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A characterization of essential pseudospectra involving polynomially compact operators

Bilel Elgabeur[a,]

[a]*University of Sfax, Faculty of Sciences of Sfax, Sfax, Tunisia*

Abstract. In this article, we study the essential pseudospectra with the class of polynomially compact operators, which is a generalization of the class of compact operators. We present some new results in essential pseudospectra for closed linear operators in Banach space with polynomially compact operators. Furthermore, we apply the obtained results to discuss the incidence of some perturbation results on left(resp. right) Weyl essential pseudospectra and left(resp. right) Fredholm essential pseudospectra. This paper aims to describe the essential pseudospectra of a sum of two bounded linear operators. As a final step, we can use the results obtained to determine the pseudo-left (right)-Fredholm spectra of 2×2 block operators matrices by measuring polynomially compact operators.

1. Introduction

Eigenvalue problems hold significant importance across numerous scientific and engineering disciplines. The primary goals when tackling these problems are to extract and localize eigenvalues. However, traditional spectral analysis falls short in achieving both objectives, as it can only identify eigenvalues without localizing them. As a solution, researchers have introduced alternative methods such as the pseudo spectrum, first proposed by Varah [27]. The pseudo spectrum has found widespread application in numerous areas of mathematical physics, including engineering (e.g., electrical engineering), aeronautics, ecology, and chemistry(see [8, 14, 26]). In engineering, for instance, eigenvalues can dictate the precision of a national power grid or an amplifier's frequency response. In aeronautics, they can reveal if airflow over an airplane wing is laminar or turbulent. In ecology, eigenvalues can determine the stability of a food web's equilibrium. In chemistry, they can establish energy states in a stable hydrogen atom. In summary, the pseudo spectrum concept has demonstrated its value in addressing eigenvalue problems, allowing researchers to accurately extract and localize eigenvalues, thus contributing to significant progress in diverse areas of science and engineering.

Inspired by the notion of pseudospectra F. Abdmouleh et al. in [2] defined the notion of pseudo Browder essential spectrum for densely closed linear operators in the Banach space. Also, F. Abdmouleh and B. Elgabeur in their works [3, 4], they declared the new concept of pseudo left (right)-Fredholm and pseudo

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Email address: bilelelgabeur@gmail.com (Bilel Elgabeur)

left(right)-Browder operator, as well as their spectra of bounded linear operators on a Banach space. Among the main results that have been studied in this work is the stability under Riesz operator perturbations of these pseudo essential spectra in Banach space. Moreover, we describe the pseudo left (right)-Fredholm and pseudo left (right)-Browder essential spectra of the sum of two bounded linear operators. Ammar and Jeribi in their works [5, 6], aimed to extend these results for the essential pseudo-spectra of bounded linear operators on a Banach space and give the definitions of pseudo-Fredholm operator and their essential pseudo spectrum.

In this paper we will continue to study these essential pseudospectra in Banach space with a big space of operators called polynomially compact operators, which are considered to be generalizations of some well-known classes Fredholm perturbations, polynomially Fredholm perturbations, polynomially strictly singular operators and polynomially compact operators. This class of operators considerably attracted the attention of many authors in order to give some useful results in spectral theory. The reader may find the following references useful: [10, 13, 15, 16] . The first goal of this paper is to generalize the results of stability of essential pseudospectra obtained in [2–6] by the polynomially compact operators perturbations for closed densely defined linear operators. The second aim of this work, is to describe the essential pseudospectrum of the sum of two bounded linear operator with the new concept of polynomially compact operator.

Let us outline the content of this paper:

The aim of Section 2 In this section, we will review some of the basic definitions and notations that relate to Fredholm operators and their essential spectra. Furthermore, we investigate polynomially compact operators with interesting results.

Throughout, Section 3 Specifically, we are interested in stability results and in a novel characterization of left Weyl (resp. right Weyl) and left Fredholm (resp. right Fredholm) essential pseudospectra in the class of polynomially compact operators.

In section 4, we establish a principal result concerning some essential pseudospectra of the sum of two bounded linear operators motivated by the notion of polynomially compact perturbations.

Finally, As an extension of the results obtained earlier, pseudo-left (right)-Fredholm spectra are defined for block operators of 2×2 by measure of polynomially compact operators .

2. Notations and definitions

Let X and Y be two Banach spaces. By an operator A from X into Y we mean a linear operator with domain $\mathcal{D}(A) \subseteq X$ and range contained in Y. We denote by $C(X,Y)$ (resp., $\mathcal{L}(X,Y)$) the set of all closed, densely defined (resp., bounded) linear operators from X to Y. The subset of all compact operators of $\mathcal{L}(X,Y)$ is designated by $\mathcal{K}(X,Y)$. If $A \in C(X,Y)$, we write $N(A) \subset X$ and $R(A) \subset Y$ for the null space and the range of A. We set $\alpha(A) := \dim N(A)$ and $\beta(A) := \operatorname{codim} R(A)$. Let $A \in C(X,Y)$ with closed range. Then A is a Φ_+-operator ($A \in \Phi_+(X,Y)$) if $\alpha(A) < \infty$, and then A is a Φ_--operator ($A \in \Phi_-(X,Y)$) if $\beta(A) < \infty$.$\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$ is the class of Fredholm operators while $\Phi_\pm(X,Y)$ denotes the set $\Phi_\pm(X,Y) = \Phi_+(X,Y) \cup \Phi_-(X,Y)$. For $A \in \Phi(X,Y)$, the index of A is defined by $i(A) = \alpha(A) - \beta(A)$. If $X = Y$, then $\mathcal{L}(X,Y), \mathcal{K}(X,Y), C(X,Y), \Phi_+(X,Y), \Phi_\pm(X,Y)$ and $\Phi(X,Y)$ are replaced, respectively, by $\mathcal{L}(X), \mathcal{K}(X), C(X), \Phi_+(X), \Phi_\pm(X)$ and $\Phi(X)$. Let $A \in C(X)$, the spectrum of A will be denoted by $\sigma(A)$. The resolvent set of $A, \rho(A)$, is the complement of $\sigma(A)$ in the complex plane. A complex number λ is in $\Phi_{+A}, \Phi_{-A}, \Phi_{\pm A}$ or Φ_A if $\lambda - A$ is in $\Phi_+(X), \Phi_-(X), \Phi_\pm(X)$ or $\Phi(X)$, respectively. Let $F \in \mathcal{L}(X,Y), F$ is called a Fredholm perturbation if $U+F \in \Phi(X,Y)$ whenever $U \in \Phi(X,Y)$. F is called an upper (resp., lower) Fredholm perturbation if $U + F \in \Phi_+(X,Y)$ (resp., $U + F \in \Phi_-(X,Y)$) whenever $U \in \Phi_+(X,Y)$ (resp., $U \in \Phi_-(X,Y)$). The set of Weyl operators is defined as $\mathcal{W}(X,Y) = \{A \in \Phi(X,Y) : i(A) = 0\}$. Sets of left and right Fredholm operators, respectively, are defined as:

$$\Phi_l(X) := \{A \in \mathcal{L}(X) : R(A) \text{ is a closed and complemented subspace of } X \text{ and } \alpha(A) < \infty\}.$$

$$\Phi_r(X) := \{A \in \mathcal{L}(X) : N(A) \text{ is a closed and complemented subspace of } X \text{ and } \beta(A) < \infty\}.$$

An operator $A \in \mathcal{L}(X)$ is left (right) Weyl if A is left (right) Fredholm operator and $i(A) \leq 0 (i(A) \geq 0)$. We use $\mathcal{W}_l(X)(\mathcal{W}_r(X))$ to denote the set of all left(right) Weyl operators. It is Known that the sets $\Phi_l(X)$ and $\Phi_r(X)$ are open satisfying the following inclusions:

$$\Phi(X) \subset \mathcal{W}_l(X) \subset \Phi_l(X) \text{ and } \Phi(X) \subset \mathcal{W}_r(X) \subset \Phi_r(X).$$

The sets of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X, Y), \mathcal{F}_+(X, Y)$ and $\mathcal{F}_-(X, Y)$, respectively. In general, we have

$$\mathcal{K}(X, Y) \subseteq \mathcal{F}_+(X, Y) \subseteq \mathcal{F}(X, Y)$$
$$\mathcal{K}(X, Y) \subseteq \mathcal{F}_-(X, Y) \subseteq \mathcal{F}(X, Y).$$

