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B-essential and B-Weyl pseudo-spectra for the sum of two commuting bounded operators

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Abstract. This paper is dedicated to investigating the pseudo-B-essential spectra of the sum of two bounded linear operators defined on a Banach space. This investigation is carried out by examining the pseudo-B-essential spectra of each individual operator, considering that the products of these operators result in finite-rank operators.

1. Introduction and Preliminaires

Recently, there has been a surge in papers exploring the concept of pseudo-spectrum. Noteworthy works on the subject include references [14, 25]. Let's begin by revisiting some well-established facts about pseudo-spectrum. The exploration of pseudo-spectra began with the observation that spectral analysis predictions diverge from numerical simulations in certain scientific and engineering problems involving nonself-adjoint operators. To address the informational gap left by the spectrum, additional sets in the complex plane, known as pseudo-spectra, have been introduced. The fundamental concept involves investigating not only points where the resolvent is large in norm (representing the spectrum) but also locations where the resolvent exhibits significant norm magnitudes.

To understand the definition of an operator U's pseudo-spectrum, denoted as $\sigma_\varepsilon(U)$, with $\varepsilon > 0$, please consult Trefethen's article [24]:

$$\sigma_\varepsilon(U) := \sigma(U) \cup \left\{\lambda \in \mathbb{C} \text{ such that } \|(\lambda - U)^{-1}\| > \frac{1}{\varepsilon}\right\}.$$

Here, $\sigma(U)$ represents the spectrum of U. As a convention, we express $\|(\lambda - U)^{-1}\| = \infty$ if $(\lambda - U)^{-1}$ is unbounded or nonexistent, i.e., if λ is in $\sigma(U)$. Another equivalent pseudo-spectrum, as defined in Davies [14] , is based on perturbations of the spectrum. For any closed operator U:

$$\sigma_\varepsilon(U) := \bigcup_{\|D\|<\varepsilon} \sigma(U + D).$$

This implies that a number λ belongs to the pseudo-spectrum of U if, and only if, it is part of the spectrum of some perturbed $U + D$ with $\|D\| < \varepsilon$.

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Ammar and Jeribi [3, 8] introduced the concept of Weyl pseudo-spectra for densely closed, linear operators in the Banach space, expressed as

$$\sigma_{W,\varepsilon}(U) = \bigcap_{K \in \mathcal{K}(X)} \sigma_{\varepsilon}(U + K).$$

This study is a continuation of our prior research [2]. Throughout this paper, we denote infinite-dimensional Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ as X and Y, respectively. The notation $\mathcal{L}(X, Y)$ (or $\mathcal{C}(X, Y)$) signifies the collection of all bounded (or closed, densely defined) linear operators from X to Y. Additionally, $\mathcal{L}(X, Y)$ represents the set of all finite-rank operators from X to Y.

When X is equal to Y, we denote $\mathcal{L}(X)$ represents the algebra of bounded linear operators on X. For a given operator U in $\mathcal{L}(X)$, we denote $N(U)$ as the null space of U and $R(U)$ as the range of U. The nullity, denoted as $\alpha(U)$, is defined as the dimension of $N(U)$, while the deficiency, denoted as $\beta(U)$, is defined as the codimension of $R(U)$ in X. If the range $R(U)$ of U is closed, and $dim(N(U)) < \infty$ (or $codim(R(U)) < \infty$), then U is termed an upper semi Fredholm (or a lower semi-Fredholm) operator, denoted by $\Phi_+(X)$ and $\Phi_-(X)$ respectively. A semi-Fredholm operator is either an upper or a lower semi-Fredholm operator, denoted by $\Phi(X)$. If both $dim(N(U))$ and $codim(R(U))$ are finite, then U is referred to as a Fredholm operator, denoted by $i(U)$. The index of U is defined as $i(U) = \alpha(U) - \beta(U)$.

