vartheta)\in\Omega$ again, the first constraint of the Mond-Weir dual problem (VD) gives

$$\sum_{i=1}^{k}\lambda_i\xi_i+\sum_{j=1}^{m}\mu_j\zeta_j+\sum_{t=1}^{q}\vartheta_t\varsigma_t=0. \tag{40}$$

By Definition 2.7, we have that $\Phi(x,y,(0,a))\geqq 0$ for any $a\in R_+$. Thus, by (40), hypothesis $\sum_{i=1}^{k}\lambda_i\rho_{f_i}+\rho_g+\rho_h\geqq 0$ implies that the inequality

$$\Phi\left(x,y,\frac{1}{3}\left(\sum_{i=1}^{k}\lambda_i\xi_i+\sum_{j=1}^{m}\mu_j\zeta_j+\sum_{t=1}^{q}\vartheta_t\varsigma_t\,,\sum_{i=1}^{k}\lambda_i\rho_{f_i}+\rho_g+\rho_h\right)\right)\geqq 0$$

holds, contradicts (40). This completes the proof of this theorem. □

If the stronger $(b,\Psi,\Phi,\rho)^w$-univexity hypothesis is imposed on the objective functions in the considered vector optimization problem (VP), then the following stronger result can be established.

Theorem 4.2. *(Weak duality): Let x and $(y,\lambda,\mu,\vartheta)$ be feasible solutions for problems (VP) and (VD), respectively. Further, assume that each function f_i, $i\in I$, is locally Lipschitz strictly $\left(b_{f_i},\Psi_{f_i},\Phi,\rho_{f_i}\right)^w$-univex at y on $D\cup Y$, g_j is locally Lipschitz $\left(b_g,\Psi_g,\Phi,\rho_g\right)^w$-univex at y on $D\cup Y$, $\sum_{t=1}^{q}\vartheta_t h_t$ is locally Lipschitz $(b_h,\Psi_h,\Phi,\rho_h)^w$-univex at y on $D\cup Y$. If $\sum_{i=1}^{k}\lambda_i\rho_{f_i}+\rho_g+\rho_h\geqq 0$,, then $f(x)\nleq f(y)$.*

Theorem 4.3. *(Strong duality). Let $\overline{x}$ be a (weak Pareto) Pareto solution in the considered nondifferentiable vector optimization problem (VP) and all the assumptions of Theorem 3.3 be fulfilled at $\overline{x}$. Then there exist $\overline{\lambda}\in R^k$, $\overline{\mu}\in R^m$ and $\overline{\vartheta}\in R^q$ such that $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\vartheta}\right)$ is feasible in (VD) and the objective functions of (VP) and (VD) are equal at these points. Further, since all the hypotheses of the weak duality theorem (Theorem 4.1) are satisfied, then $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\vartheta}\right)$ is a weakly efficient solution of a maximum type in (VD). If $\overline{\lambda}>0$ and all hypotheses of the weak duality theorem (Theorem 4.2) are satisfied, then $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\vartheta}\right)$ is an efficient solution of a maximum type in (VD).*

Proof. By assumption, $\overline{x}$ is a (weak) Pareto optimal solution in (VP) and the constraint qualification is satisfied at $\overline{x}$. Then there exist the Lagrange multipliers $\overline{\lambda}\in R^k$, $\overline{\mu}\in R^m$ and $\overline{\vartheta}\in R^q$ such that the Karush-Kuhn-Tucker necessary optimality conditions (3)-(5) are satisfied at $\overline{x}$. Thus, the feasibility of $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\vartheta}\right)$ in (VD) follows directly from these necessary optimality conditions and also from the feasibility of $\overline{x}$ in (VP). Therefore, the objective functions of problems (VP) and (VD) are equal at $\overline{x}$ and $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\vartheta}\right)$. Hence, weak efficiency of a maximum type of $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\vartheta}\right)$ in (VD) follows directly from weak duality (Theorem 4.1), whereas efficiency of maximum type follows from Theorem 4.2. □

Theorem 4.4. *(Converse duality): Let $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\vartheta}\right)$ be an efficient solution of a maximum type (weakly efficient solution of a maximum type) in the vector Mond-Weir dual problem (VD) such that $\overline{y} \in D$. Further, assume that each function f_i, $i \in I$, is locally Lipschitz $\left(b_{f_i}, \Psi_{f_i}, \Phi, \rho_{f_i}\right)^w$-univex (locally Lipschitz strictly $\left(b_{f_i}, \Psi_{f_i}, \Phi, \rho_{f_i}\right)^w$-univex) at $\overline{y}$ on $D \cup Y$, g_j is locally Lipschitz $\left(b_g, \Psi_g, \Phi, \rho_g\right)^w$-univex at $\overline{y}$ on $D \cup Y$, $\sum_{t=1}^{q} \overline{\vartheta}_t h_t$ is locally Lipschitz $(b_h, \Psi_h, \Phi, \rho_h)^w$-univex at $\overline{y}$ on $D \cup Y$. If $\sum_{i=1}^{k} \overline{\lambda}_i \rho_{f_i} + \rho_g + \rho_h \geqq 0$. Then $\overline{y}$ is a weakly efficient solution (an efficient solution) in the considered nondifferentiable vector optimization problem (VP).*

