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Topological position of a point with respect to a set in a raw bistruct

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Abstract. In this study, we investigate the topological positions of points relative to sets in various topologies induced by raw binary operations. The fact that the raw binary operation is weaker than both partial and multivalued ones allows for a relatively wider variety of topologies induced by it. This gives us the ability to determine the topological positions of a point in a raw binary structure relative to a set by considering the lo-topology, ro-topology, and o-topology induced by the raw binary operation. Through the analysis of topological positions such as interior points, limit points, and closure points, we gain a deeper understanding of the nature of raw binary operations from a topological perspective. The potential benefit of this work is to expand the conceptual framework in terms of topological evaluability of problems in algebraic structures. In this study, we introduce the concepts of raw-binary interior point, raw-binary limit point, raw-binary closure point and give some useful facts about finding raw-binary interior, raw-binary limit and raw-binary closure points of a given set in lo-topology, ro-topology and o-topology. The potential benefit of this study is to provide topological evaluability of problems in algebraic structures by extending the conceptual framework.

1. Introduction

It is only natural to seek answers to questions such as whether there exist topologies induced by a relation, and if so, how they can be obtained. It is also important to understand which properties of the relation require specific topological properties in the induced topology. Fortunately, the literature contains studies that provide affirmative answers to these questions and more.

In 1969, Smithson made significant contributions to the study of topologies generated by binary relations by demonstrating the conditions under which these spaces satisfy separation axioms as well as whether they are compact or connected [21].

Allam and colleagues further advanced this field by presenting several methods for indirectly generating topological spaces using operators such as closure, interior, and neighborhood operators given by binary relations, leading to significant findings [3].

Induràin and Knoblauch also contributed to this field by studying categories of topologies generated by a binary relation, and demonstrating that any metric topology can be generated by a binary relation [15].

In the context of algebraic structures, a group-like structure refers to a non-empty set X equipped with a (partial) binop on X that satisfies specific axioms. These axioms encompass properties such as associativity, closure, commutativity, identity, and invertibility. For further information, you can refer to the following

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sources: [4–7, 13, 16, 17, 20]. Notable examples of group-like structures that utilize a partial binary operation are groupoids, semigroupoids, and small categories.

Bending the restriction that a group-like structure contains a binop or a partial binop, we consider the class of group-like structures containing raw binops. This is because the class of topologies produced by a raw binop is larger than that produced by a partial binop. In our previous works, we have established the theoretical infrastructure of this idea and have obtained some results on the topologies induced by group-like structures [18, 19].

The literature contains numerous studies on soft and rough sets with a wide range of applications [1, 2, 8–12, 14]. In rough set theory, we can think of the indiscernibility relation $\mathrm{IND}(P) = \left\{(x_i, x_j) \in U \,\middle|\, \forall p \in P,\ p(x_i) = p(x_j)\right\}$ as a raw binary operation of the form $* \subseteq I^2 \times I$, where $I = \{0, 1, 2, ..., n\}$ is an index set, $U = \{x_i | i \in I\}$ is the indexed set of objects and P is a subset of the set A of attributes. With similar reasoning, in soft set theory, we can consider the indiscernibility relation $\mathrm{IND}(A, f) = \left\{(x_i, x_j) \in U \,\middle|\, \forall p \in A,\ x_i \in f(p) \Leftrightarrow x_j \in f(p)\right\}$ as a raw binary operation of the form $* \subseteq I^2 \times I$, where $I = \{0, 1, 2, ..., n\}$ is an index set, $U = \{x_i | i \in I\}$ is the indexed universal set of objects, A is a subset of the set E of parameters and f is a function from A to the power set of U.

The aim of this work is to enrich the conceptual framework in the literature for topologies induced by raw bistructs. The possible benefit of this is to create a ground for topological evaluation of problems considered in any algebraic structure.

This study introduces the concepts of raw-binary interior point, raw-binary limit point, and raw-binary closure point. It also presents some useful facts about finding raw-binary interior, raw-binary limit, and raw-binary closure points of a given set in low-, ro-, and o-topology.

2. Preliminaries

A raw bistruct $(X, *)$ is X equipped with a binary relation $* \subseteq X^2 \times X$ which is called a raw binary operation (or shortly a raw binop) on X. We denote $\{z \in X | \, (x, y) * z\}$ by $x * y$. Let $u, v \in X$. A *left operand* is defined by $L(u, v) = \{x \in X | v \in x * u\}$. Similarly, *right operand* are defined by $R(u, v) = \{x \in X | v \in u * x\}$ while *output* is defined by $O(u, v) = \{x \in X | x \in u * v\}$. In a raw bistruct $(X, *)$, we can use the collection of all sets $L(u, v)$ as a subbase for a topology on X which is called left operand topology (or shortly lo-topology and denoted by $\mathcal{L}$. Similarly, using sets $R(u, v)$ and $O(u, v)$, we have ro-topology $\mathcal{R}$ and o-topology $\mathcal{O}$, respectively [18, 19].