The operator U is termed a B-Fredholm, an upper (or lower) semi B-Fredholm operator, if there exists an integer n such that the range space $R(U_n)$ is closed, and U_n is a Fredholm, an upper (or lower) semi-Fredholm operator. Here, U_n represents the restriction of U to $R(U_n)$ considered as a map from $R(U_n)$ into $R(U_n)$ (specifically, $U_0 = U$) (refer to [2]). The index of a B-Fredholm operator U is defined as the index of the Fredholm operator U_n, where n is any integer such that $R(U_n)$ is closed, and U_n is a Fredholm operator. Specifically, $i(U) = \alpha(U_n) - \beta(U_n)$, where $\alpha(U_n)$ represents the dimension of the kernel $ker(U_n)$ of U_n and $\beta(U_n)$ is the codimension of the range $R(U_n) = R(U_{n+1})$ of U_n into $R(U_n)$. We use ${}^{B}\Phi(X)$, ${}^{B}\Phi_+(X)$, and ${}^{B}\Phi_-(X)$ to denote the classes of all B-Fredholm, upper semi B-Fredholm, and lower semi B-Fredholm operators, respectively.

The upper (or lower) semi B-Fredholm spectrum and the B-Fredholm spectrum of U are defined by

$${}^{B}\sigma_1(U) = \{\lambda \in \mathbb{C} \text{ such that } U - \lambda I \notin {}^{B}\Phi_+(X)\}.$$

$${}^{B}\sigma_2(U) = \{\lambda \in \mathbb{C} \text{ such that } U - \lambda I \notin {}^{B}\Phi_-(X)\}.$$

$${}^{B}\sigma_F(U) = \{\lambda \in \mathbb{C} \text{ such that } U - \lambda I \notin {}^{B}\Phi(X)\}.$$

An operator $U \in \mathcal{L}(X)$ is termed a B-Weyl operator if it is a B-Fredholm operator with an index of 0. The B-Weyl spectrum ${}^{B}\sigma_W(U)$ of U is defined by

$${}^{B}\sigma_W(U) = \{\lambda \in \mathbb{C} \text{ such that } U - \lambda I \text{ is not a B-Weyl operator}\}.$$

In recent times, B-Fredholm and B-Weyl operators have been further generalized to pseudo-B-Fredholm and pseudo-B-Weyl operators. More precisely:

Definition 1.1. ([1])

(i) The essential pseudo-B-Fredholm spectra is defined by

$${}^{B}\sigma_{F,\varepsilon}(U) = \{\lambda \in \mathbb{C} \textit{ such that } U - \lambda \textit{ is not pseudo B-Fredholm}\}.$$

(ii) The essential pseudo-B-Weyl spectra is defined by

$${}^{B}\sigma_{W,\varepsilon}(U) = \{\lambda \in \mathbb{C} \textit{ such that } U - \lambda \textit{ is not pseudo B-Weyl}\}.$$

Definition 1.2. *Let* $\varepsilon > 0$ *and* $U \in \mathcal{L}(X)$.

(i) U is called a pseudo-upper (resp. lower) semi-B-Fredholm operator if $U+D$ is an upper (resp. lower) semi-B-Fredholm operator for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$.

(ii) U is called a pseudo-semi-B-Fredholm operator if $U+D$ is an semi-B-Fredholm operator for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$.

(iii) U is called a pseudo-B-Fredholm if $U+D$ is an B-Fredholm operator for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$.

(iv) U is called a pseudo-B-Weyl if $U+D$ is an B-Weyl operator for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$.

Let ${}^{B}\Phi^{\varepsilon}(X)$ denote the set of pseudo-B-Fredholm operators, ${}^{B}\Psi^{\varepsilon}(X)$ represent the set of pseudo-B-Weyl operators and ${}^{B}\Phi^{\varepsilon}_{+}(X)$ (or ${}^{B}\Phi^{\varepsilon}_{-}(X)$) represent the set of pseudo-upper semi-B-Fredholm (or lower semi-B-Fredholm) operators. If a complex number λ belongs to ${}^{B}\Phi^{\varepsilon}_{U}(X)$, ${}^{B}\Psi^{\varepsilon}_{U}(X)$ and ${}^{B}\Phi^{\varepsilon}_{+U}(X)$, or ${}^{B}\Phi^{\varepsilon}_{-U}(X)$, it means that $\lambda - U$ is an element of ${}^{B}\Phi^{\varepsilon}(X)$, ${}^{B}\Psi^{\varepsilon}(X)$ and ${}^{B}\Phi^{\varepsilon}_{+}(X)$, or ${}^{B}\Phi^{\varepsilon}_{-}(X)$, respectively. This paper specifically focuses on investigating the essential pseudospectra given by:

