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Pseudo core invertibility in rings and its applications

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Abstract. We present new additive results regarding the pseudo core inverse in a ring with involution. We establish the necessary and sufficient conditions under which the sum of two pseudo core invertible elements in a ring is also pseudo core invertible. As an application, we investigate the pseudo core invertibility of operator matrices. These findings extend the additive result on pseudo core invertibility by Gao and Chen [Comm. Algebra, 46 (2018), 38-50].

1. Introduction

An involution of a ring R is an anti-automorphism whose square is the identity map 1. A ring R with involution $*$ is called a *-ring, e.g., C^*-algebra. Let R be a *-ring. An element $a \in R$ has p-core inverse (i.e., pseudo core inverse) if there exist $x \in R$ and $k \in \mathbb{N}$ such that

$$xa^{k+1} = a^k, ax^2 = x, (ax)^* = ax.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{D}}$. An element $a \in R$ has Drazin inverse provided that there exists $x \in R$ such that

$$xa^{k+1} = a^k, ax^2 = x, ax = xa,$$

where k is the index of a (denoted by $i(a)$), i.e., the smallest k such that the previous equations are satisfied. Such x is unique if exists, denoted by a^D, and called the Drazin inverse of a. As is well known, a square complex matrix A has group inverse if and only if $rank(A^k) = rank(A^{k+1})$. The p-core invertibility in a ring is attractive. This notion was introduced by Gao and Chen in 2018 (see [6]). This is a natural extension of the core inverse which is the first studied by Baksalary and Trenkler for a complex matrix in 2010 (see [1]). A matrix $A \in C^{n\times n}$ has core inverse $A^{\textcircled{\#}}$ if and only if $AA^{\textcircled{\#}} = P_A$ and $\mathcal{R}(A^{\textcircled{\#}}) \subseteq \mathcal{R}(A)$, where P_A is the projection on $\mathcal{R}(A)$ (see [1]). Rakic et al. (see [12]) generalized the core inverse of a complex to the case of an element in a ring. An element a in a ring R has core inverse if and only if there exist $x \in R$ such that

$$a = axa, xR = aR, Rx = Ra^*.$$

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If such x exists, it is unique, and denote it by $a^{\textcircled{\#}}$. Recently, many authors have studied core and p-core inverses from many different views, e.g., [4, 5, 7, 9–15]. An element $a \in R$ has $(1,3)$ inverse provided that there exists some $x \in R$ such that $a = axa$ and $(ax)^* = ax$. We list several characterizations of p-core inverse.

Theorem 1.1. *(see [6, Theorem 2.3 and Theorem 2.5], [14, Theorem 3.1]) Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) $a \in R^{\textcircled{D}}$.
(2) $a \in R^D$ *and* a^k *has* $(1,3)$ *inverse, where* $k = i(a)$.
(3) *There exists* $x \in R$ *such that* $a^n x a^n = a$ *and* $a^n R = xR = x^* R$ *for some* $n \in \mathbb{N}$.
(4) $a^m \in R$ *has core inverse for some positive integer* m.

Let $a, b \in R$ have p-core inverses. In [6, Theorem 4.4], Gao and Chen proved that $a+b$ has p-core inverse when $ab = ba = 0$ and $a^*b = 0$. This inspires us to investigate new additive properties for p-core invertibility in a *-ring.

In Section 2, we focus on additive findings concerning p-core invertible elements within a ring. When $ab = ba$ and $a^*b = ba^*$, we delineate the necessary and sufficient conditions that ensure $a + b$, an element of the ring R, is p-core invertible.

Let X be a Hilbert space. Let $\mathcal{B}(X)$ represent the *-ring of all bounded linear operators acting from X to itself. The involution in this context is specified as the conjugate transpose of the bounded linear operators. In Section 3, we extend our additive findings to the realm of bounded linear operators, and we deduce several criteria that govern when a block operator matrix admits a p-core inverse.

Throughout the paper, all *-rings are associative with an identity. An element $p \in R$ is a projection if $p^2 = p = p^*$. $R^D, R^{\textcircled{D}}$ and R^{nil} denote the sets of all Drazin, p-core invertible and nilpotent elements in R, respectively. Let $a \in R^D$. We use a^π to stand for the spectral idempotent of a corresponding to $\{0\}$, i.e., $a^\pi = 1 - aa^D$.

2. Key lemmas

To establish the primary findings, we require several lemmas. We commence with the following:

Lemma 2.1. *([6, Proposition 4.2])) Let $a, b \in R^{\textcircled{D}}$. If $ab = ba$ and $a^*b = ba^*$, then $a^{\textcircled{D}}b = ba^{\textcircled{D}}$.*

Lemma 2.2. *([6, Theorem 4.3])) Let $a, b \in R^{\textcircled{D}}$. If $ab = ba$ and $a^*b = ba^*$, then $ab \in R^{\textcircled{D}}$ and $(ab)^{\textcircled{D}} = a^{\textcircled{D}}b^{\textcircled{D}}$.*

Lemma 2.3. *([6, Theorem 4.4])) Let $a, b \in R^{\textcircled{D}}$. If $ab = ba = 0$ and $a^*b = 0$, then $a + b \in R^{\textcircled{D}}$.*

Lemma 2.4. *Let $a \in R^{\textcircled{D}}$ and $b \in R$. Then the following are equivalent:*

(1) $(1 - a^{\textcircled{D}}a)b = 0$.
(2) $(1 - aa^{\textcircled{D}})b = 0$.