Definition 2.1. ([18]) Given a raw bistruct $(X, *)$. F be a non-empty finite subset of X^2. Let $x \in X$. A point $x \in X$ is called a *left* (resp., *right*, *output*) *common point of F* if $v \in x * u$ (resp., $v \in u * x$, $x \in u * v$) holds for every $(u, v) \in F$. A subset $A \subseteq X$ is called a *left (*resp. *right, output) common set* of F if every element of A is a left (resp. right, output) common point of F.

Definition 2.2. ([18]) Given a raw bistruct $(X, *)$. $\mathcal{F}$ be a family of non-empty finite subsets of X^2. $\mathcal{F}$ is called a *left (resp., right*, output) *common pointer of a point* $x \in X$ if there exists a set $F \in \mathcal{F}$ having x as a left (resp. right, output) common point. $\mathcal{F}$ is called a *left* (resp. *right, output*) *common pointer of a subset* $A \subseteq X$ if it is a left (resp., right, output) common pointer of each element of A. $\mathcal{F}$ is called a *left* (*resp., right, output*) *non-common pointer of a subset* $A \subseteq X$ if it is a left (resp., right, output) common pointer of none of the elements of A.

Proposition 2.3. *Given a raw bistruct $(X, *)$. Then the following properties hold:*

1. *For every pair u, v of points in X, $L(u, v)$ (resp., $R(u, v)$, $O(u, v)$) is a $\mathcal{L}$(resp., $\mathcal{R}$, $\mathcal{O}$)-open.*
2. *For every non-empty finite subset F of X^2, $\bigcap_{(u,v)\in F} L(u, v)$ (resp. $\bigcap_{(u,v)\in F} R(u, v)$, $\bigcap_{(u,v)\in F} O(u, v)$) is a $\mathcal{L}$(resp., $\mathcal{R}$, $\mathcal{O}$)-open.*
3. *For every family $\mathcal{F}$ of non-empty finite subsets of X^2, $\bigcup_{F\in\mathcal{F}} \bigcap_{(u,v)\in F} L(u, v)$ (resp., $\bigcup_{F\in\mathcal{F}} \bigcap_{(u,v)\in F} R(u, v)$, $\bigcup_{F\in\mathcal{F}} \bigcap_{(u,v)\in F} O(u, v)$) is a $\mathcal{L}$(resp., $\mathcal{R}$, $\mathcal{O}$)-open.*

Proof. It is immediately proven from the definition of the left (resp., right, output) operand topology. □

Remark 2.4. We call a set of the form given in Proposition 2.3(1) a *initial-open*. Although not of the form given in Proposition 2.3(1), a set of the form given in Proposition 2.3(2) are called a *mid-open (by F)*. We also call a set of the form given only in Proposition 2.3(3) a *final-open (by $\mathcal{F}$)*.

Definition 2.5. Given a raw bistruct $(X, *)$. A raw binop $\circ$ on X defined by the relation

$$x \in y \circ z \Leftrightarrow x \notin y * z$$

is called the *dual of the raw binop* $*$. The raw bistruct $(X, \circ)$ is called the *dual of the raw bistruct* $(X, *)$.

Definition 2.6. ([18]) Given a raw bistruct $(X, *)$.

1. The topology for which the collection $\{(L(u, v))^c \, | u, v \in X\}$ is a subbase is called the *dual of lo-topology* induced by $(X, *)$. It is denoted by $\mathcal{L}'_{(X,*)}$, $\mathcal{L}'_*$ or simply $\mathcal{L}'$.
2. The topology for which the collection $\{(R(u, v))^c \, | u, v \in X\}$ is a subbase is called the *dual of ro-topology* induced by $(X, *)$. It is denoted by $\mathcal{R}'_{(X,*)}$, $\mathcal{R}'_*$ or simply $\mathcal{R}'$.
3. The topology for which the collection $\{(O(u, v))^c \, | u, v \in X\}$ is a subbase is called the *dual of o-topology* induced by $(X, *)$. It is denoted by $\mathcal{O}'_{(X,*)}$, $\mathcal{O}'_*$ or simply $\mathcal{O}'$.