$$ {}^{B}\sigma_{1,\varepsilon}(U) = \{\lambda \in \mathbb{C} \text{ such that } U-\lambda \notin {}^{B}\Phi^{\varepsilon}_{+}(X)\} := \mathbb{C}\backslash\, {}^{B}\Phi^{\varepsilon}_{+U}(X), $$

$$ {}^{B}\sigma_{2,\varepsilon}(U) = \{\lambda \in \mathbb{C} \text{ such that } U-\lambda \notin {}^{B}\Phi^{\varepsilon}_{-}(X)\} := \mathbb{C}\backslash\, {}^{B}\Phi^{\varepsilon}_{-U}(X), $$

$$ {}^{B}\sigma_{F,\varepsilon}(U) = \{\lambda \in \mathbb{C} \text{ such that } U-\lambda \notin {}^{B}\Phi^{\varepsilon}(X)\} := \mathbb{C}\backslash\, {}^{B}\Phi^{\varepsilon}_{U}(X), $$

$$ {}^{B}\sigma_{W,\varepsilon}(U) = \{\lambda \in \mathbb{C} \text{ such that } U-\lambda \notin {}^{B}\Psi^{\varepsilon}(X)\} := \mathbb{C}\backslash\, {}^{B}\Psi^{\varepsilon}_{U}(X). $$

It is worth noting that as ε tends to 0, the conventional definition of the essential spectra of a closed operator U is regained.

Next, we present the following lemmas, which have been established by Berkani:

Lemma 1.3. ([10, Proposition 3.3]) *Let $U \in \mathcal{L}(X)$ be a B-Fredholm operator and let F be a finite rank operator. Then $U+F$ is a B-Fredholm operator and $i(U+F) = i(U)$.*

Lemma 1.4. *Let X be a Banach space and U, V, M, $N \in \mathcal{L}(X)$ be mutually commuting operators, satisfying $UM + NV = I$. Then*

(i) $\big($[11, Proposition 3.2]$\big)$, *$UV \in {}^{B}\Phi(X)$ if and only if U and V are B-Fredholm operators on X.*

(ii) $\big($[13, Proposition 4.3]$\big)$, *$UV \in {}^{B}\Phi_{+}(X)$ if and only if U and V are upper semi-B-Fredholm operators on X.*

(iii) $\big($[13, Proposition 4.3]$\big)$, *$UV \in {}^{B}\Phi_{-}(X)$ if and only if U and V are lower semi-B-Fredholm operators on X.*

(iv) $\big($[12, Theorem 1.1]$\big)$, *if U and V are B-Fredholm operators, then UV is a B-Fredholm operator and $i(UV) = i(U) + i(V)$.*

Remark 1.5. *The condition $UM + NV = I$, in Lemma* 1.4, *is very important to prove that $i(UV) = i(U) + i(V)$* (see [12]).

The subsequent theorems, demonstrated in [5, 6], are hereby presented.

Theorem 1.6. $\big($[5, Theorem 2.3]$\big)$ *Let X be a Banach space, $\varepsilon > 0$ and consider $U, V \in \mathcal{L}(X)$. Then*

(i) If for all bounded operator D such that $\|D\| < \varepsilon$ and $U(V+D) \in \mathcal{F}(X)$, then

$$ \sigma_{e4,\varepsilon}(U+V)\backslash 0 \subset [\sigma_{e4}(U) \cup \sigma_{e4,\varepsilon}(V)]\backslash 0. $$

If, further, $(V+D)U \in \mathcal{F}(X)$, then

$$ \sigma_{e4,\varepsilon}(U+V)\backslash 0 = [\sigma_{e4}(U) \cup \sigma_{e4,\varepsilon}(V)]\backslash 0. $$

(ii) *If for all bounded operator D such that* $\|D\| < \varepsilon$ *and* $U(V + D) \in \mathcal{F}_+(X)$, *then*

$$\sigma_{e1,\varepsilon}(U + V)\backslash 0 \subset [\sigma_{e1}(U) \cup \sigma_{e1,\varepsilon}(V)]\backslash 0.$$

If, further, $(V + D)U \in \mathcal{F}_+(X)$, *then*

$$\sigma_{e1,\varepsilon}(U + V)\backslash 0 = [\sigma_{e1}(U) \cup \sigma_{e1,\varepsilon}(V)]\backslash 0.$$

