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On three-point generalizations of Banach and Edelstein fixed point theorems

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Abstract. Let X be a metric spaces. Recently in [27] it was considered a new type of mappings $T\colon X \to X$ which can be characterized as mappings contracting perimeters of triangles. These mappings are defined by the condition based on the mapping of three points of the space instead of two, as it is adopted in many fixed point theorems. In the present paper we consider so-called (F, G)-contracting mappings, which form a more general class of mappings than mappings contracting perimeters of triangles. The fixed point theorem for these mappings is proved. We also prove a fixed point theorem for mappings contracting perimeters of triangles in the sense of Edelstein.

1. Introduction

The Contraction Mapping Principle was established by S. Banach in his dissertation (1920) and published in 1922 [5]. It has been generalized in many ways over the years. It is possible to distinguish two types of generalizations of this theorem: in the first case the contractive nature of the mapping is weakened, see, e.g. [1, 21, 28, 29, 40]; in the second case the topology is weakened, see, e.g. [2–4, 7, 13, 16, 22, 23, 31, 39].

Let X be a nonempty set. Recall that a mapping $d\colon X\times X \to \mathbb{R}^+$, $\mathbb{R}^+ = [0,\infty)$ is a *metric* if for all $x, y, z \in X$ the following axioms hold:

(i) $(d(x, y) = 0) \Leftrightarrow (x = y)$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leqslant d(x, z) + d(z, y)$.

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The pair (X, d) is called a *metric space*.

Everywhere below by $|X|$ we denote the cardinality of the set X.

In [27] it was considered a new type of mappings $T: X \to X$ which can be characterized as mappings contracting perimeters of triangles.

Definition 1.1. *Let (X, d) be a metric space with $|X| \geqslant 3$. We shall say that $T: X \to X$ is a* mapping contracting perimeters of triangles *on X if there exists $\alpha \in [0, 1)$ such that the inequality*

$$d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) \leqslant \alpha(d(x, y) + d(y, z) + d(x, z)) \tag{1}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Recall that a mapping $T: X \to X$ is a *contraction* on the metric space (X, d) if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant \alpha d(x, y) \tag{2}$$

for all $x, y \in X$.

It is clear that every contraction is a mapping contractive perimeters of triangles.

Remark 1.2. *Note that the requirement for $x, y, z \in X$ to be pairwise distinct in Definition 1.1 is essential. One can see that otherwise this definition is equivalent to the definition of contraction mapping.*

In [27] it was shown that mappings contracting perimeters of triangles are continuous. The fixed point theorem for such mappings was proved and the classical Banach fixed-point theorem was obtained like a simple corollary. An example of a mapping contracting perimeters of triangles which is not a contraction mapping was constructed for a space X with $\mathrm{card}(X) = \aleph_0$.

In [19] authors noted that except Banach's fixed point theorem there are also three classical fixed point theorems against which metric extensions are usually checked. These are, respectively, Nadler's well-known set-valued extension of Banach's theorem [26], the extension of Banach's theorem to nonexpansive mappings [20], and Caristi's theorem [6]. Note that an important place in the fixed point theory is also occupied by Edelstein's [15] fixed point theorem, the scheme of the proof of which is fundamentally different from the proof of above mentioned theorems.

Theorem 1.3 (Edelstein, 1962). *Let X be a metric space and let $T: X \to X$ be a mapping satisfying*

$$d(Tx, Ty) < d(x, y) \tag{3}$$

for all $x \neq y$, $x, y \in X$. Assume that there exists $x \in X$ such that the sequence of iterates $(T^n x)$ contains a subsequence $(T^{n_k} x)$ convergent to a point $\xi \in X$. Then ξ is a unique fixed point of T.

Clearly, that if X is a compact metric space and $T: X \to X$ satisfies (3) for all $x \neq y$, $x, y \in X$, then there exists a unique fixed point. Recall that mappings of type (3) are called *contractive*.

One new interesting proof of Edelstein's theorem was given in [10]. Note that generalizations of this theorem are not as numerous as generalizations of Banach's theorem. One of the most famous generalizations is Suzuki's [33] theorem. Let us mention also generalizations of Theorem 1.3 in topological spaces [17, 24], ν-generalized metric spaces [34, 36, 38], complete metric spaces [35], compact metric spaces [37], Cartesian product of metric spaces [8]. See [9, 12, 14, 18, 25, 30, 32] for further developments in this direction.

In Section 2 we consider so-called (F, G)-contracting mappings, which form a more general class of mappings than mappings contracting perimeters of triangles, see Definition 2.1. We show that (F, G)-contracting mappings are continuous and prove the fixed point theorem for these mappings.

In Section 3 we show continuity and prove a fixed point theorem for mappings contracting perimeters of triangles in the sense of Edelstein.

2. Mappings with controlled contraction

In this section we consider a more general class of mappings in ordinary metric spaces than mappings contracting perimeters of triangles and prove a fixed point theorem for this class.