Proposition 2.7. *Given a raw bistruct $(X, *)$. Let $\circ$ be the dual of $*$. Then the following properties hold:*

1. *A subset $A \subseteq X$ is $\mathcal{L}_*$(resp., $\mathcal{R}_*$, $\mathcal{O}_*$)-open if and only if A is $\mathcal{L}'_\circ$(resp., $\mathcal{R}'_\circ$, $\mathcal{O}'_\circ$)-open.*
2. *A subset $A \subseteq X$ is $\mathcal{L}_*$(resp., $\mathcal{R}_*$, $\mathcal{O}_*$)-closed if and only if A is $\mathcal{L}'_\circ$(resp., $\mathcal{R}'_\circ$, $\mathcal{O}'_\circ$)-closed.*

Proof. Let $(X, \circ)$ be the dual of a raw bistruct $(X, *)$ and $A \subseteq X$.

1. Let $A \subseteq X$ be a $\mathcal{L}_*$-open. From the definition of lo-topology, we have the equality $A = \bigcup_{F \in \mathcal{F}} \bigcap_{(u,v) \in F} L_* (u, v)$ for some family $\mathcal{F}$ of non-empty finite subsets of X^2. Then $x \in A$ if and only if for some $F \in \mathcal{F}$ such that $x \in L_* (u, v)$, or equivalently, $x \notin L_\circ (u, v)$ for every $(u, v) \in F$. It follows immediately that $x \in A$ if and only if there exists $F \in \mathcal{F}$ such that $x \notin \bigcup_{(u,v) \in F} L_\circ (u, v)$. It implies that $x \in A \Leftrightarrow x \notin \bigcap_{F \in \mathcal{F}} \bigcup_{(u,v) \in F} L_\circ (u, v)$. Applying De Morgan's rules, we get $A = \bigcup_{F \in \mathcal{F}} \bigcap_{(u,v) \in F} (L_\circ (u, v))^c$. Thus A is a $\mathcal{L}'_\circ$-open. A similar argument works for $\mathcal{R}$ and $\mathcal{O}$.

2. Let $A \subseteq X$ be a $\mathcal{L}_*$-closed. Then $X \setminus A$ is a $\mathcal{L}_*$-open. From Proof 2, $X \setminus A$ is a $\mathcal{L}'_\circ$-open which implies A is a $\mathcal{L}'_\circ$-closed. A similar argument works for $\mathcal{R}$ and $\mathcal{O}$. □

Definition 2.8. Given a raw bistruct $(X, *)$. F be a non-empty finite subset of X^2 $x \in X$. A subset A of X is said to be *raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, $\mathcal{O}$)-covered with respect to F* if it is a subset of $\bigcup_{(u,v) \in F} L(u, v)$ (resp., $\bigcup_{(u,v) \in F} R(u, v)$, $\bigcup_{(u,v) \in F} O(u, v)$).

Definition 2.9. Given a raw bistruct $(X, *)$. A point $x \in X$ is called a *raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, $\mathcal{O}$)-interior point of a subset $A \subseteq X$* if there exists a non-empty finite subset F of X^2 having x as a left (resp., right, output) common point such that A^c *is* raw-binarily $\mathcal{L}'$(resp., $\mathcal{R}'$, $\mathcal{O}'$)-covered with respect to F. A *raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, $\mathcal{O}$)-interior* of a subset A of X consists of all raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, $\mathcal{O}$)-interior points of A.

Definition 2.10. Given a raw bistruct $(X, *)$. A point $x \in X$ is called a *raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, $\mathcal{O}$)-limit point of a subset $A \subseteq X$* if every non-empty finite subsets of X^2 having x as a left (resp., right, output) common point has also a point $y \in A$ other than x as a left (resp., right, output) common point. A *raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, $\mathcal{O}$)-derived set* of a subset A of X consists of all raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, $\mathcal{O}$)-limit points of A.

Definition 2.11. Given a raw bistruct $(X, *)$. A point $x \in X$ is called a *raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, $\mathcal{O}$)-closure point of a subset $A \subseteq X$* if for every non-empty finite subsets F of X^2 having x as a left (resp., right, output) common point, A is not raw-binarily $\mathcal{L}'$(resp., $\mathcal{R}'$, $\mathcal{O}'$)-covered with respect to F. A *raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, $\mathcal{O}$)-closure* of a subset A of X consists of all raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, $\mathcal{O}$)-closure points of A.

3. Main results

Theorem 3.1. *Given a raw bistruct $(X,*)$. A point x in a subset $A \subseteq X$ is a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, $\mathcal{O}$)-interior point of A if and only if x is an interior point of A with respect to the lo-topology $\mathcal{L}$ (resp., the ro-topology $\mathcal{R}$, the o-topology $\mathcal{O}$).*

Proof. Given a raw bistruct $(X,*)$. Let $A \subseteq X$ and $x \in A$.