(iii) *If for all bounded operator D such that* $\|D\| < \varepsilon$ *and* $U(V + D) \in \mathcal{F}_-(X)$, *then*

$$\sigma_{e2,\varepsilon}(U + V)\backslash 0 \subset [\sigma_{e2}(U) \cup \sigma_{e2,\varepsilon}(V)]\backslash 0.$$

If, further, $(V + D)U \in \mathcal{F}_-(X)$, *then*

$$\sigma_{e2,\varepsilon}(U + V)\backslash 0 = [\sigma_{e2}(U) \cup \sigma_{e2,\varepsilon}(V)]\backslash 0.$$

(iv) *If for all bounded operator D such that* $\|D\| < \varepsilon$ *and* $U(V + D) \in \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$, *then* $\sigma_{e3,\varepsilon}(U + V)\backslash 0 \subset [(\sigma_{e3}(U) \cup \sigma_{e3,\varepsilon}(V)) \cup (\sigma_{e3}(U) \cap \sigma_{e2,\varepsilon}(V)) \cup (\sigma_{e2}(U) \cap \sigma_{e1,\varepsilon}(V))]\backslash 0$. *Moreover, if* $(V + D)U \in \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$, *then* $\sigma_{e3,\varepsilon}(U + V)\backslash 0 = [(\sigma_{e3}(U) \cup \sigma_{e3,\varepsilon}(V)) \cup (\sigma_{e3}(U) \cap \sigma_{e2,\varepsilon}(V)) \cup (\sigma_{e2}(U) \cap \sigma_{e1,\varepsilon}(V))]\backslash 0$.

Theorem 1.7. $\big($[6, Theorem 2.11]$\big)$ *Let X be a Banach space,* $\varepsilon > 0$, *U and V two elements of* $\mathcal{L}(X)$. *If for all bounded operators D such that* $\|D\| < \varepsilon$ *we have* $U(V + D) \in \mathcal{F}(X)$, *then*

$$\sigma_{w,\varepsilon}(U + V)\backslash 0 \subseteq [\sigma_w(U) \cup \sigma_{w,\varepsilon}(V)]\backslash 0.$$

If, further, $(V + D)U \in \mathcal{F}(X)$, *then*

$$\sigma_{w,\varepsilon}(U + V)\backslash 0 = [\sigma_w(U) \cup \sigma_{w,\varepsilon}(V)]\backslash 0.$$

The primary objective of this paper is to broaden the applicability of the theorems presented in [2] to include pseudo-B-Fredholm operators, as well as pseudo-upper (or pseudo-lower) semi B-Fredholm operators within a Banach space X.
To begin, we establish the connection between the pseudo-B-essential spectra of U and its inverse, denoted as U^{-1}. Subsequently, we delve into an exploration of the pseudo-B-essential spectra of the sum of two bounded linear operators defined on a Banach space. This investigation is conducted by examining the pseudo-B-essential spectra of each of the two operators, particularly when their products result in finite-rank operators.

The organization of the paper unfolds as follows: Section 2 is dedicated to the various pseudo-B-essential spectra of bounded linear operators on a Banach space.

2. MAIN RESULTS

Theorem 2.1. *Let* $\varepsilon > 0$. *Let U, T and S be commuting operators an a Banach space X. If for all operator D such that* $\|D\| < \varepsilon$, $DU + U^{-1}D = -D^2$ *and if* $0 \in \rho(U) \cap \rho(U + D)$ *then, for every* $\lambda \neq 0$ *satisfying* $(U^{-1} + D)S + T(U + D) = I + \lambda^{-1}S$, *we have*

$$\lambda \in {}^{B}\sigma_{W,\varepsilon}(U) \text{ if and only if } \frac{1}{\lambda} \in {}^{B}\sigma_{W,\varepsilon}(U^{-1}). \tag{1}$$

Proof. Let λ be a non-zero complex number. Since $0 \in \rho(U)$ and $DU + U^{-1}D = -D^2$, then we can write

$$U + D - \lambda I = -\lambda(U^{-1} - \lambda^{-1}I + D)(U + D). \tag{2}$$

First, let $\lambda \notin {}^{B}\sigma_{W,\varepsilon}(U)$. Then $U-\lambda I$ is pseudo-B-Weyl operator, so $U-\lambda I+D \in {}^{B}\Phi_{\varepsilon}(X)$ and $i(U-\lambda I+D) = 0$ for all bounded operator D such that $\|D\| < \varepsilon$. By Eq (2) we obtain $(U^{-1} - \lambda^{-1}I + D)(U + D) \in {}^{B}\Phi(X)$. In the other hand, we have