If $X = Y$ we write $\mathcal{F}(X), \mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ for $\mathcal{F}(X, X), \mathcal{F}_+(X, X)$ and $\mathcal{F}_-(X, X)$, respectively. Let $\Phi^b(X, Y), \Phi^b_+(X, Y)$ and $\Phi^b_-(X, Y)$ denote the sets $\Phi(X, Y) \cap \mathcal{L}(X, Y), \Phi_+(X, Y) \cap \mathcal{L}(X, Y)$ and $\Phi_-(X, Y) \cap \mathcal{L}(X, Y)$, respectively. If in Definition 1.1 we replace $\Phi(X, Y), \Phi_+(X, Y)$ and $\Phi_-(X, Y)$ by $\Phi^b(X, Y), \Phi^b_+(X, Y)$ and $\Phi^b_-(X, Y)$ we obtain the sets $\mathcal{F}^b(X, Y), \mathcal{F}^b_+(X, Y)$ and $\mathcal{F}^b_-(X, Y)$. These classes of operators were introduced and investigated in [6]. In particular, it is shown that $\mathcal{F}^b(X, Y)$ is a closed subset of $\mathcal{L}(X, Y)$ and $\mathcal{F}^b(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$. In general we have

$$\mathcal{K}(X, Y) \subseteq \mathcal{F}^b_+(X, Y) \subseteq \mathcal{F}^b(X, Y)$$
$$\mathcal{K}(X, Y) \subseteq \mathcal{F}^b_-(X, Y) \subseteq \mathcal{F}^b(X, Y)$$

Let $A \in \mathcal{C}(X)$. It follows from the closeness of A that $\mathcal{D}(A)$ endowed with the graph norm $\|\cdot\|_A$ ($\|x\|_A = \|x\| + \|Ax\|$) is a Banach space denoted by X_A. Clearly, for $x \in \mathcal{D}(A)$ we have $\|Ax\| \leqslant \|x\|_A$, so $A \in \mathcal{L}(X_A, X)$. Furthermore, we have the obvious relations

$$\begin{cases} \alpha(\hat{A}) = \alpha(A), \quad \beta(\hat{A}) = \beta(A), \quad R(\hat{A}) = R(A) \\ \alpha(\hat{A} + \hat{B}) = \alpha(A + B), \\ \beta(\hat{A} + \hat{B}) = \beta(A + B) \text{ and } R(\hat{A} + \hat{B}) = R(A + B) \end{cases} \tag{1}$$

In this paper we are concerned with the following essential spectra of $A \in C(X)$:

$\sigma_e(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi(X)\}$: the Fredholm spectrum of A.

$\sigma^l_e(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi_l(X)\}$: the left Fredholm spectrum of A.

$\sigma^r_e(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi_r(X)\}$: the right Fredholm spectrum of A.

$\sigma_w(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \mathcal{W}(X)\}$: the Weyl spectrum of A.

$\sigma^l_w(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \mathcal{W}_l(X)\}$: the left Weyl spectrum of A.

$\sigma^r_w(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \mathcal{W}_r(X)\}$: the right Weyl spectrum of A.

$\sigma_{\text{eap}}(A) := \mathrm{C} \backslash \rho_{\text{eap}}(A)$: the essential approximate point spectrum of A.

$\sigma_{e\delta}(A) := \mathrm{C} \backslash \rho_{e\delta}(T)$: the essential defect spectrum of A.

where

$$\rho_{\text{eap}}(A) := \{\lambda \in \mathrm{C} \text{ such that } \lambda - A \in \Phi_+(X) \text{ and } i(\lambda - A) \leq 0\},$$

and

$$\rho_{e\delta}(A) := \{\lambda \in \mathrm{C} \text{ such that } \lambda - A \in \Phi_-(X) \text{ and } i(\lambda - A) \geq 0\}$$

The definition of pseudo spectrum of a closed densely linear operator A for every $\varepsilon > 0$ is given by:

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{\lambda \in \mathbb{C} : \left\|(\lambda - A)^{-1}\right\| > \frac{1}{\varepsilon}\right\}. \tag{2}$$

By convention, we write $\left\|(\lambda - A)^{-1}\right\| = \infty$ if $(\lambda - A)^{-1}$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma(A)$. In [8], Davies defined another equivalent of the pseudo spectrum, one that is in terms of perturbations of the spectrum. In fact for $A \in C(X)$, we have

$$\sigma_{\varepsilon}(A) := \bigcup_{\|D\|<\varepsilon} \sigma(A + D). \tag{3}$$

Inspired by the notion of pseudospectra, Ammar and Jeribi in their works [5, 6], aimed to extend these results for the essential pseudo-spectra of bounded linear operators on a Banach space and give the definitions of pseudo-Fredholm operator as follows: for $A \in \mathcal{L}(X)$ and for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have A is called a pseudo-upper (resp. lower) semi-Fredholm operator if $A + D$ is an upper (resp. lower) semi-Fredholm operator and it is called a pseudo semi-Fredholm operator if $A + D$ is a semi-Fredholm operator. A is called a pseudo-Fredholm operator if $A + D$ is a Fredholm operator. They are noted by $\Phi^{\varepsilon}(X)$ the set of pseudo-Fredholm operators, by $\Phi^{\varepsilon}_{\pm}(X)$ the set of pseudo-semi-Fredholm operator and by $\Phi^{\varepsilon}_{+}(X)$ (resp. $\Phi^{\varepsilon}_{-}(X)$) the set of pseudo-upper semi-Fredholm (resp. lower semi-Fredholm) operator. A complex number λ is in $\Phi^{\varepsilon}_{\pm A}, \Phi^{\varepsilon}_{+A}, \Phi^{\varepsilon}_{-A}$ or Φ^{ε}_{A} if $\lambda - A$ is in $\Phi^{\varepsilon}_{\pm}(X), \Phi^{\varepsilon}_{+}(X), \Phi^{\varepsilon}_{-}(X)$ or $\Phi^{\varepsilon}(X)$.
F. Abdmouleh and B. Elgabeur in [4] defining the concept of pseudo left (resp. right)-Fredholm, for $A \in \mathcal{L}(X)$ and for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have A is called a pseudo left (resp. right) Fredholm operator if $A + D$ is an left (resp. right) Fredholm operator they are denoted by $\Phi^{\varepsilon}_{l}(X)$ (resp. $\Phi^{\varepsilon}_{r}(X)$).

In this paper we are concerned with the following essential pseudospectra of $A \in C(X)$:

$$\begin{aligned}
\sigma_{e1,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^{\varepsilon}_{+}(X)\} = \mathbb{C}\backslash\Phi^{\varepsilon}_{+A},\\
\sigma_{e2,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^{\varepsilon}_{-}(X)\} = \mathbb{C}\backslash\Phi^{\varepsilon}_{-A},\\
\sigma_{e3,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^{\varepsilon}_{\pm}(X)\} = \mathbb{C}\backslash\Phi^{\varepsilon}_{\pm A},\\
\sigma_{e,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^{\varepsilon}(X)\} = \mathbb{C}\backslash\Phi^{\varepsilon}_{A},\\
\sigma_{eap,\varepsilon}(A) &:= \sigma_{e1,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) > 0, \forall \|D\| < \varepsilon\},\\
\sigma_{e\delta,\varepsilon}(A) &:= \sigma_{e2,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) < 0, \forall \|D\| < \varepsilon\},\\
\sigma^{l}_{e,\varepsilon}(A) &:= \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^{\varepsilon}_{l}(X)\right\},\\
\sigma^{r}_{e,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^{\varepsilon}_{r}(X)\},\\
\sigma^{l}_{w,\varepsilon}(A) &:= \sigma^{l}_{e,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) > 0, \forall \|D\| < \varepsilon\},\\
\sigma^{r}_{w,\varepsilon}(A) &:= \sigma^{r}_{e,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) < 0, \forall \|D\| < \varepsilon\},\\
\sigma_{w,\varepsilon}(A) &:= \sigma_{e,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) = 0, \forall \|D\| < \varepsilon\}.
\end{aligned}$$

Note that if ε tends to 0, we recover the usual definition of the essential spectra of a closed operator A. The subsets σ_{e1} and σ_{e2} are the Gustafson and Weidmann essential spectra [12], σ_{e3} is the Kato essential spectrum,[19] σ_{e} is the Wolf essential spectrum[12], σ_{e5} is the Schechter essential spectrum[24], σ_{eap} is the essential approximate point spectrum [22], $\sigma_{e\delta}$ is the essential defect spectrum [23], $\sigma^{l}_{e}(A)$(resp.$\sigma^{r}_{e}(A)$) is the left (resp. right) Fredholm essential spectra and $\sigma^{l}_{w}(A)$(resp.$\sigma^{r}_{w}(A)$) is the left (resp. right) Weyl essential spectra [11, 28, 29].
As a concept, pseudospectra and essential pseudospectra are interesting because they offer more information than spectra, especially about transients rather than just asymptotic behavior. Moreover, they perform more efficiently than spectra in terms of convergence and approximation. These include the existence of approximate eigenvalues far from the spectrum and the instability of the spectrum even under small perturbations. Various applications of pseudospectra and essential pseudospectra have been developed as a result of the analysis of pseudospectra and essential pseudospectra.