$(\Rightarrow)$: We show that there exists a $\mathcal{L}$-open subset U of A containing x. From the definition of lo-topology, there exists a family $\mathcal{F}$ of non-empty finite subsets of X^2 such that $x \in \bigcup_{F\in\mathcal{F}} \bigcap_{(u,v)\in F} L(u,v)$. It means that there exists a non-empty finite subset of X^2, say F, $x \in \bigcap_{(u,v)\in F} L(u,v)$. So x is a left common point of F. From the hypothesis, F satisfies the condition in Definition 2.9, that is, A^c *is* raw-binarily $\mathcal{L}'$-covered with respect to F. Then from Definition 2.8, A^c is a subset of $\bigcup_{(u,v)\in F} (L(u,v))^c$. Therefore it follows immediately that

$$A = (A^c)^c \supseteq \left[\bigcup_{(u,v)\in F} (L(u,v))^c\right]^c = \bigcap_{(u,v)\in F} L(u,v).$$

Set $U = \bigcap_{(u,v)\in F} L(u,v)$. Thus U is a $\mathcal{L}$-open set such that $x \in U \subseteq A$.

$(\Leftarrow)$: We can reverse the steps of the first part of the proof.

Using similar arguments, it can be proven that the implication holds for the ro-topology $\mathcal{R}$ and the o-topology $\mathcal{O}$. $\square$

Example 3.2. Let $*$ be a raw binop on $X = \{1,2,3,4,5\}$ defined by

$$x * y = \left\lfloor \frac{x}{y+1} \right\rfloor$$

where $\lfloor\ \rfloor$ is the floor function.

For example, let us find the left operand set $L(1,2)$. To do this, we need to find the values x satisfying

$$2 \in x * 1 = \left\lfloor \frac{x}{2} \right\rfloor.$$

Then we have $L(1,2) = \{4,5\}$.

Thus, all left operand sets for the raw bistruct $(X,*)$ are as follows:

L	1	2	3	4	5
1	$\{2,3\}$	$\{4,5\}$	$\emptyset$	$\emptyset$	$\emptyset$
2	$\{3,4,5\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	$\{4,5\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	$\{5\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

.

So, we have the lo-topology $\mathcal{L}$ induced by $(X,*)$ as follows

$$\mathcal{L} = \{X, \emptyset, \{3\}, \{5\}, \{2,3\}, \{3,5\}, \{4,5\}, \{2,3,5\}, \{3,4,5\}, \{2,3,4,5\}\}.$$

The set of interior points of $A = \{1,3,5\}$ with respect to the lo-topology $\mathcal{L}$ is $\{5\}$.

On the other hand, the dual of $\mathcal{L}$ is

$$\mathcal{L}' = \{X, \emptyset, \{1\}, \{1,2\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,4,5\}, \{1,2,3,4\}, \{1,2,4,5\}\}.$$

Let $F = \{(1,2),(4,1)\} \subseteq X^2$. It is easy to see that a left common point of F is 5. Furthermore, we have

$$\bigcup_{(u,v)\in F} (L(u,v))^c = (L(1,2))^c \cup (L(2,4))^c = \{4,5\}^c \cup \{5\}^c = \{1,2,3,4\} \supseteq \{2,4\} = A^c$$

which means that 5 is a raw-binarily $\mathcal{L}$-interior point of A. It can be checked that A has only one raw-binarily $\mathcal{L}$-interior point.

Theorem 3.3. *Given a raw bistruct $(X,*)$ and two subsets $A, B \subseteq X$. A point x is a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-interior point of $A \cap B$ if and only if x is a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-interior point of both A and B.*

Proof. Given a raw bistruct $(X,*)$. Let $A, B \subseteq X$.

$(\Rightarrow)$: By the hypothesis, we have a non-empty finite subset F of X^2 having x as a left common point such that $(A \cap B)^c$ *is* raw-binarily $\mathcal{L}'$-covered with respect to F. Then from Definition 2.8, $(A \cap B)^c \subseteq \bigcup_{(u,v)\in F} (L(u,v))^c$ which implies that $\bigcap_{(u,v)\in F} L(u,v) \subseteq A \cap B$. It follows from this that $\bigcap_{(u,v)\in F} L(u,v) \subseteq A$ and $\bigcap_{(u,v)\in F} L(u,v) \subseteq B$. Thus we have $A^c \subseteq \bigcup_{(u,v)\in F} (L(u,v))^c$ and $B^c \subseteq \bigcup_{(u,v)\in F} (L(u,v))^c$ which means that x is a raw-binarily $\mathcal{L}$-interior point of both A and B.