$$(U^{-1} + D)S + T(U + D) = I + \lambda^{-1}S.$$

Hence, we have that

$$(U + D - \lambda^{-1}I)S + T(U + D) = I. \tag{3}$$

Observe from Lemma 1.2 (i) that $(U + D - \lambda^{-1}I) \in {}^{B}\Phi(X)$ and $(U + D) \in {}^{B}\Phi(X)$. Again, using Lemma 1.2 (iv) and Eq 2, we obtain

$$i(U^{-1} - \lambda^{-1}I + D)(U + D) = i(U^{-1} - \lambda^{-1}I + D) + i(U + D) = 0.$$

Since $0 \in \rho(U + D)$, then $U + D \in {}^{B}\Phi(X)$ and $i(U + D) = 0$. Therefore $i(U^{-1} - \lambda^{-1}I + D) = 0$, we deduce that $(U + D - \lambda^{-1}I) \in BW_{\varepsilon}(X)$ for all bounded operator D such that $\|D\| < \varepsilon$. Furthermore, $(U - \lambda^{-1}I)$ is pseudo-B-Weyl operator. Consequently, $\frac{1}{\lambda} \notin {}^{B}\sigma_{W,\varepsilon}(U)$. Therefore

$$\frac{1}{\lambda} \in {}^{B}\sigma_{W,\varepsilon}(U^{-1}) \text{ implies that } \lambda \in {}^{B}\sigma_{W,\varepsilon}(U). \tag{4}$$

We now show the inverse implication of (4). Suppose that $\lambda^{-1} \notin {}^{B}\sigma_{W,\varepsilon}(U^{-1})$, then $(U - \lambda(U^{-1} - \lambda^{-1}I)$ is pseudo-B-Weyl operator, so $(U^{-1} - \lambda^{-1}I + D) \in {}^{B}\Phi_{\varepsilon}(X)$ and $i(U^{-1} - \lambda^{-1}I + D) = 0$ for all bounded operator D such that $\|D\| < \varepsilon$. Since $0 \in \rho(U + D)$, then $U + D \in {}^{B}\Phi(X)$ and $i(U + D) = 0$. Observe from Lemma 1.2 (i) that $(U^{-1} - \lambda^{-1} + D)(U + D) \in {}^{B}\Phi(X)$. So, by Eq (2), Eq (3) and Lemma 1.2 (iv), we deduce that

$$(U + D - \lambda I) \in {}^{B}\Phi(X) \text{ and } i(U + D - \lambda I) = 0, \ \forall \|D\| < \varepsilon,$$

i.e $(U - \lambda^{-1}I)$ is pseudo-B-Weyl operator. Consequently, we deduce that $\lambda \notin {}^{B}\sigma_{W,\varepsilon}(U)$. Therefore

$$\lambda \in {}^{B}\sigma_{W,\varepsilon}(U) \text{ implies that } \frac{1}{\lambda} \in {}^{B}\sigma_{W,\varepsilon}(U^{-1}),$$

and that concludes the proof. □

Corollary 2.2. *Let $\varepsilon > 0$. Let U, T and S be commuting operators an a Banach space X. If for all operator D such that $\|D\| < \varepsilon$, $DU + U^{-1}D = -D^2$ and if $0 \in \rho(U) \cap \rho(U + D)$ then, for every $\lambda \neq 0$ satisfying $(U^{-1} + D)S + T(U + D) = I + \lambda^{-1}S$, we have*

$$\lambda \in {}^{B}\sigma_{F,\varepsilon}(U) \text{ if and only if } \frac{1}{\lambda} \in {}^{B}\sigma_{F,\varepsilon}(U^{-1}), \tag{5}$$

and

$$\lambda \in {}^{B}\sigma_{i,\varepsilon}(U) \text{ if and only if } \frac{1}{\lambda} \in {}^{B}\sigma_{i,\varepsilon}(U^{-1}), \text{ for } i = 1, 2. \tag{6}$$

Theorem 2.3. *Let $\varepsilon > 0$. Let U, T and S be commuting operators an a Banach space X. For all operator D such that $\|D\| < \varepsilon$ satisfying $(U + D)T + SV = I + \lambda(T + S)$, for every $\lambda \neq 0$. Then,*

