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Fixed point results for θ-ϕ interpolative mappings in super metric space with an application

Swati Saxena[a,*], U. C. Gairola[a]

[a]*Department of Mathematics, H. N. B. Garhwal University, BGR Campus, Pauri Garhwal-246001, Uttarakhand, India*

Abstract. This paper aims at introducing the notion of generalized interpolative θ-ϕ contraction, interpolative Matkowski type contraction and an interpolative θ-ϕ Kannan type contraction in frame of Super metric space. The result proved in this paper improve and extend the corresponding results due to Matkowski and Karapinar, Kannan and Karapinar. Examples are given to demonstrate the relevance of our results. As application, we obtain the solution for the non-linear matrix equations.

1. Introduction

Fixed point theory is a fundamental concept in mathematics, particularly in the field of functional analysis and related areas. It has significant applications in various branches of mathematics and have practical implications in many real-world problems. Applications of fixed point theory are vast and diverse, ranging from economics and computer science to physics. For example in the study of phase transitions, finding solutions to equations and optimization problems, nash equilibrium in game theory and engineering, control theory and stability analysis etc. Key theorems in fixed point theory, such as the Banach fixed point theorem [3] and the Brouwer fixed point theorem provide crucial insights and tools to determine the existence and properties of fixed points for different types of mappings. The Banach Contraction Principle also known as the Banach Fixed Point Theorem, is a fundamental result in the field of metric space theory and functional analysis.
The generalization of Banach contraction principle in many spaces such as b-metric space [6], fuzzy metric space [31], partial metric space [26], modular metric space [5], cone metric space [8] etc has been done by several authors. In 2007, Huang and Jhang [8] proposed the concept of cone metric space which is a generalization of metric space. Many authors have obtained fixed point results for different types of contractions in this space([2], [9], [12], [23], [29]). In 2015, Jleli and Samet [11] introduced a new generalization of metric spaces that recovers a large class of topological spaces including standard metric spaces, b-metric spaces, dislocated metric spaces, and modular spaces. Recently a new generalization of metric space known as super metric space was introduced by Karapinar and Khojasteh [21] in 2022.

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* Corresponding author: Swati Saxena
Email addresses: swatisaxena567@gmail.com (Swati Saxena), ucgairola@rediffmail.com (U. C. Gairola)
ORCID iDs: https://orcid.org/0000-0003-0797-7095 (Swati Saxena), https://orcid.org/0000-0001-5517-0021 (U. C. Gairola)

As it is known that a mapping which satisfies Banach contraction mapping is necessarily continuous. A pertinent query arises: Can a discontinuous mapping with comparable contractive conditions possess a fixed point? In 1968, Kannan [13] provided an affirmative response to this inquiry. According to Kannan, a mapping U is said to be Kannan contraction if there exists $\alpha \in [0, 1/2)$ such that for any $x, y \in X$ we have

$$d(Ux, Uy) \leq \alpha\{d(x, Ux) + d(y, Uy)\}$$

where U is not a continuous map. He proved that if X is a complete metric space and U is a Kannan contraction mapping then it has a unique fixed point. In 1999, Pant [27] obtained the first result on discontinuity at fixed point. Recently some new results to this problem has been obtained by Pant et al. ([4], [28]).

Further in 2018, Karapinar [14] revisit the Kannan type contraction and used the interpolation technique. An interpolative contraction mapping is a contraction mapping in which the contraction factor k can be chosen to interpolate or "squeeze" the distance between points at any desired rate between 0 and 1. For standard metric space (M, b) Karapinar gave the following generalization of Kannan type contraction by interpolative approach

$$b(Ux, Uy) \leq \lambda([b(x, Ux)]^q [b(y, Uy)]^{1-q}),$$

for all $x, y \in M \sim Fix(U)$ and $\lambda \in [0, 1)$.

Many authors have done work in this direction recently ([1], [7], [17], [30]). Several recent studies in interpolative Ćirić-Reich-Rus contractions ([1], [22]), Meir-Keeler type contraction ([15], [20]), Hardy Roger type contraction [7] may be reffered to. Continuing this, in 2020 Karapinar et al. [18] introduced the notion of interpolative Boyd-Wong type contraction and Matkowski type contraction for a standard metric space and partial metric space. They proved fixed point theorems for these contraction mappings.

Motivated by recent results, in this paper we introduced the notion of generalized interpolative θ-ϕ contraction, interpolative Matkowski type contraction and an interpolative θ-ϕ Kannan type contraction in frame of Super metric space. We also provide examples to illustrate how our results are relevant compared to some existing ones in the literature. Conclusion of our paper is demonstrated with the help of an application of our primary finding in solving nonlinear matrix equations.