We now list some of the known facts about left and right Fredholm operators in Banach space which will be used in the sequel.

Proposition 2.1. [18, propositon 2.3] *Let X, Y and Z be three Banach spaces.*

(i) *If* $A \in \Phi^b(Y,Z)$ *and* $T \in \Phi_l^b(X,Y)$ *(resp.* $T \in \Phi_r^b(X,Y)$*), then* $AT \in \Phi_l^b(X,Z)$ *(resp.* $AT \in \Phi_r^b(X,Z)$*).*

(ii) *If* $A \in \Phi^b(Y,Z)$ *and* $T \in \Phi_l^b(X,Y)$ *(resp.* $T \in \Phi_r^b(X,Y)$*), then* $TA \in \Phi_l^b(X,Z)$ *(resp.* $TA \in \Phi_r^b(X,Z)$*).* ◊

Theorem 2.2. [21, 24] *Let* X, Y *and* Z *be three Banach spaces,* $A \in \mathcal{L}(Y,Z)$ *and* $T \in \mathcal{L}(X,Y)$*.*
(i) *If* $A \in \Phi^b(Y,Z)$ *and* $T \in \Phi^b(X,Y)$*, then* $AT \in \Phi^b(X,Z)$ *and* $i(AT) = i(A) + i(T)$*.*

(ii) *If* $X = Y = Z$*,* $AT \in \Phi^b(X)$ *and* $TA \in \Phi^b(X)$*, then* $A \in \Phi^b(X)$ *and* $T \in \Phi^b(X)$*.* ◊

Lemma 2.3. [11, Theorem 2.3] *Let* $A \in \mathcal{L}(X)$*, then*

(i) $A \in \Phi_l^b(X)$ *if and only if, there exist* $A_l \in \mathcal{L}(X)$ *and* $K \in \mathcal{K}(X)$ *such that* $A_lA = I - K$*.*

(ii) $A \in \Phi_r^b(X)$ *if and only if, there exist* $A_r \in \mathcal{L}(X)$ *and* $K \in \mathcal{K}(X)$ *such that* $AA_r = I - K$*.* ◊

Lemma 2.4. [11, Theorem 2.7] *Let* $A \in \mathcal{L}(X)$*.*

If $A \in \Phi_l^b(X)$*(resp.*$\Phi_r^b(X)$*) and* $K \in \mathcal{K}(X)$*, then* $A + K \in \Phi_l^b(X)$*(resp.*$\Phi_r^b(X)$*) and* $i(A + K) = i(A)$*.* ◊

Lemma 2.5. [11, Theorem 2.5] *Let* $A, B \in \mathcal{L}(X)$*,*

If $A \in \Phi_l^b(X)$*(resp.*$\Phi_r^b(X)$*) and* $B \in \Phi_l^b(X)$*(resp.*$\Phi_r^b(X)$*) then* $AB \in \Phi_l^b(X)$*(resp.*$\Phi_r^b(X)$*) and*

$$i(A + B) = i(A) + i(B). \quad ◊$$

We close with the following Lemma.

Lemma 2.6. [7, Lemma 3.4] *Let* $A \in \mathcal{L}(X)$*.*

(i) *If* $AB \in \Phi_l^b(X)$ *then* $B \in \Phi_l^b(X)$*.*

(ii) *If* $AB \in \Phi_r^b(X)$ *then* $A \in \Phi_r^b(X)$*.*

Definition 2.7. *Let* X *be a Banach space.*

(i) *An operator* $A \in \mathcal{L}(X)$ *is said to have a left Fredholm inverse if there exists* $A_l \in \mathcal{L}(X)$ *such that* $I - A_lA \in \mathcal{K}(X)$*.*

(ii) *An operator* $A \in \mathcal{L}(X)$ *is said to have a right Fredholm inverse if there exists* $A_r \in \mathcal{L}(X)$ *such that* $I - AA_r \in \mathcal{K}(X)$*.*
◊

We know by the classical theory of Fredholm operators, see for example [19], that A belong to $\Phi(X)$ if it possesses a left, right or two-sided Fredholm inverse, respectively.

We define these sets $\mathcal{I}nvF_A^l(X)$ and $\mathcal{I}nvF_A^r(X)$ by:

$$\mathcal{I}nv_{A,l}^F(X) := \{A_l \in \mathcal{L}(X) : A_l \text{ is a left Fredholm inverse of } A\},$$

$$\mathcal{I}nv_{A,r}^F(X) := \{A_r \in \mathcal{L}(X) : A_r \text{ is a right Fredholm inverse of } A\}.$$

Definition 2.8. *An minimal polynomial* P *is the unitary polynomial of smaller degree which cancels an endomorphism, that is to say a linear application of a vector space in itself.*

We say that $A \in \mathcal{L}(X)$ is polynomially compact if there exists a nonzero complex polynomial $p(.)$ such that the operator $p(A) \in \mathcal{K}(X)$. The set of polynomially operators will be denoted by $\mathcal{P}_{\mathcal{K}}(X)$.

If A belongs $\mathcal{P}_{\mathcal{K}}(X)$, then there exists a nonzero polynomial $p(.)$ such that $p(A) \in \mathcal{K}(X)$.

In the following, $\mathcal{E}_{\mathcal{P}\mathcal{K}}(X)$ will denote the subset of $\mathcal{P}_{\mathcal{K}}(X)$ defined by:

$$\mathcal{E}_{\mathcal{P}\mathcal{K}}(X) := \{A \in \mathcal{P}_{\mathcal{K}}(X) \text{ such that the minimal polynomial } p(.) \text{ of } A \text{ satisfies } p(-1) \neq 0\}.$$

Let us recall the following results which are fundamental for the proofs of the main results.

Proposition 2.9. [16, Theorem 2.1]
If $F \in \mathcal{E}_{\mathcal{P}\mathcal{K}}(X)$*, then* $I + F \in \Phi(X)$ *and* $i(I + F) = 0$*.* ◊

3. Stability of essential pseudospectra by means of polynomially compact perturbations operators

The following theorem provides a practical criterion for the stability of some essential pseudospectra for perturbed linear operators.

Theorem 3.1. *Let $\varepsilon > 0$ and consider $A, B \in \mathcal{C}(X)$. Assume that there are A_0, B_0, K_1 and $K_2 \in \mathcal{L}(X)$ and there exists a nonzero polynomial $p(.)$ such that $p(-1) \neq 0$ satisfies:*

$$K_1,\ K_2 \in \mathcal{E}_{\mathcal{PK}}(X), \tag{4}$$

$$AA_0 = I - p(K_1), \tag{5}$$

$$BB_0 = I - p(K_2). \tag{6}$$

(*i*) *If $0 \in \Phi_{+A} \cap \Phi_{+B}, A_0 - B_0 \in \mathcal{F}_+(X)$ and $i(A) = i(B)$ then*

$$\sigma_{\mathrm{eap},\varepsilon}(A) = \sigma_{\mathrm{eap},\varepsilon}(B). \tag{7}$$

(*ii*) *If $0 \in \Phi_{-A} \cap \Phi_{-B}, A_0 - B_0 \in \mathcal{F}_-(X)$ and $i(A) = i(B)$ then*

$$\sigma_{e\delta,\varepsilon}(A) = \sigma_{e\delta,\varepsilon}(B). \tag{8}$$

(*iii*) *If $0 \in \Phi_A \cap \Phi_B$ and $A_0 - B_0 \in \mathcal{F}(X)$ then*

$$\sigma_{e,\varepsilon}(A) = \sigma_{e,\varepsilon}(B). \tag{9}$$

Furthermore, if $i(A) = i(B)$, then

$$\sigma_{w,\varepsilon}(A) = \sigma_{w,\varepsilon}(B). \tag{10}$$

Proof. Let λ be a complex number, Equations (5) and (6) imply

$$(\lambda - A - D)A_0 - (\lambda - B - D)B_0 = p(K_1) - p(K_2) + (\lambda - D)(A_0 - B_0). \tag{11}$$