(i) if $(U + D)V \in \mathcal{F}_0(X)$, then

$${}^{B}\sigma_{F,\varepsilon}(U + V)\backslash\{0\} = [\,{}^{B}\sigma_{F,\varepsilon}(U) \cup {}^{B}\sigma_{F}(V)]\backslash\{0\}.$$

(ii) if $(U + D)V \in \mathcal{F}_0(X)$, then

$${}^{B}\sigma_{i,\varepsilon}(U + V)\backslash\{0\} = [\,{}^{B}\sigma_{i,\varepsilon}(U) \cup {}^{B}\sigma_{i}(V)]\backslash\{0\}, \text{ for } i = 1, 2.$$

Proof. Let λ be a non-zero complex number. Since $0 \in \rho(U)$, then we can write

$$(U + D - \lambda I)(V - \lambda I) = (U + D)V - \lambda(U + V + D - \lambda I). \quad (7)$$

(i) First, let $\lambda \notin [\,{}^{B}\sigma_{F,\varepsilon}(U) \cup {}^{B}\sigma_{F}(V)] \cup \{0\}$. Then $U - \lambda I \in {}^{B}\Phi_{\varepsilon}(X)$ and $V - \lambda I\, {}^{B}\Phi(X)$, so $U - \lambda I + D \in {}^{B}\Phi(X)$ for all bounded operator D such that $\|D\| < \varepsilon$ and $(V - \lambda I) \in {}^{B}\Phi(X)$. In the other hand, we have $(U + D)T + SV = I + \lambda(T + S)$ Hence, we have that

$$(U + D - \lambda I)T + S(V - \lambda I) = I. \quad (8)$$

Observe from Lemma 1.2 (i) that $(U+D-\lambda I)(V-\lambda I) \in {}^{B}\Phi(X)$. Since $(U+D)V \in \mathcal{F}_0(X)$, by Eq (7) and Lemma 1.1, we see that $(U + V - \lambda I + D) \in {}^{B}\Phi(X)$ for all bounded operator D such that $\|D\| < \varepsilon$ i.e $(U + V - \lambda I)$ is pseudo-B-Fredholm operator. Consequently, we deduce that $\lambda \notin {}^{B}\sigma_{F,\varepsilon}(U + V)$. Therefore

$${}^{B}\sigma_{F,\varepsilon}(U + V)\backslash\{0\} \subseteq [\,{}^{B}\sigma_{F,\varepsilon}(U) \cup {}^{B}\sigma_{F}(V)]\backslash\{0\}. \quad (9)$$

We now show the inverse inclusion of (9). Let $\lambda \notin {}^{B}\sigma_{F,\varepsilon}(U + V)\backslash \cup \{0\}$.Then $U + V - \lambda I$ is pseudo-B-fredholm operator, so $U + V - \lambda I + D \in {}^{B}\Phi_{\varepsilon}(X)$ for all bounded operator D such that $\|D\| < \varepsilon$. Since $(U + D)V \in \mathcal{F}_0(X)$, then applying Eq (7) and Lemma 1.1 we have

$$(U + D - \lambda I)(V + D) \in {}^{B}\Phi_{\varepsilon}(X). \quad (10)$$

By taking account of (10), Eq (7) and Lemma 1.2 (i), we see that $(U+D-\lambda I) \in {}^{B}\Phi_{\varepsilon}(X)$ and $(V-\lambda I) \in {}^{B}\Phi_{\varepsilon}(X)$. Therefore $\lambda \notin {}^{B}\sigma_{F}(U+D) \cup {}^{B}\sigma_{F}(V)$ for all bounded operator D such that $\|D\| < \varepsilon$, i.e $\lambda \notin {}^{B}\sigma_{F,\varepsilon}(U) \cup {}^{B}\sigma_{F}(V)$. Consequently, we deduce that

$$[\,{}^{B}\sigma_{F,\varepsilon}(U) \cup {}^{B}\sigma_{F}(V)]\backslash\{0\} \subseteq {}^{B}\sigma_{F,\varepsilon}(U + V)\backslash\{0\}.$$