$(\Leftarrow)$: By the hypothesis, there exists a non-empty finite subset F of X^2 having x as a left common point such that both A^c and B^c are raw-binarily $\mathcal{L}'$-covered with respect to F. From Definition 2.8, $A^c \subseteq \bigcup_{(u,v)\in F} (L(u,v))^c$ and $B^c \subseteq \bigcup_{(u,v)\in F} (L(u,v))^c$. Then $A \cap B$ is a superset of $\bigcap_{(u,v)\in F} L(u,v)$. It follows immediately that $(A \cap B)^c \subseteq \bigcup_{(u,v)\in F} (L(u,v))^c$. Thus $(A \cap B)^c$ *is* raw-binarily $\mathcal{L}'$-covered with respect to F.

Using similar arguments, it can be proven that the implication holds for the ro-topology $\mathcal{R}$ and the o-topology O. □

Theorem 3.4. *Given a raw bistruct $(X,*)$ and two subsets $A, B \subseteq X$. If a point x is a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-interior point of at least one of A and B, then x is also a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-interior point of $A \cup B$.*

Proof. Given a raw bistruct $(X,*)$. Let $A, B \subseteq X$. By the hypothesis, there exists a non-empty finite subset F of X^2 having x as a left common point such that at least one of A^c and B^c are raw-binarily $\mathcal{L}'$-covered with respect to F. From Definition 2.8, we get $A^c \subseteq \bigcup_{(u,v)\in F} (L(u,v))^c$ or $B^c \subseteq \bigcup_{(u,v)\in F} (L(u,v))^c$ which implies that $(A \cup B)^c = A^c \cap B^c \subseteq \bigcup_{(u,v)\in F} (L(u,v))^c$. Thus $(A \cup B)^c$ *is* raw-binarily $\mathcal{L}'$-covered with respect to F which completes the proof. Using similar arguments, it can be proven that the implication holds for the ro-topology $\mathcal{R}$ and the o-topology O. □

The converse of Theorem 3.4 does not hold in general, as we will see from the following example.

Example 3.5. Let $X = \{a,b,c,d,e\}$ and $A = \{a,b\}, B = \{b,c\} \subseteq X$. Let $*$ be a raw binop on a set X as in the operation table below.

$*$	a	b	c	d	e
a	$\emptyset$	$\emptyset$	$\{d,e\}$	$\emptyset$	$\emptyset$
b	$\{a\}$	$\emptyset$	$\{d\}$	$\emptyset$	$\emptyset$
c	$\emptyset$	$\emptyset$	$\{e\}$	$\emptyset$	$\emptyset$
d	$\{a\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
e	$\{a\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

.

Then non-empty left operand sets with respect to $*$ are only $L(a,a) = \{b,d,e\}$, $L(c,d) = \{a,b\}$ and $L(c,e) = \{a,c\}$. Therefore the lo-topology induced by $(X,*)$ is as follows

$$\mathcal{L} = \{\{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{b,d,e\}, \{a,b,d,e\}, X, \emptyset\}.$$

Set $F = \{(c,e)\}$. Then c is a left common point of F since $c \in \{a,c\} = L(c,e)$. Furthermore, note that there is no set other than F having c as the left common point. Also, we get

$$(A \cup B)^c = \{a,b,c\}^c = \{d,e\} \subseteq \{b,d,e\} = \{a,c\}^c = (L(c,e))^c = \bigcup_{(x,y)\in F} (L(x,y))^c$$

which means that $(A \cup B)^c$ *is* raw-binarily $\mathcal{L}'$-covered with respect to F. So c is a raw-binarily $\mathcal{L}$-interior point of $A \cup B$.

On the other hand, for the only set F having c as a left common point it holds

$$A^c = \{a,b\}^c = \{c,d,e\} \nsubseteq \{b,d,e\} = \{a,c\}^c = (L(c,e))^c = \bigcup_{(x,y)\in F} (L(x,y))^c$$

and

$$B^c = \{b, c\}^c = \{a, d, e\} \nsubseteq \{b, d, e\} = \{a, c\}^c = (L(c, e))^c = \bigcup_{(x,y)\in F} (L(x, y))^c.$$

It follows from this that neither of A^c and B^c is raw-binarily $\mathcal{L}'$-covered with respect to F. Thus c is a raw-binarily $\mathcal{L}$-interior point of neither A and B.

Theorem 3.6. *Given a raw bistruct* $(X, *)$*. A point x in X is a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, $\mathcal{O}$)-limit point of a subset $A \subseteq X$ if and only if x is a limit point of A with respect to the lo-topology $\mathcal{L}$ (resp., the ro-topology $\mathcal{R}$, the o-topology $\mathcal{O}$).*

Proof. Given a raw bistruct $(X, *)$. Let $x \in X$ and $A \subseteq X$.