2. Preliminary

Within this section we recall the basic definitions, lemma, results which will help to develop our main results. We begin with recalling the definition of super metric space.

Definition 2.1. [21] *Let X be a nonempty set and $m : X \times X \to [0, \infty)$, then m is said to be super metric if*

(m_1) *for all $x, y \in X$, if $m(x, y) = 0$, then $x = y$;*

(m_2) *$m(x, y) = m(y, x)$ for all $x, y \in X$;*

(m_3) *there exists $s \geq 1$ such that for every $y \in X$, there exist distinct sequences $x_n, y_n \subset X$, with $m(x_n, y_n) \to 0$ when $n \to \infty$ such that*

$$\lim_{n\to\infty} sup\ m(y_n, y) \leq s \lim_{n\to\infty} sup\ m(x_n, y)$$

Also, (X, m, s) is called a super metric space.

Example 2.2. *Let $X = [0, \infty)$, $m : X \times X \to [0, \infty)$ defined by*

$$m(x, y) = \begin{cases} |x^3 - y^3|, & for\ \ x, y \in \mathbb{R},\ x \neq y \\ 0, & for\ \ x = y \end{cases}$$

Let $y \in X$ and $\{x_n\}, \{y_n\}$ are two distinct sequences in X such that $m(x_n, y_n) \to 0$ as $n \to \infty$. Since the sequence are distinct, we have

$$m(x_n, y_n) = |x_n^3 - y_n^3| \to 0 \quad as \quad n \to \infty$$

Thus, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a$ *and*

$$\lim_{n\to\infty} \sup m(y_n, y) = \lim_{n\to\infty} \sup |x_n^3 - y_n^3| \leq s|a^3 - y_n^3| = \lim_{n\to\infty} \sup m(x_n, y)$$

when $y = 0$ the proof is simple and direct. Thus, (X, m) is a super metric space.

Definition 2.3. [21] *On a super metric space (X, m, s), a sequence $\{x_n\}$ is said to*

(i) convergent and converges to x in X iff $\lim_{n\to\infty} m(x_n, x) = 0$.

(ii) A cauchy sequence in X iff $\lim_{n\to\infty} \sup\{m(x_n, x_p) : p > n\} = 0$.

Proposition 2.4. [19] *The limit of a convergent sequence is unique in a super metric space.*

Definition 2.5. [21] *A super metric space is said to be complete if and only if every cauchy sequence is convergent in itself.*

In 2014, θ- contraction was defined by Jleli and Samet [10] as following:

Definition 2.6. [10] *Let Θ be the family of all functions $\theta : (0, \infty) \to [1, \infty)$ such that*

(θ_1) *θ is non-decreasing;*

(θ_2) *for each sequence $x_n \in (0, \infty)$ $\lim_{n\to\infty} x_n = 0$ iff $\lim_{n\to\infty} \theta(x_n) = 1$;*

(θ_3) *θ is continuous.*

From the above definitions we can categorise the Banach contraction as a specific type of θ contraction while vice-versa need not be true. Jleli and Samet proved a fixed point theorem for θ-contraction. According to them:

Theorem 2.7. [10] *Let (X, d) be a complete metric space and $T : X \to X$ be a θ-contraction. Then T has a unique fixed point.*

Recently, Zheng et al. [32] defined the new type of contractive mappings as follows:

Definition 2.8. [32] *Let Φ be the family of all functions $\phi : [1, \infty) \to [1, \infty)$ such that*

(ϕ_1) *ϕ is non-decreasing;*

(ϕ_2) *for each $t > 1$, $\lim_{n\to\infty} \phi^n(t) = 1$;*

(ϕ_3) *ϕ is continuous on $[1, \infty)$.*

Lemma 2.9. [32] *If $\phi \in \Phi$, then $\phi(1) = 1$ and $\phi(t) < t$ for each $t \in [1, \infty)$.*

Zheng et al. [32] gave the definition of θ-ϕ contraction as follow:

Definition 2.10. [32] *Let (X, d) be a metric space and $T : X \to X$ be a self-map, then T is said to be a θ-ϕ contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,*

$$\theta(d(Tx, Ty)) \leq \phi[\theta(N(x, y))]$$

where $N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.

3. Main Result

Motivated and inspired by Zheng et al. [32] we define the notion of generalized interpolative θ-ϕ contraction on the super metric space and obtained the fixed point result.

