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Fuzzyfication of new open set types

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Abstract. In this study, we introduce fuzzy *AF*-open sets in a fuzzy topological space (X, τ). Some properties and characterizations of the fuzzy *AF*-open set are studied. Also we investigate and research the notions of fuzzy *AF*-interior and fuzzy *AF*-cluster points in a fuzzy topological space. Further fuzzy *AF*-compactness is defined. Its properties and characterizations are examined.

1. Introduction

The concept of an ordinary set was become general with the emergence of fuzzy sets in Zadeh's 1965 classical study [35]. In [15], this was further generalized by Goguen's description of L-fuzzy sets. Subsequently, Chang [6] led to an increase in the number of different fuzzy topology concepts. An alternative definition of fuzzy topology was made by Lowen [24]. The algebraic properties of fuzzy sets were studied by Luca and Termini [26]. The concept of fuzzy set, which is useful, used and has increasing applications in many different fields including information theory, pattern recognition, probability theory, actually corresponds to the physical situation where there is no definite criterion for membership value. Studies in abstract mathematics based on the fuzzy set idea have solid foundations. At the same time the concepts of fuzzy topological spaces [6], fuzzy groups [28], fuzzy regular spaces [25], fuzzy normed linear spaces ([21], [22], [29]), fuzzy vector spaces ([18], [13]), fuzzy metric spaces ([1], [20]) and fuzzy proximity spaces [19] were given by the respective authors. Fuzzy topological spaces have been found to be useful in solving many problems in different fields. For example; geographic information theory ([8], [9], [10]), quantum physics ([26], [27]), modeling [32] etc. Many mathematicians generalized many concepts in general topology by examining them in fuzzy topological spaces. In 1981, Azad [3] studied in fuzzy topology the concept of semi-open set given by Levine in [23]. This led to the study of weak versions of many concepts in these spaces ([2], [17]). In this paper, we introduce fuzzy *AF*-open sets in a fuzzy topological space (X, τ). Some properties and characterizations of the fuzzy *AF*-open set are studied. Also we investigate and research the notions of fuzzy *AF*-interior and fuzzy *AF*-cluster points in a fuzzy topological space. Moreover fuzzy *AF*-compactness is defined. Its properties and characterizations are examined.

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2. Preliminaries

We recall some well-known definitions.

Definition 2.1. ([35] Let X be a non-empty set a fuzzy set λ in X is characterized by its membership function $\mu_\lambda : X \to [0,1]$ and $\mu_\lambda(x)$ is interpreted as the degree of membership of element x in fuzzy set λ, for each $x \in X$. It is clear that λ is completely determined by the tipping set

$$\lambda = \{(x, \mu_\lambda(x)) : x \in X\}.$$

Definition 2.2. ([35]) Let $\lambda = \{(x, \mu_\lambda(x)) : x \in X\}$ and $\beta = \{(x, \mu_\beta(x)) : x \in X\}$ be two fuzzy sets in X. Then their union $\lambda \vee \beta$, intersection $\lambda \wedge \beta$ and complement λ^c are also fuzzy sets with the membership functions defined as follows:

(i) $\mu_{(\lambda\vee\beta)}(x) = max\{\mu_\lambda(x), \mu_\beta(x)\}$, every $x \in X$,

(ii) $\mu_{(\lambda\wedge\beta)}(x) = min\{\mu_\lambda(x), \mu_\beta(x)(x)\}$, every $x \in X$,

(iii) $\mu^c_\lambda(x) = 1 - \mu_\lambda(x), every\ x \in X.$

Definition 2.3. ([6]) Let X be a non-empty set and I the unit interval $[0,1]$. A fuzzy set in X is an element of the set I^x of all functions from X to I. 0_X and l_X denote the fuzzy sets given by $0_X(x) = 0$, for all $x \in X$ and $1_X(x) = 1$, for all $x \in X$. Equality of two fuzzy sets λ and μ on X is determined by the usual equality condition for mappings, i.e. $\lambda = \mu \Rightarrow$ (for all $x \in X$) $\lambda(x) = \mu(x)$. A fuzzy set λ on X is said to be a subset of a fuzzy set μ on X written $\lambda \leq \mu$, if $\lambda(x) \leq \mu(x)$, for all $x \in X$. The complement of λ fuzzy set a on X is given by 1 - λ. As outlined by Bellman and Giertz [2] the elementary operations on fuzzy sets λ_i on X are given by

$$\bigvee_{i\in I} \lambda_i(x) = sup\{\lambda_i(x) : i \in I\}, \text{ for all } x \in X,$$
$$\bigwedge_{i\in I} \lambda_i(x) = inf\{\lambda_i(x) : i \in I\}, \text{ for all } x \in X,$$

where I denotes an arbitrary index set.

Definition 2.4. ([6]) A fuzzy topology is a family τ of fuzzy sets in X, which satisfies the following conditions:

(i) $0_X, l_X \in \tau$,

(ii) If λ, $\mu \in \tau$, then $\lambda \wedge \mu \in \tau$,

(iii) If $\lambda_i \in \tau$ for each $i \in I$, then $\bigvee_i \lambda_i \in \tau$.

τ is called a fuzzy topology for X, and the pair (X, τ) is a fuzzy topological space. Every member of τ is called τ-open fuzzy set (or simply fuzzy open set). A fuzzy set is τ-closed if and only if its complement is τ-open.
In the sequel, we write a fuzzy topological space X (or (X, τ)) in place of 'a space X with fuzzy topology τ'. For a fuzzy set λ of X, the closure $Cl\lambda$ and the interior $Int\lambda$ of λ are defined respectively, as

$$Cl(\lambda) = inf\{\mu : \mu \geq \lambda, 1 - \mu \in \tau\}, \text{ and } Int\lambda = sup\{\mu : \mu \leq \lambda, \mu \in \tau\}.$$

Definition 2.5. ([33]) A fuzzy set which is a fuzzy point with support $x \in X$ and the value $\lambda \in (0,1]$ will be denoted by x_λ . The value of a fuzzy set β for some $x \in X$ will be denoted by $\beta(x)$. Also, for a fuzzy point x_λ and a fuzzy set β we shall write $x_\lambda \in \beta$ to mean that $\lambda \leq \beta(x)$.

Definition 2.6. ([7]) Let (X, τ) fuzzy topological space and λ, β two fuzzy sets then $\lambda \leq \beta$ if and only if $\lambda(x) \leq \beta(x)$ for all $x \in X$, and λ is said to be quasi-coincident with a fuzzy set β, denoted by $\lambda q \beta$, if there exists $x \in X$ such that $\lambda(x) + \beta(x) > 1$.

Definition 2.7. ([15]) A fuzzy set on X is called a fuzzy singleton if it takes the value zero (0) for all points x in X except one point. The point at which a fuzzy singleton takes the non-zero value is called the support and the corresponding element of [0,1] its value. A fuzzy singleton with value 1 is called a crisp singleton.

Definition 2.8. ([7]) A fuzzy set V in (X,τ) is called a $q-neighborhood$ ($q-nbd$, for short) of a fuzzy point x_λ if and only if there exists a fuzzy open set U such that $x_\lambda qU \leq V$. We will denote the set of all q-nbd of x_λ in (X,τ) by $Nq(x_\lambda)$.

Definition 2.9. ([6]) Let f be a function from X to Y. Let B be a fuzzy set in Y with membership function $\mu_B(y)$. Then the inverse of B, written as $f^{-l}(B)$, is a fuzzy set in X whose membership function is defined by

$$\mu_{f^{-l}(B)}(x) = \mu_B(f(x)) \text{ for all } x \text{ in } X.$$

Conversely, let A be a fuzzy set in X with membership function $\mu_A(x)$. The image of A, written as $f(A)$, is a fuzzy set in Y whose membership function is given by

$$\mu_{f(A)}(y) = \begin{cases} \sup\{\mu_A(z) : z \in f^{-l}(y)\} & \textit{if } f^{-l}(y) \textit{ is not empty,} \\ 0 & \textit{otherwise,} \end{cases}$$

for all y in Y, where $f^{-1}(y) = \{x : f(x) = y\}$.