Proof. (1) ⇒ (2) Since $(1 - a^{\textcircled{D}}a)b = 0$, we have $b = a^{\textcircled{D}}ab$. Hence, $(1 - aa^{\textcircled{D}})b = (1 - aa^{\textcircled{D}})a^{\textcircled{D}}ab = 0$.

(2) ⇒ (1) Let $m = i(a)$. Then $a^{\textcircled{D}} = a^m a^D (a^m)^{(1,3)}$ (see [6, Theorem 2.3]). Since $(1 - aa^{\textcircled{D}})b = 0$, we get $b = aa^{\textcircled{D}}b$. Therefore

$$\begin{array}{rcl}
(1 - a^{\textcircled{D}}a)b & = & (1 - a^{\textcircled{D}}a))aa^{\textcircled{D}}b \\
& = & (1 - a^{\textcircled{D}}a))a[a^m a^D (a^m)^{(1,3)}]b \\
& = & (1 - a^{\textcircled{D}}a))a^{m+1} a^D (a^m)^{(1,3)}b \\
& = & (a^m - a^{\textcircled{D}}a^{m+1})aa^D (a^m)^{(1,3)}b \\
& = & [a^m - a^D (a^m (a^m)^{(1,3)} a^m) a] aa^D (a^m)^{(1,3)}b \\
& = & (a^m - a^D a^{m+1}) aa^D (a^m)^{(1,3)}b \\
& = & 0,
\end{array}$$

as desired. □

Let $a, p^2 = p \in R$ and $p^\pi = 1 - p$. Then a has the Pierce decomposition relative to p, i.e., $p = pap + pap^\pi + p^\pi ap + p^\pi ap^\pi$. We denote it by a matrix form: $\begin{pmatrix} pap & app^\pi \\ p^\pi ap & p^\pi ap^\pi \end{pmatrix}_p$. We now derive

Lemma 2.5.

(1) Let $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Then $x \in M_2(R)^{\mathbb{D}}$ and $x^{\mathbb{D}} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ if and only if $a, d \in R^{\mathbb{D}}$ and $\sum\limits_{i=1}^{m} a^{i-1}a^\pi bd^{m-i} = 0$ for some $m \geq i(a)$.

(2) Let p be a projection and $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}_p$. Then $x \in R^{\mathbb{D}}$ and $p^\pi x^{\mathbb{D}} p = 0$ if and only if $a \in (pRp)^{\mathbb{D}}, d \in (p^\pi Rp^\pi)^{\mathbb{D}}$ and $\sum\limits_{i=1}^{m} a^{i-1}a^\pi bd^{m-i} = 0$ for some $m \geq i(a)$.

Proof. (1). " $\Rightarrow$ " Since $x \in R^{\mathbb{D}}$, it follows by [6, Theorem 2.5] that $x^m \in R^{\#}$, where $m = i(x)$. Then $m \geq i(a)$. In this case, $(x^m)^{\#} = (x^{\mathbb{D}})^m$ and $x^{\mathbb{D}} = x^{m-1}(x^m)^{\#}$. By hypothesis, we can write $x^{\mathbb{D}} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, and so $(x^{\mathbb{D}})^m = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. This implies that $(x^m)^{\#} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. Obviously, we have $x^m = \begin{pmatrix} a^m & b_m \\ 0 & d^m \end{pmatrix}_p$, where $b_1 = b, b_m = ab_{m-1} + bd^{m-1}$. By induction, we get $b_m = \sum\limits_{i=1}^{m} a^{i-1}bd^{m-i}$. In light of [13, Theorem 2.5], $a^m, d^m \in R^{\#}$ and $(a^m)^\pi b_m = 0$. By virtue of [6, Theorem 2.5], $a, d \in R^{\mathbb{D}}$.

Since $a \in R^D$, we see that $a^m \in R^D$ and $(a^m)^D = (a^D)^m$. Hence, $(a^m)^\pi = 1 - a^m(a^m)^D = 1 - a^m(a^D)^m = 1 - (aa^D)^m = 1 - aa^D = a^\pi$. Therefore $\sum\limits_{i=1}^{m} a^{i-1}a^\pi bd^{m-i} = (a^m)^\pi b_m = 0$, as required.