Definition 2.1. *Let (X, d) be a metric space with $|X| \geqslant 3$ and let functions $F, G \colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be such that for all $\xi, \eta, \zeta \in \mathbb{R}^+$ the following conditions hold:*

$$\begin{aligned}&F \text{ and } G \text{ are symmetric in all variables, i.e., } F(\xi,\eta,\zeta) = F(\pi(\xi),\pi(\eta),\pi(\zeta)),\\ &G(\xi,\eta,\zeta) = G(\pi(\xi),\pi(\eta),\pi(\zeta)), \text{ where } \pi \text{ is any permutation of the set } \{\xi,\eta,\zeta\};\end{aligned} \tag{4}$$

$$G(\xi,\eta,\zeta) \geqslant \xi; \tag{5}$$

$$F(\xi,\eta,\zeta) \geqslant G(\xi,\eta,\zeta); \tag{6}$$

$$G(0,0,0) = 0 \text{ and } G \text{ is continuous at } (0,0,0); \tag{7}$$

$$\text{The function } G \text{ is monotone increasing in all of its arguments.} \tag{8}$$

*We shall say that $T \colon X \to X$ is an (F, G)-*contracting mapping *on X if there exists $\alpha \in [0, 1)$ such that the inequality*

$$F(d(Tx,Ty), d(Ty,Tz), d(Tx,Tz)) \leqslant \alpha\, G(d(x,y), d(y,z), d(x,z)) \tag{9}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Note that the proposed (F, G)-contracting mappings are trivial extension of the known generalized contractive-type mappings studied in [29].

Remark 2.2. *Inequalities (5) and (6) for all $\xi, \eta, \zeta \in \mathbb{R}^+$ imply*

$$F(\xi,\eta,\zeta) \geqslant \xi. \tag{10}$$

Remark 2.3. *Note that the requirement for $x, y, z \in X$ to be pairwise distinct is essential. One can see that, otherwise, in the case*

$$F(\xi,\eta,\zeta) = G(\xi,\eta,\zeta) = \xi + \eta + \zeta$$

(when F and G are such that T is a mapping contracting perimeters of triangles) this definition is equivalent to Definition 1.1.

Proposition 2.4. *If in Definition 2.1 the function G additionally satisfies the condition*

$$G(\xi,\eta,\zeta) \leqslant K\xi \ \text{ with } \ \alpha K < 1, \tag{11}$$

then T is a mapping contracting perimeters of triangles.

Proof. By condition (4) we have $F(\xi,\eta,\zeta) = F(\eta,\xi,\zeta)$ and $F(\xi,\eta,\zeta) = F(\zeta,\eta,\xi)$, which together with (10) gives

$$F(\xi,\eta,\zeta) \geqslant \eta, \quad F(\xi,\eta,\zeta) \geqslant \zeta.$$

Summarizing the left and right parts of previous two inequalities and of inequality (10) we get

$$\frac{\xi+\eta+\zeta}{3} \leqslant F(\xi,\eta,\zeta). \tag{12}$$

Analogously, by (4) and (11) we have $G(\xi,\eta,\zeta) = G(\eta,\xi,\zeta) \leqslant K\eta$, $G(\xi,\eta,\zeta) = G(\zeta,\xi,\eta) \leqslant K\zeta$, which gives

$$G(\xi,\eta,\zeta) \leqslant K\frac{\xi+\eta+\zeta}{3}. \tag{13}$$

Finally, combining inequalities (12) and (13) with (9), we get

$$\frac{1}{3}(d(Tx,Ty) + d(Ty,Tz) + d(Tx,Tz)) \leqslant \frac{\alpha K}{3}(d(x,y) + d(y,z) + d(x,z)),$$

which is equivalent to (1) since $\alpha K < 1$. This completes the proof. □

Proposition 2.5. *Let (X, d) be a metric space, $|X| \geqslant 3$, and let $T: X \to X$ be an (F, G)-contracting mapping on X. Then T is continuous.*

Proof. Let (X, d) be a metric space with $|X| \geqslant 3$, $T: X \to X$ be a mapping contracting perimeters of triangles on X and let x_0 be an isolated point in X. Then, clearly, T is continuous at x_0. Let now x_0 be an accumulation point. Let us show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(Tx_0, Tx) < \varepsilon$ whenever $d(x_0, x) < \delta$. Suppose that $x \neq x_0$, otherwise this assertion is evident. Since x_0 is an accumulation point, for every $\delta > 0$ there exists $y \in X$ such that $x_0 \neq y \neq x$ and $d(x_0, y) < \delta$. Since the points x_0, x and y are pairwise distinct by (10) and (9) we have

$$\begin{aligned} d(Tx_0, Tx) &\leqslant F(d(Tx_0, Tx), d(Tx_0, Ty), d(Tx, Ty)) \\ &\leqslant \alpha G(d(x_0, x), d(x_0, y), d(x, y)). \end{aligned}$$

Further, using the triangle inequality $d(x, y) \leqslant d(x_0, x) + d(x_0, y)$, we obtain

$$d(Tx_0, Tx) \leqslant \alpha G(d(x_0, x), d(x_0, y), d(x_0, x) + d(x_0, y)). \tag{14}$$

By (7) we get that for every $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality $\alpha G(\delta, \delta, 2\delta) < \varepsilon$ holds. Using this inequality, inequalities $d(x_0, x) < \delta$, $d(x_0, y) < \delta$, the monotonicity of G and inequality (14), we get the desired inequality $d(Tx_0, Tx) < \varepsilon$, which completes the proof. □

Let T be a mapping on the metric space X. A point $x \in X$ is called a *periodic point of period n, $n \in \mathbb{N}^+$*, if $T^n(x) = x$. The least positive integer n for which $T^n(x) = x$ is called the prime period of x, see, e.g., [11, p. 18]. In particular, the point x is of prime period 2 if $T(T(x)) = x$ and $Tx \neq x$.