Definition 3.1. *Let (X, m) be a super metric space. A self map $U : X \to X$ is generalized interpolative θ-ϕ contraction, if there exists $p, q, r \in (0, 1/s)$ and $\phi \in \Phi, \theta \in \Theta$ s.t.*

$$\theta(m(Ux, Uy)) \leq \phi[\theta([m(x, y)]^q[m(x, Ux)]^p[m(y, Uy)]^r[\frac{1}{3s}[m(x, Uy) + m(y, Ux)]^{1-p-q-r}])] \tag{1}$$

$\forall\ x, y \in X$.

Theorem 3.2. *Let (X, m, s) be a complete super metric space and $U : X \to X$ be a self mapping satisfying* (1)*, then $Fix(U)$ is non-empty.*

Proof. Consider a sequence $\{x_n\} \subset X$ and let $x_0 \in X$ be such that $x_n = U^n x_0$ for all $n \geq 0$. If $x_n = x_{n+1}$, for some n, then $Fix(U)$ is non empty. If $x_n \neq x_{n+1}$ from (1), we have

$$\begin{aligned}\theta(m(Ux_n, Ux_{n-1})) &\leq \phi[\theta\{[m(x_n, x_{n-1})]^q[m(x_n, Ux_n)]^p[m(x_{n-1}, Ux_{n-1})]^r \\ &\qquad [\frac{1}{3s}[m(x_n, Ux_{n-1}) + m(x_{n-1}, Ux_n)]]^{1-p-q-r}\}] \\ &= \phi[\theta\{[m(x_n, x_{n-1})]^q[m(x_n, x_{n+1})]^p[m(x_{n-1}, x_n)]^r \\ &\qquad [\frac{1}{3s}[m(x_n, x_n) + m(x_{n-1}, x_{n+1})]]^{1-p-q-r}\}] \\ &\leq \phi[\theta\{[m(x_n, x_{n-1})]^q[m(x_n, x_{n+1})]^p[m(x_{n-1}, x_n)]^r \\ &\qquad [\frac{1}{3s}[2sm(x_n, x_n) + s(m(x_{n-1}, x_n) + m(x_n, x_{n+1}))]]^{1-p-q-r}\}].\end{aligned} \tag{2}$$

Let

$$m(x_n, x_{n-1}) \leq m(x_n, x_{n+1})$$

for some $n \geq 1$, so that

$$\frac{1}{2}(m(x_{n-1}, x_n) + m(x_n, x_{n+1})) \leq m(x_n, x_{n+1}).$$

Thus, (2) yield

$$\begin{aligned}0 < \theta(m(x_{n+1}, x_n)) &\leq \phi[\theta\{[m(x_n, x_{n-1})]^q[m(x_n, x_{n+1})]^p[m(x_{n-1}, x_n)]^r[m(x_n, x_{n+1})]^{1-p-q-r}\}] \\ &= \phi[\theta\{[m(x_n, x_{n-1})]^{q+r}[m(x_n, x_{n+1})]^{1-q-r}\}].\end{aligned} \tag{3}$$

we obtain,

$$\theta(m(x_{n+1}, x_n)) \leq \theta\{[m(x_n, x_{n-1})]^{q+r}[m(x_n, x_{n+1})]^{1-q-r}\}$$
$$\theta(m(x_{n+1}, x_n)) \leq \theta\{m(x_n, x_{n-1})\}.$$

but θ is non-decreasing hence contradiction occurs. Therefore, $m(x_n, x_{n-1})$ is a non-increasing sequence. Thus

$$\begin{aligned}\theta(m(x_{n+1}, x_n)) &\le \phi[\theta\{[m(x_n, x_{n-1})]^{q+r}[m(x_{n+1}, x_n)]^r[m(x_{n+1}, x_n)]^{1-p-q-r}\}]\\ &= \phi[\theta\{[m(x_n, x_{n-1})]^{q+r}[m(x_{n+1}, x_n)]^{1-p-q}\}]\\ &\le \phi[\theta\{m(x_n, x_{n-1})\}].\end{aligned}$$

we conclude that

$$\theta(m(x_{n+1}, x_n)) \le \phi[\theta\{m(x_n, x_{n-1})\}] \le ... \le \phi^n[\theta\{m(x_1, x_0)\}]. \tag{4}$$