Theorem 2.10. ([6]) *Let f be a function from X to Y. Then,*

(1) $f^{-l}(\lambda^c)=(f^{-l}(\lambda))^c$ *for any fuzzy set λ in Y,*

(2) $f(\lambda^c) \geq (f(\lambda))^c$,

(3) $\lambda_1 \leq \lambda_2 \Rightarrow f^{-1}(\lambda_1) \leq f^{-1}(\lambda_2)$, *where λ_1 and λ_2 are fuzzy sets in Y,*

(4) $\mu_1 \leq \mu_2 \Rightarrow f(\mu_1) \leq f(\mu_2)$, *where μ_1 and μ_2 are fuzzy sets in X,*

(5) $\lambda \geq f(f^{-1}(\lambda))$ *for any fuzzy set λ in Y,*

(6) $\mu \leq f^{-1}(f(\mu))$ *for any fuzzy set μ in X,*

(7) *Let f be a function from X to Y and g be a function from Y to Z.Then $(gof)^{-1}(\beta) = f^{-1}(g^{-1}(\beta))$ for any fuzzy set β in Z, where gof is the composition of g and f.*

Definition 2.11. A subset λ of a fuzzy topological space (X,τ) is said to be

(i) Fuzzy $\alpha-open$([5]) $\lambda \leq Int(Cl(Int(\lambda)))$,

(ii) Fuzzy $pre-open$([5]) $\lambda \leq Int(Cl(\lambda))$,

(iii) Fuzzy $semi-open$([3]) $\lambda \leq Cl(Int(\lambda))$,

(iv) Fuzzy $\beta-open$([25]) $\lambda \leq Cl(Int(Cl(\lambda)])$.

By Definition 2.11, the following diagram is obtained:

$$\begin{array}{ccccc} fuzzy-open & \rightarrow & fuzzy\ \alpha-open & \rightarrow & fuzzy\ pre-open \\ & & \downarrow & & \downarrow \\ & & fuzzy\ semi-open & \rightarrow & fuzzy\ \beta-open \end{array}$$

Diagram I

The fuzzy α-interior [31] $f\alpha Int(\lambda)$, of λ is defined as follows:
$f\alpha\text{Int}(\lambda) = \bigvee\{\mu : \mu \leq \lambda,\ \mu \text{ is fuzzy } \alpha-open\}$. The fuzzy pre-interior [30], $fpInt(\lambda)$, fuzzy semi-interior [34] $fsInt(\lambda)$, fuzzy β interior [16] $f\beta Int(\lambda)$ are similarly defined.

3. *AF*-open sets with fuzzification

Definition 3.1. Let (X, τ) be a fuzzy topological space. A subset λ of X is said to be *fuzzy AF-open* set if $\lambda \leq Int(\lambda \vee \mu)$ for every μ is fuzzy open set such that $0_X \neq \mu \neq 1_X$. The complement of the *fuzzy AF-open* set is called *fuzzy AF-closed*. We denote the family of all fuzzy *AF*-open (resp. fuzzy *AF*-closed) sets of a fuzzy topological spece (X, τ) by $FAFO(X)(resp.\ FAFC(X))$.

Problem 3.2. *Let (X, τ) be a fuzzy topological space. In Definition 3.1, for every $\mu \in \tau$ such that $0_X \neq \mu \neq 1_X$, can we obtain a new type of fuzzy AF-open sets by taking the fuzzy closure of μ instead of μ?*

Theorem 3.3. *Every fuzzy open set in a fuzzy topological space (X, τ) is fuzzy AF-open set.*

Proof. Let (X, τ) be any fuzzy topological space and let $\lambda \leq X$ be any fuzzy open set. Therefore, $\lambda = Int(\lambda) \leq Int(\lambda \vee \mu)$ for every μ is fuzzy open set such that $0_X \neq \mu \neq 1_X$. Thus, A is fuzzy *AF*-open set. Then for the collection of $FAFO(X)$, $\tau \leq FAFO(X)$. □

Remark 3.4. The converse of Theorem 3.3 is not always true as shown by the following example.

Example 3.5. $X = \{a, b, c\}, \tau = \{0_X, \lambda, 1_X\}, \lambda, \mu : X \to I$ be two fuzzy sets in X, defined as: $\lambda = \{(a, 0.5), (b, 0.7), (c, 0.9)\}$ and $\mu = \{(a, 0.4), (b, 0.3), (c, 0.9)\}$. Then $\mu \in FAFO(X)$ and but the set μ is not fuzzy open.

Theorem 3.6. *Let (X, τ) be any fuzzy topological space and λ, μ be two fuzzy AF-open sets. Then, the following properties are hold:*

(1) *$\lambda \wedge \mu$ is fuzzy AF-open set.*

(2) *$\lambda \vee \mu$ is fuzzy AF-open set.*

Proof. (1) Let λ and μ be two fuzzy *AF*-open sets. Then from Definition 3.1, $\lambda \leq Int(\lambda \vee \beta)$ and $\mu \leq Int(\mu \vee \beta)$ for every β is fuzzy open set and $0_X \neq \beta \neq 1_X$. Then $\lambda \wedge \mu \leq Int(\lambda \vee \beta) \wedge Int(\mu \vee \beta) = Int((\lambda \vee \beta) \wedge (\mu \vee \beta)) \leq Int((\lambda \wedge \mu) \vee \beta)$.
(2) Let λ and μ be two fuzzy *AF*-open sets. Then from Definition 3.1, $\lambda \leq Int(\lambda \vee \beta)$ and $\mu \leq Int(\mu \vee \beta)$ for every β is fuzzy open set and $0_X \neq \beta \neq 1_X$. Then $\lambda \vee \mu \leq Int(\lambda \vee \beta) \vee Int(\mu \vee \beta) = Int((\lambda \vee \beta) \vee (\mu \vee \beta)) \leq Int((\lambda \vee \mu) \vee \beta)$. □

Proposition 3.7. *Let (X, τ) be any fuzzy topological space. If for every $\alpha \in \Delta$, $\lambda_\alpha \in FAFO(X)$, then $\bigvee_{\alpha\in\Delta} \lambda_\alpha \in FAFO(X)$.*

Proof. Let $\lambda_\alpha \in FAFO(X)$ for every $\alpha \in \Delta$. Then $\lambda_\alpha \leq \bigvee_{\alpha\in\Delta} \lambda_\alpha$, for every $\alpha \in \Delta$. For any β is fuzzy open $(0_X \neq \mu \neq 1_X)$ and each $\alpha \in \Delta$, $\lambda_\alpha \leq Int(\lambda_\alpha \vee \beta) \leq Int[(\bigvee_{\alpha\in\Delta} \lambda_\alpha) \vee \beta]$. Hence, we have $\bigvee_{\alpha\in\Delta} \lambda_\alpha \leq Int[(\bigvee_{\alpha\in\Delta} \lambda_\alpha) \vee \beta]$. Therefore $\bigvee_{\alpha\in\Delta} \lambda_\alpha \in FAFO(X)$. □

Theorem 3.8. *Let (X, τ) be any fuzzy topological space and $\tau_{FAFO} = \{\lambda \leq X \ / \ \lambda\ is\ a\ fuzzy\ AF-open\ set\ of\ (X, \tau)\}$. Then is a τ_{FAFO} a fuzzy topology such that $\tau \leq \tau_{FAFO}$.*

Proof. According to Theorem 3.3, we have $\tau \leq \tau_{FAFO}$. We show that τ_{FAFO} is a fuzzy topology:
(1) It is clear that $0_X, 1_X \in \tau_{FAFO}$.
(2) and (3) are seen that from Theorem 3.6 and Proposition 3.7. □

4. Generalizations of fuzzy *AF*-open sets

Definition 4.1. A subset λ of a fuzzy topological space (X, τ) is said to be

(*i*) Fuzzy $AF\alpha - open$ if $\lambda \leq f\alpha Int(\lambda \vee \mu)$ for every β is fuzzy open and $0_X \neq \mu \neq 1_X$,

(*ii*) Fuzzy $AFpre - open$ if $\lambda \leq fpInt(\lambda \vee \mu)$ for every β is fuzzy open and $0_X \neq \mu \neq 1_X$,

(iii) Fuzzy $AFsemi-open$ if $\lambda \leq fsInt(\lambda \vee \mu)$ for every β is fuzzy open and $0_X \neq \mu \neq 1_X$,

(iv) Fuzzy $AF\beta - open$ if $\lambda \leq f\beta Int(\lambda \vee \mu)$ for every β is fuzzy open and $0_X \neq \mu \neq 1_X$.