" $\Leftarrow$ " Since $a, d \in R^{\mathbb{D}}$, it follows by [6, Theorem 2.5] that $a^k, d^k \in R^{\#}$, where $k = max\{i(a), i(d)\}$. Write $x^k = \begin{pmatrix} a^k & b_k \\ 0 & d^k \end{pmatrix}$, where $b_1 = b, b_k = ab_{k-1} + bd^{k-1}$. Then $b_m = \sum\limits_{i=1}^{m} a^{i-1}bd^{m-i}$. By hypothesis, we have $a^\pi b_m = 0$. As in the preceding discussion, we prove that $a^\pi b_k = 0$. In light of [13, Theorem 2.5], $x^k \in R^{\#}$. According to [6, Theorem 2.5], we get $x \in R^{\mathbb{D}}$. Further, $p^\pi(x^k)^{\#}p = 0$, and so $p^\pi(x^{\mathbb{D}})^k p = 0$, Write $(x^{\mathbb{D}})^k = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. Then $x^{\mathbb{D}} = x^{k-1}(x^{\mathbb{D}})^k = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. This implies that $p^\pi x^{\mathbb{D}} p = 0$, as desired.

(2). " $\Rightarrow$ " Since $x \in R^{\mathbb{D}}$, we have $x^m \in R^{\#}$, where $m = i(x) \geq i(a)$. Moreover, $(x^m)^{\#} = (x^{\mathbb{D}})^m$ and $x^{\mathbb{D}} = x^{m-1}(x^m)^{\#}$. Since $p^\pi x^{\mathbb{D}} p = 0$, we can write $x^{\mathbb{D}} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}_p$, and so $(x^{\mathbb{D}})^m = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}_p$. As in the proof in (1), $a^m, d^m \in pR^{\#}p \subseteq (pRp)^{\#}$ and $(a^m)^\pi b_m = 0$. In view of [6, Theorem 2.5], $a \in (pRp)^{\mathbb{D}}$ and $d \in (p^\pi Rp^\pi)^{\mathbb{D}}$. Moreover, we have $\sum\limits_{i=1}^{m} a^{i-1}a^\pi bd^{m-i} = 0$, as required.

" $\Leftarrow$ " Since $a \in (pRp)^{\mathbb{D}}, d \in (p^\pi Rp^\pi)^{\mathbb{D}}$, we easily check that $a^k \in (pRp)^{\#}$, $d^k \in (p^\pi Rp^\pi)^{\#}$, where $k = max\{i(a), i(b)\}$. Write $x^k = \begin{pmatrix} a^k & b_k \\ 0 & d^k \end{pmatrix}_p$, where $b_1 = b, b_k = ab_{k-1} + bd^{k-1}$. Similarly to the discussion in (1), $x \in R^{\mathbb{D}}$ and $x^{\mathbb{D}} = x^{k-1}(x^{\mathbb{D}})^k = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}_p$, as asserted. □

3. Main Results

This section is dedicated to examining the p-core inverse of the sum of two p-core invertible elements within a ring. We are now prepared to demonstrate the following:

Theorem 3.1. *Let $a, b \in R^{\mathbb{D}}$. If $ab = ba$ and $a^*b = ba^*$, then the following are equivalent:*

(1) $a + b \in R^{\mathbb{D}}$ and $a^{\pi}(a + b)^{\mathbb{D}}aa^{\mathbb{D}} = 0$.
(2) $1 + a^{\mathbb{D}}b \in R^{\mathbb{D}}$ and

$$\sum_{i=1}^{n}[1 + a^{\mathbb{D}}b]^{i-1}a^{i-1}[1 + a^{\mathbb{D}}b]^{\pi}a[aa^{\mathbb{D}} - a^{\mathbb{D}}a](a + b)^{n-i} = 0$$

for some $n \geq i(1 + a^{\mathbb{D}}b)$.

Proof. Since $ab = ba$ and $a^*b = ba^*$, it follows by Lemma 2.1 that $a^{\mathbb{D}}b = ba^{\mathbb{D}}$. Let $p = aa^{\mathbb{D}}$. Then $p^{\pi}bp = (1 - aa^{\mathbb{D}})baa^{\mathbb{D}} = (1 - aa^{\mathbb{D}})aa^{\mathbb{D}}b = 0$. Moreover, we have $pbp^{\pi} = aa^{\mathbb{D}}b(1 - aa^{\mathbb{D}}) = aba^{\mathbb{D}}(1 - aa^{\mathbb{D}}) = 0$. Evidently, $a^D = a^D aa^D$ and $a^{\mathbb{D}} = a^m a^D (a^m)^{(1,3)}$ with $m = i(a)$. Then we compute that

$$\begin{array}{rcl} p^{\pi}ap & = & (1 - aa^{\mathbb{D}})aaa^{\mathbb{D}} \\ & = & (1 - aa^{\mathbb{D}})aa^D(a^m)(a^m)^{(1,3)} \\ & = & (1 - aa^{\mathbb{D}})a^{m+1}(a^D)^m(a^m)(a^m)^{(1,3)} \\ & = & (a^{m+1} - aa^{\mathbb{D}}a^{m+1})(a^D)^m(a^m)(a^m)^{(1,3)} \\ & = & 0. \end{array}$$

So we get

$$a = \begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & 0 \\ 0 & b_4 \end{pmatrix}_p.$$