($\Rightarrow$) : Let U be an arbitrary $\mathcal{L}$-open containing x. We show that U contains a point in A other than x. From the definition of lo-topology, there exists a family $\mathcal{F}$ of non-empty finite subsets of X^2 such that $U = \bigcup_{F\in\mathcal{F}} \bigcap_{(u,v)\in F} L(u, v)$. It means that there exists a non-empty finite subsets of X^2, say F, $x \in \bigcap_{(u,v)\in F} L(u, v)$. So F has x as a left common point. From the hypothesis, F satisfies the condition in Definition 2.10, that is, F has also a point $y \in A$ other than x as a left common point. Then from Definition 2.1, for every $(u, v) \in F$, $v \in y * u$ and so $y \in L(u, v)$. Therefore it follows immediately that $y \in \bigcap_{(u,v)\in F} L(u, v)$ and thus $y \in \bigcup_{F\in\mathcal{F}} \bigcap_{(u,v)\in F} L(u, v) = U$.

($\Leftarrow$) : We can reverse the steps of the first part of the proof.

By similar arguments, it can be proven that the implication holds for the ro-topology $\mathcal{R}$ and the o-topology $\mathcal{O}$. □

Example 3.7. We consider the bistruct $(X, *)$ in Example 3.5. Set $F = \{(c, d), (c, e)\}$. Then

$$\bigcap_{(x,y)\in F} L(x, y) = L(c, d) \cap L(c, e) = \{a\}$$

which means that F has none point other than a as a left common point and so a is not a raw-binarily $\mathcal{L}$-limit point of A. By similar reasoning, it is shown that b and d are not raw-binarily $\mathcal{L}$-limit points of A. On the other hand, for each non-empty finite subset F of X^2 having c as a left common point, $\bigcap_{(u,v)\in F} L(u, v)$ is a set of form either $\{a, c\}$ or X which means that F has also a as a left common point. Hence c is a raw-binarily $\mathcal{L}$-limit point of A. Similarly, it is easy to see that e is a raw-binarily $\mathcal{L}$-limit point of A. Indeed, the derived set of A with respect to the lo-topology $\mathcal{L}$ is $A' = \{c, e\}$.

Theorem 3.8. *Given a raw bistruct* $(X, *)$*. A subset A of X is a $\mathcal{L}$(resp., $\mathcal{R}$, $\mathcal{O}$)-closed if and only if it contains all of its raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, $\mathcal{O}$)-limit points.*

Proof. Given a raw bistruct $(X, *)$ and a subset $A \subseteq X$.

($\Rightarrow$) : Let $x \in X$ be a raw-binarily $\mathcal{L}$-limit point of A. Assume that $x \notin A$. Then $x \in X \setminus A$ and from the hypothesis $X \setminus A$ is open. From the definition of lo-topology, there exists a family $\mathcal{F}$ of non-empty finite subsets of X^2 such that $X \setminus A = \bigcup_{F\in\mathcal{F}} \bigcap_{(u,v)\in F} L(u, v)$. Then for some $F \in \mathcal{F}$, $x \in \bigcap_{(u,v)\in F} L(u, v)$ which means that F has x as a left common point. Since x is a raw-binarily $\mathcal{L}$-limit point of A, F has also a left common point $y \in A$ other than x as a left common point. It follows immediately that $X \setminus A$ contains a point $y \in A$ other than x. On the other hand, since $x \notin A$, $A \setminus \{x\} = A$ and so

$$(X \setminus A) \cap (A \setminus \{x\}) = (X \setminus A) \cap A = \emptyset$$

which contradicts our assumption. Thus $x \in A$.

($\Leftarrow$) : Let $x \in X \setminus A$. Then $x \notin A$. Therefore from the hypothesis, x is not a raw-binarily $\mathcal{L}$-limit point of A. Hence there exists a non-empty finite subset F of X^2 having x as a left common point does not have any left common point in A. It follows that for some $(u, v) \in F$, $v \notin y * u$, or equivalently $y \notin L(u, v)$ for every $y \in A$. Therefore $L(u, v) \cap A = \emptyset$ and so $L(u, v) \subseteq X \setminus A$. By Proposition 2.3(1), $X \setminus A$ is a $\mathcal{L}$-open since $x \in X \setminus A$ is chosen arbitrarily. Thus A is a $\mathcal{L}$-closed. □

Theorem 3.9. *Given a raw bistruct $(X, *)$. A point x in a subset $A \subseteq X$ is a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-closure point of A if and only if x is a closure point of A with respect to the lo-topology $\mathcal{L}$ (resp., the ro-topology $\mathcal{R}$, the o-topology O).*

Proof. Given a raw bistruct $(X, *)$. Let $A \subseteq X$ and $x \in A$.