(*i*) Let $\lambda \notin \sigma_{\mathrm{eap},\varepsilon}(B)$, then $\lambda \in \Phi^{\varepsilon}_{+B}$ such that $i(\lambda - B - D) \leq 0$ for all $\|D\| < \varepsilon$. Since $B + D$ is closed and $\mathcal{D}(B + D) = \mathcal{D}(B)$ endowed with the graph norm is a Banach space denoted by X_{B+D}. We can regard $B + D$ an operator from X_{B+D} into X. This will be denoted by $\widehat{B+D}$. Using Equation (1) we can show that

$$\lambda - \widehat{B+D} \in \Phi^b_+(X_B, X) \text{ and } i(\lambda - \widehat{B+D}) \leq 0.$$

Moreover, since $K_2 \in \mathcal{E}_{\mathcal{PK}}(X)$, applying Proposition 2.9, we obtain $I - p(K_2) \in \Phi(X)$.
Applying [[25], Theorem 2.7, p.171] and Equation (6), we get $B_0 \in \Phi^b_+(X, X_B)$.

That is $(\lambda - \widehat{B+D})B_0 \in \Phi^b_+(X)$. By taking into account Equation (11) we have:

$$(\lambda - A - D)A_0 = p(K_1) - p(K_2) + (\lambda - D)(A_0 - B_0) + (\lambda - B - D)B_0.$$

Since $A_0 - B_0 \in \mathcal{F}_+(X)$, then $(\lambda - D)(A_0 - B_0) \in \mathcal{F}_+(X)$ and by [[17], Lemma 2.1], we obtain

$$(\lambda - D)(A_0 - B_0) + (\lambda - B - D)B_0 \in \Phi^b_+(X). \tag{12}$$

Remembering that $p(K_1), p(K_2) \in \mathcal{K}(X)$ and by using [24], we obtain

$$p(K_1) - p(K_2) + (\lambda - D)(A_0 - B_0) + (\lambda - B - D)B_0 \in \Phi^b_+(X). \tag{13}$$

Therefore by Equations (11), (12) and (13) asserts that $(\lambda - \widehat{A+D})A_0 \in \Phi_+^b(X)$, and

$$i((\lambda - \widehat{A+D})A_0) = i((\lambda - \widehat{B+D})B_0). \tag{14}$$

A similar reasoning as before combining Equations (1) and (5), Proposition 2.9 and [[25], Corollary 1.6, p. 166], [[25], Theorem 2.6, p. 170] shows that $A_0 \in \Phi_+^b(X, X_A)$ where $X_A := (\mathcal{D}(A), \|.\|_A)$. By [[25], Theorem 1.4, p. 108] one sees that

$$A_0 S = I - F \ \ on\ X_A, \tag{15}$$

where $S \in \mathcal{L}(X_A, X)$ and $F \in \mathcal{K}(X_A)$, by Equation (6) we have

$$(\lambda - \widehat{B+D})A_0 S = (\lambda - \widehat{A+D}) - (\lambda - \widehat{A+D})F. \tag{16}$$

Combining the fact that $S \in \Phi^b(X_A, X)$ with [[25], Theorem 6.6, p. 129], we show that $(\lambda - \widehat{A+D})A_0 S \in \Phi_+^b(X_A, X)$. Following [[25], Theorem 6.3, p. 128], we derive $(\lambda - \widehat{A+D}) \in \Phi_+^b(X_A, X)$. Thus, Equation (1) asserts that

$$(\lambda - A - D) \in \Phi_+(X), \forall D \in \mathcal{L}(X), \|D\| < \varepsilon. \tag{17}$$

On the other hand, the assumptions $p(K_1), p(K_2) \in \mathcal{K}(X)$, Equations (5), (6) and Proposition 1, [[25], Theorem 2.3 , p. 111] reveals that

$$i(A) + i(A_0) = i(I - p(K_1)) = 0 \ \text{ and } \ i(B) + i(B_0) = i(I - p(K_2)) = 0,$$

since $i(A) = i(B)$. That is $i(A_0) = i(B_0)$.

Using Equation (14) and [[21], Theorem 2.3, p. 111], we can write

$$i(\lambda - A - D) + i(A_0) = i(\lambda - B - D) + i(B_0).$$

Therefore

$$i(\lambda - A - D) \leq 0, \forall D \in \mathcal{L}(X), \|D\| < \varepsilon. \tag{18}$$

Using Equations (17) and (18), we conclude that

$$\lambda \notin \sigma_{eap,\varepsilon}(A).$$

Therefore we prove the inclusion

$$\sigma_{eap,\varepsilon}(A) \subset \sigma_{eap,\varepsilon}(B).$$

The opposite inclusion follows from symmetry and we obtain Equation (7).

(*ii*) The proof of Equation (8) may be checked in a similar way to that in (*i*). It suffices to replace $\sigma_{eap,\varepsilon}(.)$, $\Phi_+(.)$, $i(.) \leq 0$, [[25], Theorem 6.6, p. 129], [[25], Theorem 6.3, p. 128] by $\sigma_{e\delta,\varepsilon}(.)$, $\Phi_-(.)$, $i(.) \geq 0$, [[21], Theorem 5 (i), p. 150], [[25], Theorem 6.7, p. 129] respectively. The details are therefore omitted.

(*iii*) If $\lambda \notin \sigma_{e,\varepsilon}(B)$, then $\lambda - B - D \in \Phi(X)$ for all $\|D\| < \varepsilon$. Since B is closed, its domain $\mathcal{D}(B)$ becomes a Banach space X_B for the graph norm $\|.\|_B$. The use of Equation (1) leads to $\lambda - \widehat{B+D} \in \Phi^b(X_B, X)$. Moreover, Equation (6), Proposition 1 and [[25], Theorem 5.13] reveals that $B_0 \in \Phi^b(X, X_B)$ and consequently $(\lambda - \widehat{B+D})B_0 \in \Phi^b(X)$. Following with the assumption, Equation (11), [[17], Lemma2.1] and [24] leads to estimate $(\lambda - \widehat{A+D})A_0 \in \Phi^b(X)$ with

$$i[(\lambda - \widehat{A+D})A_0] = i[(\lambda - \widehat{B+D})B_0]. \tag{19}$$

Since $A \in \mathcal{C}(X)$, proceeding as above, Equation (5) implies that $A_0 \in \Phi^b(X, X_A)$. By [[25], Theorem 5.4] we can write

$$A_0 S = I - F \ \ on\ X_A, \tag{20}$$

where $S \in \mathcal{L}(X_A, X)$ and $F \in \mathcal{F}(X_A)$. Taking into account Equation (20) we infer that

$$(\lambda - \widehat{A + D})A_0 S = (\lambda - \widehat{A + D}) - (\lambda - \widehat{A + D})F. \tag{21}$$

Therefore, since $S \in \Phi^b(X_A, X)$, the use of [25, Theorem 6.6] amounts to

$$(\lambda - \widehat{A + D})A_0 S \in \Phi^b(X_A, X).$$

Applying [[25],Theorem 6.3], we prove that $(\lambda - \widehat{A + D}) \in \Phi^b(X_A, X)$ and consequently

$$(\lambda - A - D) \in \Phi(X), \forall D \in \mathcal{L}(X), \|D\| < \varepsilon.$$

Thus $\lambda \notin \sigma_{e,\varepsilon}(A)$. This implies that $\sigma_{e,\varepsilon}(A) \subset \sigma_{e,\varepsilon}(B)$. Conversely, if $\lambda \notin \sigma_{e,\varepsilon}(A)$, we can easily derive the opposite inclusion.

