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2−conformal and conformal vector fields on Riemannian manifolds

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Abstract. In this paper, we study 2−conformal vector fields on Riemannian manifolds. Some relations between 2−conformal vector fields, conformal vector fields, Killing and 2−Killing vector fields have been obtained. Also, relation between monotone vector fields and 2−conformal vector fields is investigated. We give the characterization of 2−conformal vector fields on standard Euclidean spaces. Finally, examples of such vector fields on some Riemannian manifolds is presented.

1. Introduction

Geometric vector fields have been discussed for geometrical and physical purposes on Riemannian and semi-Riemannian manifolds. Among of these geometric vector fields, Killing, 2−Killing and conformal vector fields have long been of interest to mathematicians and physicists.

Killing vector fields are the infinitesimal generators of isometries, that is, flows generated by Killing fields are isometries of their underlying manifolds [9]. A smooth vector field X on a (semi-)Riemannian manifold (N, g) is called Killing vector field if it satisfies $\mathcal{L}_X g = 0$, where $\mathcal{L}$ denotes Lie derivative. These vector fields have significant links with both the topology and geometry of a Riemannian manifold. Also, in application to general theory of relativity, Killing vector fields show symmetries of spacetime manifolds.

A vector field X on a (semi-)Riemannian manifold (N, g) is said to be a conformal vector field if there exists a smooth function f (known as potential function) on N such that $\mathcal{L}_X g = 2fg$. If the potential function f be constant, we call X is homothetic vector field. When f is identically zero, X reduces to Killing vector field. Flow generated by a conformal vector field defines conformal transformations, that is, preserve the metric tensor g of the manifold upto scalar tensor and preserve the conformal structure (for more details see [2, 9]).

In this paper, we introduce the notion of 2−conformal vector fields and obtain some results. A smooth vector field $X \in \mathcal{X}(N)$ on a (semi-)Riemannian manifold (N, g) is said to be 2−conformal vector field if it satisfies $\mathcal{L}_X \mathcal{L}_X g = 2\sigma g$ for some smooth function σ. If σ is constant the vector field is called 2−homothetic.

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Also, when σ be identically zero, the vector field X is said to be 2−Killing vector field (see [1, 3, 7]).

2. Preliminaries

Consider an n-dimensional Riemannian manifold (N, g) with Lie algebra $\mathcal{X}(N)$ of smooth vector fields and Riemannian connection ∇. Recall that the Riemannian curvature (0,4)-tensor for $X, U \in \mathcal{X}(N)$ satisfies the following equation [8]

$$g(\nabla_X \nabla_U X, U) - g(\nabla_{[X,U]} X, U] = g(\nabla_U \nabla_X X, U) - R(X, U, X, U), \tag{1}$$

also, the Ricci curvature is given by

$$\mathrm{Ric}(X, U) = \sum_{i=1}^{n} g(\bar{R}(e_i, X)U, e_i) = \Sigma_{i=1}^{n} R(e_i, X, e_i, U) = \sum_{i}^{n} R(X, e_i, U, e_i), \tag{2}$$

where $\bar{R}$ is the Riemannian curvature tensor of type $(1, 3)$ and $\{e_1, ..., e_n\}$ is a local orthonormal frame on N. As mentioned before, a smooth vector field X is said to be a 2−conformal vector field if it satisfies $\mathcal{L}_X \mathcal{L}_X g = 2\sigma g$, where $\mathcal{L}_X$ is the Lie derivative operator along to X.
Remind the definition of Lie derivative operator. For $X, Y, Z \in \mathcal{X}(N)$, we have

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = X.g(Y, Z) - g([X, Y], Z) - g([X, Z], Y). \tag{3}$$

Let ϕ be the 1-form dual to X in the Riemannian manifold (N, g), so that

$$\phi(Y) = g(Y, X). \tag{4}$$

We can define a skew symmetric tensor field ψ on N by

$$d\phi(Y, Z) = 2g(\psi Y, Z), \qquad Y, Z \in \mathcal{X}(N). \tag{5}$$

Using equalities (4) and (3) in Koszula's formula we have the following relation

$$2g(\nabla_Y X, Z) = (\mathcal{L}_X g)(Y, Z) + d\phi(Y, Z), \qquad Y, Z \in \mathcal{X}(N). \tag{6}$$

Directly, from the formula (6), we obtain the following lemma.