$(\Rightarrow)$: Let U be a $\mathcal{L}$-open set containing x. We show that U intersects A. From the definition of lo-topology, there exists a family $\mathcal{F}$ of non-empty finite subsets of X^2 such that $U = \bigcup_{F\in\mathcal{F}} \bigcap_{(u,v)\in F} L(u, v)$. Then there exists a member F of $\mathcal{F}$ such that $x \in \bigcap_{(u,v)\in F} L(u, v)$. So F has x as a left common point. From the hypothesis, F satisfies the condition in Definition 2.11, that is, A *is not* raw-binarily $\mathcal{L}'$(resp. $\mathcal{R}'$, O')-covered with respect to F. Then from Definition 2.8, A is not a subset of $\bigcup_{(u,v)\in F} (L(u, v))^c$. Therefore it follows immediately that

$$A^c \not\supseteq \left[\bigcup_{(u,v)\in F} (L(u, v))^c\right]^c = \bigcap_{(u,v)\in F} L(u, v)$$

which implies that $A^c \not\supseteq U$ and so $U \cap A \neq \emptyset$.

$(\Leftarrow)$: We can reverse the steps of the first part of the proof.

Using similar arguments, it can be proven that the implication holds for the ro-topology $\mathcal{R}$ and the o-topology O. □

Theorem 3.10. *Given a raw bistruct $(X, *)$ and two subsets $A, B \subseteq X$. A point x is a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-closure point of $A \cup B$ if and only if x is a raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, O)-closure point of at least one of A and B.*

Proof. Given a raw bistruct $(X, *)$. Let $A, B \subseteq X$.

$(\Rightarrow)$: From the hypothesis, for every non-empty finite subset F of X^2 having x as a left common point, $A \cup B$ *is* not raw-binarily $\mathcal{L}'$-covered with respect to F. Then from Definition 2.8, $A \cup B \not\subseteq \bigcup_{(u,v)\in F} (L(u, v))^c$. It follows immediately that $A \not\subseteq \bigcup_{(u,v)\in F} (L(u, v))^c$ or $B \not\subseteq \bigcup_{(u,v)\in F} (L(u, v))^c$. Thus we say that x is a raw-binarily $\mathcal{L}$-closure point of at least one of A and B.

$(\Leftarrow)$: By the hypothesis, for every non-empty finite subset F of X^2 having x as a left common point, at least one of A and B are raw-binarily $\mathcal{L}'$-covered with respect to F. From Definition 2.8, $A \not\subseteq \bigcup_{(u,v)\in F} (L(u, v))^c$ or $B \not\subseteq \bigcup_{(u,v)\in F} (L(u, v))^c$. It follows immediately that $A \not\subseteq \bigcup_{(u,v)\in F} (L(u, v))^c$ or $A \cup B \not\subseteq \bigcup_{(u,v)\in F} (L(u, v))^c$. Thus $A \cup B$ *is* raw-binarily $\mathcal{L}'$-covered with respect to F.

Using similar arguments, it can be proven that the implication holds for the ro-topology $\mathcal{R}$ and the o-topology O. □

Theorem 3.11. *Given a raw bistruct $(X, *)$ and two subsets $A, B \subseteq X$. If a point x is a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-closure of $A \cap B$, then x is also a raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, O)-closure point of both A and B.*

Proof. Given a raw bistruct $(X, *)$. Let $A, B \subseteq X$. By the hypothesis, for every non-empty finite subset F of X^2 having x as a left common point such that $A \cap B$ are raw-binarily $\mathcal{L}'$-covered with respect to F. From Definition 2.8, we get $A \cap B \not\subseteq \bigcup_{(u,v)\in F} (L(u, v))^c$ which implies that $A \not\subseteq \bigcup_{(u,v)\in F} (L(u, v))^c$ and $B \not\subseteq \bigcup_{(u,v)\in F} (L(u, v))^c$. Thus both of A and B are raw-binarily $\mathcal{L}'$-covered with respect to F which completes the proof. Using similar arguments, it can be proven that the implication holds for the ro-topology $\mathcal{R}$ and the o-topology O. □

The converse of Theorem 3.11 does not hold in general, as we will see from the following example.

Example 3.12. We consider the bistruct $(X, *)$ in Example 3.5. Given $A = \{a, d, e\}, B = \{b, c, d\} \subseteq X$. Set $F = \{(c, e)\}$. Then F has c as a left common point since $c \in \{a, c\} = L(c, e)$. Furthermore, note that there is no set other than F, which has c as the left common point. Also, we get

$$A = \{a, d, e\} \not\subseteq \{b, d, e\} = \{a, c\}^c = (L(c, e))^c = \bigcup_{(x,y)\in F} (L(x, y))^c$$

and

$$B = \{b, c, d\} \nsubseteq \{b, d, e\} = \{a, c\}^c = (L(c, e))^c = \bigcup_{(x,y)\in F} (L(x, y))^c$$

which means that neither of A and B is raw-binarily $\mathcal{L}'$-covered with respect to F. So c is a raw-binarily $\mathcal{L}$-closure point of both A and B.