Now, we prove Equation (10). If $\lambda \notin \sigma_{w,\varepsilon}(B)$, then, $\lambda \in \Phi_B^{\varepsilon}$ and $i(\lambda - B - D) = 0$, for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$. On the other hand, since $p(K_1), p(K_2) \in \mathcal{K}(X)$ and $i(A) = i(B) = 0$, using the Atkinson theorem, we obtain $i(A_0) = i(B_0) = 0$. This together with Equation (19) gives $i(\lambda - \widehat{A + D}) = i(\lambda - \widehat{B + D})$. Consequently $i(\lambda - A - D) = 0$, for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$. Hence $\lambda \notin \sigma_{w,\varepsilon}(A)$, which proves the inclusion $\sigma_{w,\varepsilon}(A) \subset \sigma_{w,\varepsilon}(B)$. The opposite inclusion follows by symmetry. □

In the following theorems we give some perturbation results of the pseudo left, pseudo right Fredholm and pseudo left, pseudo right Weyl spectra for bounded linear operator in Banach space.

Theorem 3.2. *Let A and B be two operators in $\mathcal{C}(X)$ and $\lambda \in \mathbb{C}$. The following statements hold:*

(i) Assume that $\lambda - A \in \Phi_l(X)$ and for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, there exists $A_l \in \mathcal{I}nv^F_{\lambda-A-D,l}(X)$ such that $BA_l \in \mathcal{E}_{\mathcal{PK}}(X)$, then

$$\sigma^l_{e,\varepsilon}(A + B) \subseteq \sigma^l_{e,\varepsilon}(A).$$

(ii) Assume that $\lambda - A \in \Phi_r(X)$ and for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, there exists $A_r \in \mathcal{I}nv^F_{\lambda-A-D,r}(X)$ such that $A_r B \in \mathcal{E}_{\mathcal{PK}}(X)$, then

$$\sigma^r_{e,\varepsilon}(A + B) \subseteq \sigma^r_{e,\varepsilon}(A).$$

Proof. (i) Let $\lambda \notin \sigma^{\text{left}}_{\text{e},\varepsilon}(A)$, $\lambda - A - D \in \Phi^{\varepsilon}_l(X)$ for all $\|D\| < \varepsilon$. As A_l is a left Fredholm inverse of $\lambda - A - D$, then by Lemma 2.3 there exists a compact operator $K \in \mathcal{K}(X)$ such that

$$A_l(\lambda - A - D) + K = I.$$

Then, we can write

$$\lambda - A - B - D = (I - BA_l)\,(\lambda - A - D) - BK. \tag{22}$$

Using the fact that $BA_l \in \mathcal{E}_{\mathcal{PK}}(X)$ and according to Proposition 2.9, we have $I - BA_l \in \Phi(X)$. Consequently, by Lemma 2.5 we get

$$(I - BA_l)\,(\lambda - A - D) \in \Phi_l(X),\ \forall D \in \mathcal{L}(X),\ \|D\| < \varepsilon.$$

Thus, combining the fact that $BK \in \mathcal{K}(X)$ with the use of Equation 22 and Lemma 2.4, we have $\lambda - A - B - D \in \Phi_l(X)$, for all $\|D\| < \varepsilon$.
Therefore, $\lambda \notin \sigma^l_{\text{e},\varepsilon}(A + B)$ as required.

(ii) Let $\lambda \notin \sigma^{r}_{e,\varepsilon}(A)$, then $\lambda - A - D \in \Phi_r(X)$ for all $\|D\| < \varepsilon$. Since A_r is a right Fredholm inverse of $\lambda - A - D$. From Lemma 2.3 we infer there exists a compact operator $K \in \mathcal{K}(X)$ such that

$$(\lambda - A - D)A_r = I - K \; \forall D \in \mathcal{L}(X), \; \|D\| < \varepsilon.$$

Then, we can write $\lambda - A - B - D$ with the following form

$$\lambda - A - B - D = (\lambda - A - D)\,(I - A_r B) - KB, \; \forall D \in \mathcal{L}(X), \; \|D\| < \varepsilon. \tag{23}$$

Since $A_r B \in \mathcal{E}_{PK}(X)$ then, according to Proposition 2.9, we have $I - A_r B \in \Phi(X)$. Consequently, by Lemma 2.5, we get

$$(\lambda - A - D)\,(I - A_r B) \in \Phi_r(X), \; \forall D \in \mathcal{L}(X), \; \|D\| < \varepsilon.$$

On the other hand, from Equation 23 and Lemma 2.4 and the fact $BK \in \mathcal{K}(X)$ we show that $\lambda - A - B - D \in \Phi_r(X)$, for all $D \in \mathcal{L}(X)$ and $\|D\| < \varepsilon$. We deduce that, $\lambda \notin \sigma^{r}_{e,\varepsilon}(A + B)$.
□

Theorem 3.3. *Let A and B be two operators in $\mathcal{L}(X)$ and $\lambda \in \mathbb{C}$. The following statements hold:*

(i) Assume that $\lambda - A \in \Phi_l(X)$ and for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, there exists $A_l \in \mathcal{I}nv^{F}_{\lambda-A-D,l}(X)$ such that $BA_l \in \mathcal{E}_{\mathcal{PK}}(X)$, then

$$\sigma^{l}_{w,\varepsilon}(A + B) \subseteq \sigma^{l}_{w,\varepsilon}(A).$$

(ii) Assume that $\lambda - A \in \Phi_r(X)$ and and for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, there exists $A_r \in \mathcal{I}nv^{F}_{\lambda-A-D,r}(X)$ such that $A_r B \in \mathcal{E}_{\mathcal{PK}}(X)$, then

$$\sigma^{r}_{w,\varepsilon}(A + B) \subseteq \sigma^{r}_{w,\varepsilon}(A).$$

Proof. (i) Assume that $\lambda \notin \sigma^{l}_{w,\varepsilon}(A)$, then we have $\lambda - A - D \in \Phi_l(X)$ and $i(\lambda - A - D) \leq 0$. A similar reasoning as above gives $\lambda - A - B - D \in \Phi_l(X)$ and it suffices to prove that $i(\lambda - A - B - D) \leq 0$. Since $BK \in \mathcal{K}(X)$ then, Using Equation 22 together with Lemmas 2.4 and 2.5, we obtain that

$$i(\lambda - A - B - D) = i\,(I - BA_l) + i(\lambda - A - D).$$

Now, Since $BA_l \in \mathcal{E}_{PK}(X)$, we get by Proposition 2.9, that $i\,(I - BA_l) = 0$. We deduce that

$$i(\lambda - A - B - D) = i\,(\lambda - A - D) \leq 0.$$

Finally, we conclude that $\lambda - A - B - D \in \mathcal{W}_l(X)$, which entails that $\lambda \notin \sigma^{\mathrm{l}}_{\mathrm{w},\varepsilon}(A + B)$.

(ii) With the same reasoning of (i). Let $\lambda \notin \sigma^{\mathrm{r}}_{\mathrm{w},\varepsilon}(A)$, then we have $\lambda - A - D \in \Phi_r(X)$ and $i(\lambda - A - D) \geq 0$. Proceeding as the proof above, we establish that $\lambda - A - B - D \in \Phi_r(X)$ and $i(\lambda - A - B - D) \geq 0$. Therefore, $\lambda - A - B - D \in \mathcal{W}_r(X)$ and we deduce that $\lambda \notin \sigma^{\mathrm{r}}_{\mathrm{w},\varepsilon}(A + B)$.
□

Remark 3.4. The results of Theorems 3.1, 3.2 and 3.3 is an extension and an improvement of the results of in [2–6] to a large class of polynomially compact perturbations operators. ◊

4. Characterization essential spectrum of two linares bounded operators

The aim of this section is to carry out a new criterions allowing to investigate some spectral analysis of sum of two linear bounded operators. The results of this section are the extension of work wrinting by F. Abdmouleh in [1]. We beginning by give the following lemma when we need in the sequel.