Since, $\phi \in \Phi$ and $\lim_{n\to\infty} \phi^n(t) = 1$ for $t > 0$, we have

$$\lim_{n\to\infty} m(x_n, x_{n+1}) = 0 \qquad (by\ \ \Theta_2) \tag{5}$$

Now, let $m, n \in \mathbb{N}$ and $m > n$. If $x_n = x_m$, we have $U^m(x_0) = U^n(x_0)$. Thus we have, $U^{m-n}(U^n(x_0)) = U^n(x_0)$. Hence $U^n(x_0)$ is the fixed point of U^{m-n}. Also,

$$U(U^{m-n}(U^n(x_0))) = U^{m-n}(U(U^n(x_0))) = U(U^n(x_0))$$

It implies that, $U(U^n(x_0))$ is the fixed point of U^{m-n}. Thus, $U(U^n(x_0)) = U^n(x_0)$. So $U^n(x_0)$ is the fixed point of U. Therefore, we can assume without losing any generality that $x_n \ne x_m$. Hence,

$$\lim_{n\to\infty} sup\ m(x_{n+1}, x_{n-1}) \le s \lim_{n\to\infty} sup\ m(x_{n+1}, x_n). \tag{6}$$

Thus, since $\lim_{n\to\infty} sup\ m(x_{n+1}, x_{n-1}) = 0$, we have

$$\lim_{n\to\infty} sup\ m(x_{n+2}, x_{n-1}) \le s \lim_{n\to\infty} sup\ m(x_{n+2}, x_{n+1}) = 0. \tag{7}$$

By applying induction, one can deduce that $\lim_{n\to\infty} sup\{m(x_n, x_m); m > n\} = 0$ which implies that $\{x_n\}$ is a Cauchy sequence. Since (X, m) is a complete super metric space, the sequence $\{x_n\}$ converges to $u \in X$. We assert that u is the fixed point of U. Conversely, assume $m(u, Uu) > 0$. Note that

$$\begin{aligned}\theta(m(x_{n+1}, Uu)) = \theta(m(Ux_n, Uu)) &\le \phi[\theta\{[m(x_n, u)]^q[m(x_n, Ux_n)]^p[m(u, Uu)]^r\\ &\qquad [\frac{1}{3}[m(x_n, Uu) + m(u, Ux_n)]]^{1-p-q-r}\}]\\ &\le \theta\{[m(x_n, u)]^q[m(x_n, x_{n+1})]^p[m(u, Uu)]^r[\frac{1}{3}[m(x_n, Uu) + m(u, x_{n+1})]]^{1-p-q-r}\}.\end{aligned}$$

Letting $\lim_{n\to\infty}$, we obtain $m(u, Uu) = 0$ which is a contradiction. Therefore, u is a fixed point of U and $Fix(U)$ is nonempty. □

Example 3.3. *Let* $X = \{1, 3, 5, 7\}$ *and* $m : X \times X \to [0, \infty)$ *such that*
$m(x, y) = |\frac{1}{x} - \frac{1}{y}|,\ x, y \in \{1, 3, 5\}/(1, 1),\ \ m(1, 1) = 7, m(7, x) = m(x, 7) = 7x, \forall\ \ x \in X.$
Here, clearly if we put $x = 7$ *and* $y = 7$, *then* $m(7, 7) = 9 \ne 0$ *also* $m(1, 1) = 7 \ne 0$.
Hence if $x = y \nRightarrow m(x, y) = 0$.
Now, let $U : X \to X$ *such that*

$$U(1) = U(7) = 7, \quad U(3) = U(5) = 1$$

(X, m, s) is a complete super metric space w.r.t. U.
Take

$$\theta(x) = \sqrt{x} + 2, \quad \forall \ x \in (0, \infty)$$

and

$$\phi(u) = \begin{cases} e^u, u > 0, u \neq 1 \\ 1, u = 1 \end{cases}$$

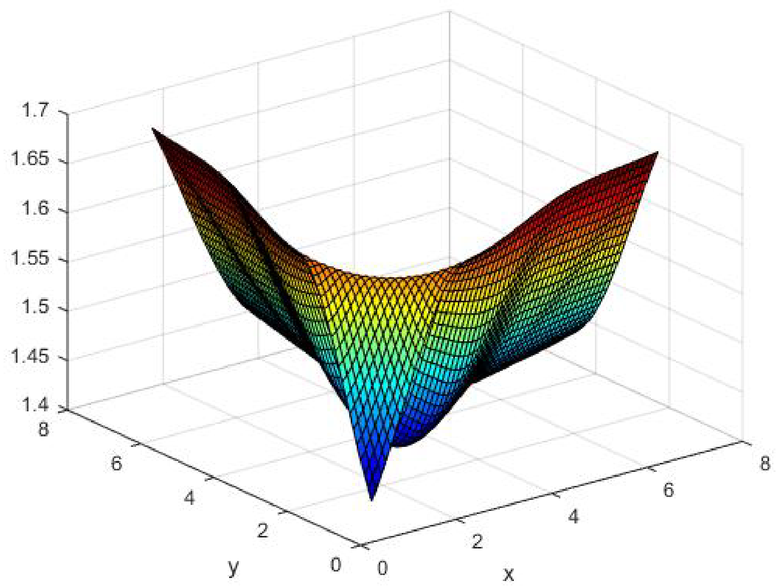

Figure 1: image representation of fixed point

By considering all possible cases, condition (1) *satisfies in every case. Here $x = 7$ is a fixed point.*