Theorem 2.6. *Let (X, d), $|X| \geqslant 3$, be a complete metric space and let $T: X \to X$ be a mapping satisfying the following two conditions:*

(i) T does not possess periodic points of prime period 2.

(ii) T is an (F, G)-contracting mapping on X.

Then T has a fixed point. The number of fixed points is at most two.

Proof. Let $x_0 \in X$. Consider the sequence $x_1 = Tx_0$, $x_2 = Tx_1$, $\cdots$, $x_{n+1} = Tx_n$, $\cdots$. Suppose first that x_i is not a fixed point of the mapping T for every $i = 0, 1, \ldots$. Let us show that all x_i are different. Since x_i is not fixed, then $x_i \neq x_{i+1} = Tx_i$. By condition (i) $x_{i+2} = T(T(x_i)) \neq x_i$ and by the assumption that x_{i+1} is not fixed we have $x_{i+1} \neq x_{i+2} = Tx_{i+1}$. Hence, x_i, x_{i+1} and x_{i+2} are pairwise distinct.

Further, set

$$\tilde{p}_0 = F(d(x_0, x_1), d(x_1, x_2), d(x_2, x_0)),\ \ p_0 = G(d(x_0, x_1), d(x_1, x_2), d(x_2, x_0)),$$

$$\tilde{p}_1 = F(d(x_1, x_2), d(x_2, x_3), d(x_3, x_1)),\ \ p_1 = G(d(x_1, x_2), d(x_2, x_3), d(x_3, x_1)),$$

$$\cdots$$

$$\tilde{p}_n = F(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_{n+2}, x_n)),$$

$$p_n = G(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_{n+2}, x_n)),$$

$$\cdots .$$

Then by (9) we have $\tilde{p}_1 \leqslant \alpha p_0$, $\tilde{p}_2 \leqslant \alpha p_1, \ldots, \tilde{p}_n \leqslant \alpha p_{n-1}$. Hence and by condition (6) we have

$$p_0 > \alpha p_0 \geqslant \tilde{p}_1 \geqslant p_1 > \alpha p_1 \geqslant \tilde{p}_2 \geqslant p_2 \cdots . \tag{15}$$

Consequently,

$$p_0 > p_1 > \cdots > p_n > \cdots . \tag{16}$$

Suppose now that $j \geqslant 3$ is a minimal natural number such that $x_j = x_i$ for some i such that $0 \leqslant i < j-2$. Then $x_{j+1} = x_{i+1}$, $x_{j+2} = x_{i+2}$. Hence, $p_i = p_j$ which contradicts to (16). Thus, all x_i are different.

Further, let us show that $\{x_i\}$ is a Cauchy sequence. By (5) and (15) we have

$$d(x_0, x_1) \leqslant p_0,$$

$$d(x_1, x_2) \leqslant p_1 \leqslant \alpha p_0,$$

$$d(x_2, x_3) \leqslant p_2 \leqslant \alpha p_1 \leqslant \alpha^2 p_0,$$

$$\cdots$$

$$d(x_{n-1}, x_n) \leqslant p_{n-1} \leqslant \alpha^{n-1} p_0,$$

$$d(x_n, x_{n+1}) \leqslant p_n \leqslant \alpha^n p_0,$$

$$\cdots.$$

By the triangle inequality for every $n, p \in \mathbb{N}^+$,

$$d(x_n,\ x_{n+p}) \leqslant d(x_n,\ x_{n+1}) + d(x_{n+1},\ x_{n+2}) + \cdots + d(x_{n+p-1},\ x_{n+p})$$

$$\leqslant \alpha^n p_0 + \alpha^{n+1} p_0 + \cdots + \alpha^{n+p-1} p_0 = \alpha^n (1 + \alpha + \cdots + \alpha^{p-1}) p_0 = \alpha^n \frac{1-\alpha^p}{1-\alpha} p_0.$$

Since $0 \leqslant \alpha < 1$ we have $0 \leqslant \alpha^p < 1$ and

$$d(x_n,\ x_{n+p}) \leqslant \frac{\alpha^n p_0}{1-\alpha}.$$

Hence, $d(x_n,\ x_{n+p}) \to 0$ as $n \to \infty$ for every $p \in \mathbb{N}^+$. Thus, $\{x_n\}$ is a Cauchy sequence. By completeness of (X, d), this sequence has a limit $x^* \in X$.

Let us prove that $Tx^* = x^*$. Since $x_n \to x^*$, and by Proposition 2.5 the mapping T is continuous, we have $x_{n+1} = Tx_n \to Tx^*$. By triangle inequality we have

$$d(x^*, Tx^*) \leqslant d(x^*, x_n) + d(x_n, Tx^*) \to 0$$

as $n \to \infty$, which means that x^* is the fixed point.