(ii) For i = 1, you can apply a similar proof as in (i) by substituting ${}^{B}\Phi(X)$ with ${}^{B}\Phi_1(X)$ and ${}^{B}\sigma_F(.)$ with ${}^{B}\sigma_1(.)$, and making use of part (ii) of Lemma l.2.
For $i = 2$, the same argument as in (i) can be employed, but this time, replace ${}^{B}\Phi(X)$ with ${}^{B}\Phi_2(X)$ and ${}^{B}\sigma_F(.)$ with ${}^{B}\sigma_2(.)$, and utilize part (iii) of Lemma l.2. □

Corollary 2.4. *Let $\varepsilon > 0$. Let U, T and S be commuting operators an a Banach space X. For all operator D such that $\|D\| < \frac{\varepsilon}{2}$ satisfying $(U + D)T + S(D + V) = I + \lambda(T + S)$, for every $\lambda \neq 0$. Then,*

(i) if $(U + D)(V + D) \in \mathcal{F}_0(X)$, then

$${}^{B}\sigma_{F,\varepsilon}(U + V)\backslash\{0\} = [\,{}^{B}\sigma_{F,\frac{\varepsilon}{2}}(U) \cup {}^{B}\sigma_{F,\frac{\varepsilon}{2}}(V)]\backslash\{0\}.$$

(ii) if $(U + D)(V + D) \in \mathcal{F}_0(X)$, then

$${}^{B}\sigma_{i,\varepsilon}(U + V)\backslash\{0\} = [\,{}^{B}\sigma_{i,\frac{\varepsilon}{2}}(U) \cup {}^{B}\sigma_{i,\frac{\varepsilon}{2}}(V)]\backslash\{0\}, \text{ for } i = 1,\ 2.$$

Theorem 2.5. *Let $\varepsilon > 0$. Let U, T and S be commuting operators an a Banach space X. For all operator D such that $\|D\| < \varepsilon$ satisfying $(U + D)T + SV = I + \lambda(T + S)$, for every $\lambda \neq 0$. If $(U + D)V \in \mathcal{F}_0(X)$, then*

$${}^{B}\sigma_{W,\varepsilon}(U + V)\backslash\{0\} \subset [\,{}^{B}\sigma_{W,\varepsilon}(U) \cup {}^{B}\sigma_{W}(V)]\backslash\{0\}. \quad (11)$$

Moreover, if $i(V - \lambda I) = 0$, then

$${}^{B}\sigma_{W,\varepsilon}(U + V)\backslash\{0\} = [\,{}^{B}\sigma_{W,\varepsilon}(U) \cup {}^{B}\sigma_{W}(V)]\backslash\{0\}.$$

Proof. (i) First, let $\lambda \notin [\ {}^{B}\sigma_{W,\varepsilon}(U) \cup\ {}^{B}\sigma_{W}(V)] \cup \{0\}$. Then $(U - \lambda I + D) \in\ {}^{B}\Phi(X)$ and $(V - \lambda I) \in\ {}^{B}\Phi(X)$ and $i(U + D - \lambda I) = i(V - \lambda I) = 0$ for all bounded operator D such that $\|D\| < \varepsilon$. In the other hand, we have $(U + D)T + SV = I + \lambda(T + S)$ Hence, we have that

$$(U + D - \lambda I)T + S(V - \lambda I) = I. \tag{12}$$

Observe from Lemma 1.2 (iv) that $(U+D-\lambda I)(V-\lambda I) \in\ {}^{B}\Phi(X)$ and $i(U+D-\lambda I)(V-\lambda I)i(U+D-\lambda I)+i(V-\lambda I) = 0$. Since $(U + D)V \in \mathcal{F}_0(X)$, by Eq (7) and Lemma 1.1, we see that

$$(U + V + D - \lambda I) \in\ {}^{B}\Phi(X) \ \text{ and } \ i(U + V + D - \lambda I) = 0 \ \ \forall \|D\| < \varepsilon.$$

i.e $(U + V - \lambda I)$ is pseudo-B-Weyl operator. Therefore, $\lambda \notin\ {}^{B}\sigma_{W,\varepsilon}(U + V)$. Consequently, we deduce that

$${}^{B}\sigma_{W,\varepsilon}(U + V)\backslash\{0\} \subseteq [\ {}^{B}\sigma_{W,\varepsilon}(U) \cup\ {}^{B}\sigma_{W}(V)]\backslash\{0\}. \tag{13}$$