Lemma 2.1. *Let X be a conformal vector field on N with the potential function f. Then, any vector field Y satisfies*

$$\nabla_Y X = fY + \psi Y. \tag{7}$$

Furthermore, if X be closed (i.e. its dual 1-form ϕ is closed) we have

$$\nabla_Y X = fY. \tag{8}$$

Also, we can prove the following lemma.

Lemma 2.2. *Let (N, g) be a Riemannian manifold. A vector field X on N is a 2−conformal with potential function σ, if and only if*

$$R(X, U, X, U) = g(\nabla_U \nabla_X X, U) + g(\nabla_U X, \nabla_U X) - \sigma g(U, U),$$

where R is the Riemannian curvature tensor of (N, g).

Proof. As $\mathcal{L}_X\mathcal{L}_X g - \sigma g$ is symmetric tensor, we know

$$\mathcal{L}_X\mathcal{L}_X g - \sigma g = 0 \Longleftrightarrow \forall U \in \mathcal{X}(N), \quad (\mathcal{L}_X\mathcal{L}_X g)(U, U) = \sigma g(U, U).$$

On the other hand, by definition we have

$$(\mathcal{L}_X\mathcal{L}_X g)(U, U) = X.(\mathcal{L}_X g)(U, U) - 2(\mathcal{L}_X g)([X, U], U).$$

Now, we can write

$$\begin{aligned}(\mathcal{L}_X\mathcal{L}_X g)(U, U) &= X.(\mathcal{L}_X g)(U, U) - 2(\mathcal{L}_X g)([X, U], U)\\ &= 2X.g(\nabla_U X, U) - 2g(\nabla_{[X,U]} X, U) - 2g(\nabla_U X, [X, U])\\ &= 2g(\nabla_X\nabla_U X, U) + 2g(\nabla_U X, \nabla_X U) - 2g(\nabla_{[X,U]} X, U)\\ &- 2g(\nabla_U X, \nabla_X U - \nabla_U X)\\ &= 2g(\nabla_X\nabla_U X, U) - 2g(\nabla_{[X,U]} X, U) + 2g(\nabla_U X, \nabla_U X).\end{aligned}$$

Using equation (1), we get

$$R(X, U, X, U) = g(\nabla_U\nabla_X X, U) + g(\nabla_U X, \nabla_U X) - \sigma g(U, U). \tag{9}$$

□

According to (2), we obtain the following result.

Corollary 2.3. *For a $2-$conformal vector field X on a Riemannian manifold (N, g) with potential function σ, we have*

$$\mathrm{Ric}(X, X) = \mathrm{div}(\nabla_X X) + \|\nabla X\|^2 - \sigma n.$$

3. Main results

In the previous section, two necessary and sufficient conditions for a vector field to be $2-$conformal were given in terms of the curvature tensor R and the Ricci tensor. In this section, we investigate the relation between conformal and $2-$conformal vector fields.

Obviously, every conformal vector field is $2-$conformal. We present a condition under which a $2-$conformal vector field is conformal vector field.

Proposition 3.1. *Let X be a $2-$conformal vector field with a potential function σ on a Riemannian manifold (N, g). Then, if X be conformal with potential function f, we have*

$$2f^2 + Xf = \sigma. \tag{10}$$

Proof. Since, $\mathcal{L}_X g = 2fg$, we get

$$\mathcal{L}_X\mathcal{L}_X g = \mathcal{L}_X(2fg) = 2f\mathcal{L}_X g + 2(Xf)g = 2(2f^2 + Xf)g.$$

On the other hand, we know $\mathcal{L}_X\mathcal{L}_X g = 2\sigma g$. Hence,

$$2f^2 + Xf = \sigma.$$

□

Note that for a conformal vector field X with a potential function f, we have $\mathrm{div}X = nf$.
Also, we can prove the following results for compact Riemannian manifolds.