Hence

$$a + b = \begin{pmatrix} a_1 + b_1 & a_2 \\ 0 & a_4 + b_4 \end{pmatrix}_p.$$

(i) Clearly, we check that

$$\begin{array}{rcl} (1 - a^{\mathbb{D}}a)a^2a^{\mathbb{D}} & = & (1 - a^{\mathbb{D}}a)a^2a^Da^m(a^m)^{(1,3)} \\ & = & (1 - a^{\mathbb{D}}a)a^{m+2}a^D(a^m)^{(1,3)} \\ & = & (a^m - a^{\mathbb{D}}a^{m+1})a^2a^D(a^m)^{(1,3)} \\ & = & 0. \end{array}$$

In view of Lemma 2.4, $(1 - aa^{\mathbb{D}})a^2a^{\mathbb{D}} = 0$. Hence

$$a_1 = aa^{\mathbb{D}}a^2a^{\mathbb{D}} = a^2a^{\mathbb{D}}.$$

Obviously, $(1 - aa^{\mathbb{D}})baa^{\mathbb{D}} = b(1 - aa^{\mathbb{D}})aa^{\mathbb{D}} = 0$. It follows by Lemma 2.4 that $(1 - a^{\mathbb{D}}a)baa^{\mathbb{D}} = 0$. Hence we have $b_1 = aa^{\mathbb{D}}baa^{\mathbb{D}} = baa^{\mathbb{D}} = a^{\mathbb{D}}abaa^{\mathbb{D}} = a^{\mathbb{D}}ba^2a^{\mathbb{D}}$, and then

$$\begin{array}{rcl} a_1 + b_1 & = & a^2a^{\mathbb{D}} + a^{\mathbb{D}}ba^2a^{\mathbb{D}} \\ & = & (1 + a^{\mathbb{D}}b)a^2a^{\mathbb{D}} \\ & \in & R^{\mathbb{D}}. \end{array}$$

This implies that

$$(a_1 + b_1)^{i-1} = (1 + a^{\mathbb{D}}b)^{i-1}(a^2a^{\mathbb{D}})^{i-1} = (1 + a^{\mathbb{D}}b)^{i-1}a^ia^{\mathbb{D}}.$$

Since $(1 + a^{\mathbb{D}}b)a^2a^{\mathbb{D}} = a^2a^{\mathbb{D}}(1 + a^{\mathbb{D}}b)$, we verify that

$$\begin{array}{rcl} (a_1 + b_1)^D & = & [(1 + a^{\mathbb{D}}b)a^2a^{\mathbb{D}}]^D \\ & = & (1 + a^{\mathbb{D}}b)^D[a^2a^{\mathbb{D}}]^D \\ & = & (1 + a^{\mathbb{D}}b)^Da^{\mathbb{D}}. \end{array}$$

Thus

$$\begin{array}{rcl} (a_1+b_1)^{\pi} & = & 1-(a_1+b_1)(a_1+b_1)^D \\ & = & 1-(1+a^{\mathbb{D}}b)a^2a^{\mathbb{D}}(1+a^{\mathbb{D}}b)^Da^{\mathbb{D}} \\ & = & (1+a^{\mathbb{D}}b)(1+a^{\mathbb{D}}b)^D[a^2a^{\mathbb{D}}a^{\mathbb{D}}] \\ & = & 1-(1+a^{\mathbb{D}}b)(1+a^{\mathbb{D}}b)^Daa^{\mathbb{D}}. \end{array}$$

(ii) Obviously, we have $(1-aa^{\mathbb{D}})aaa^{\mathbb{D}} = a^2a^{\mathbb{D}} - aa^{\mathbb{D}}aaa^{\mathbb{D}} = 0$. Then

$$a_4 = (1-aa^{\mathbb{D}})a(1-aa^{\mathbb{D}}) = (1-aa^{\mathbb{D}})a.$$

Since $(1-aa^D)a^{m+1} = 0$, we deduce that

$$\begin{array}{rcl} a_4^{m+1} & = & (1-aa^{\mathbb{D}})a^{m+1} \\ & = & [1-a(a^ma^D(a^m)^{(1,3)})]a^{m+1} \\ & = & [a^m-aa^D(a^m(a^m)^{(1,3)})a^m)]a \\ & = & (a^m-a^Da^{m+1})a \\ & = & (1-aa^D)a^{m+1} = 0, \end{array}$$

and so $a_4\in R^{nil}$. Moreover,

$$b_4 = (1-aa^{\mathbb{D}})b(1-aa^{\mathbb{D}}) = (1-aa^{\mathbb{D}})b.$$

Since $bp^{\pi} = p^{\pi}b, b^*p^{\pi} = (p^{\pi}b)^* = (bp^{\pi})^* = p^{\pi}b^*$. In light of Lemma 2.2, $b_4 = p^{\pi}b\in R^{\mathbb{D}}$ and $b_4^{\mathbb{D}} = p^{\pi}b^{\mathbb{D}}$.

$$a_4+b_4 = (1-aa^{\mathbb{D}})(a+b)$$

$$(a_4+b_4)^{m-i} = (1-aa^{\mathbb{D}})(a+b)^{m-i}.$$

(1) ⇒ (2) We have

$$(a+b)^{\mathbb{D}} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}_p.$$

Then $[p(a+b)p]^{\mathbb{D}} = \alpha$. That is, $(a+b)aa^{\mathbb{D}}\in R^{\mathbb{D}}$.