The complement of a fuzzy $AF\alpha - open$ (resp. $fuzzy\ AFp - open$, $fuzzy\ AFs - open$, $fuzzy\ AF\beta - open$) set is said to be $fuzzy\ AF\alpha - closed$ (resp. $fuzzy\ AFp - closed$, $fuzzy\ AFs - closed$, $fuzzy\ AF\beta - closed$). The family of all fuzzy $AF\alpha$-open (fuzzy $AF\alpha$-closed) (resp. fuzzy AFp-open (fuzzy AFp-closed), fuzzy AFs-open (fuzzy AFs-closed), fuzzy $AF\beta$-open (fuzzy $AF\beta$-closed)) sets in a fuzzy topological space (X, τ) is denoted by F$AF\alpha$O(X) (F$AF\alpha$C(X)) (resp. FAFPO(X) (FAFPC(X)), FAFSO(X) (FAFSC(X)), F$AF\beta$O(X) (F$AF\beta$C(X))).

From Definition 4.1, we have the following diagram:

$$\begin{array}{ccccccc} fuzzy-open & \rightarrow & fuzzy\ AF-open & \rightarrow & fuzzy\ AF\alpha-open & \rightarrow & fuzzy\ AFp-open \\ & & & & \downarrow & & \downarrow \\ & & & & fuzzy\ AFs-open & \rightarrow & fuzzy\ AF\beta-open \end{array}$$

Diagram II

Problem 4.2. *In the above definition, for every $\mu \in \tau$ such that $0_X \neq \mu \neq 1_X$, can a new types of fuzzy AF-open set be given by taking the fuzzy closure of μ instead of μ?*

Remark 4.3. The inverses of the requirements in the diagram above may not always be true.

Example 4.4. It can be seen from Example 3.5 that not every fuzzy AF-open set is a fuzzy open set.

Example 4.5. $X = \{a, b, c\}, \tau = \{0, \lambda, 1\}, \lambda, \mu : X \rightarrow I$ be two fuzzy sets in X, defined as: $\lambda = \{(a, 0.2), (b, 0.7), (c, 0.4)\}$ and $\mu = \{(a, 0.7), (b, 0.9), (c, 0.1)\}$. Then $\mu \in FAF\alpha(X)$ and but the set μ is not fuzzy AF-open.

Example 4.6. $X = \{a, b, c\}, \tau = \{0, \mu, 1\}, \lambda, \mu : X \rightarrow I$ be two fuzzy sets in X, defined as: $\lambda = \{(a, 0.2), (b, 0.3), (c, 0.7)\}$ and $\mu = \{(a, 0.1), (b, 0.2), (c, 0.2)\}$. Then $\lambda \in FAFSO(X)$ and but the set λ is neither fuzzy $AF\alpha$-open nor fuzzy AFp-open.

Example 4.7. $X = \{a, b, c\}, \tau = \{0, \mu, 1\}, \lambda, \mu : X \rightarrow I$ be two fuzzy sets in X, defined as: $\lambda = \{(a, 0.3), (b, 0.8), (c, 0.7)\}$ and $\mu = \{(a, 0.1), (b, 0.3), (c, 0.4)\}$. Then $\lambda \in FAFPO(X)$ and but the set λ is neither fuzzy $AF\alpha$-open nor fuzzy AFs-open.

Remark 4.8. From Example 4.6 and Example 4.7, fuzzy AFp-open sets and fuzzy AFs-open sets are independent of each other.

Example 4.9. $X = \{a, b, c\}, \tau = \{0, \lambda, 1\}, \lambda, \mu : X \rightarrow I$ be two fuzzy sets in X, defined as: $\lambda = \{(a, 0.1), (b, 0.3), (c, 0.1)\}$ and $\mu = \{(a, 0.3), (b, 0.5), (c, 0.7)\}$. Then $\mu \in Fh\beta O(X)$ and but the set μ is not fuzzy hp-open.

Example 4.10. $X = \{a, b, c\}, \tau = \{0, \lambda, 1\}, \lambda, \mu : X \rightarrow I$ be two fuzzy sets in X, defined as: $\lambda = \{(a, 0.2), (b, 0.8), (c, 0.5)\}$ and $\mu = \{(a, 0.6), (b, 0.5), (c, 0.4)\}$. Then $\mu \in Fh\beta O(X)$ and but the set μ is not fuzzy hs-open.

5. Fuzzy *AF*-interior and fuzzy *AF*-closure operators

Definition 5.1. Let (X, τ) be a fuzzy topological space and a fuzzy subset λ of X. The fuzzy AF-interior, $Int_{AF}(\lambda)$, is defined as follows :

$$Int_{AF}(\lambda) = \bigvee\{\mu : \mu \in FAFO(X),\ \mu \leq \lambda\} = sup\{\mu : \mu \in FAFO(X),\ \mu \leq \lambda\}.$$

Theorem 5.2. *Let (X, τ) be a fuzzy topological space and λ, μ fuzzy subsets of X. Then the following statements are hold:*

(1) *$Int_{AF}(\lambda)$ is fuzzy AF-open set,*

(2) $Int_{AF}(\lambda) \leq \lambda$,

(3) *$Int_{AF}(\lambda)$ is the largest fuzzy AF-open subset contained in the set λ,*

(4) $Int_{AF}(Int_{AF}(\lambda)) = Int_{AF}(\lambda)$,

(5) *If $\lambda \leq \mu$, $Int_{AF}(\lambda) \leq Int_{AF}(\mu)$,*

(6) $Int_{AF}(\lambda) \vee Int_{AF}(\mu) \leq Int_{AF}(\lambda \vee \mu)$,

(7) $Int_{AF}(\lambda) \wedge Int_{AF}(\mu) = Int_{AF}(\lambda \wedge \mu)$.

Proof. (1) $Int_{AF}(\lambda)$ is fuzzy AF-open set. Indeed, the union of fuzzy AF-open sets belonging to the fuzzy topological space τ is fuzzy AF-open from the Proposition 3.7.
(2) It is clear from Definition 5.1.
(3) Let's assume the opposite, that is, a fuzzy AF-open set β that is larger than the set $Int_{AF}(\lambda)$ that the set λ contains. That is, $Int_{AF}(\lambda) \leq \beta \leq \lambda$. On the other hand, for every $\mu \leq \lambda$ fuzzy AF-open set from Definition 5.1, $\mu \leq Int_{AF}(\lambda)$. If we take $\mu = \beta$ specifically, we find $\beta \leq Int_{AF}(\lambda)$. Then $\beta = Int_{AF}(\lambda)$ is obtained. Thus, the fuzzy set $Int_{AF}(\lambda)$ is the largest fuzzy AF-open subset contained in the set λ.
(4) Let $\beta = Int_{AF}(\lambda)$. By (2) and Definition 5.1, $\beta = Int_{AF}(\beta)$. Then $Int_{AF}(\lambda) = Int_{AF}(Int_{AF}(\lambda))$.
(5) Since $\lambda \leq \mu$ and $Int_{AF}(\lambda) \leq \lambda$, $Int_{AF}(\lambda) \leq \mu$. By (2), $Int_{AF}(\mu) \leq \mu$. From (3), since $Int_{AF}(\mu)$ is the largest fuzzy open set contained in μ fuzzy sets, $Int_{AF}(\lambda) \leq Int_{AF}(\mu) \leq \mu$. In that case $Int_{AF}(\lambda) \leq Int_{AF}(\mu)$.
(6) $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$ always hold. From (5), $Int_{AF}(\lambda) \leq Int_{AF}(\lambda \vee \mu)$ and $Int_{AF}(\mu) \leq Int_{AF}(\lambda \vee \mu)$, respectively. Therefore $Int_{AF}(\lambda) \vee Int_{AF}(\mu) \leq Int_{AF}(\lambda \vee \mu)$.
(7) It is always hold that $\lambda \wedge \mu \leq \lambda$ and $\lambda \wedge \mu \leq \mu$. From (5), we obtain $Int_{AF}(\lambda \wedge \mu) \leq Int_{AF}(\lambda)$ and $Int_{AF}(\lambda \wedge \mu) \leq Int_{AF}(\mu)$, respectively. Hence $Int_{AF}(\lambda \wedge \mu) \leq Int_{AF}(\lambda) \wedge Int_{AF}(\mu)$. On the other hand $Int_{AF}(\lambda) \leq \lambda$ and $Int_{AF}(\mu) \leq \mu$. From here $Int_{AF}(\lambda) \wedge Int_{AF}(\mu) \leq \lambda \wedge \mu$. Since $Int_{AF}(\lambda) \wedge Int_{AF}(\mu)$ are fuzzy AF-open sets and $Int_{AF}(\lambda \wedge \mu)$ is the largest fuzzy AF-open set contained in the $\lambda \wedge \mu$ fuzzy set, we have $Int_{AF}(\lambda) \wedge Int_{AF}(\mu) \leq Int_{AF}(\lambda \wedge \mu) \leq \lambda \wedge \mu$. Thus $Int_{AF}(\lambda) \wedge Int_{AF}(\mu) = Int_{AF}(\lambda \wedge \mu)$. □