Definition 3.4. [24] *Let Ψ be the set of functions, $\psi : [0, \infty) \to [0, \infty)$ such that*

(i) *ψ is non decreasing.*

(ii) $\lim_{n\to\infty} \psi^n(t) = 0$ *for each $t > 0$.*

Lemma 3.5. [25] *Let $\psi \in \Psi$ then $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$.*

Definition 3.6. *Let (X, m) be a complete super metric space. A self map $U : X \to X$ is an interpolative Matkowski type contraction, if there exist $p, q, r \in (0, 1)$ and $\psi \in \Psi$ such that*

$$m(Ux, Uy) \leq \psi([m(x, y)]^q [m(x, Ux)]^p [m(y, Uy)]^r [\frac{1}{3}(m(x, Uy) + m(y, Ux)]^{1-p-q-r}) \tag{8}$$

$\forall \ x, y \in X.$

Theorem 3.7. *Let (X, m) be a complete super metric space with a continuous metric m and $U : X \to X$ be a self map satisfying (8), then Fix(U) is non empty.*

Proof. Consider a sequence $\{x_n\} \subset X$ and let $x_0 \in X$ be such that $x_n = U^n x_0 \ \forall \ n \geq 0$. If $x_n = x_{n+1}$ for some n then $Fix(U)$ is non empty. If $x_n \neq x_{n+1}$ from (8), we have

$$\begin{aligned} m(Ux_n, Ux_{n-1}) &\leq \psi([m(x_n, x_{n-1})]^q [m(x_n, Ux_n)]^p [m(x_{n-1}, Ux_{n-1})]^r \\ &\qquad [\frac{1}{3}[m(x_n, Ux_{n-1}) + m(x_{n-1}, Ux_n)]]^{1-p-q-r}) \\ &= \psi([m(x_n, x_{n-1})]^q [m(x_n, x_{n+1})]^p [m(x_{n-1}, x_n)]^r [\frac{1}{3}[m(x_n, x_n) + m(x_{n-1}, x_{n+1})]]^{1-p-q-r}) \\ &\leq \psi([m(x_n, x_{n-1})]^q [m(x_n, x_{n+1})]^p [m(x_{n-1}, x_n)]^r \\ &\qquad [\frac{1}{3}[m(x_n, x_n) + m(x_{n-1}, x_n) + m(x_n, x_{n+1})]]^{1-p-q-r}). \end{aligned} \tag{9}$$

Suppose,

$$m(x_{n-1}, x_n) \leq m(x_n, x_{n+1})$$

for some $n \geq 1$, so that

$$\frac{1}{3}[m(x_n, x_n) + m(x_{n-1}, x_n) + m(x_n, x_{n+1})] \leq m(x_n, x_{n+1})$$

Thus, (9) yields

$$\begin{aligned} 0 < m(x_n, x_{n+1}) &\leq \psi([m(x_n, x_{n-1})]^q [m(x_n, x_{n+1})]^p [m(x_{n-1}, x_n)]^r [m(x_n, x_{n+1})]^{1-p-q-r}) \\ &= \psi([m(x_n, x_{n-1})]^{q+r} [m(x_n, x_{n+1})]^{1-q-r}). \end{aligned} \tag{10}$$

we obtain

$$\begin{aligned} &m(x_n, x_{n+1}) \leq [m(x_n, x_{n-1})]^{q+r} [m(x_n, x_{n+1})]^{1-q-r} \\ &m(x_n, x_{n+1}) \leq m(x_n, x_{n-1}). \end{aligned}$$

which is a contradiction. Thus, $\{m(x_n, x_{n-1})\}$ is a non-increasing sequence of non-negative real numbers, which is convergent due to monotone convergence theorem. Using (10), we obtain

$$\begin{aligned} m(x_n, x_{n+1}) &\leq \psi([m(x_n, x_{n-1})]^{q+r} [m(x_n, x_{n+1})]^{1-q-r}) \\ &\leq \psi(m(x_n, x_{n-1})) \end{aligned}$$

We conclude that

$$m(x_n, x_{n+1}) \leq \psi(m(x_n, x_{n-1})) \leq ... \leq \psi^n(m(x_0, x_1)). \tag{11}$$