Suppose that there exist at least three pairwise distinct fixed points x, y and z. Then $Tx = x$, $Ty = y$ and $Tz = z$, which contradicts to (9) combined with (6). □

Example 2.7. *It is easy to see that the class of functions F and G satisfying the conditions of Definition 2.1 is enough wide. As an example we can take $F(\xi, \eta, \zeta) = \xi + \eta + \zeta$ and $G(\xi, \eta, \zeta) = (\xi^q + \eta^q + \zeta^q)^{\frac{1}{q}}$, $q \geqslant 1$.*

Example 2.8. *Let $G(\xi, \eta, \zeta) = \xi + \eta + \zeta$ and let*

$$F(\xi, \eta, \zeta) = 3\varphi^{-1}\left(\frac{\varphi(\xi) + \varphi(\eta) + \varphi(\zeta)}{3}\right),$$

where $\varphi \colon [0, \infty) \to [0, \infty)$ is continuous, strictly increasing and convex function. Since φ is convex by Jensen's inequality we have

$$\varphi\left(\frac{\xi + \eta + \zeta}{3}\right) \leqslant \frac{\varphi(\xi) + \varphi(\eta) + \varphi(\zeta)}{3}$$

for all $\xi, \eta, \zeta \in [0, \infty)$, which implies condition (6). The other conditions of Definition 2.1 can be easily verified by the reader.

Example 2.9. *Let the function H satisfy all conditions of Definition 2.1 that satisfies the function G, then as F we can take $F(\xi, \eta, \zeta) = G(\xi, \eta, \zeta) + H(\xi, \eta, \zeta)$.*

Remark 2.10. *Let the supposition of Theorem 2.6 holds, let additionally F and G be continuous at their domain and let the mapping T have a fixed point x^* which is a limit of some iteration sequence $x_0, x_1 = Tx_0, x_2 = Tx_1, \cdots$ such that $x_n \neq x^*$ for all $n = 1, 2, \ldots$. Then x^* is a unique fixed point.*

Indeed, suppose that T has another fixed point $x^{**} \neq x^*$. It is clear that $x_n \neq x^{**}$ for all $n = 1, 2, \ldots$. Hence, we have that the points x^*, x^{**} and x_n are pairwise distinct for all $n = 1, 2, \ldots$. Consider the ratio

$$R_n = \frac{F(d(Tx^*, Tx^{**}), d(Tx^*, Tx_n), d(Tx^{**}, Tx_n))}{G(d(x^*, x^{**}), d(x^*, x_n), d(x^{**}, x_n))}$$

$$= \frac{F(d(x^*, x^{**}), d(x^*, x_{n+1}), d(x^{**}, x_{n+1}))}{G(d(x^*, x^{**}), d(x^*, x_n), d(x^{**}, x_n))}.$$

It is clear that $d(x^*, x_{n+1}) \to 0$, $d(x^*, x_n) \to 0$, $d(x^{**}, x_{n+1}) \to d(x^{**}, x^*)$ and $d(x^{**}, x_n) \to d(x^{**}, x^*)$. Taking into consideration condition (6), suppose first that

$$F(d(x^*, x^{**}), 0, d(x^{**}, x^*)) = G(d(x^*, x^{**}), 0, d(x^{**}, x^*)).$$

In this case by continuity of F and G we get $R_n \to 1$ as $n \to \infty$. Now suppose that

$$F(d(x^*, x^{**}), 0, d(x^{**}, x^*)) > G(d(x^*, x^{**}), 0, d(x^{**}, x^*)).$$

In this case by continuity of F and G we obtain $R_n > 1$ for some sufficiently large n. Anyway both cases contradict to condition (9).

Proposition 2.11. *Let (X, d) be a metric space and let $T\colon X \to X$ be an (F, G)-contracting mapping on X with continuous F and G and with*

$$G(\xi, \xi, 0) \leqslant k\xi, \tag{17}$$

where k is such that $\alpha k < 1$ and α is as in (9). If x is an accumulation point of X, then inequality (2) holds for all points $y \in X$ with the coefficient αk.

Proof. Let $y \in X$ and let $x \in X$ be an accumulation point. Then there exists a sequence (z_n) such that $z_n \to x$, $z_n \neq y$, $z_n \neq x$, and all z_n are different. Hence, by (9) the inequality

$$F(d(Tx, Ty), d(Ty, Tz_n), d(Tx, Tz_n)) \leqslant \alpha G(d(x, y), d(y, z_n), d(x, z_n))$$

holds for every $n \in \mathbb{N}^+$. Since every metric is continuous, we have $d(y, z_n) \to d(x, y)$, $d(x, z_n) \to 0$, $d(Ty, Tz_n) \to d(Tx, Ty)$ and $d(Tx, Tz_n) \to 0$. Letting $n \to \infty$ and using the continuity of F and G we obtain

$$F(d(Tx, Ty), d(Tx, Ty), 0) \leqslant \alpha G(d(x, y), d(x, y), 0).$$

By (10) we have

$$d(Tx, Ty) \leqslant \alpha G(d(x, y), d(x, y), 0).$$

Using (17), we obtain (2) with another coefficient αk. □

Corollary 2.12. *Let the supposition of Proposition 2.11 hold. If all points of X are accumulation points, then T is a contraction mapping.*

3. Edelstein type fixed point theorem

The main result of this section is an analogue of Edelstein's fixed point theorem for mappings contracting perimeters of triangles.