Theorem 5.3. *Let (X, τ) be a fuzzy topological space and and a fuzzy subset λ of X. Then, λ fuzzy set to be AF-open set if and only if $Int_{AF}(\lambda) = \lambda$.*

Proof. ⇒ Let λ be a fuzzy AF-open set. From Theorem 5.2 (2), $Int_{AF}(\lambda) \leq \lambda$. On the other hand, since λ is a fuzzy AF-open set, $\lambda \leq \lambda$ and by Definition 5.1, $\lambda \leq Int_{AF}(\lambda)$. In that case $\lambda = Int_{AF}(\lambda)$.
⇐ According to the hypothesis, let's take $\lambda = Int_{AF}(\lambda)$. Since $Int_{AF}(\lambda)$ is a fuzzy AF-open set and $Int_{AF}(\lambda) = \lambda$, so λ is a fuzzy AF-open set. □

Lemma 5.4. *For 1_X and 0_X fuzzy AF-open sets, then $Int_{AF}(1_X) = 1_X$ and $Int_{AF}(0_X) = 0_X$.*

Definition 5.5. Let (X, τ) be a fuzzy topological space and a fuzzy subset λ of X. The fuzzy AF-closure of λ, $Cl_{AF}(\lambda)$, is defined as follows:

$$Cl_{AF}(\lambda) = \bigwedge\{\beta : \beta \in FAFC(X),\ \lambda \leq \beta\} = inf\{\beta : (1_X - \beta) \in FAFO(X),\ \lambda \leq \beta\}.$$

It is obvious that $Cl_{AF}(\lambda)$ is fuzzy AF-closed set for any $\lambda \leq X$.

Theorem 5.6. *Let (X, τ) be a fuzzy topological space and λ, μ fuzzy subsets of X. Then the following properties hold:*

(1) *$Cl_{AF}(\lambda)$ is fuzzy AF-closed set,*

(2) $\lambda \leq Cl_{AF}(\lambda)$,

(3) *$Cl_{AF}(\lambda)$ is the smallest fuzzy closed set containing λ,*

(4) *$Cl_{AF}(Cl_{AF}(\lambda)) = Cl_{AF}(\lambda)$,*

(5) *If $\lambda \leq \mu$, $Cl_{AF}(\lambda) \leq Cl_{AF}(\mu)$,*

(6) *$Cl_{AF}(\lambda \wedge \mu) \leq Cl_{AF}(\lambda) \wedge Cl_{AF}(\mu)$,*

(7) *$Cl_{AF}(\lambda \vee \mu) = Cl_{AF}(\lambda) \vee Cl_{AF}(\mu)$,*

(8) *$Cl_{AF}(1_X)$=1_X and $Cl_{AF}(0_X)$=0_X.*

Theorem 5.7. *Let λ be any fuzzy set in a fuzzy topological space (X, τ). Then $Cl_{AF}(l - \lambda)$ =1 $- Int_{AF}(\lambda)$ and $Int_{AF}(l - \lambda) = 1 - Cl_{AF}(\lambda)$.*

Proof. We see that a fuzzy AF-open set $\beta \leq \lambda$ is precisely the complement of a fuzzy AF-closed set $\nu = 1-\beta \geq 1 - \lambda$. Thus
$Int_{AF}(\lambda) = \bigvee\{l - \nu : \nu \text{ is fuzzy } AF - closed \text{ and } \nu \geq 1 - \lambda\}$
$=1 - \bigwedge\{\nu : \nu \text{ is fuzzy } AF - closed \text{ and } \nu \geq 1 - \lambda\}$
$= 1 - Cl_{AF}(l - \lambda)$
whence
$Cl_{AF}(l - \lambda) = 1 - Int_{AF}(\lambda)$.
Next let β be any fuzzy AF-open set. Then for a fuzzy AF-closed set $\mu \geq \lambda$, $\beta = 1 - \mu \leq 1 - \lambda$.
$Cl_{AF}(\lambda) = \bigwedge\{1 - \beta : \beta \text{ is fuzzy } AF - open \text{ and } \beta \leq 1 - \lambda\}$
$= 1 - \bigvee\{\beta : \beta \text{ is fuzzy } AF - open \text{ and } \beta \leq 1 - \lambda\}$
$= 1 - Int_{AF}(l - \lambda)$.
As a result
$Int_{AF}(l - \lambda) = 1 - Cl_{AF}(\lambda)$. □

Definition 5.8. Let β be a fuzzy set in a fuzzy topological space (X, τ) and x_α is a fuzzy point of X. β is called:

(*i*) *$AF - neighbourhood$* of x_α if there exists a fuzzy set $\mu \in FAFO(X)$ such that $x_\alpha \in \mu \leq \beta$.

(*ii*) *$AF - Q - neighbourhood$* of x_α if there exists a fuzzy set $\mu \in FAFO(X)$ such that $x_\alpha q\mu \leq \beta$.

Theorem 5.9. *A fuzzy set $\beta \in FAFO(X)$ if and only if β is a AF-neighbourhood of x_α, for every fuzzy point $x_\alpha \in \beta$.*

Proof. Straightforward. □

Definition 5.10. Let (X, τ) be the fuzzy topological space, $\lambda \leq 1_X$ and x_α the fuzzy point. If every AF-Q-neighborhood of x_α fuzzy point is quasi-coincident with λ, the x_α fuzzy point is called a AF-cluster point of the fuzzy set λ. The notation $\nu q\mu$ $(\nu\tilde{q}\mu)$ will sense that it is quasi-coincident (not quasi-coincident) with ν and μ.

Theorem 5.11. *Let β be a fuzzy set of a fuzzy topological space X. Then a fuzzy point $x_\alpha \in Cl_{AF}(\beta)$ if and only if every AF-Q-neighbourhood of x_α is quasi-coincident with β.*

Proof. $\Rightarrow$ Suppose $x_\alpha \in Cl_{AF}(\beta)$ and if possible let there exist a AF-Q-neighbourhood μ of x_α such that $\mu\tilde{q}\beta$. Then there exists a fuzzy set $\mu_1 \in FAFO(X)$ such that $x_\alpha q\mu_l \leq \mu$ which shows that $\mu_l\tilde{q}\beta$ and hence $\beta \leq (\mu_l)^c$. As $(\mu_l)^c \in FAFC(X)$, $Cl_{AF}(\beta) \leq (\mu_l)^c$. Since $x_\alpha \in (\mu_l)^c$, $x_\alpha\tilde{q}\mu_l$. From this contradiction, $\mu q\beta$.
$\Leftarrow$ Suppose every AF-Q-neighbourhood of x_α is quasi-coincident with β. If $x_\alpha \notin Cl_{AF}(\beta)$ then there exists a fuzzy AF-closed set $\mu \geq \beta$ such that $x_\alpha \notin \mu$. So $\mu^c \in FAFO(X)$ such that $x_\alpha q\mu^c$ and $(\mu^c\tilde{q}\beta)$ a contradiction. □

6. Fuzzy *AF*-continous functions

Definition 6.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be fuzzy AF-continuous if for each $\lambda \in \sigma$, $f^{-1}(\lambda)$ is fuzzy AF-open in (X, τ).