Since, $\psi \in \Psi$ and $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$, and we have

$$\lim_{n\to\infty} m(x_{n-1}, x_n) = 0. \tag{12}$$

Now, let $m, n \in \mathbb{N}$ and $m > n$. If $x_n = x_m$, we have $U^m(x_0) = U^n(x_0)$. Thus, we have $U^{m-n}(U^n(x_0)) = U^n(x_0)$. Thus, we have $U^n(x_0)$ is the fixed point of U^{m-n}. Also,

$$U(U^{m-n}(U^n(x_0))) = U^{m-n}(U(U^n(x_0))) = U(U^n(x_0))$$

It means that, $U(U^n(x_0))$ is the fixed point of U^{m-n}. Thus, $U(U^n(x_0)) = U^n(x_0)$. So, $U^n(x_0)$ is the fixed point of U. Therefore, we can assume without losing any generality that $x_n \neq x_m$. Hence,

$$\lim_{n\to\infty} sup \ m(x_{n+1}, x_{n-1}) \leq s \lim_{n\to\infty} sup \ m(x_{n+1}, x_n). \tag{13}$$

Thus, since $\lim_{n\to\infty} sup\ m(x_{n+1}, x_{n-1}) = 0$, we have

$$\lim_{n\to\infty} sup\ m(x_{n+2}, x_{n-1}) \leq s \lim_{n\to\infty} sup\ m(x_{n+2}, x_{n+1}) = 0. \tag{14}$$

By applying induction, one can deduce that $\lim_{n\to\infty} sup\{m(x_n, x_m); m > n\} = 0$. It implies that $\{x_n\}$ is a Cauchy sequence. Since (X, m) is a complete super metric space, the sequence $\{x_n\}$ converge to $u \to X$. We assert that u is the fixed point of U. Conversely, assume $m(u, Uu) > 0$. Note that

$$m(x_{n+1}, Uu) \leq \psi([m(x_n, u)]^q [m(x_n, Ux_n)]^p [m(u, Uu)]^r [\frac{1}{3}[m(x_n, Uu) + m(u, Ux_n)]]^{1-p-q-r})$$

letting $n \to \infty$ we obtain $m(u, Uu) = 0$ which is a contradiction. Therefore, u is a fixed point of U and $Fix(U)$ is non empty. □

Example 3.8. *Let $X = [0, 2]$ and $m : X \times X \to [0, \infty)$ such that $m(x, y) = (x - y)^2$, $\forall\ x, y \in X$*
Now, let $U : X \to X$ such that

$$U(x) = \frac{x}{10} \quad \forall\ x \in X$$

(X, m, s) is a complete super metric space w.r.t. U.
Take

$$\psi(t) = \begin{cases} t, & t \in [0, 1) \\ \frac{t}{2}, & t \in [1, 2] \end{cases}$$

Let $x, y \in X$, clearly U satisfies (8). Now, $u = 0 \in X$ such that $U(u) = 0$. Here u is the fixed point of U.

Now, we obtain fixed point for an interpolative θ-ϕ Kannan type contraction mapping.

Definition 3.9. *Let (X, m) be a super metric space. A mapping $U : X \to X$ is said to be an interpolative θ-ϕ Kannan type contraction mapping if there exist some $\alpha \in (0, 1)$ such that*

$$\theta(m(Ux, Uy)) \leq \phi[\theta([m(x, Ux)]^\alpha [m(y, Uy)]^{1-\alpha})] \tag{15}$$

for all $x, y \in X$ with $x \neq Ux$.

Theorem 3.10. *Let (X, m) be a complete super metric space and U be an interpolative θ-ϕ Kannan type contraction. Then U has a unique fixed point in X.*

Proof. Consider a sequence $\{x_n\} \subset X$ and let $x_0 \in X$ be such that $x_n = U^n x_0\ \forall\ n \geq 0$. If $x_n = x_{n+1}$ for some n then $Fix(U)$ is non empty. If $x_n \neq x_{n+1}$, from (15), we have

$$\begin{aligned} \theta(m(Ux_n, Ux_{n-1})) &\leq \phi[\theta([m(x_n, Ux_n)]^\alpha [m(x_{n-1}, Ux_{n-1})]^{1-\alpha})] \\ &= \phi[\theta([m(x_n, x_{n+1})]^\alpha [m(x_{n-1}, x_n)]^{1-\alpha})] \\ &\leq \theta([m(x_n, x_{n+1})]^\alpha [m(x_{n-1}, x_n)]^{1-\alpha}) \\ \theta(m(x_{n+1}, x_n)) &\leq \theta(m(x_n, x_{n-1})). \end{aligned}$$

but θ is non-decreasing function. Hence, contradiction occurs. Therefore, $m(x_n, x_{n-1})$ is a non increasing sequence. Thus,

$$\theta(m(x_{n+1}, x_n)) \leq \phi[\theta([m(x_n, x_{n-1})]^{1-\alpha})]$$

we deduce that

$$\theta(m(x_{n+1}, x_n)) \leq \phi[\theta([m(x_n, x_{n-1})]^{1-\alpha})] \leq ... \leq \phi^n[\theta([m(x_n, x_{n-1})]^{1-\alpha})]. \tag{16}$$