We now show the inverse inclusion of (13). Let $\lambda \notin\ {}^{B}\sigma_{W,\varepsilon}(U + V) \cup \{0\}$. Then $(U + V - \lambda I)$ is pseudo-B-Weyl operator, so $(U + V - \lambda I + D) \in\ {}^{B}\Phi(X)$ and $i(U + V - \lambda I + D) = 0$ for all bounded operator D such that $\|D\| < \varepsilon$. Since $(U + D)V \in \mathcal{F}_0(X)$, then applying Eq (7) and Lemma 1.1 we have

$$(U + D - \lambda I)(V - \lambda I) \in\ {}^{B}\Phi(X) \ \text{ and } \ i(U + D - \lambda I)(V - \lambda I) = i(U + D - \lambda I) + i(V - \lambda I) = 0. \tag{14}$$

Now, by taking account of (14), Eq (12) and Lemma 1.2 (i), we have that $(U + D - \lambda I) \in\ {}^{B}\Phi(X)$ and $(V - \lambda I) \in\ {}^{B}\Phi(X)$. Indeed, since $i(V - \lambda I) = 0$,by (14), we see that $i(U + D - \lambda I) = 0$. So we conclude $\lambda \notin\ {}^{B}\sigma_{W,\varepsilon}(U) \cup\ {}^{B}\sigma_{W}(V)$. Therefore,

$$[\ {}^{B}\sigma_{W,\varepsilon}(U) \cup\ {}^{B}\sigma_{W}(V)]\backslash\{0\} \subset\ {}^{B}\sigma_{W,\varepsilon}(U + V)\backslash\{0\}.$$

Thus, we obtain

$${}^{B}\sigma_{W,\varepsilon}(U + V)\backslash\{0\} = [\ {}^{B}\sigma_{W,\varepsilon}(U) \cup\ {}^{B}\sigma_{W}(V)]\backslash\{0\}.$$

□

Corollary 2.6. *Let $\varepsilon > 0$. Let U, T and S be commuting operators an a Banach space X. For all operator D such that $\|D\| < \frac{\varepsilon}{2}$ satisfying $(U + D)T + S(D + V) = I + \lambda(T + S)$, for every $\lambda \neq 0$. If $(U + D)(V + D) \in \mathcal{F}_0(X)$, then*

$${}^{B}\sigma_{W,\varepsilon}(U + V)\backslash\{0\} \subset [\ {}^{B}\sigma_{W,\frac{\varepsilon}{2}}(U) \cup\ {}^{B}\sigma_{W,\frac{\varepsilon}{2}}(V)]\backslash\{0\}. \tag{15}$$

Moreover, if $i(U + D - \lambda) = 0$, then

$${}^{B}\sigma_{W,\varepsilon}(U + V)\backslash\{0\} = [\ {}^{B}\sigma_{W,\frac{\varepsilon}{2}}(U) \cup\ {}^{B}\sigma_{W,\frac{\varepsilon}{2}}(V)]\backslash\{0\}. \tag{16}$$

Remark 2.7. *(i) If $0 \in \overline{\left[[\ {}^{B}\sigma_{W,\varepsilon}(U) \cup\ {}^{B}\sigma_{W}(V)]\backslash\{0\}\right]}$, then $0 \in\ {}^{B}\sigma_{W,\varepsilon}(U+V)$ and ${}^{B}\sigma_{W,\varepsilon}(U+V) =\ {}^{B}\sigma_{W,\varepsilon}(U) \cup {}^{B}\sigma_{W}(V)$.*

(ii) If $0 \in \overline{\left[[\ {}^{B}\sigma_{W,\frac{\varepsilon}{2}}(U) \cup\ {}^{B}\sigma_{W,\frac{\varepsilon}{2}}(V)]\backslash\{0\}\right]}$, then $0 \in\ {}^{B}\sigma_{W,\varepsilon}(U + V)$ and ${}^{B}\sigma_{W,\varepsilon}(U + V) =\ {}^{B}\sigma_{W,\frac{\varepsilon}{2}}(U) \cup\ {}^{B}\sigma_{W,\frac{\varepsilon}{2}}(V)$. The proof of this claim stems from the closed nature of the pseudo-B-essential spectrum.

(iii) The identical outcome as in (i) holds true for Theorem 2.3 when substituting ${}^{B}\sigma_{W,\varepsilon}(.)$ with ${}^{B}\sigma_{F,\varepsilon}(.)$ or ${}^{B}\sigma_{i,\varepsilon}(.)$ for $i = 1, 2$.

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