Theorem 6.2. *Every fuzzy continuous function is fuzzy AF-continuous.*

Proof. By Theorem 3.3, every fuzzy open set is fuzzy AF-open and the proof is obvious. □

Example 6.3. $X = \{a, b\}$, $Y = \{0.1, 0.4\}$, $\lambda, \mu : X \to I$ be two fuzzy sets in X and $\beta : Y \to I$ be fuzzy set in Y defined as follows: $\lambda = \{(a, 0.2), (b, 0.2)\}$, $\mu = \{(a, 0.3), (b, 0.7)\}$ and $\beta = \{(0.1, 0.2), (0.4, 0.2)\}$. Let $\tau = \{0_X, \mu, 1_X\}$, $\sigma = \{0_Y, \beta, 1_Y\}$. Then the function $f : X \to Y$ defined by f(a)=0.1, f(b)=0.4 is a fuzzy AF-continuous, but not fuzzy continuous.

Definition 6.4. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $fuzzy\ AF\alpha - continuous$ (resp. $fuzzy\ AFp - continuous$, $fuzzy\ AFs - continuous$, $fuzzy\ AF\beta - continuous$) if for each $\lambda \in \sigma$, $f^{-1}(\lambda)$ is fuzzy $AF\alpha$-open (resp. fuzzy AFp-open, fuzzy AFs-open, fuzzy $AF\beta$-open) in (X, τ).

By Definitions 6.1 and 6.4, the following implications hold:

$$\begin{array}{ccccccc} fuzzy - cont & \rightarrow & fuzzy\ AF - cont & \rightarrow & fuzzy\ AF\alpha - cont & \rightarrow & fuzzy\ AFp - cont \\ & & & & \downarrow & & \downarrow \\ & & & & fuzzy\ AFs - cont & \rightarrow & fuzzy AF\beta - cont \end{array}$$

Diagram III

Remark 6.5. None of the implications in DIAGRAM III is reversible as shown by examples stated below.

Example 6.6. It can be seen from Example 6.3 that not every fuzzy AF-continuous function is a fuzzy continuous.

Example 6.7. $X = \{a, b\}$, $Y = \{0.2, 0.5\}$, $\lambda, \mu : X \to I$ be two fuzzy sets in X and $\beta : Y \to I$ be fuzzy set in Y defined as follows: $\lambda = \{(a, 0.7), (b, 0.4)\}$, $\mu = \{(a, 0.9), (b, 0.1)\}$ and $\beta = \{(0.2, 0.9), (0.5, 0.1)\}$. Let $\tau = \{0_X, \lambda, 1_X\}$, $\sigma = \{0_Y, \beta, 1_Y\}$. Then the function $f : X \to Y$ defined by f(a)=0.2, f(b)=0.5 is a fuzzy $AF\alpha$-continuous, but not fuzzy AF-continuous.

Example 6.8. $X = \{a, b\}$, $Y = \{0.1, 0.4\}$, $\lambda, \mu : X \to I$ be two fuzzy sets in X and $\beta : Y \to I$ be fuzzy set in Y defined as follows: $\lambda = \{(a, 0.3), (b, 0.7)\}$, $\mu = \{(a, 0.2), (b, 0.2)\}$ and $\beta = \{(0.1, 0.3), (0.4, 0.7)\}$. Let $\tau = \{0_X, \mu, 1_X\}$, $\sigma = \{0_Y, \beta, 1_Y\}$. Then the function $f : X \to Y$ defined by f(a)=0.1, f(b)=0.4 is a fuzzy AFs-continuous, but neither fuzzy $AF\alpha$-continuous nor fuzzy AFp-continuous.

Example 6.9. $X = \{a, b, c\}$, $Y = \{0.1, 0.3, 0.5\}$, $\lambda, \mu : X \to I$ be two fuzzy sets in X and $\beta : Y \to I$ be fuzzy set in Y defined as follows: $\lambda = \{(a, 0.4), (b, 0.9), (c, 0.8)\}$, $\mu = \{(a, 0.2), (b, 0.4), (c, 0.5)\}$ and $\beta = \{(0.1, 0.4), (0.3, 0.9), (0.5, 0.8)\}$. Let $\tau = \{0_X, \mu, 1_X\}$, $\sigma = \{0_Y, \beta, 1_Y\}$. Then the function $f : X \to Y$ defined by f(a)=0.1, f(b)=0.3, f(c)=0.5 is a fuzzy AFp-continuous, neither fuzzy $AF\alpha$-continuous nor fuzzy AFs-continuous.

Example 6.10. $X = \{a, b, c\}$, $Y = \{0.2, 0.5, 0.6\}$, $\lambda, \mu : X \to I$ be two fuzzy sets in X and $\beta : Y \to I$ be fuzzy set in Y defined as follows: $\lambda = \{(a, 0.1), (b, 0.4), (c, 0.1)\}$, $\mu = \{(a, 0.4), (b, 0.5), (c, 0.8)\}$ and $\beta = \{(0.2, 0.3), (0.5, 0.5), (0.6, 0.8)\}$. Let $\tau = \{0_X, \lambda, 1_X\}$, $\sigma = \{0_Y, \beta, 1_Y\}$. Then the function $f : X \to Y$ defined by f(a)=0.2, f(b)=0.5, f(c)=0.6 is a fuzzy $AF\beta$-continuous, but not fuzzy AFp-continuous.

Example 6.11. $X = \{a, b, c\}$, $Y = \{0.3, 0.5, 0.7\}$, $\lambda, \mu : X \to I$ be two fuzzy sets in X and $\beta : Y \to I$ be fuzzy set in Y defined as follows: $\lambda = \{(a, 0.2), (b, 0.8), (c, 0.5)\}$, $\mu = \{(a, 0.6), (b, 0.5), (c, 0.4)\}$ and $\beta = \{(0.3, 0.6), (0.5, 0.5), (0.7, 0.4)\}$. Let $\tau = 0_X, \lambda, 1_X$, $\sigma = 0_Y, \beta, 1_Y$. Then the function $f : X \to Y$ defined by f(a)=0.3, f(b)=0.5, f(c)=0.7 is a fuzzy $AF\beta$-continuous, but not fuzzy AFs-continuous.

Corollary 6.12. *A function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy AF-continuous if and only if $f : (X, \tau) \to (Y, \sigma)$ is fuzzy continuous.*

Proof. This is an immediate consequence of Theorem 3.8. □

Theorem 6.13. *A function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy AF-continuous and $g : (Y, \sigma) \to (R, \eta)$ is fuzzy continuous, then $gof : (X, \tau) \to (R, \eta)$ is fuzzy AF-continuous.*

Proof. It is clear. □

By using fuzzy *AF*-neighborhood, fuzzy *AF*-open sets, fuzzy *AF*-closed sets, fuzzy *AF*-interior and fuzzy *AF*-closure, we obtain characterizations of fuzzy *AF*-continuous functions.