Proof. The proof of this theorem follows directly from weak duality (Theorem 4.1 or Theorem 4.2). □

Theorem 4.5. *(Restricted converse duality): Let $\overline{x}$ be a feasible solution in the considered multiobjective programming problem (VP) and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\vartheta}\right)$ be a feasible solution of a maximum type in the vector Mond-Weir dual problem (VD) such that $f(\overline{x}) = f(\overline{y})$. Further, assume that each function f_i, $i \in I$, is locally Lipschitz $\left(b_{f_i}, \Psi_{f_i}, \Phi, \rho_{f_i}\right)^w$-univex at $\overline{y}$ on $D \cup Y$, $\sum_{j=1}^{m} \overline{\mu}_j g_j(\cdot)$ is locally Lipschitz $\left(b_g, \Psi_g, \Phi, \rho_g\right)^w$-univex at $\overline{y}$ on $D \cup Y$, $\sum_{t=1}^{q} \overline{\vartheta}_t h_t(\cdot)$ is locally Lipschitz $(b_h, \Psi_h, \Phi, \rho_h)^w$-univex at $\overline{y}$ on $D \cup Y$. and $\sum_{i=1}^{k} \overline{\lambda}_i \rho_{f_i} + \rho_g + \rho_{h_t} \geqq 0$. Then $\overline{x}$ is a weak Pareto solution in (VP) and $\overline{y}$ a weakly efficient solution of a maximum type in (VD). If each function f_i, $i \in I$, is locally Lipschitz strictly $\left(b_{f_i}, \Psi_{f_i}, \Phi, \rho_{f_i}\right)^w$-univex at $\overline{y}$ on $D \cup Y$, then $\overline{x}$ is a Pareto solution in (VP) and $\overline{y}$ an efficient solution of a maximum type in (VD).*

Proof. We proceed by contradiction. If $\overline{x}$ is not a weak efficient solution in (VP), then there exists $\widetilde{x} \in D$ such that $f(\widetilde{x}) < f(\overline{x})$. By assumption, $f(\overline{x}) = f(\overline{y})$. Thus,

$$f(\widetilde{x}) < f(\overline{y}). \tag{41}$$

By assumption, f_i, $i \in I$, is locally Lipschitz $\left(b_{f_i}, \Psi_{f_i}, \Phi, \rho_{f_i}\right)^w$-univex at $\overline{y}$ on $D \cup Y$. Hence, by Definition 2.7, the inequalities

$$b_{f_i}(\widetilde{x}, \overline{y})\, \Psi_{f_i}(f_i(\widetilde{x}) - f_i(\overline{y})) \geqq \Phi\left(\widetilde{x}, \overline{y}, \left(\xi_i, \rho_{f_i}\right)\right), \; i \in I \tag{42}$$

hold for for any $\xi_i \in \partial f_i(\overline{y})$. By (41) and the assumptions imposed on functions b_{f_i} and Ψ_{f_i}, $i \in I$, one has

$$b_{f_i}(\widetilde{x}, \overline{y})\, \Psi_{f_i}(f_i(\widetilde{x}) - f_i(\overline{y})) < 0.$$

Thus, the inequalities above yield

$$\Phi\left(\widetilde{x}, \overline{y}, \left(\xi_i, \rho_{f_i}\right)\right) < 0, \; i \in I \tag{43}$$

hold for for any $\xi_i \in \partial f_i(\overline{y})$. By assumptions, $\sum_{j=1}^{m} \overline{\mu}_j g_j$ is locally Lipschitz $\left(b_g, \Psi_g, \Phi, \rho_g\right)^w$-univex at $\overline{y}$ on $D \cup Y$, $\sum_{t=1}^{q} \overline{\vartheta}_t h_t(\cdot)$ is locally Lipschitz $(b_h, \Psi_h, \Phi, \rho_h)^w$-univex at $\overline{y}$ on $D \cup Y$. Hence, by Definition 2.7, the inequalities

$$b_g(\widetilde{x}, \overline{y})\, \Psi_g\left(\sum_{j=1}^{m} \overline{\mu}_j g_j(\widetilde{x}) - \sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y})\right) \geqq \Phi\left(\widetilde{x}, \overline{y}, \left(\sum_{j=1}^{m} \overline{\mu}_j \zeta_j, \rho_g\right)\right), \tag{44}$$

$$b_h(\widetilde{x}, \overline{y})\, \Psi_h\left(\sum_{t=1}^{q} \overline{\vartheta}_t h_t(\widetilde{x}) - \sum_{t=1}^{q} \overline{\vartheta}_t h_t(\overline{y})\right) \geqq \Phi\left(\widetilde{x}, \overline{y}, \left(\sum_{t=1}^{q} \overline{\vartheta}_t \varsigma_t, \rho_h\right)\right) \tag{45}$$

hold for any $\zeta_j \in \partial g_j(y)$, $j \in J$ and any $\varsigma_t \in \partial h_t(y)$, $t \in T$, respectively. From the assumptions imposed on the functions b_g, Ψ_g, b_h, Ψ_h, one has, respectively,

$$b_g(\widetilde{x}, \overline{y})\, \Psi_g\left(\sum_{j=1}^{m} \overline{\mu}_j g_j(\widetilde{x}) - \sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y})\right) \leqq 0, \tag{46}$$

$$b_h(\widetilde{x},\overline{y})\,\Psi_h\left(\sum_{t=1}^{q}\overline{\vartheta}_t h_t(\widetilde{x})-\sum_{t=1}^{q}\overline{\vartheta}_t h_t(\overline{y})\right)\leqq 0. \tag{47}$$