Since, $\phi \in \Phi$ and $\lim_{n\to\infty} \phi^n(t) = 1$ for each $t > 0$ and we have

$$\lim_{n\to\infty} m(x_n, x_{n-1}) = 0 \qquad (by\Theta_2)$$

Now let $m, n \in \mathbb{N}$ and $m > n$. If $x_n = x_m$, we have $U^m(x_0) = U^n(x_0)$. Thus, we have $U^{m-n}(U^n(x_0)) = U^n(x_0)$. Hence $U^n(x_0)$ is the fixed point of U^{m-n}. Also,

$$U(U^{m-n}(U^n(x_0))) = U^{m-n}(U(U^n(x_0))) = U(U^n(x_0))$$

It implies that, $U(U^n(x_0))$ is the fixed point of U^{m-n}. Thus, $U(U^n(x_0)) = U^n(x_0)$. So $U^n(x_0)$ is the fixed point of U. Therefore, we can assume without losing any generality that $x_n \neq x_m$. Hence,

$$\lim_{n\to\infty} sup\ m(x_{n+1}, x_{n-1}) \leq s \lim_{n\to\infty} sup\ m(x_{n+1}, x_n). \tag{17}$$

Thus, since $\lim_{n\to\infty} sup\ m(x_{n+1}, x_{n-1}) = 0$, we have

$$\lim_{n\to\infty} sup\ m(x_{n+2}, x_{n-1}) \leq s \lim_{n\to\infty} sup\ m(x_{n+2}, x_{n+1}) = 0. \tag{18}$$

By applying induction, one can deduce that $\lim_{n\to\infty} sup\{m(x_n, x_m); m > n\} = 0$. It implies that $\{x_n\}$ is a Cauchy sequence. Since (X, m) is a complete super metric space, the sequence $\{x_n\}$ converge to $u \to X$. We assert that u is the fixed point of U. Conversely, assume $m(u, Uu) > 0$. Note that

$$\theta(m(x_{n+1}, Uu)) = \theta(m(Ux_n, Uu)) \leq \phi[\theta([m(x_n, Ux_n)]^{\alpha}[m(u, Uu)]^{1-\alpha})]$$

Letting $n \to \infty$, we obtain $m(u, Uu) = 0$ since θ is non decreasing which is a contradiction. Therefore, u is a fixed point of U and $Fix(U)$ is non empty. □

Example 3.11. *Let* $X = [0, 1]$ *and* $m : X \times X \to [0, \infty)$ *such that* $m(x, y) = (x - y)^2,\ x, y \in [0, 1]$
Now, let $U : X \to X$ *such that*

$$U(x) = \frac{1 - x}{2}, \quad \forall\ x \in [0, 1]$$

(X, m, s) *is a complete super metric space w.r.t.* U.
Take

$$\theta(x) = 1 + x, \quad \forall\ \ x \in (0, \infty)$$

and

$$\phi(t) = 2^t, \quad \forall\ \ t \in [1, \infty)$$

For any $x, y \in X$, *clearly* U *satisfies* (15). *Now,* $u = \frac{1}{3} \in X$ *such that* $U(u) = \frac{1}{3}$. *Here* u *is the fixed point of* U.

4. Application

In the present section we prove a theorem for solving non linear matrix equations as an application of our main result. We use the following notations:
$\|.\|_{tr}$ represents the trace norm. $\|C\|_{tr}$ or $tr(C)$ is the sum of eigen values of C^*C.
$\|.\|$ represents spectral norm. $\|C\| = \sqrt{\lambda^+(C^*C)}$ where $\lambda^+(C^*C)$ is the largest eigen value of C^*C.
M_n denotes set of $n \times n$ matrices.
H_n denotes set of $n \times n$ hermitian matrices.
A_n denotes set of $n \times n$ positive definite matrices.
H_n^+ denotes set of $n \times n$ positive definite hermitian matrices.
$C \succeq 0 \implies C \in H_n^+$.
$C \succ 0 \implies C \in A_n$.
$C \succ D \implies C - D \succ 0$.
$C \succeq D \implies C - D \succeq 0$.