Lemma 6.14. *Let (X, τ) be a fuzzy topological space. A fuzzy subset μ is fuzzy AF-closed if and only if $Cl(\mu \wedge \beta) \leq \mu$ for every fuzzy closed set β of X such that $0_X \neq \beta \neq 1_X$.*

Proof. μ is fuzzy *AF*-closed if and only if $1_X - \mu$ is fuzzy *AF*-open. By Definition 3.1, $(1_X - \mu) \leq Int[(1_X - \mu) \vee \alpha]$ for every $\alpha \in \tau$ such that $0_X \neq \alpha \neq 1_X$.
This is equivalent to $1_X - Int[(1_X - \mu) \vee \alpha] \leq \mu$. Now, we have $1_X - Int[(1_X - \mu) \vee \alpha] = Cl(1_X - [(1_X - \mu) \vee \alpha])$ $= Cl(\mu \wedge (1_X - \alpha))$.
Therefore, we obtain $Cl(\mu \wedge \beta) \leq \mu$ for every fuzzy closed set β of X such that $0_X \neq \beta \neq 1_X$. □

Theorem 6.15. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is fuzzy AF-continuous,*

(2) *For each point $x \in X$ and each fuzzy open $\mu \leq Y$ containing $f(x)$, there exists $\alpha \in FAFO(X)$ such that $x \in \alpha$, $f(\alpha) \leq \mu$,*

(3) *For each point $x \in X$ and each fuzzy open set μ of Y containing $f(x)$, there exists a fuzzy AF-neighorhood λ of x such that $f(\lambda) \leq \mu$,*

(4) *The inverse image of each fuzzy closed set in Y is fuzzy AF-closed,*

(5) *For each fuzzy closed set μ of Y, $Cl(f^{-1}(\mu) \wedge \beta) \leq f^{-1}(\mu)$ for every closed set in X such that $0_X \neq \beta \neq 1_X$,*

(6) *For each fuzzy subset μ of Y, $Cl(f^{-1}(Cl(\mu)) \wedge \beta) \leq f^{-1}(Cl(\mu))$ for every fuzzy closed set β in X such that $0_X \neq \beta \neq 1_X$,*

(7) *For each fuzzy subset λ of X, $f(Cl[\lambda \wedge \beta]) \leq Cl(f(\lambda))$ for every fuzzy closed set β in X such that $0_X \neq \beta \neq 1_X$,*

(8) *For each fuzzy subset μ of Y, $Cl_{AF}(f^{-1}(\mu)) \leq f^{-1}(Cl(\mu))$,*

(9) *For each fuzzy subset μ of Y, $f^{-1}(Int(\mu)) \leq Int_{AF}(f^{-1}(\mu))$.*

Proof. (1) ⇒ (2): Let $x \in X$ and μ be any fuzzy open set of Y containing $f(x)$. Set $\alpha = f^{-1}(\mu)$, then by Definition 6.1, α is a fuzzy *AF*-open set containing x and $f(\alpha) \leq \mu$.
(2) ⇒ (3): Every fuzzy *AF*-open set containing x is a fuzzy *AF*-neighborhood of x and the proof is obvious.
(3) ⇒ (1): Let μ be any fuzzy open set in Y. For each $x \in f^{-1}(\mu)$, $f(x) \in \mu \in \sigma$. By (3) there exists a fuzzy **AF**-neighborhood ν of x such that $f(\nu) \leq \mu$; hence $x \in \nu \leq f^{-1}(\mu)$. There exists $\alpha_x \in FAFO(X)$ such that $x \in \alpha_x \leq \nu \leq f^{-1}(\mu)$. Hence $f^{-1}(\mu) = \bigvee\{\alpha_x : x \in f^{-1}(\mu)\} \in FAFO(X)$. This shows that f is fuzzy *AF*-continuous.
(1) ⇒ (4) ⇒ (5) ⇒ (1): By Lemma 6.14, the proof is obvious.
(5) ⇒ (6): For each fuzzy subset μ of Y, $Cl(\mu)$ is fuzzy closed in Y and the proof is obvious.
(6) ⇒ (7): Let λ be any fuzzy subset of X. Set $\mu = f(\lambda)$, then by (6) $Cl[\lambda \wedge \beta] \leq Cl[f^{-1}(Cl(f(\lambda))) \wedge \beta] \leq f^{-1}(Cl(f(\lambda)))$ for every fuzzy closed set β in X such that $0_X \neq \beta \neq 1_X$. Therefore, we obtain for each fuzzy subset λ of X, $f(Cl[\lambda \wedge \beta]) \leq Cl(f(\lambda))$ for every fuzzy closed set β in X such that $0_X \neq \beta \neq 1_X$.

(7) ⇒ (1): Let μ be any open set of Y. Then $1_Y - \mu$ is fuzzy closed in Y. Set $\alpha = f^{-1}(1_Y - \mu)$, then by (7) $f(Cl[f^{-1}(1_Y - \mu) \wedge \beta]) \le Cl(f(f^{-1}(1_Y - \mu)))) = 1_Y - \mu$ for every fuzzy closed set β in X such that $0_X \ne \beta \ne 1_X$. Therefore, we have
$Cl[f^{-1}(1_Y - \mu) \wedge \beta]$
$\le f^{-1}(f(Cl[f^{-1}(1_Y - \mu) \wedge \beta])$
$\le f^{-1}(1_Y - \mu) = 1_X - f^{-1}(\mu)$.
Therefore, $f^{-1}(\mu) \le 1_X - Cl[f^{-1}(1_Y - \mu) \wedge \beta]$
$= Int[1_X - f^{-1}(1_Y - \mu) \wedge \beta]$
$= Int[f^{-1}(\mu) \vee (1_X - \beta)]$
$= Int[f^{-1}(\mu) \vee \alpha]$
for every fuzzy open set α of X such that $0_X \ne \beta \ne 1_X$.
(4) ⇒ (8): Let μ be any fuzzy subset of Y. By (4) $f^{-1}(Cl(\mu))$ is fuzzy AF-closed in X and
$f^{-1}(\mu) \le f^{-1}(Cl(\mu))$. Therefore, $Cl_{AF}(f^{-1}(\mu)) \le f^{-1}(Cl(\mu))$.
(8) ⇒ (9): Let μ be any fuzzy subset of Y. Then,
$f^{-1}(Int(\mu)) = f^{-1}(1_Y - Cl(1_Y - \mu))$
$=1_X - f^{-1}(Cl(1_Y - \mu)) \le 1_X - Cl_{AF}(f^{-1}(1_Y - \mu))$
$= 1_X - Cl_{AF}(1_X - f^{-1}(\mu))$
$= Int_{AF}(f^{-1}(\mu))$.
(9) ⇒ (1): Let μ be any fuzzy open set of Y. By (9), $f^{-1}(\mu) \le Int_{AF}(f^{-1}(\mu)) \le f^{-1}(\mu)$. Therefore, we have $Int_{AF}(f^{-1}(\mu)) = f^{-1}(\mu)$ and hence f is fuzzy AF-continuous. □

Definition 6.16. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be fuzzy AF-irresolute if for each fuzzy AF-open set μ in Y, $f^{-1}(\mu)$ is fuzzy AF-open in X.

Theorem 6.17. *If a function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy AF-irresolute, then f is fuzzy AF-continuous.*

The converse of Theorem 6.17 is not always true as shown by the following example.

Example 6.18. $X = \{a, b, c\}$, $Y = \{0.1, 0.7, 0.5\}$, $\lambda : X \to I$ be two fuzzy sets in X and $\mu, \beta : Y \to I$ be fuzzy set in Y defined as follows: $\lambda = \{(a, 0.3), (b, 0.2), (c, 0.5)\}$, $\mu = \{(0.1, 0.4), (0.7, 0.5), (0.5, 0.7)\}$ and $\beta = \{(0.1, 0.3), (0.7, 0.2), (0.5, 0.5)\}$. Let $\tau = \{0_X, 1_X, \lambda\}$, $\sigma = \{0_Y, 1_Y, \beta\}$. Then the function $f : X \to Y$ defined by f(a)=0.1, f(b)=0.7, f(c)=0.5 is a fuzzy AF-continuous, but not fuzzy AF-irresolute.

Definition 6.19. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $fuzzy\ AF-open$ (resp. $fuzzy\ AF\alpha-open$, $fuzzy\ AFp-open$, $fuzzy\ AFs-open$, $fuzzy\ AF\beta-open$), if $f(\lambda)$ is $fuzzy\ AF-open$ (resp. $fuzzy\ AF\alpha-open$, $fuzzy\ AFp-open$, $fuzzy\ AFs-open$, $fuzzy\ AF\beta-open$) in Y for every fuzzy open set λ in X.

Proposition 6.20. *Every fuzzy open function is fuzzy AF-open.*

Proof. It is obvious. □

Remark 6.21. As can be seen from Example 3.5, the converse of Proposition 6.20 may not always be true.