Definition 3.1. *Let (X, d) be a metric space with $|X| \geqslant 3$. We shall say that $T\colon X \to X$ is a mapping* contracting perimeters of triangles in the sense of Edelstein *on X if the inequality*

$$d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) < d(x, y) + d(y, z) + d(z, x) \tag{18}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Proposition 3.2. *Mappings contracting perimeters of triangles in the sense of Edelstein are continuous.*

Proof. Let (X, d) be a metric space with $|X| \geqslant 3$, $T\colon X \to X$ be a mapping contracting perimeters of triangles on X and let x_0 be an isolated point in X. Then, clearly, T is continuous at x_0. Let now x_0 be an accumulation point. Let us show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(Tx_0, Tx) < \varepsilon$ whenever $d(x_0, x) < \delta$. Let $x \neq x_0$. Since x_0 is an accumulation point, for every $\delta > 0$ there exists $y \in X$, $x_0 \neq y \neq x$, such that $d(x_0, y) < \delta$. By (18) we have

$$\begin{aligned} d(Tx_0, Tx) &\leqslant d(Tx_0, Tx) + d(Tx_0, Ty) + d(Tx, Ty) \\ &< d(x_0, x) + d(x_0, y) + d(x, y). \end{aligned}$$

Using the triangle inequality $d(x, y) \leqslant d(x_0, x) + d(x_0, y)$, we have

$$d(Tx_0, Tx) < 2(d(x_0, x) + d(x_0, y)) < 2(\delta + \delta) = 4\delta.$$

Setting $\delta = \varepsilon/4$, we obtain the desired inequality. □

Theorem 3.3. *Let (X, d), $|X| \geqslant 3$, be a metric space and let the mapping $T\colon X \to X$ satisfy the following three conditions:*

(i) T is a mapping contracting perimeters of triangles in the sense of Edelstein.

(ii) T does not possess periodic points of prime period 2.

(iii) There exists an $x \in X$ such that the sequence (ξ_n), $\xi_n = T^n x$, has a subsequence (ξ_{n_i}) which converges to $\xi \in X$.

Then ξ is a fixed point of T. The general number of fixed points of T is at most two.

Proof. Suppose that ξ is not a fixed point of T. Then, by condition (ii), there are only two possibilities: 1) the points ξ, $T\xi$ and $T^2\xi$ are pairwise distinct; 2) $T\xi$ is fixed.

Suppose that possibility 1) holds. Throughout the text below for any $x, y, z \in X$ denote the perimeter $p(x, y, z)$ on the points x, y, z by

$$p(x, y, z) = d(x, y) + d(y, z) + d(z, x).$$

The mapping $r(q, s, t)$ of

$$Y := X \times X \times X \setminus (\{(x, y, z) : x = y\} \cup \{(x, y, z) : y = z\} \cup \{(x, y, z) : z = x\})$$

into the real line, defined by

$$r(p, q, s) := \frac{p(Tq, Ts, Tt)}{p(q, s, t)}$$

is continuous on Y since every metric is continuous and the denominator of the fraction is nonzero. Hence, there exists a neighbourhood U of $(\xi, T(\xi), T^2(\xi))$ and R such that $(q, s, t) \in U$ implies

$$0 \leq r(q, s, t) < R < 1. \tag{19}$$

Let S_1, S_2 and S_3 be open balls centered at ξ, $T\xi$ and $T^2\xi$, respectively, of radius $\rho > 0$ small enough so as to have

$$\rho < \frac{1}{7} p(\xi, T\xi, T^2\xi) \tag{20}$$

and $S_1 \times S_2 \times S_3 \subseteq U$. Consider the sequences (ξ_{n_i+1}) and (ξ_{n_i+2}). Since T is continuous, by condition (iii) the first sequence converges to $T\xi$ and the second one converges to $T^2\xi$. Hence, and since $\xi_{n_i} \to \xi$, there exists a positive integer N such that $i > N$ implies $\xi_{n_i} \in S_1$, $\xi_{n_i+1} \in S_2$ and $\xi_{n_i+2} \in S_3$. Then

$$p(\xi_{n_i}, \xi_{n_i+1}, \xi_{n_i+2}) > \rho, \quad (i > N). \tag{21}$$

Indeed, assume the opposite. Then, by the triangle inequality,

$$\begin{aligned} p(\xi, T\xi, T^2\xi) &\leq d(\xi, \xi_{n_i}) + d(\xi_{n_i}, \xi_{n_i+1}) + d(\xi_{n_i+1}, T\xi) \\ &+d(T\xi, \xi_{n_i+1})) + d(\xi_{n_i+1}, \xi_{n_i+2}) + d(\xi_{n_i+2}, T^2\xi) \\ &+d(T^2\xi_{n_i+2}), \xi_{n_i+2}) + d(\xi_{n_i+2}, \xi_{n_i}) + d(\xi_{n_i}, \xi) \\ &\leq 6\rho + p(\xi_{n_i}, \xi_{n_i+1}, \xi_{n_i+2}) \leq 7\rho, \end{aligned}$$

which contradicts to (20).

Suppose first that ξ_i is a fixed point of T for some i. Clearly, in this case every subsequence of (ξ_n) converges to $\xi_i = \xi$. Suppose now that ξ_i is not a fixed point of the mapping T for every $i = 1, 2, \ldots$. Since ξ_i is not fixed, then $\xi_i \neq \xi_{i+1} = T\xi_i$. Since T have no periodic points of prime period 2 we have $\xi_{i+2} = T(T\xi_i) \neq \xi_i$ and by the supposition that ξ_{i+1} is not fixed we have $\xi_{i+1} \neq \xi_{i+2} = T\xi_{i+1}$. Hence, every three consecutive points ξ_i, ξ_{i+1} and ξ_{i+2} are pairwise distinct.