On the other hand, for the only set F having c as a left common point it holds

$$A \cap B = \{d\} \subseteq \{b, d, e\} = \{a, c\}^c = (L(c, e))^c = \bigcup_{(x,y)\in F} (L(x, y))^c .$$

It follows from this that $A \cap B$ is not raw-binarily $\mathcal{L}'$-covered with respect to F. Thus c is not a raw-binarily $\mathcal{L}$-closure point of $A \cap B$.

Theorem 3.13. *Given a raw bistruct $(X, *)$. A point x in a subset $A \subseteq X$ is a raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, $\mathcal{O}$)-interior point of A if and only if x is not a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, $\mathcal{O}$)-closure point of A^c.*

Proof. Given a raw bistruct $(X, *)$. Let $A \subseteq X$.

($\Rightarrow$) : Assume that x is a raw-binarily $\mathcal{L}$-interior point of A. Let F be a non-empty finite subset of X^2 having x as a left common point. From the hypothesis, F satisfies the condition in Definition 3.1, that is, A^c *is* raw-binarily $\mathcal{L}'$-covered with respect to F. Then from Definition 2.11, x is not a raw-binarily $\mathcal{L}$-closure point of A^c.

($\Leftarrow$) : Assume that x is a raw-binarily $\mathcal{L}$-closure point of A^c. Let F be a non-empty finite subset of X^2 having x as a left common point. From the hypothesis, F satisfies the condition in Definition 2.11, that is, A^c *is* not raw-binarily $\mathcal{L}'$-covered with respect to F. Then from Definition 3.1, x is not a raw-binarily $\mathcal{L}$-interior point of A.

Using similar arguments, it can be proven that the implication holds for the ro-topology $\mathcal{R}$ and the o-topology $\mathcal{O}$. □

Corollary 3.14. *Given a raw bistruct $(X, *)$. A point x in a subset $A \subseteq X$ is a raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, $\mathcal{O}$)-closure point of A if and only if x is not a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, $\mathcal{O}$)-interior point of A^c.*

Proof. We complete the proof by substituting A^c for A in the contrapositive of the statement in Theorem 3.13. □

4. Further Work

The objective of our upcoming study is to conduct a thorough investigation into the salient characteristics of topologies that are generated through the utilization of a raw binop satisfying one or more group-like axioms, namely, associativity, totality (closureness), commutativity, identity, and invertibility.

5. Conclusion

The primary objective of this study is to enhance and expand the existing conceptual framework within the literature concerning topologies induced by raw bistructs. By doing so, this study aims to establish a more robust foundation that facilitates the topological evaluation and analysis of problems encountered within various algebraic structures. This enriched conceptual framework has the potential to provide a encouraging basis for addressing and resolving complex issues in algebraic contexts from a topological perspective.

Below, we provide several useful conclusions concerning the topological position of a point relative to a set in the lo-topology, ro-topology, and o-topology derived from a raw binary operation. These findings hold significant practical value.

Given a raw bistruct X and two subsets A, B of X. A point $x \in A$ is a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-interior point of A, then x is an interior point of A with respect to the lo-topology $\mathcal{L}$ (resp., the ro-topology $\mathcal{R}$, the o-topology O). A point x is a raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, O)-interior point of $A \cap B$ if and only if x is a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-interior point of both A and B. If a point x is a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-interior point of at least one of A and B, then x is also a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-interior point of $A \cup B$.

If a point x in X is a raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, O)-limit point of a subset $A \subseteq X$, then x is a limit point of A with respect to the lo-topology $\mathcal{L}$ (resp., the ro-topology $\mathcal{R}$, the o-topology O). A subset A of X is a $\mathcal{L}$(resp., $\mathcal{R}$, O)-closed if and only if it contains all of its raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-limit points.

If a point x in a subset $A \subseteq X$ is a raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, O)-closure point of A, then x is a closure point of A with respect to the lo-topology $\mathcal{L}$ (resp., the ro-topology $\mathcal{R}$, the o-topology O). A point x is a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-closure point of $A \cup B$ if and only if x is a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-closure point of at least one of A and B. If a point x is a raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, O)-closure of $A \cap B$, then x is also a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-closure point of both A and B.

A point x in a subset $A \subseteq X$ is a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-interior point of A if and only if x is not a raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, O)-closure point of A^c. A point x in a subset $A \subseteq X$ is a raw-binarily $\mathcal{L}$(resp. $\mathcal{R}$, O)-closure point of A if and only if x is not a raw-binarily $\mathcal{L}$(resp., $\mathcal{R}$, O)-interior point of A^c.

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