We observe that

$$\begin{array}{rcl} 1+a^{\mathbb{D}}b & = & [1-aa^{\mathbb{D}}]+[aa^{\mathbb{D}}+a^{\mathbb{D}}b] \\ & = & [1-aa^{\mathbb{D}}]+[aa^{\mathbb{D}}+ba^{\mathbb{D}}] \\ & = & [1-aa^{\mathbb{D}}]+[a+b]a^{\mathbb{D}} \end{array}$$

By hypothesis and $(aa^{\mathbb{D}})^* = aa^{\mathbb{D}}$, we easily check that

$$\begin{array}{rcl} [(a+b)aa^{\mathbb{D}}]a^{\mathbb{D}} & = & a^{\mathbb{D}}[(a+b)aa^{\mathbb{D}}], \\ \big((a+b)aa^{\mathbb{D}}\big)^*a^{\mathbb{D}} & = & a^{\mathbb{D}}\big((a+b)aa^{\mathbb{D}}\big)^*. \end{array}$$

In view of Lemma 2.2, $(a+b)a^{\mathbb{D}} = [(a+b)aa^{\mathbb{D}}]a^{\mathbb{D}}\in R^{\mathbb{D}}$. Then $(a+b)a^{\mathbb{D}}\in R^{(1,3)}$, and so we have $k\in\mathbb{N}$ and $y\in R$ such that

$$[(a+b)aa^{\mathbb{D}}]^k = [(a+b)aa^{\mathbb{D}}]^ky[(a+b)aa^{\mathbb{D}}]^k,$$

$$\big([(a+b)aa^{\mathbb{D}}]^ky\big)^* = [(a+b)aa^{\mathbb{D}}]^ky.$$

By induction, we have

$$[(a+b)a^{\mathbb{D}}]^k[a^2a^{\mathbb{D}}]^k = [(a+b)aa^{\mathbb{D}}]^k.$$

We verify that

$$\begin{array}{rl} & [(a+b)a^{\mathbb{D}}]^k[(a^2a^{\mathbb{D}})^ky][(a+b)a^{\mathbb{D}}]^k[a^2a^{\mathbb{D}}]^k \\ = & [(a+b)aa^{\mathbb{D}}]^ky[(a+b)aa^{\mathbb{D}}]^k \\ = & [(a+b)aa^{\mathbb{D}}]^k \\ = & [(a+b)a^{\mathbb{D}}]^k[a^2a^{\mathbb{D}}]^k. \end{array}$$

Clearly, $[a^2a^{\circledD}]^k(a^{\circledD})^k = aa^{\circledD}$. Then

$$\begin{aligned}
[(a+b)a^{\circledD}]^k[(a^2a^{\circledD})^k y][(a+b)a^{\circledD}]^k &= [(a+b)a^{\circledD}]^k,\\
(((a+b)a^{\circledD})^k(a^2a^{\circledD})^k y)^* &= [((a+b)aa^{\circledD})^k y]^*\\
&= ((a+b)aa^{\circledD})^k y\\
&= [(a+b)a^{\circledD}]^k(a^2a^{\circledD})^k y.
\end{aligned}$$

Therefore $[(a+b)a^{\circledD}]^k$ has $(1,3)$-inverse $(a^2a^{\circledD})^k y$. By virtue of [6, Theorem 2.3], $(a+b)a^{\circledD} \in R^{\circledD}$. Obviously, we have

$$(1-aa^{\circledD})(a+b)a^{\circledD} = (1-aa^{\circledD})^*(a+b)a^{\circledD} = 0.$$

According to Lemma 2.3, $1 + a^{\circledD}b \in R^{\circledD}$. Moreover, we have

$$\sum_{i=1}^{n}[a_1+b_1]^{i-1}(a_1+b_1)^{\pi}a_2(a_4+b_4)^{n-i} = 0$$

for some $n \geq i(a_1+b_1)$. Therefore

$$\sum_{i=1}^{n}[1+a^{\circledD}b]^{i-1}a^{i-1}[1+a^{\circledD}b]^{\pi}a[aa^{\circledD}-a^{\circledD}a](a+b)^{n-i} = 0$$

for some $n \geq i(1+a^{\circledD}b)$.

(2) $\Rightarrow$ (1) Let $x = (1+a^{\circledD}b)^{\circledD}$. Since $(1+a^{\circledD}b)aa^{\circledD} = aa^{\circledD}(1+a^{\circledD}b)$ and $(aa^{\circledD})^* = aa^{\circledD}$, we have

$$aa^{\circledD}(1+a^{\circledD}b)^* = (1+a^{\circledD}b)^*aa^{\circledD}.$$

In light of Lemma 2.1, we get $aa^{\circledD}x = xaa^{\circledD}$.