Let us show that all ξ_i are different. Set

$$p_0 = p(x_0, x_1, x_2), \quad p_1 = p(x_1, x_2, x_3), \cdots, p_n = p(x_n, x_{n+1}, x_{n+2}), \cdots.$$

Since ξ_i, ξ_{i+1} and ξ_{i+2} are pairwise distinct by (18) we have

$$p_0 > p_1 > \cdots > p_n > \cdots. \tag{22}$$

Suppose that $j \geqslant 3$ is a minimal natural number such that $\xi_j = \xi_i$ for some i such that $0 \leqslant i < j - 2$. Then $\xi_{j+1} = \xi_{i+1}$, $\xi_{j+2} = \xi_{i+2}$. Hence, $p_i = p_j$ which contradicts to (22).

Since ξ_{n_i}, ξ_{n_i+1} and ξ_{n_i+2} are pairwise distinct points of (ξ_i), for $i > N$ we can use (19):

$$p(\xi_{n_i+1}, \xi_{n_i+2}, \xi_{n_i+3}) < R \cdot p(\xi_{n_i}, \xi_{n_i+1}, \xi_{n_i+2}).$$

Repeated use of condition (18) and this inequality gives for $\ell > j > N$

$$\begin{aligned} p(\xi_{n_\ell}, \xi_{n_\ell+1}, \xi_{n_\ell+2}) &\leqslant p(\xi_{n_{\ell-1}+1}, \xi_{n_{\ell-1}+2}, \xi_{n_{\ell-1}+3}) \\ &< R \cdot p(\xi_{n_{\ell-1}}, \xi_{n_{\ell-1}+1}, \xi_{n_{\ell-1}+2}) \leqslant \cdots \\ &< R^{\ell-j} \cdot p(\xi_{n_j}, \xi_{n_j+1}, \xi_{n_j+2}) \to 0, \quad \ell \to \infty, \end{aligned}$$

which contradicts to (21). Thus, $T(\xi) = \xi$.

Suppose that possibility 2) holds. Since $T\xi_{n_i} = \xi_{n_i+1} \to T\xi$, for every $\varepsilon > 0$ there exist $N_1 > 0$ such that for every $i, j > N_1$, $i \neq j$, inequalities $d(\xi_{n_i+1}, T\xi) < \varepsilon$ and $d(\xi_{n_j+1}, T\xi) < \varepsilon$ hold. Hence, using the triangle inequality $d(\xi_{n_i+1}, \xi_{n_j+1}) \leqslant d(\xi_{n_i+1}, T\xi) + d(T\xi, \xi_{n_j+1})$, we get

$$p(\xi_{n_i+1}, \xi_{n_j+1}, T\xi) < 4\varepsilon. \tag{23}$$

Since there is no fixed points in the sequence (ξ_n) and $T\xi$ is fixed for T, we have that $T\xi \neq \xi_n$ for all $n \geqslant 1$. Hence, using the fact that all ξ_i are different we see that the points ξ_{n_i+1}, ξ_{n_j+1} and $T\xi$ are pairwise distinct and for any $k \geqslant 1$ we can consecutively apply k times inequality (18) to inequality (23):

$$p(T^k\xi_{n_i+1}, T^k\xi_{n_j+1}, T^kT\xi) = p(\xi_{n_i+k+1}, \xi_{n_j+k+1}, T\xi) < 4\varepsilon. \tag{24}$$

On the other hand, since $\xi_{n_i} \to \xi$, for every $\varepsilon > 0$ there exist $N_2 > 0$ such that for every $m > N_2$ the inequality $d(\xi_{n_m}, \xi) < \varepsilon$ holds. It is clear that it is always possible to choose $m > N_2$ such that $n_m > n_i + 1$. Let $k = n_m - n_i - 1$. Hence, $n_m = n_i + k + 1$ and $\xi_{n_m} = \xi_{n_i+k+1}$. Further, using the inequality $d(\xi, T\xi) \leqslant d(\xi, \xi_{n_m}) + d(\xi_{n_m}, \xi_{n_j+k+1}) + d(\xi_{n_j+k+1}, T\xi)$, we obtain

$$p(\xi_{n_i+k+1}, \xi_{n_j+k+1}, T\xi) = p(\xi_{n_m}, \xi_{n_j+k+1}, T\xi)$$

$$= d(\xi_{n_m}, \xi_{n_j+k+1}) + d(\xi_{n_j+k+1}, T\xi) + d(T\xi, \xi_{n_m})$$

$$\geqslant d(\xi, T\xi) - d(\xi, \xi_{n_m}) - d(\xi_{n_j+k+1}, T\xi) + d(\xi_{n_j+k+1}, T\xi) + d(T\xi, \xi_{n_m})$$

$$> d(\xi, T\xi) - \varepsilon.$$

Setting $\varepsilon = d(\xi, T\xi)/5$ we obtain that this inequality contradicts to (24).

Suppose that there exists at least three pairwise distinct fixed points x, y and z. Then $Tx = x$, $Ty = y$ and $Tz = z$, which contradicts to (18). □

Corollary 3.4. *Let (X, d), $|X| \geqslant 3$, be a compact metric space and let the mapping $T\colon X \to X$ satisfy the following two conditions:*

(i) T is a mapping contracting perimeters of triangles in the sense of Edelstein.

(ii) T does not possess periodic points of prime period 2*.*

Then T has a fixed point. The general number of fixed points of T is at most two.