Combining (44) and (46), (45) and (47), we obtain, respectively,

$$\Phi\left(\widetilde{x},\overline{y},\left(\sum_{j=1}^{m}\overline{\mu}_j\zeta_j,\rho_g\right)\right)\leqq 0, \tag{48}$$

$$\Phi\left(\widetilde{x},\overline{y},\left(\sum_{t=1}^{q}\overline{\vartheta}_t\varsigma_t,\rho_h\right)\right)\leqq 0. \tag{49}$$

Then, by Definition 2.7, it follows that $\Phi(\widetilde{x},\overline{y},\cdot)$ is strictly quasi-convex on R^{n+1}. Hence, by Proposition 2.5, (43), (48) and (49) imply

$$\Phi\left(\overline{x},\overline{y},\frac{1}{3}\left(\sum_{i=1}^{k}\overline{\lambda}_i\xi_i+\sum_{j=1}^{m}\overline{\mu}_j\zeta_j+\sum_{t=1}^{q}\overline{\vartheta}_t\varsigma_t\,,\sum_{i=1}^{k}\overline{\lambda}_i\rho_{f_i}+\rho_g+\rho_h\right)\right)<0. \tag{50}$$

Using the first constraint of dual problem (VD), we have

$$\sum_{i=1}^{k}\lambda_i\xi_i+\sum_{j=1}^{m}\mu_j\zeta_j+\sum_{t=1}^{q}\vartheta_t\varsigma_t=0. \tag{51}$$

Then, by Definition 2.7, one has $\Phi(x,y,(0,a))\geqq 0$ for any $a\in R_+$. Thus, (51) together with the assumption $\sum_{i=1}^{k}\overline{\lambda}_i\rho_{f_i}+\rho_g+\rho_{h_t}\geqq 0$ imply that the inequality

$$\Phi\left(x,y,\frac{1}{3}\left(\sum_{i=1}^{k}\overline{\lambda}_i\xi_i+\sum_{j=1}^{m}\overline{\mu}_j\zeta_j+\sum_{t=1}^{q}\overline{\vartheta}_t\varsigma_t\,,\sum_{i=1}^{k}\overline{\lambda}_i\rho_{f_i}+\rho_g+\rho_h\right)\right)\geqq 0$$

holds, contradicts (50). This completes the proof of this theorem. □

5. Conclusions

In the paper, a new class of nonconvex nondifferentiable vector optimization problems has been investigated in which every component of the functions involved is a locally Lipschitz $(b,\Psi,\Phi,\rho)^w$-univex function (with respect to, not necessarily, the same b, Ψ and ρ). Then, the sufficient optimality conditions for a weak Pareto solution and a Pareto solution and duality theorems in the sense of Mond-Weir have been established for the considered nonconvex nonsmooth multiobjective programming problem under nondifferentiable $(b,\Psi,\Phi,\rho)^w$-univexity hypotheses. Moreover, as we noticed, the definition of nondifferentiable $(b,\Psi,\Phi,\rho)^w$-univexity unifies many generalized convex notions, earlier introduced in the literature (see Remark 2.9). In order to illustrate the sufficient optimality conditions established in the paper, an example of a nonconvex nonsmooth multiobjective programming problem with nondifferentiable $(b,\Psi,\Phi,\rho)^w$-univex function has been given. It is interesting that not all functions constituting the considered nonsmooth vector optimization problem posses the fundamental property of the most classes of generalized convex functions, namely that a stationary point of such a function is also its global minimum. Thus, we have also shown that many generalized convexity notions existing in the literature (that is, invexity [23], b-invexity [10], r-invexity [3], G-invexity [4], ρ-(η,θ)-invexity [36], V-r-invexity [5], univexity [32]) may fail in proving the sufficiency of the Karush-Kuhn-Tucker necessary optimality conditions and various Mond-Weir duality theorems for the nonconvex nonsmooth vector optimization problem considered in the paper. Thus, the class of nonconvex nonsmooth multiobjective programming problems, for which the Generalized Karush-Kuhn-Tucker necessary optimality conditions are also sufficient for (weakly) efficiency of a feasible solution and several duality theorems in the sense of Mond-Weir are fulfilled, have been extended in the paper, in comparison to similarly results earlier established in the literature under other generalized convexity assumptions.

It seems that the techniques employed in this paper can be used in proving similarly results for various classes of nondifferentiable optimization problems with $(b, \Psi, \Phi, \rho)^w$-univexity functions. We shall investigate these results for other extremum problems in the subsequent papers.

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