Step 1. It is easy to verify that

$$\begin{aligned}
(a^2a^{\circledD})a^{\circledD} &= aa^{\circledD} = a^{\circledD}(a^2a^{\circledD}),\\
a^{\circledD}(a^2a^{\circledD})a^{\circledD} &= a^{\circledD}(aa^{\circledD}) = a^{\circledD},\\
(a^2a^{\circledD})a^{\circledD}(a^2a^{\circledD}) &= (aa^{\circledD})(a^2a^{\circledD}) = a^2a^{\circledD}
\end{aligned}$$

Thus $a^2a^{\circledD} \in R^D$. As $1+a^{\circledD}b \in R^{\circledD}$, it follows by Theorem 1.1 that $1+a^{\circledD}b \in R^D$. Since $(1+a^{\circledD}b)a^2a^{\circledD} = (a+b)aa^{\circledD} = aa^{\circledD}(a+b) = a^2a^{\circledD}(1+a^{\circledD}b)$, it follows by [16, Lemma 2] that $(1+a^{\circledD}b)a^2a^{\circledD} \in R^D$ and

$$\begin{aligned}
[(a+b)aa^{\circledD}]^{\pi} &= [(1+a^{\circledD}b)a^2a^{\circledD}]^{\pi}\\
&= 1-(1+a^{\circledD}b)a^2a^{\circledD}(1+a^{\circledD}b)^Da^{\circledD}\\
&= 1-(1+a^{\circledD}b)(1+a^{\circledD}b)^Daa^{\circledD}.
\end{aligned}$$

Step 2. We check that

$$\begin{aligned}
[(1+a^{\circledD}b)a^2a^{\circledD}]^k[(a^{\circledD})^k x] &= [(1+a^{\circledD}b)]^k[a^2a^{\circledD}]^k[(a^{\circledD})^k x]\\
&= [(1+a^{\circledD}b)]^k x][aa^{\circledD}]
\end{aligned}$$

Hence,

$$\begin{aligned}
\Big([(1+a^{\circledD}b)a^2a^{\circledD}]^k[(a^{\circledD})^k x]\Big)^* &= [(1+a^{\circledD}b)]^k x]^*[aa^{\circledD}]^*\\
&= [(1+a^{\circledD}b)]^k x][aa^{\circledD}]\\
&= [(1+a^{\circledD}b)a^2a^{\circledD}]^k[(a^{\circledD})^k x].
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
[(1+a^{\circledD}b)a^2a^{\circledD}]^k[(a^{\circledD})^k x][(1+a^{\circledD}b)a^2a^{\circledD}]^k &= [(1+a^{\circledD}b)]^k x][aa^{\circledD}][(1+a^{\circledD}b)a^2a^{\circledD}]^k\\
&= [(1+a^{\circledD}b)]^k x[(1+a^{\circledD}b]^k[aa^{\circledD}][a^2a^{\circledD}]^k\\
&= [(1+a^{\circledD}b)]^k[a^2a^{\circledD}]^k\\
&= [(1+a^{\circledD}b)a^2a^{\circledD}]^k.
\end{aligned}$$

Accordingly, $(a+b)aa^{D} = (1+a^{D}b)a^2a^{D} \in R^{D}$.

Step 3. Case 1. $b_4 \in R^{nil}$. It is easy to verify that $a_4b_4 = (1-aa^{D})ab = (1-aa^{D})ba = b_4a_4$. Therefore $a_4+b_4 \in R^{nil} \subseteq R^{D}$.

Case 2. $b_4 \notin R^{nil}$.

Let $q = b_4b_4^{D}$. Then $p^{\pi}bp = (1-aa^{D})baa^{D} = (1-aa^{D})aba^{D} = 0$. Similarly, $pbp^{\pi} = 0$. Moreover,

$$\begin{aligned} p^{\pi}ap &= (1-aa^{D})aaa^{D} \\ &= (1-aa^{D})a(aa^{D})^m \\ &= (1-aa^{D})a^{m+1}a^{D} \\ &= 0. \end{aligned}$$

So we get

$$a_4 = \begin{pmatrix} c_1 & 0 \\ 0 & c_4 \end{pmatrix}_q, b_4 = \begin{pmatrix} d_1 & d_2 \\ 0 & d_4 \end{pmatrix}_q.$$

Hence $a_4+b_4 = \begin{pmatrix} c_1+d_1 & d_2 \\ 0 & c_4+d_4 \end{pmatrix}_q$. Here $a_1+b_1 = (a+b)aa^{D}, a_4+b_4 = p^{\pi}(a+b)p^{\pi} = p^{\pi}(a+b)$.

Step 1. $c_1 \in R^{nil}, b_1 \in R^{D}$. By the preceding discussion, we have $c_1+d_1 \in R^{D}$.

Step 2. Clearly, $(1-aa^{D})a(1-aa^{D}) = (1-aa^{D})a$. Hence, $a_4 \in R^{nil}$. As $(1-aa^{D})b(1-aa^{D}) = (1-aa^{D})b$, we see that $b_4 \in R^{nil}$. Moreover, $a_4b_4 = (1-aa^{D})ab = (1-aa^{D})ba = b_4a_4$. Therefore $a_4+b_4 \in R^{nil}$.