Theorem 3.2. *Let X be a* $2-$*conformal vector field on a compact Riemannian manifold* (N, g) *of dimension* $n \geq 2$ *with* $\sigma \leq 0$*. If* $\mathrm{Ric}(X, X) \leq 0$ *then, X is a parallel vector field and consequently is a Killing vector field.*

Proof. By Corollary 2.3, we have $\mathrm{Ric}(X, X) = \mathrm{div}(\nabla_X X) + \|\nabla X\|^2 - \mathrm{n}\sigma \leq 0$. Integrating from both sides of this equality and taking into account that the integral of divergence vanishes, we obtain

$$\int_N Ric(X, X)\, dN = \int_N (\|\nabla X\|^2 - n\sigma) dN \leq 0,$$

where dN stands for the volume element related to g. But, we know $\int_N (\|\nabla X\|^2 - n\sigma) dN \geq 0$. Hence, $\nabla X = \sigma = 0$, i.e. X is a parallel vector field and consequently, $\mathcal{L}_X g = 0$. □

Theorem 3.3. *Let X be a conformal closed vector field with a potential function f on a compact Riemannian manifold* (N, g) *with* $n \geq 2$*. If* $\mathrm{Ric}(X, X) \leq 0$*, then, X is a parallel field.*

Proof. Consider a local orthonormal frame $\{e_1, ..., e_n\}$ to compute $Ric(Y, X)$ for any vector field Y. By (8), we have

$$\nabla_Y X = fY.$$

Now, we can write

$$\begin{aligned}
\mathrm{Ric}(Y, X) &= \sum_{i=1}^{n} g(\nabla_{e_i}\nabla_Y X - \nabla_Y \nabla_{e_i} X - \nabla_{[e_i, Y]} X, e_i) \\
&= \sum_{i=1}^{n} g(\nabla_{e_i} fY - \nabla_Y f e_i - f\nabla_{e_i} Y + f\nabla_Y e_i, e_i) \\
&= \mathrm{div}(fY) - f\mathrm{div}(Y) - (Y.f)n \\
&= -(n-1)Y.f.
\end{aligned}$$

Thus, we have

$$\mathrm{Ric}(X, X) = -(n-1)Xf \leq 0. \tag{11}$$

Consequently $X.f \geq 0$ and $2f^2 + Xf \geq 0$. Therefore by equation (10) we have $\sigma \geq 0$.

$$X\mathrm{div}X + \frac{2}{n}(\mathrm{div}X)^2 = n\sigma.$$

One knows that for any arbitrary vector field X, we have

$$\mathrm{div}(X\mathrm{div}(X)) = X\mathrm{div}X + (\mathrm{div}X)^2.$$

Hence,

$$\mathrm{div}(X\mathrm{div}(X)) = \frac{n-2}{n}(\mathrm{div}X)^2 + n\sigma,$$

and by Stoke's theorem, integrating the above equality on N, we obtain

$$n\int_N \sigma dN + \frac{n-2}{n}\int_N (\mathrm{div}X)^2 dN = 0.$$

Therefore $\sigma = \mathrm{div}X = 0$, so $f = 0$. Now, (8) indicates that X is a parallel vector field (we can also get this result here using Theorem 3.3). □

4. Monotone Vector Field

Monotone vector field was first introduced by Nemeth in Riemannian manifolds [5, 6]. In this section, we summarize the facts on such vector fields and then, we study the relation between 2−conformal vector fields and monotone fields.

Consider a vector field X on a Riemannian manifold (N, g). Let I be an interval of real numbers and $\alpha : I \to N$ be a geodesic, then the function

$$\phi_{X,\alpha} : I \to \mathbb{R}, \qquad \phi_{X,\alpha}(t) \mapsto g(X_{\alpha(t)}, \alpha'(t)),$$

can be defined. The vector field X is said to be monotone, if for every geodesic α on N, the function $\phi_{X,\alpha}$ is monotone. One can see that Killing vector fields are monotone. Also, for a convex real smooth function f on (N, g), the gradient field ∇f is monotone.
If, for every geodesic α on N, $\phi_{X,\alpha}$ is a increasing (decreasing) function, the vector field X is called increasing (decreasing) field. A characterization of monotone vector field is given by the following theorem.