Lemma 4.1. [7, Lemma 4.1] *Let $A \in \mathcal{L}(X)$.*
(*i*) *If $C\sigma_e^l(A)$ is connected, then*

$$\sigma_e^l(A) = \sigma_w^l(A).$$

(*ii*) *If $C\sigma_e^r(A)$ is connected, then*

$$\sigma_e^r(A) = \sigma_w^r(A).$$

Theorem 4.2.
Let $A, B \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}^*$. The following statements hold:

(i) *Assume that the subsets $C\sigma_e^l(A)$ and $C\sigma_e^l(B)$ are connected, $-\lambda^{-1}ABQ_l \in \mathcal{E}_{\mathcal{PK}}(X)$ and $-\lambda^{-1}BAQ_l \in \mathcal{E}_{\mathcal{PK}}(X)$, for every $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, then we have:*

$$\left[\sigma_w^l(A) \cup \sigma_{w,\varepsilon}^l(B)\right] \setminus \{0\} \subseteq \sigma_{w,\varepsilon}^l(A+B) \setminus \{0\}.$$

(ii) *Assume that the subsets $C\sigma_e^r(A)$ and $C\sigma_e^r(B)$ are connected, $-\lambda^{-1}Q_rAB \in \mathcal{E}_{\mathcal{PK}}(X)$ and $-\lambda^{-1}Q_rBA \in \mathcal{E}_{\mathcal{PK}}(X)$, for every $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, then we have:*

$$\left[\sigma_w^r(A) \cup \sigma_{w,\varepsilon}^r(B)\right] \setminus \{0\} \subseteq \sigma_{w,\varepsilon}^r(A+B) \setminus \{0\}.$$

(iii) *Assume that the subsets $C\sigma_e^l(A)$, $C\sigma_e^l(B)$, $C\sigma_e^r(A)$ and $C\sigma_e^r(B)$ are connected, $-\lambda^{-1}ABQ_l \in \mathcal{E}_{\mathcal{PK}}(X)$, $-\lambda^{-1}BAQ_l \in \mathcal{E}_{\mathcal{PK}}(X)$, $-\lambda^{-1}Q_rAB \in \mathcal{E}_{\mathcal{PK}}(X)$ and $-\lambda^{-1}Q_rBA \in \mathcal{E}_{\mathcal{PK}}(X)$, for $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$ and $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, then we have:*

$$\left[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\right] \setminus \{0\} \subseteq \sigma_{w,\varepsilon}(A+B) \setminus \{0\}. \qquad \diamond$$

Proof.
Firstly we note two equality which is used repeatedly

$$(\lambda - A)(\lambda - B - D) = A(B + D) + \lambda(\lambda - A - B - D). \tag{24}$$

$$(\lambda - B - D)(\lambda - A) = (B + D)A + \lambda(\lambda - A - B - D). \tag{25}$$

(*i*) Let $\lambda \notin \sigma_{w,\varepsilon}^l(A+B) \cup \{0\}$ so we have $\lambda - A - B - D \in \Phi_l(X)$ and $i(\lambda - A - B - D) \leq 0$. Then following to the Lemma 2.3 there exist $Q_l \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $Q_l(\lambda - A - B - D) = I - K$.
So when we use Equation (24) we obtain

$$\begin{aligned}
(\lambda - A)(\lambda - B - D) &= A(B + D) + \lambda(\lambda - A - B - D),\\
&= AB[Q_l(\lambda - A - B - D) + K] + \lambda(\lambda - A - B - D),\\
&= [ABQ_l + \lambda I](\lambda - A - B - D) + ABK,\\
&= \lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABK.
\end{aligned}$$

Since $\lambda[\lambda^{-1}ABQ_l + I] \in \Phi(X)$ and $(\lambda - A - B - D) \in \Phi_l(X)$ it follows from Proposition 2.1 that $\lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) \in \Phi_l(X)$. Since $ABK \in \mathcal{K}(X)$, this implies by the use of Lemma 2.4 that

$$\lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABQ_lK \in \Phi_l(X).$$

So $(\lambda - A)(\lambda - B - D) \in \Phi_l(X)$ and as a direct consequence of Lemma 2.6 we obtain

$$\lambda - B - D \in \Phi_l(X), \forall D \in \mathcal{L}(X), \|D\| < \varepsilon. \tag{26}$$

In the other hand, when we use the Equation (25) we have

$$\begin{aligned}
(\lambda - B - D)(\lambda - A) &= BA + \lambda(\lambda - A - B - D),\\
&= BA[Q_l(\lambda - A - B - D) + K] + \lambda(\lambda - A - B - D),\\
&= [BAQ_l + \lambda I](\lambda - A - B - D) + BAK,\\
&= \lambda[\lambda^{-1}BAQ_l + I](\lambda - A - B - D) + BAK.
\end{aligned}$$

Since $\lambda[\lambda^{-1}BAQ_l + I] \in \Phi(X)$ and $(\lambda - A - B - D) \in \Phi_l(X)$ it follows from Proposition 2.1 that

$$\lambda[\lambda^{-1}BAQ_l + I](\lambda - A - B - D) \in \Phi_l(X).$$

Obviously, since $BAK \in \mathcal{K}(X)$ and applying Lemma 2.4, we find that

$$\lambda[\lambda^{-1}BAQ_l + I](\lambda - A - B - D) + BAK \in \Phi_l(X).$$

So $(\lambda - B - D)(\lambda - A) \in \Phi_l(X)$. Therefore using Lemma 2.6 we obtain

$$\lambda - A \in \Phi_l(X). \tag{27}$$

Now, to check the index we must have a discussion according to the sign, thus using the above we have

$$i(\lambda - A) + i(\lambda - B - D) = i(\lambda - A - B - D) \leq 0.$$

Case1: If $i(\lambda - A) \leq 0$.

Using Lemma 4.1 the index $i(\lambda - B - D)$ must be negative.Therefore adding this condition to Equations (26) and (27) we obtain

$$\lambda \notin \left[\sigma^l_w(A) \cup \sigma^l_{w,\varepsilon}(B)\right] \cup \{0\}.$$

Case2: If $i(\lambda - B - D) \leq 0$.

Following to Lemma 4.1 the index $i(\lambda - A)$ must be negative.

Then adding this condition to Equations (26) and (27) we assert

$$\lambda \notin \left[\sigma^l_w(A) \cup \sigma^l_{w,\varepsilon}(B)\right] \cup \{0\}.$$

Case3: If $i(\lambda - A) > 0$.

Following to Lemma 4.1 the index $i(\lambda-B-D)$ should be positif which contradicts the fact that $i(\lambda-A-B-D) \leq 0$.

Case4: If $i(\lambda - B - D) > 0$.

Following to Lemma 4.1 the index $i(\lambda - A)$ must be positif which contradicts the fact that $i(\lambda - A - B - D) \leq 0$.

(ii) Let $\lambda \notin \sigma^r_{w,\varepsilon}(A + B) \cup \{0\}$ then $\lambda - A - B - D \in \Phi_r(X)$ and $i(\lambda - A - B - D) \leq 0$. So by Lemma 2.3 there exist $Q_r \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $(\lambda - A - B - D)Q_r = I - K$

So following to the Equation (24) we have

$$\begin{aligned}
(\lambda - A)(\lambda - B - D) &= AB + \lambda(\lambda - A - B - D),\\
&= [(\lambda - A - B - D)Q_r + K]AB + \lambda(\lambda - A - B - D),\\
&= (\lambda - A - B - D)[Q_rAB + \lambda I] + ABK,\\
&= \lambda(\lambda - A - B - D)[\lambda^{-1}Q_rAB + I] + ABK.
\end{aligned}$$

Since $\lambda[\lambda^{-1}Q_rAB + I] \in \Phi(X)$ and $(\lambda - A - B - D) \in \Phi_r(X)$ it follows by Proposition 2.1 that

$$\lambda[\lambda^{-1}Q_rAB + I](\lambda - A - B - D) \in \Phi_r(X).$$

Since $ABK \in \mathcal{K}(X)$ then

$$\lambda[\lambda^{-1}Q_rAB + I](\lambda - A - B - D) + ABK \in \Phi_r(X).$$

So $(\lambda - A)(\lambda - B - D) \in \Phi_r(X)$, following to Lemma 2.6 we infer that

$$\lambda - A \in \Phi_r(X). \tag{28}$$

In the other hand, the use of Equation (25) assert

$$\begin{array}{rcl}(\lambda - B - D)(\lambda - A) & = & BA + \lambda(\lambda - A - B - D),\\ & = & BA[(\lambda - A - B - D)Q_r + K]BA + \lambda(\lambda - A - B - D),\\ & = & (\lambda - A - B - D)[Q_rBA + \lambda I] + KBA,\\ & = & \lambda(\lambda - A - B - D)[\lambda^{-1}Q_rBA + I] + KBA.\end{array}$$

Since by hypothesis $[\lambda^{-1}Q_rBA + I] \in \Phi(X)$ and $(\lambda - A - B - D) \in \Phi_r(X)$ we have by Proposition 2.1

$$\lambda(\lambda - A - B - D)[\lambda^{-1}Q_rBA + I] \in \Phi_r(X).$$

Since $KBA \in \mathcal{K}(X)$ we obtain

$$\lambda(\lambda - A - B - D)[\lambda^{-1}Q_rBA + I] + KBA \in \Phi_r(X).$$

So $(\lambda - B - D)(\lambda - A) \in \Phi_r(X)$ then the use of Lemma 2.6 infer that

$$\lambda - B - D \in \Phi_r(X),\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{29}$$