Step 3.

$$\begin{aligned} & \sum_{i=1}^{n}(c_1+d_1)^{i-1}(c_1+d_1)^{\pi}d_2(c_4+d_4)^{n-i} \\ = \ & (1-bb^{D})(1-aa^{D})\sum_{i=1}^{n}[1+a^{D}b]^{i-1}a^{i-1}[1+a^{D}b]^{\pi} \\ & a[aa^{D}-a^{D}a](a+b)^{n-i} \\ = \ & 0. \end{aligned}$$

Therefore $a_4+b_4 \in R^{D}$. Moreover, we have

$$\begin{aligned} & \sum_{i=1}^{n}(a_1+b_1)^{i-1}(a_1+b_1)^{\pi}a_2(a_4+b_4)^{n-i} \\ = \ & \sum_{i=1}^{n}[(1+a^{D}b)^{i-1}a^{i}a^{D}][1-(1+a^{D}b)(1+a^{D}b)^{\#}aa^{D}]aa^{D}a \\ & (1-aa^{D})(a+b)^{n-i} \\ = \ & -\sum_{i=1}^{n}[1+a^{D}b]^{i-1}a^{i-1}[1+a^{D}b]^{\pi}a[aa^{D}-a^{D}a](a+b)^{n-i} \\ = \ & 0. \end{aligned}$$

Therefore $a+b \in R^{D}$. Moreover, we have $p^{\pi}(a+b)^{D}p = 0$. In view of Lemma 2.4, $a^{\pi}(a+b)^{D}aa^{D} = 0$. This completes the proof. □

Recall that $a \in R$ be *-DMP, if there exists some $n \in \mathbb{N}$ such that a^n has More-Penrose inverse and group inverse and $(a^n)^{\dagger} = (a^n)^{\#}$ (see [3]). We now derive

Corollary 3.2. *Let $a \in R$ be *-DMP, $b \in R^{D}$. If $ab = ba$ and $a^*b = ba^*$, then the following are equivalent:*

(1) $a+b \in R^{D}$.
(2) $1+a^{D}b \in R^{D}$.

Proof. Since $a \in R$ is *-DMP, it follows by [8, Theorem 2.10] that $aa^{D} = a^{D}a$. The result follows by Theorem 3.1. □

The preceding conditions $ab = ba$ and $a^*b = ba^*$ are necessary as the following shows.

Example 3.3. *Let $R = \mathbb{C}^{2\times 2}$ be the ring of 2×2 complex matrices, with conjugate transpose as the involution. Choose*

$$a = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in R.$$

*Then a is *-DMP, $a^{\textcircled{D}} = \begin{pmatrix} -i & 0 \\ 0 & 0 \end{pmatrix}$, $b^{\textcircled{D}} = 0$ and $1 + a^{\textcircled{D}}b = 1 \in R^{\textcircled{D}}$. But $a + b = \begin{pmatrix} i & 0 \\ 1 & 0 \end{pmatrix} \notin R^{\textcircled{D}}$. In this case, $ab \neq ba$.*

4. Applications

Let X and Y be Hilbert spaces. We denote by $\mathcal{B}(X, Y)$ the set of all bounded linear operators from X to Y. The objective of this section is to delve into the p-core invertibility of a block operator matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \mathcal{B}(X)^{\textcircled{D}}, B \in \mathcal{B}(X, Y), C \in \mathcal{B}(Y, X), D \in \mathcal{B}(Y)^{\textcircled{D}}$ and $BC \in \mathcal{B}(X)^{\textcircled{D}}, CB \in \mathcal{B}(Y)^{\textcircled{D}}$. Here, M is a bounded linear operator on $X \oplus Y$. For the detailed formula of $M^{\textcircled{D}}$, we leave to the readers as they can be derived by the straightforward computation according to our proof.

Theorem 4.1. *If $AB = BD, DC = CA, A^*B = BD^*, D^*C = CA^*$ and $A^{\textcircled{D}}BD^{\textcircled{D}}C$ is nilpotent, then M has p-core inverse.*

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, Q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Since A and D have p-core inverses, so has P, and that

$$P^{\textcircled{D}} = \begin{pmatrix} A^{\textcircled{D}} & 0 \\ 0 & D^{\textcircled{D}} \end{pmatrix}.$$

By hypothesis, $Q^2 = \begin{pmatrix} BC & 0 \\ 0 & CB \end{pmatrix}$ has p-core inverse. In light of [6, Theorem 2.6], Q has p-core inverse. We easily check that

$$PQ = \begin{pmatrix} 0 & 0 \\ DC & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ CA & 0 \end{pmatrix} = QP.$$

Likewise, we verify that $P^*Q = QP^*$. Moreover, we check that

$$I_{X\oplus Y} + P^{\textcircled{D}}Q = \begin{pmatrix} I_X & A^{\textcircled{D}}B \\ D^{\textcircled{D}}C & I_Y \end{pmatrix}.$$

Since $A^{\textcircled{D}}BD^{\textcircled{D}}C$ is nilpotent, we prove that $I_{X\oplus Y} + P^{\textcircled{D}}Q$ is invertible, and so it has p-core inverse. Additionally, $[I_{X\oplus Y} + P^{\textcircled{D}}Q]^{\pi} = 0$. According to Theorem 3.1, M has p-core inverse, as asserted. □

Let $T \in \mathcal{B}(X, Y)$. The conjugate transpose of T is an operator $T^* \in \mathcal{B}(Y, X)$. It is easy to see that if $T \in \mathcal{B}(X)^{\textcircled{D}}$, then $T^* \in \mathcal{B}(X)^{\textcircled{D}}$.