Theorem 4.1. *[5] The below assertions are equivalent for a vector field X on a Riemannian manifold (N, g):*
a) X is increasing (decreasing) vector field on N.
b) For any arbitrary vector field $U \in \mathcal{X}(N)$, $g(\nabla_U X, U) \geq 0 (\leq 0)$.

Also, we have the following characterization result for vector fields with the property that $\phi_{X,\alpha}$ is a constant function.

Theorem 4.2. *[7] With the assumptions as in the previous theorem, the following statements are equivalent*
a) For any geodesic α on N, the function $\phi_{X,\alpha}$ is constant,
b) For any vector field $U \in \mathcal{X}(N)$, $g(\nabla_U X, U) = 0$,
c) X is a Killing vector field.

In the next theorem, we study the relation between 2−conformal vector fields and monotone vector fields.

Theorem 4.3. *Let (N, g) is a Riemannain manifold and X is a 2−conformal vector field on N with $\sigma \leq 0$ which has the property that $\nabla_X X$ is increasing field. If there exists a point $p \in N$ and a tangent vector $0 \neq u \in T_p N$ such that $R(X_p, u, X_p, u) \leq 0$, then*
a) $\nabla_u X = 0$ and $R(X_p, u, X_p, u) = 0$,
b) The vector field $\nabla_X X$ is not strictly increasing,
c) X is 2−Killing.

Proof. Since, X is 2−conformal, one gets

$$R(X, U, X, U) = g(\nabla_U \nabla_X X, U) + g(\nabla_U X, \nabla_U X) - \sigma g(U, U), \quad \forall U \in \mathcal{X}(N).$$

At the point p, we have

$$g(\nabla_u X, \nabla_u X) = R(X_p, u, X_p, u) - g(\nabla_u \nabla_X X, u) + \sigma g(u, u).$$

As $\nabla_X X$ is a increasing vector field, we have

$$g(\nabla_u \nabla_X X, u) geq 0,$$

which implies

$$g(\nabla_u X, \nabla_u X) \leq 0.$$

But, $g(\nabla_u X, \nabla_u X) \geq 0$. Therefore, $g(\nabla_u X, \nabla_u X) = 0$ which shows

$$\nabla_u X = 0, \quad R(X_p, u, X_p, u) = 0, \quad g(\nabla_u \nabla_X X, u) = 0, \quad \sigma = 0.$$

The above equalities lead us to conclude that $\nabla_X X$ is not strictly increasing and, X is a 2−Killing vector field. □

Theorem 4.4. *Let X be a 2−conformal vector field with $\sigma \leq 0$ on a Riemannain manifold (N, g) of negative sectional curvature. Then,*
a) $\nabla_X X$ is a decreasing vector field,
b) If $\nabla_X X$ is non vanishing and the sectional curvature be strictly negative on N, then $\nabla_X X$ is strictly decreasing vector field.

Proof. a) As mentioned before, we have

$$g(\nabla_U \nabla_X X, U) = R(X, U, X, U) - g(\nabla_U X, \nabla_U X) + \sigma g(U, U) \leq 0,$$

for all $U \in \mathcal{X}(N)$. Hence, $\nabla_X X$ is a decreasing vector field.
b) If $\nabla_X X$ is not a strictly decreasing field, then there must be a point $p \in N$ equipped a non-zero tangent vector $u \in T_p N$, such that

$$g(\nabla_u \nabla_X X, u) = 0.$$

Therefore, we get

$$R(X_p, u, X_p, u) = 0, \quad \nabla_u X = 0, \quad \sigma = 0.$$

By hypothesis, the sectional curvature of N is strictly negative, so the equality $R(X_p, u, X_p, u) = 0$ shows that u and X_p are dependent vectors, therefore

$$\exists \lambda \in \mathbb{R}, \quad X_p = \lambda u.$$

Note that we have shown that $\nabla_u X = 0$, so one can deduce $\nabla_{X_p} X = 0$, which is contrast the fact that $\nabla_X X$ is not vanishing on N. As a result, $\nabla_X X$ is a strictly decreasing vector field. □