Now, to check the index we must have a discussion according to the sign, thus using the above we have

$$i(\lambda - A) + i(\lambda - B - D) = i(\lambda - A - B - D) \geq 0.$$

Case 1: If $i(\lambda - A) \geq 0$.
Using Lemma 4.1 the index $i(\lambda - B - D)$ must be positif. Therefore adding this condition to Equations (28) and (29) we get

$$\lambda \notin \left[\sigma_w^r(A) \cup \sigma_{w,\varepsilon}^r(B)\right] \cup \{0\}.$$

Case 2: If $i(\lambda - B - D) \geq 0$.
Following to Lemma 4.1 the index $i(\lambda - A)$ must be positif.
Then adding this condition to Equations (26) and (27) we obtain

$$\lambda \notin \left[\sigma_w^r(A) \cup \sigma_{w,\varepsilon}^r(B)\right] \cup \{0\}.$$

Case 3: If $i(\lambda - A) < 0$.
Following to Lemma 4.1 the index $i(\lambda - B - D)$ should be negative which contradicts the fact that $i(\lambda - A - B - D) \geq 0$.
Case 4: If $i(\lambda - B - D) < 0$.
Following to Lemma 4.1 the index $i(\lambda - A)$ should be negative which contradicts the fact that $i(\lambda - A - B - D) \geq 0$.
(iii) Let $\lambda \notin \sigma_{w,\varepsilon}(A + B) \cup \{0\}$ therefore $\lambda - A - B - D \in \Phi(X)$ and $i(\lambda - A - B - D) = 0$ then there exist $Q_l, Q_r \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $Q_l(\lambda - A - B - D) = I - K$ and $(\lambda - A - B - D)Q_r = I - K$.
Now, according to items (i) and (ii) we get

$$\left[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\right] \setminus \{0\} \subseteq \sigma_{w,\varepsilon}(A + B) \setminus \{0\}.$$

□

Theorem 4.3.
Let $A, B \in \mathcal{L}(X)$ such that $AB = BA$ and $\lambda \in \mathbb{C}^*$. The following statements hold:
(i) *If there exists $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, such that $-\lambda^{-1}ABQ_l \in \mathcal{E}_{\mathcal{PK}}(X)$ then*

$$\sigma_{e,\varepsilon}^l(A + B) \setminus \{0\} = \left[\sigma_e^l(A) \cup \sigma_{e,\varepsilon}^l(B)\right] \setminus \{0\}.$$

(ii) *If there exists $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}Q_rAB \in \mathcal{E}_{\mathcal{PK}}(X)$ then*

$$\sigma_{e,\varepsilon}^r(A + B) \setminus \{0\} = \left[\sigma_e^r(A) \cup \sigma_{e,\varepsilon}^r(B)\right] \setminus \{0\}.$$

(iii) *If there exists* $Q \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, *such that* $-\lambda^{-1}QAB \in \mathcal{E}_{\mathcal{PK}}(X)$ *and* $-\lambda^{-1}ABQ \in \mathcal{E}_{\mathcal{PK}}(X)$ *then*

$$\sigma_{e,\varepsilon}(A+B) \setminus \{0\} = \Big[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\Big] \setminus \{0\}. \qquad \diamond$$

Proof.
(i) Let $\lambda \notin \sigma^l_{e,\varepsilon}(A+B) \cup \{0\}$ then $\lambda - A - B - D \in \Phi_l(X)$.
We assume there exists $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, thus, using Equation (24) we have

$$\begin{aligned}(\lambda - A)(\lambda - B - D) &= A(B+D) + \lambda(\lambda - A - B - D),\\ &= AB[Q_l(\lambda - A - B - D) + K] + \lambda(\lambda - A - B - D),\\ &= [ABQ_l + \lambda I](\lambda - A - B - D) + ABK,\\ &= \lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABK.\end{aligned}$$

Obviously, $-\lambda^{-1}ABQ_l \in \mathcal{E}_{\mathcal{PK}}(X)$ then by Proposition 2.9 we infer that $\lambda^{-1}ABQ_l + I \in \Phi(X)$. Therefore, by Lemma 2.5 we obtain $[\lambda^{-1}ABQ_l + \lambda I](\lambda - A - B - D) \in \Phi_l(X)$.
Since $ABK \in \mathcal{K}(X)$ and by applying Lemma 2.4 we obtain

$$\lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABK \in \Phi_l(X).$$

We conclude that

$$(\lambda - A)(\lambda - B - D) \in \Phi_l(X), \ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon.$$

Hence, by Lemma 2.6 we deduce that

$$(\lambda - B - D) \in \Phi_l(X), \ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{30}$$

On the other hand, using the fact that $AB = BA$ and according to the Equation (25) we observe that

$$\begin{aligned}(\lambda - B - D)(\lambda - A) &= BA + \lambda(\lambda - A - B - D),\\ &= AB + \lambda(\lambda - A - B - D),\\ &= AB[Q_l(\lambda - A - B - D) + K] + \lambda(\lambda - A - B - D),\\ &= [ABQ_l + \lambda I](\lambda - A - B - D) + ABK,\\ &= \lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABK.\end{aligned}$$

Using the same reasoning we conclude that $(\lambda - B - D)(\lambda - A) \in \Phi_l(X)$. Therefore, by Lemma 2.6 we deduce that

$$(\lambda - A) \in \Phi_l(X). \tag{31}$$

Finally, the two Equations (30) and (31) imply that $\lambda \notin \Big[\sigma^l_e(A) \cup \sigma^l_{e,\varepsilon}(B)\Big] \cup \{0\}$.
So, we obtain

$$\Big[\sigma^l_e(A) \cup \sigma^l_{e,\varepsilon}(B)\Big] \setminus \{0\} \subset \sigma^l_{e,\varepsilon}(A+B) \setminus \{0\}.$$

The other inclusion is allows us to achieve equality is in [7, Theorem 4.3].
(ii) Let $\lambda \notin \sigma^r_{e,\varepsilon}(A+B) \cup \{0\}$ then $\lambda - A - B - D \in \Phi_r(X)$, for all $\|D\| < \varepsilon$.
We assume there exists $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$ thus,

$$\begin{aligned}(\lambda - A)(\lambda - B - D) &= AB + \lambda(\lambda - A - B - D),\\ &= [(\lambda - A - B - D)Q_r + K]AB + \lambda(\lambda - A - B - D),\\ &= (\lambda - A - B - D)\lambda[\lambda^{-1}Q_rAB + I] + KAB.\end{aligned}$$

Evidently, $-\lambda^{-1}Q_rAB \in \mathcal{E}_{\mathcal{PK}}(X)$ and by applying Proposition 2.9 we deduce that $\lambda^{-1}Q_rAB + I \in \Phi(X)$. Since, KAB is compact, then by Lemma 2.4 we obtain

$$(\lambda - A - B - D)\lambda[\lambda^{-1}Q_rAB + I] + KAB \in \Phi_l(X).$$

Consequently, we have $(\lambda - A)(\lambda - B - D) \in \Phi_r(X)$ and by Lemma 2.6 we infer that

$$(\lambda - A) \in \Phi_r(X). \tag{32}$$

Further, we have $AB = BA$ so,

$$\begin{aligned}(\lambda - B - D)(\lambda - A) &= BA + \lambda(\lambda - A - B - D),\\ &= AB + \lambda(\lambda - A - B - D),\\ &= [(\lambda - A - B - D)Q_r + K]AB + \lambda(\lambda - A - B - D),\\ &= (\lambda - A - B - D)\lambda[\lambda^{-1}Q_rAB + I] + KAB.\end{aligned}$$

Using the same reasoning we conclude that $(\lambda - B - D)(\lambda - A) \in \Phi_r(X)$. Then, by Lemma 2.6 we deduce that

$$(\lambda - B - D) \in \Phi_r(X),\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{33}$$

Finally, the two Equations (32) and (33) imply that

$$\lambda \notin \left[\sigma_e^r(A) \cup \sigma_{e,\varepsilon}^r(B)\right] \cup \{0\}.$$

So, we obtain

$$\left[\sigma_e^r(A) \cup \sigma_{e,\varepsilon}^r(B)\right] \setminus \{0\} \subset \sigma_{e,\varepsilon}^r(A + B) \setminus \{0\}.$$

The other inclusion is allows us to achieve equality is in [7, Theorem 4.3].