Theorem 6.22. *A function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy AF-open if and only if for each fuzzy subset $\mu \le Y$ each fuzzy closed set β of X containing $f^{-1}(\mu)$, there exists a fuzzy AF-closed set $\nu \le Y$ containing μ such that $f^{-1}(\nu) \le \beta$.*

Proof. ⇒ Let $\nu = 1_Y - f(1_X - \beta)$. Since $f^{-1}(\mu) \le \beta$, we have $f(1_X - \beta) \le 1_Y - \mu$. Since f is fuzzy AF-open, then ν is fuzzy AF-closed and $f^{-1}(\nu) = 1_X - f^{-1}(f(1_X - \beta)) \le 1_X - (1_X - \beta) = \beta$.
⇐ Let α be any fuzzy open set of X and $\mu = 1_Y - f(\alpha)$. Then $f^{-1}(\mu) = 1_X - f^{-1}(f(U)) \le 1_X - \alpha$ and $1_X - \alpha$ is fuzzy closed. By the hypothesis, there exists a fuzzy AF- closed set ν of Y containing μ such that $f^{-1}(\nu) \le 1_X - \alpha$. Then, we have $\nu \le 1_Y - f(\alpha)$. Therefore, we obtain $1_Y - f(\alpha) \le \nu \le 1_Y - f(\alpha)$ and $f(\alpha)$ is fuzzy AF-open in Y. This shows that f is fuzzy AF-open. □

Proposition 6.23. *A function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy open and $g : (Y, \sigma) \to (Z, \eta)$ is fuzzy AF-open, then $gof : (X, \tau) \to (Z, \eta)$ is fuzzy AF-open.*

7. Fuzzy *AF*-compact

Definition 7.1. ([14]) A family of fuzzy subsets ξ of a fuzzy topological space X is called *form a fuzzy filterbases* if and only if for every finite family $\{\lambda_j : j = 1, 2, ..., n\}$, $\bigwedge_{j=1}^{n} \lambda_j \neq 0_X$.

Definition 7.2. Let (X, τ) be a fuzzy topological space and $x_\alpha \in X$ for $\alpha \in]0, 1]$. A point x_α is called a *AF-cluster point of filterbases* β if and only if $x_\alpha \in Cl_{AF}(F)$ such that for every $F \in \beta$.

Definition 7.3. A family ζ of fuzzy *AF*-open sets in a fuzzy topological space X is called *a AF-open cover* of a fuzzy set ς of X if and only if $(\bigvee_{\omega\in\zeta} \omega)(y) = 1_X$, for each $y \in X(\varsigma)$. A fuzzy *AF*-open cover ζ of a fuzzy set ς in a fuzzy topological space X is called have a *finite subcover* if and only if there exists a finite subfamily $\nu = \{\omega_1, ..., \omega_n\}$ of ζ such that $(\bigvee_{j=1}^{n} \omega_j)(y) \geq \varsigma(y)$, for every $y \in X(\varsigma)$ (The notation $X(\varsigma)$ stands for the support of the fuzzy set ς).

Definition 7.4. ([6]) A fuzzy topological space X is called *compact* if and only if every open cover of X has a finite subcover.

Definition 7.5. A fuzzy topological space (X, τ) is called *fuzzy AF-compact* if and only if for every filter base β that the finite intersection of its elements to be quasi-coincident with ζ, $(\bigwedge_{F\in\beta} Cl_{AF}(F)) \wedge \zeta \neq 0_X$.

Theorem 7.6. A fuzzy topological space (X, τ) is fuzzy *AF*-compact if and only if each family A_α ($\alpha \in]0, 1]$) of *AF*-open fuzzy sets such that $\bigvee_{\zeta\in A_\alpha} \zeta = 1_X$ there is a finite subfamily $\mu \leq A_\alpha$ such that $\bigvee_{\zeta\in\mu} \zeta = 1_X$.

Proof. Let the *AF*-open cover of A_α be ζ. Suppose that ζ does not have a finite subcover. Then there exists an $y \in X$ such that $\xi_j(y) < \alpha$ $(j = 1, ..., n)$ for every finite subfamily $\{\xi_1, ..., \xi_n\}$ of ζ. From here $\xi_j^c(y) > 1 - \alpha$. Hence $\bigwedge_{j=1}^{n} \xi_j^c(y) \neq 0_X$ and it includes a fuzzy point y_λ for any $y \in X$. By the hypothesis, $\bigvee_{\xi\in\zeta} \xi(y) = 1_X$ for every $y \in X$, thus we have $\bigwedge\{\xi^c : \xi \in \zeta\} = 0_X$. This is contradiction. Therefore every *AF*-open cover of A_α has a finite subcover.
On the other hand assume that X is not *AF*-compact. Then It has a filter base as follows: $\bigwedge_{F\in\beta} Cl_{AF}(F) = 0_X$ and $\bigvee_{F\in\beta}(Cl_{AF}(F))^c(y) = 1_X$ for every $y \in X$. It follows from $\{(Cl_{AF}(F))^c : F \in \beta\}$ is a *AF*-open cover of A_α for every $0 < \alpha \leq 1$ and it has a finite subcover. Thus, $\bigvee_{j=1}^{n}(Cl_{AF}(F_j))^c(y) = 1_X$ from here $\bigwedge_{j=1}^{n}(Cl_{AF}(F_j))(y) = 0_X$ for every $y \in X$. Hence we obtain that $\bigwedge_{j=1}^{n} F_j = 0_X$. This is a contradiction, since members of a filter basis are B_j′ s. In that case X is *AF*-compact. □

Remark 7.7. Since each open fuzzy set implies *AF*-open, so every fuzzy *AF*-compact space implies compact space. But the converse need not be true.

Theorem 7.8. *Every fuzzy AF-closed set in a fuzzy AF-compact space is fuzzy AF-compact.*

Proof. Straightforward. □

Theorem 7.9. *Let (X, τ) be a fuzzy topological space and let $\{K_j\}_{1\leq j\leq n}$ be a collection of AF-closed subsets of X. If K_j is AF-compact for every j=1,...n, then $K = \bigvee_{1\leq j\leq n} K_j$ is AF-compact.*

Proof. It is clear. □

Theorem 7.10. *The image of a AF-compact space under a AF-irresolute function is AF-compact.*

Proof. It is obvious. □

Definition 7.11. A fuzzy set ν in a fuzzy topological space (X, τ) is called *fuzzy AF-compact relative* to X if and only if for each family λ of *AF*-open fuzzy sets such that $\bigvee_{\zeta\in\lambda} \zeta \geq \nu(y)$ there is a finite subfamily $\mu \leq \lambda$ such that $\bigvee_{\zeta\in\mu} \zeta \geq \nu(y)$ for each $y \in X(\nu)$.

Theorem 7.12. *A fuzzy topological space X is AF-compact if and only if for every family $\{\zeta_j : j \in J\}$ of AF-closed fuzzy sets of X, $\bigwedge_{j\in J} \zeta_j \neq 0_X$.*

Proof. Let $\{\zeta_j : j \in J\}$ be a family of AF-closed fuzzy sets with the finite intersection property. Assume that $\bigwedge_{j\in J} \zeta_j = 0_X$. From here $\bigvee_{j\in J}(\zeta)^c_j = 1_X$. Since $\{(\zeta)^c_j : j \in J\}$ is a family of AF-open fuzzy sets cover of X, by the hypotesis, $\bigvee_{j\in J}(\zeta)^c_j = 1_X$ for a finite subset $K \subset J$. Then $\bigwedge_{j\in K} \zeta_j = 0_X$. This a contradiction. Thus we have $\bigwedge_{j\in J} \zeta_j \neq 0_X$.
On the other hand, let $\{\zeta_j : j \in J\}$ be a family of AF-open fuzzy sets cover of X. Assume that $\bigvee_{j\in K} \zeta_j \neq 1_X$ for every finite subset $K \subset J$. Then $\bigwedge_{j\in K}(\zeta)^c_j \neq 0_X$. Hence $\{(\zeta)^c_j : j \in J\}$ provides the finite intersection property. Then from the hypothesis $\bigwedge_{j\in J}(\zeta)^c_j \neq 0_X$ it follows from $\bigvee_{j\in K} \zeta_j \neq 1_X$. This is contradiction. Thus $\{\zeta_j : j \in J\}$ is a AF-open cover of X. Therefore, we have X is fuzzy AF-compact. □