Lemma 4.1. *If $C, D \in H_n^+$ then $0 \le tr(CD) \le \theta(\|C\|.tr(D))$.*

Proof. The eigen value of the product of two positive semi definite matrices is non negative. Particularly we have $tr(CD) \ge 0$. Moreover, since $C \le \|C\| I_n$ and θ is non decreasing, we have

$$0 \le \theta(tr((\|C\| - C)D)) = \theta(tr(\|C\|D - CD)) = \theta(\|C\|tr(D) - tr(CD)) \le \theta(\|C\|tr(D))$$

□

Lemma 4.2. *If $C \in H_n$ and $C \prec I$ then $\theta(\|C\|) \le 1$.*

Consider the following non linear matrix equation

$$C = J + \sum_{i=1}^{m} V_i^* F(C) V_i \tag{19}$$

where each V_i is an arbitrary $n \times n$ matrix for each $i = 1, 2, ..., m$.
J is a positive definite hermitian matrix.
$F : H_n \to A_n$ such that $F(0) = 0$ is an order preserving continuous map.
H_n endowed with trace norm is a normed Banach Space hence it is a complete super metric space.
Let $T : H_n \to H_n$ such that

$$T(C) = J + \sum_{i=1}^{m} V_i^* F(C) V_i \quad \forall \quad X \in H_n \tag{20}$$

T is a continuous order preserving self mapping. Clearly solution of (19) is a fixed point of T.

Theorem 4.3. *Considering the equation* (19)*, let $\exists\, k \ge 1$ where k is a real number and M such that for $X, Y \in H_n$ with $X \prec Y$ having following properties*

(a) $\sum_{i=1}^{m} V_i^* V_i \le M I_n$ *and* $\sum_{i=1}^{m} V_i^* F(J) V_i \succ 0$

(b) $\theta(tr(F(Y) - F(X))^k) \le \dfrac{1}{M}\phi(\theta\{\|X - Y\|_{tr}^q \|X - T(X)\|_{tr}^p \|Y - T(Y)\|_{tr}^r \delta\})$

where $\delta = \dfrac{1}{3}(\|X - T(Y)\|_{tr} + \|Y - T(X)\|_{tr})^{1-p-q-r}$ *and $p, q, r \in (0, 1)$ with $p + q + r < 1$ and $\phi \in \Phi$, then the matrix equation defined by* (19) *has a solution.*

Proof. Consider a super metric $P : H_n \times H_n \to [0, \infty)$ as $P(C, D) = \|C - D\|_{tr}^k$. Thus, (H_n, P, s) is a complete super metric space with $s = 2^{k-1}$.
Let $X, Y \in H_n$ with $X \prec Y$ and consider

$$\begin{aligned}
P(T(Y), T(X)) &= \|J + \sum_{i=1}^{m} V_i^* F(Y) V_i - J - \sum_{i=1}^{m} V_i^* F(X) F_i\|_{tr} \\
&= [tr(\sum_{i=1}^{m} V_i^* F(Y) V_i - V_i^* F(X) V_i)]^p \\
&= (\sum_{i=1}^{m} (tr(V_i^* (F(Y) - F(X)) V_i)))^p \\
&= (tr(\sum_{i=1}^{m} (V_i^* (F(Y) - (F(X)) V_i)))^p \\
&= (tr(\sum_{i=1}^{m} (V_i^* V_i)(F(Y) - F(X))))^p \\
&\le \|(\sum_{i=1}^{m} (V_i^* V_i))\|^p \|(F(Y) - F(X))\|_{tr}^p
\end{aligned}$$

using (i), lemma (4.1) and (4.2) we have

$$P(T(Y), T(X)) \leq M^p \theta(|tr(F(X) - F(Y)|^p) \tag{21}$$

Since $\sum_{i=1}^{m} V_i^* F(J) V_i < 0$, consequently we have $J > T(J)$. Now using (b), (20) and theorem (3.2), we get T has a fixed point. Hence, the solution for non-linear matrix equation (19) exists. □

5. Conclusion

In this paper, we presented a novel framework for exploring the presence of fixed points. The notion of generalized interpolative θ-ϕ contraction, interpolative Matkowski type contraction and an interpolative θ - ϕ Kannan type contraction in frame of Super metric space is introduced. Our results generalized and extended the result presented in ([14], [18]) and some other results in the existing literature. We have included pertinent examples to validate our findings. Additionally, we created an application that utilizes our primary result to address non-linear matrix equations. The result presented in this paper will provide insights for future research, enabling the exploration of fixed point existence for these contractions within various metric space settings and their applications across diverse fields.

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