Proof. Indeed, condition (iii) of Theorem 3.3 holds if X is compact. □

Recall that for a given metric space X, a point $x \in X$ is said to be an *accumulation point* of X if every open ball centered at x contains infinitely many points of X. In [27] in Corollary 2.8 it was shown that mappings contracting perimeters of triangles, in metric spaces such that every point of the space is an accumulation point, are contraction mappings. The following proposition shows that similar effect does not hold for mappings contracting perimeters of triangles in the sense of Edelstein.

Proposition 3.5. *Let (X, d), $|X| \geqslant 3$, be a metric space and let $T\colon X \to X$ be a mapping contracting perimeters of triangles in the sense of Edelstein. If all points of X are accumulation points, then the mapping T is not obligatory contractive.*

Proof. Let (X, d) be a metric space such that $X = [-2, -1] \cup [1, 2]$ and d is the Euclidean metric. It is clear that all points of the space X are accumulation points. Define a mapping $T\colon X \to X$ as follows: $T(x) = -1$ if $x \in [-2, -1]$ and $T(x) = 1$ if $x \in [1, 2]$. If for three pairwise distinct points x, y, z we have $x, y, z \in [-2, -1]$ or $x, y, z \in [1, 2]$, then clearly (18) holds. If $x \in [-2, -1]$ and $y, z \in [1, 2]$ or $x, y \in [-2, -1]$ and $z \in [1, 2]$ then (18) also holds, since $d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) = 2 + 2 + 0 = 4$ and $d(x, y) + d(y, z) + d(z, x) > 4$ because at least one of the points x, y, z does not equal -1 or 1. It remains to note that inequality (3) does not hold since $d(-1, 1) = d(T(-1), T(1)) = 2$. □

Example 3.6. *Let us construct an example of a mapping T in a finite metric space contracting perimeters of triangles in the sense of Edelstein, which is not a contractive mapping. Let (X, d) be a metric space such that $X = \{x, y, z\}$, $d(x, y) = d(y, z) = d(x, z) = 1$, and let $T\colon X \to X$ be such that $Tx = x$, $Ty = y$ and $Tz = x$. One can easily see that (18) holds and (3) does not hold since x and y are fixed points.*

Example 3.7. *Let us construct an example of a mapping $T\colon X \to X$ contracting perimeters of triangles in the sense of Edelstein that is not a contractive mapping and that is not a mapping contracting perimeters of triangles for a metric space X with $|X| = \aleph_0$. Let $X = \{x^*, x_0, x_0', x_1, x_1', \dots\}$ and let ε be a positive real number.*

Define the metric d on X as follows:

$$d(x,y) = \begin{cases} \frac{1}{i^2}, & \text{if } x = x_{i-1} \text{ or } x = x'_{i-1}, \\ & \text{and } y = x_i, \text{ or } y = x'_i,\ i = 1,2,3,\dots, \\ \frac{\varepsilon}{i^2}, & \text{if } x = x_i,\ y = x'_i \\ & \text{or } x = x_{i-1}, y = x'_{i-1},\ i = 1,3,5,\dots, \\ d(x_i,x_{i+1}) + \cdots + d(x_{j-1},x_j), & \text{if } x = x_i \text{ or } x = x'_i \\ & \text{and } y = x_j \text{ or } y = x'_j,\ i+1<j, \\ \frac{\pi^2}{6} - d(x_0,x_i), & \text{if } x = x_i \text{ or } x = x'_i \text{ and } y = x^*, \\ 0, & \text{if } x = y, \end{cases}$$

see Figure 1.

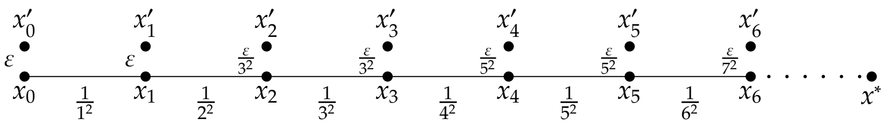

Figure 1: The points of the space (X,d) with distances between them.

The reader can easily verify that for sufficiently small ε the metric d is well defined. For this recall only the well-known fact $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$. *Moreover, the space is complete with the single accumulation point* x^*.

Define a mapping $T\colon X \to X$ *as* $Tx_i = x_{i+1}$, $Tx'_i = x'_{i+1}$ *for all* $i = 0,1,\dots$ *and* $Tx^* = x^*$. *Since* $d(x_i,x'_i) = d(Tx_i,Tx'_i)$, $i = 0,2,4,\dots$, *using (3), we see that T is not a contractive mapping. Suppose that there exists* $0 \leqslant \alpha < 1$ *such that (1) holds for all pairwise distinct* $x,y,z \in X$. *Let* $x = x_i$, $y = x_{i+1}$, $z = x_{i+2}$. *Consider the ratio*

$$\frac{d(Tx,Ty) + d(Ty,Tz) + d(Tx,Tz)}{d(x,y) + d(y,z) + d(x,z)}$$

$$= \frac{d(x_{i+1},x_{i+2}) + d(x_{i+2},x_{i+3}) + d(x_{i+1},x_{i+3})}{d(x_i,x_{i+1}) + d(x_{i+1},x_{i+2}) + d(x_i,x_{i+2})}$$

$$= \left(\frac{2}{(i+2)^2} + \frac{2}{(i+3)^2}\right) : \left(\frac{2}{(i+1)^2} + \frac{2}{(i+2)^2}\right) \to 1$$

as $i \to \infty$, *which contradicts to (1). Thus, T is not a mapping contracting perimeters of triangles. Verifying inequality (18) for all pairwise distinct* $x,y,z \in X$ *is almost evident. Thus, T is a mapping contractive perimeters of triangles in the sense of Edelstein.*

References

[1] M. Asadi. Discontinuity of control function in the (F,φ,θ)-contraction in metric spaces. *Filomat*, 31(17):5427–5433, 2017.