5. Examples

In this section, we show that there are Riemannian manifolds which admit 2−conformal vector fields as well as there are spaces which admit no non-trivial 2−conformal vector field. By trivial we mean that the vector field not to be Killing.

Example 5.1. *A vector field $X = X^i \partial_i$ on the Euclidean manifold $(\mathbb{R}^n, g_{can})$ is a 2−conformal vector field with potential σ, if and only if for all $i, j = 1 \cdots n$,*

$$\partial_i X^l \partial_l X^j + \partial_j X^l \partial_l X^i + X^l \partial_l \Big(\partial_i X^j + \partial_j X^i \Big) + 2 \sum_{k=1}^{n} (\partial_i X^k \partial_j X^k) = \sigma.$$

For $n = 1$ and any differentiable function f, the vector field $0 \neq X = f \partial_t$ is a 2−conformal vector field with potential $\sigma = 2f'^2$ on $\mathbb{R}$ if it satisfies

$$f f'' = 0,$$

where f' denotes the regular derivative with respect to t. The solution of this equation is $f(t) = at + b$, for some constants a, b.

An example of 2−conformal Killing vector field which is not a conformal vector field, is $X = (x^k)^{\frac{1}{3}} \partial_k$ on $(\mathbb{R}^*)^n$

Example 5.2. *In this example, we show that there is no non-zero 2−conformal left invariant vector field on simply connected two-step nilpotent Lie group of dimension five.*
Let N is a simply connected two-step nilpotent Lie group of dimension five and $\mathfrak{n}$ is its Lie algebra. In order to examine 2−conformal vector fields on these spaces we recall the classification of these spaces which is given in [4].

Case 1: Lie algebras with one dimensional center: *In this case there exists an orthonormal basis* $\{e_1, e_2, e_3, e_4, e_5\}$ *of* $\mathfrak{n}$ *such that*

$$[e_1, e_2] = \lambda e_5, \quad [e_3, e_4] = \mu e_5, \tag{12}$$

where, $\{e_5\}$ *is a basis for the center of* $\mathfrak{n}$*, and* $\lambda \geq \mu > 0$*. Also, it is considered that the other commutators are zero. Direct computations show that a* $2-$*conformal vector field in this case is of the form* $X = ae_5$*, for some constant* a*.*

Case 2: Lie algebras with two dimensional center: *In this type,* $\mathfrak{n}$ *admits an orthonormal basis* $\{e_1, e_2, e_3, e_4, e_5\}$ *such that*

$$[e_1, e_2] = \lambda e_4, \quad [e_1, e_3] = \mu e_5, \tag{13}$$

where, $\{e_4, e_5\}$ *is a basis for the center of* $\mathfrak{n}$*, the other commutators are zero and* $\lambda \geq \mu > 0$*.*
Direct computations show that a $2-$*conformal vector field in this case is of the form* $X = ae_4 + be_5$*, for some constants* a, b*.*

Case 3: Lie algebras with three dimensional center: *The Lie algebra structure of this case is as follows: The Lie algebra,* $\mathfrak{n}$ *admits an orthonormal basis* $\{e_1, e_2, e_3, e_4, e_5\}$ *such that for* $\lambda > 0$

$$[e_1, e_2] = \lambda e_3, \tag{14}$$

where, $\{e_3, e_4, e_5\}$ *is a basis for the center of* $\mathfrak{n}$*, the other commutators are zero.*
Direct computations indicate that a $2-$*conformal vector field in this case is of the form* $X = ae_3 + be_4 + ce_5$*, for some constants* a, b, c*.*

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