Theorem 7.13. *A fuzzy topological space X is fuzzy AF-compact if and only if every filterbases β in X, $\bigwedge_{F\in\beta} Cl_{AF}(F) \neq 0_X$.*

Proof. Let ξ be the cover of fuzzy AF-open set X and let ξ not has a finite subcover. Then for every finite subcollection $\{\zeta_1, \zeta_1, ..., \zeta_n\}$ of ξ, there exists $y \in X$ such that $\zeta_j(y) < 1$ for every $1 \leq j \leq n$. Then $(\zeta_j)^c(y) > 0$, from here $\bigwedge_{1\leq j\leq n}(\zeta_j)^c(y) \neq 0_X$. Hence $\{(\zeta_j)^c(y) : \zeta_j \in \xi\}$ forms a filterbases in X. Since ξ is the cover of fuzzy AF-open set X, then $(\bigvee_{\zeta_j\in\xi} \zeta_j)(y) = 1_X$ for every $y \in X$ and $\bigwedge_{\zeta_j\in\xi} Cl_{AF}(\zeta_j)^c(y) = \bigwedge_{\zeta_j\in\xi}(\zeta_j)^c(y) = 0_X$, which is a contradiction. Then every the cover of fuzzy AF-open set X has a finite subcover and thus X is fuzzy AF-compact.
On the other hand, assume there exists a filterbases β such that $\bigwedge_{F\in\beta} Cl_{AF}(F) = 0_X$, from here $(\bigvee_{F\in\beta}(Cl_{AF}(F))^c)(y) = 1_X$ for every $y \in X$ and thus $\xi = \{(Cl_{AF}(F))^c) : F \in \beta\}$ is a cover of fuzzy AF-open set X. Since X is fuzzy AF-compact, then ξ has a finite subcover. In that case $(\bigvee_{1\leq j\leq n}(Cl_{AF}(F_j))^c)(y) = 1_X$ and it is obtained $(\bigvee_{1\leq j\leq n}(F_j)^c)(y) = 1_X$. We have $\bigwedge_{1\leq j\leq n} F_j(y) = 0_X$. Since the elements of the β filterbases are F_j , this is a contradiction. In that case $\bigwedge_{F\in\beta} Cl_{AF}(F) \neq 0_X$. □

Theorem 7.14. *A fuzzy setv ν in a fuzzy topological space X is fuzzy AF-compact relative to X if and only if for every filterbases β such that every finite of members of β is quasi coincident with ν, $(\bigwedge_{F\in\beta} Cl_{AF}(F)) \wedge \nu \neq 0_X$.*

Proof. Suppose that ν not be fuzzy AF-compact relative to X, then there exists a AF-open fuzzy set λ cover of ν such that λ has not finite subcover μ. Then $(\bigvee_{\zeta_j\in\mu} \zeta_j)(y) < \nu(y)$ for some $y \in X(y)$, hence $(\bigwedge_{\zeta_j\in\mu}(\zeta_j)^c)(y) > (\nu)^c(y) \geq 0$ and thus $\{(\zeta_j)^c : \zeta_j \in \lambda\}$ forms a filterbases and $\bigwedge_{\zeta_j\in\mu}(\zeta_j)^c q\nu$. By hypotesis $(\bigwedge_{\zeta_j\in\mu} Cl_{AF}(\zeta_j)^c) \wedge \nu \neq 0_X$ and so that $(\bigwedge_{\zeta_j\in\mu}(\zeta_j)^c) \wedge \nu \neq 0_X$. Then for any $y \in X(\nu)$, $(\bigwedge_{\zeta_j\in\lambda}(\zeta_j)^c)(y) > 0_X$, so that $(\bigvee_{\zeta_j\in\lambda} \zeta_j)(y) < 1_X$. This is a contradiction. Therefore ν is fuzzy AF-compact relative to X.
On the other hand, assume that there exists a filterbases β such that every finite of members of β is quasi coincident with ν and $(\bigwedge_{F\in\beta} Cl_{AF}(F)) \wedge \nu \neq 0_X$. Then for every $y \in X(\nu)$, $(\bigwedge_{F\in\beta} Cl_{AF}(F))(y) = 0_X$ and thus $(\bigvee_{F\in\beta}(Cl_{AF}(F))^c)(y) = 1_X$ for every $y \in X(\nu)$. Hence $\lambda = \{(Cl_{AF}(F))^c : F \in \beta\}$ is AF-open fuzzy set cover ν. Since ν is fuzzy AF-compact relative to X, then there exists a finite subcover, consider $\{(Cl_{AF}(F_1))^c, (Cl_{AF}(F_2))^c, ..., (Cl_{AF}(F_n))^c\}$, such that $(\bigvee_{1\leq j\leq n}(Cl_{AF}(F_j))^c)(y) \geq \nu(y)$ for every $y \in X(\nu)$. So that $(\bigwedge_{1\leq j\leq n}(Cl_{AF}(F_j)))(y) \leq \nu^c(y)$ for every $y \in X(\nu)$, thus $\bigwedge_{1\leq j\leq n}(Cl_{AF}(F_j))\tilde{q}\nu$. This is a contradiction. Therefore for every filterbases β such that every finite of members of β is quasi coincident with ν, $(\bigwedge_{F\in\beta} Cl_{AF}(F))\wedge\nu \neq 0_X$. □

Theorem 7.15. *Every AF-closed fuzzy subset of a fuzzy AF-compact space is fuzzy AF-compact relative to X .*

Proof. Let β be a fuzzy filterbases in X and a AF-closed fuzzy set ν. For each finite subfamily μ of β, it is provided that $\nu q \wedge \{F : F \in \mu\}$. Suppose that $\beta^\star = \{\nu\} \cup \beta$. For every finite subfamily $\mu^\star$ of $\beta^\star$, if $\nu \notin \mu^\star$, then $\bigwedge \mu^\star \neq 0_X$. If $\nu \in \mu^\star$ and since $\nu q \bigwedge\{F : F \in \mu^\star - \nu\}$, then $\bigwedge \mu^\star \neq 0_X$. Hence $\mu^\star$ is a fuzzy filterbases in X. Since X is fuzzy AF-compact, then $\bigwedge_{F\in\beta^\star} Cl_{AF}(F) \neq 0_X$. It follows from $(\bigwedge_{F\in\beta} Cl_{AF}(F)) \wedge \nu = (\bigwedge_{F\in\beta} Cl_{AF}(F)) \wedge Cl_{AF}\nu \neq 0_X$. By Theorem 7.14, ν is fuzzy AF-compact relative to X. □

Theorem 7.16. *If a function $f : X \to Y$ is fuzzy AF-irresolute and ν is fuzzy AF-compact relative to X, then $f(\nu)$ is fuzzy AF-compact relative to Y.*

Proof. Let the AF-open set cover of $X(f(\nu))$ be family $\{\zeta_j\}_{j\in J}$. For $y \in X(\nu)$, $f(y) \in f(X(\nu)) = X(f(\nu))$. Since f fuzzy AF-irresolute, then $\{f^{-1}(\zeta_j)\}_{j\in J}$ is fuzzy AF-open set cover of $X(\nu)$. Since ν is fuzzy AF-compact relative to X, we have $X(\nu) \leq \bigvee_{j=1}^{n} f^{-1}(\zeta_j)$. From here $X(\nu) \leq f^{-1}(\bigvee_{j=1}^{n} \zeta_j)$ and then $X(f(\nu)) = f(X(\nu)) \leq ff^{-1}(\bigvee_{j=1}^{n} \zeta_j) \leq \bigvee_{j=1}^{n} \zeta_j$. We obtain that $f(\nu)$ is fuzzy AF-compact relative to Y. □

8. Conclusions

We define fuzzy AF-open sets in a fuzzy topological space (X, τ). We obtain some properties and of fuzzy AF-open sets. We introduce and investigate fuzzy AF-continuous functions on a fuzzy topological space. And also, we examine the notion of fuzzy AF-continuous functions and fuzzy AF-irresolute functions. Further fuzzy AF-compactness is defined. Its properties and characterizations are examined. Moreover, we offer two open problems in this study.

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