Corollary 4.2. *If $AB = BD, DC = CA, D^*C = CA^*, A^*B = BD^*$ and $BD^{\textcircled{D}}CA^{\textcircled{D}}$ is nilpotent, then M has p-core inverse.*

Proof. Obviously, $M^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}$. By hypothesis, we have $A^*C^* = C^*D^*, D^*B^* = B^*A^*, AC^* = C^*D, DB^* = B^*A$ and $(A^*)^{\textcircled{D}}C^*(D^*)^{\textcircled{D}}B^*$ is nilpotent. Applying Theorem 4.1 to the operator M^*, we prove that M^* has p-core inverse. Therefore M has p-core inverse, as desired. □

We are now ready to prove:

Theorem 4.3. *If $AB = BD, DC = CA, B^*A = DB^*$ and $B(CB)^{D}DC(BC)^{D}A$ is nilpotent, then M has p-core inverse.*

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, Q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

As is the proof of Theorem 4.1, P and Q have p-core inverses. Moreover, we check that

$$\begin{aligned} Q^*P &= \begin{pmatrix} 0 & C^* \\ B^* & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \\ &= \begin{pmatrix} 0 & C^*D \\ B^*A & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & AC^* \\ DB^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 & C^* \\ B^* & 0 \end{pmatrix} \\ &= PQ^*. \end{aligned}$$

Similarly, $QP = PQ$. Since $(Q^2)^{D} = \begin{pmatrix} (BC)^{D} & 0 \\ 0 & (CB)^{D} \end{pmatrix}$, it follows by [6, Theorem 2.6] that

$$\begin{aligned} Q^{D} &= Q(Q^2)^{D} \\ &= \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\begin{pmatrix} (BC)^{D} & 0 \\ 0 & (CB)^{D} \end{pmatrix} \\ &= \begin{pmatrix} 0 & B(CB)^{D} \\ C(BC)^{D} & 0 \end{pmatrix}. \end{aligned}$$

Further, we verify that

$$\begin{aligned} I_{X\oplus Y} + Q^{D}P &= I_{X\oplus Y} + \begin{pmatrix} 0 & B(CB)^{D} \\ C(BC)^{D} & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \\ &= \begin{pmatrix} I_X & B(CB)^{D}D \\ C(BC)^{D}A & I_Y \end{pmatrix}. \end{aligned}$$

Since $B(CB)^{D}DC(BC)^{D}A$ is nilpotent, we prove that $I_{X\oplus Y} + Q^{D}P$ is invertible; hence, it has p-core inverse. Additionally, $[I_{X\oplus Y} + Q^{D}P]^{\pi} = 0$. In light of Theorem 3.1, M has p-core inverse. □

Corollary 4.4. *If $AB = BD, DC = CA, C^*A = DC^*$ and $A(CB)^{D}BD(BC)^{D}\ C$ is nilpotent is nilpotent, then M has p-core inverse.*

Proof. Similarly to Corollary 4.2, it is enough to apply Theorem 4.3 to the operator M^*. □

Theorem 4.5. *If $BC = 0, CB = 0, CA = DC, AC^* = C^*D$ and*

$$\sum_{i=1}^{i(A)} A^{i-1}A^{\pi}BD^{m-i} = 0,$$

then M has p-core inverse.

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}, Q = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

Clearly, P has p-core inverse. According to Lemma 2.5, Q has p-core inverse. We check that

$$\begin{aligned} PQ &= \begin{pmatrix} 0 & 0 \\ CA & CB \end{pmatrix} \\ &= \begin{pmatrix} BC & 0 \\ DC & 0 \end{pmatrix} \\ &= QP; \\ P^*Q &= \begin{pmatrix} 0 & C^*D \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & AC^* \\ 0 & 0 \end{pmatrix} \\ &= QP^*. \end{aligned}$$

Clearly $P^{\mathcal{D}} = 0$, and so $I_{X\oplus Y} + P^{\mathcal{D}}Q = I_{X\oplus Y}$ has p-core inverse. Therefore M has p-core inverse by Theorem 3.1. □

Corollary 4.6. *If $BC = 0, CB = 0, AB = BD, A^*B = BD^*$ and*

$$\sum_{i=1}^{i(A)} CA^{i-1}A^{\pi} = 0,$$

then M has p-core inverse.

Proof. As in the discussion in Corollary 4.2, we may apply Theorem 4.5 to the bounded linear operator M^*. □

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