(*iii*) Let $\lambda \notin \sigma_{e,\varepsilon}(A + B) \cup \{0\}$. Then $\lambda - A - B - D \in \Phi(X)$ means that $\lambda - A - B - D \in \Phi_l(X) \cap \Phi_r(X)$.

Now, by the hypothesis there exists $Q \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, and by applying the results in statements (*i*) and (*ii*) we infer that $(\lambda - A - B - D) \in \Phi_r(X)$ and $(\lambda - A - B - D) \in \Phi_l(X)$, therefore $(\lambda - A - B - D) \in \Phi(X)$.

Also, using the hypothesis that $-\lambda^{-1}QAB \in \mathcal{E}_{\mathcal{PK}}(X)$, $-\lambda^{-1}ABQ \in \mathcal{E}_{\mathcal{PK}}(X)$ and $AB = BA$ we give us this two condition:

$$(\lambda - A)(\lambda - B - D) \in \Phi(X) \text{ and } (\lambda - B - D)(\lambda - A) \in \Phi(X).$$

Therefore, following Theorem 2.2 we obtain $(\lambda - A) \in \Phi(X)$ and $(\lambda - B - D) \in \Phi(X)$ means that $\lambda \notin \left[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\right] \cup \{0\}$. Then we get the following inclusion

$$\left[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\right] \setminus \{0\} \subseteq \sigma_{e,\varepsilon}(A + B) \setminus \{0\}.$$

The other inclusion is allows us to achieve equality is in [7, Theorem 4.3]. □

The same reasoning of the above theorem, we allow to obtain the result of the following result.

Theorem 4.4. *Let $A, B \in \mathcal{L}(X)$ such that $AB = BA$ and $\lambda \in \mathbb{C}^*$. The following statements hold:*

(i) *If there exists $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, such that $-\lambda^{-1}ABQ_l \in \mathcal{E}_{\mathcal{PK}}(X)$, then*

$$\sigma_{w,\varepsilon}^l(A + B) \setminus \{0\} = \left[\sigma_{w,\varepsilon}^l(A) \cup \sigma_{w,\varepsilon}^l(B)\right] \setminus \{0\}.$$

(ii) *If there exists $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}Q_rAB \in \mathcal{E}_{\mathcal{PK}}(X)$, then*

$$\sigma_{w,\varepsilon}^r(A + B) \setminus \{0\} = \left[\sigma_w^r(A) \cup \sigma_{w,\varepsilon}^r(B)\right] \setminus \{0\}.$$

(iii) *If there exists $Q \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}QAB \in \mathcal{E}_{\mathcal{PK}}(X)$ and $-\lambda^{-1}ABQ \in \mathcal{E}_{\mathcal{PK}}(X)$ then*

$$\sigma_{w,\varepsilon}(A + B) \setminus \{0\} = \left[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\right] \setminus \{0\}. \qquad \diamond$$

5. Application to bounded 2 × 2 block operator matrices forms

The objective of this section is to utilize Theorem 4.3 from Section 4 in order to analyze the pseudo left (right)-Fredholm essential spectra of the given operator matrix.

Let X_1 and X_2 be two Banach spaces and consider the 2×2 block operator matrices defined on $X_1 \times X_2$ by:

$$\mathcal{M} := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

where $A \in \mathcal{L}(X_1)$, $B \in \mathcal{L}(X_2)$, $C \in \mathcal{L}(X_2, X_1)$ and $D \in \mathcal{L}(X_1, X_2)$.

Next, we define the following matrix:

$$\mathfrak{D} = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix},$$

where $D_1 \in \mathcal{L}(X_1)$, $D_4 \in \mathcal{L}(X_2)$, $D_2 \in \mathcal{L}(X_2, X_1)$, $D_3 \in \mathcal{L}(X_1, X_2)$ and $\|\mathfrak{D}\| = \max\{\|D_i\|, \forall 1 \leq i \leq 4\}$.

Our goal is to find the pseudo-left (right)-Fredholm essential spectra of Matrix $\mathcal{M}$.

Theorem 5.1. *Let the 2×2 block operator matrix $\mathcal{M}_C$ and $\varepsilon > 0$. In all that follows we will make the following assumptions:*

$$\mathcal{H} : \begin{cases} \|\mathfrak{D}\| < \varepsilon, \\ AC = CB, \\ A + B \in \Phi(X), \\ CB \in \mathcal{K}(X_1 \times X_2). \end{cases}$$

Then, we have that

(i) $\sigma_{e,\varepsilon}^{left}(\mathcal{M}_C)\backslash\{0\} \subseteq \left[\sigma_{e,\varepsilon}^{left}(A) \cup \sigma_{e,\varepsilon}^{left}(B)\right]\backslash\{0\}$.

(ii) $\sigma_{e,\varepsilon}^{right}(\mathcal{M}_C)\backslash\{0\} \subseteq \left[\sigma_{e,\varepsilon}^{right}(A) \cup \sigma_{e,\varepsilon}^{right}(B)\right]\backslash\{0\}$.

Proof. We begin by presenting the polynomial P in the specified format:

$$\begin{array}{cccc} P & : & \mathbb{R}^2 & \rightarrow & \mathbb{R} \\ & & (x, y) & \mapsto & P(x, y) = x.y \end{array}$$

We can write

$$\begin{aligned} \mathcal{M} := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} &= \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \\ &= \mathcal{M}_C + \mathcal{M}_{A,B}. \end{aligned}$$

We have:

$$P(\mathcal{M}_C, \mathcal{M}_{A,B}) = \mathcal{M}_C.\mathcal{M}_{A,B} = \begin{pmatrix} 0 & CB \\ 0 & 0 \end{pmatrix}.$$

it follows from the hypothesis (H) that:

$$P(\mathcal{M}_C, \mathcal{M}_{A,B}) \in \mathcal{K}(X_1 \times X_2), \text{and } \mathcal{M}_C.\mathcal{M}_{A,B} \in \mathcal{E}_{\mathcal{PK}}(X).$$

Moreover we have $A+B \in \Phi(X)$ then there exist $A_0 \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $A_0(A+B) = I - K$. Then

$$A_0(A+B+D) = I - K', \text{with } K' \in \mathcal{K}(X).$$

Using Theorem 4.3, we obtain that

(i) $\sigma_{e,\varepsilon}^{left}(\mathcal{M})\backslash\{0\} = \sigma_{e,\varepsilon}^{left}(\mathcal{M}_C + \mathcal{M}_{A,B})\backslash\{0\} = \left[\sigma_{e}^{left}(\mathcal{M}_C) \cup \sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B})\right]\backslash\{0\}$.

(ii) $\sigma_{e,\varepsilon}^{right}(\mathcal{M})\backslash\{0\} = \sigma_{e,\varepsilon}^{right}(\mathcal{M}_C + \mathcal{M}_{A,B})\backslash\{0\} = \left[\sigma_{e}^{right}(\mathcal{M}_C) \cup \sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B})\right]\backslash\{0\}$.

Furthermore, we can readily demonstrate $\sigma_{e}^{left}(\mathcal{M}_C) = \sigma_{e}^{right}(\mathcal{M}_C) = \{0\}$. Consequently, applying [[3], Theorem 4 (i)], we show that

$$\begin{aligned}
\sigma_{e,\varepsilon}^{left}(\mathcal{M})\backslash\{0\} &= \left[\sigma_{e}^{left}(\mathcal{M}_C) \cup \sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B})\right]\backslash\{0\} \\
&= \left[\{0\} \cup \sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B})\right]\backslash\{0\} \\
&= \sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B}) \\
&\subseteq \left[\sigma_{e,\varepsilon}^{left}(A) \cup \sigma_{e,\varepsilon}^{left}(B)\right]\backslash\{0\},
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{e,\varepsilon}^{right}(\mathcal{M})\backslash\{0\} &= \left[\sigma_{e}^{right}(\mathcal{M}_C) \cup \sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B})\right]\backslash\{0\} \\
&= \left[\{0\} \cup \sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B})\right]\backslash\{0\} \\
&= \sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B}) \\
&\subseteq \left[\sigma_{e,\varepsilon}^{right}(A) \cup \sigma_{e,\varepsilon}^{right}(B)\right]\backslash\{0\}.
\end{aligned}$$

□

Compliance with ethical standards

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