[2] M. Asadi, E. Karapınar, and A. Kumar. α-ψ-Geraghty contractions on generalized metric spaces. *J. Inequal. Appl.*, pages 1–21,, 2014.

[3] M. Asadi, E. Karapınar, and P. Salimi. A new approach to G-metric and related fixed point theorems. *J. Inequal. Appl.*, pages 1–14,, 2013.

[4] M. Asadi and A. Khalesi. Lower semi-continuity in a generalized metric space. *Adv. Theory Nonlinear Anal. Appl.*

[5] S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.*, 3:133–181, 1922.

[6] J. Caristi. Fixed point theorems for mappings satisfying inwardness conditions. *Trans. Amer. Math. Soc.*, 215:241–251, 1976.

[7] M. Cherichi and B. Samet. Fixed point theorems on ordered gauge spaces with applications to nonlinear integral equations. *Fixed Point Theory Appl.*, 2012:1–19, 2012. Paper No. 13.

[8] S. Czerwik. Generalization of Edelstein's fixed point theorem. *Demonstratio Math.*, 9(2):281–285, 1976.
[9] S. Czerwik. Some inequalities, characteristic roots of a matrix and Edelstein's fixed point theorem. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 24(10):827–828, 1976.
[10] P. Z. Daffer and H. Kaneko. A new proof of a fixed point theorem of Edelstein. *Sci. Math.*, 1(1):97–101, 1998.
[11] R. L. Devaney. *An introduction to chaotic dynamical systems*. Studies in Nonlinearity. Westview Press, Boulder, CO, 2003.
[12] S. Dhompongsa, W. Inthakon, and A. Kaewkhao. Edelstein's method and fixed point theorems for some generalized nonexpansive mappings. *J. Math. Anal. Appl.*, 350(1):12–17, 2009.
[13] C. Di Bari and P. Vetro. Common fixed points in generalized metric spaces. *Appl. Math. Comput.*, 218(13):7322–7325, 2012.
[14] D. Dorić, Z. Kadelburg, and S. Radenović. Edelstein-Suzuki-type fixed point results in metric and abstract metric spaces. *Nonlinear Anal.*, 75(4):1927–1932, 2012.
[15] M. Edelstein. On fixed and periodic points under contractive mappings. *J. London Math. Soc.*, 37:74–79, 1962.
[16] S. Janković, Z. Kadelburg, and S. Radenović. On cone metric spaces: a survey. *Nonlinear Anal.*, 74(7):2591–2601, 2011.
[17] L. Janos. On the Edelstein contractive mapping theorem. *Canad. Math. Bull.*, 18(5):675–678, 1975.
[18] E. Karapınar. Edelstein type fixed point theorems. *Fixed Point Theory Appl.*, 2012:1–12, 2012. Article number 107.
[19] W. Kirk and N. Shahzad. *Fixed point theory in distance spaces*. Springer, Cham, 2014.
[20] W. A. Kirk. A fixed point theorem for mappings which do not increase distances. *Amer. Math. Monthly*, 72:1004–1006, 1965.
[21] W. A. Kirk. Fixed points of asymptotic contractions. *J. Math. Anal. Appl.*, 277(2):645–650, 2003.
[22] W. A. Kirk and N. Shahzad. Generalized metrics and Caristi's theorem. *Fixed Point Theory Appl.*, 2013:1–9, 2013. Paper No. 129.
[23] H. Lakzian and B. Samet. Fixed points for (ψ, ϕ)-weakly contractive mappings in generalized metric spaces. *Appl. Math. Lett.*, 25(5):902–906, 2012.
[24] A. Liepinš. Edelstein's fixed point theorem in topological spaces. *Numer. Funct. Anal. Optim.*, 2(5):387–396, 1980.
[25] F. Moradlou, P. Salimi, and P. Vetro. Some new extensions of Edelstein-Suzuki-type fixed point theorem to G-metric and G-cone metric spaces. *Acta Math. Sci. Ser. B (Engl. Ed.)*, 33(4):1049–1058, 2013.
[26] S. B. Nadler, Jr. Multi-valued contraction mappings. *Pacific J. Math.*, 30:475–488, 1969.
[27] E. Petrov. Fixed point theorem for mappings contracting perimeters of triangles. *J. Fixed Point Theory Appl.*, 25(3):1–11, 2023. Paper No. 74.
[28] O. Popescu. Some remarks on the paper "Fixed point theorems for generalized contractive mappings in metric spaces". *J. Fixed Point Theory Appl.*, 23:1–10, 2021. Paper No. 72.
[29] P. D. Proinov. Fixed point theorems for generalized contractive mappings in metric spaces. *J. Fixed Point Theory Appl.*, 22:1–27, 2020. Paper No. 21.
[30] J. Reinermann, G. H. Seifert, and V. Stallbohm. Two further applications of the Edelstein fixed point theorem to initial value problems of functional equations. *Numer. Funct. Anal. Optim.*, 1(3):233–254, 1979.
[31] N. Saleem, I. Iqbal, B. Iqbal, and S. Radenovíc. Coincidence and fixed points of multivalued F-contractions in generalized metric space with application. *J. Fixed Point Theory Appl.*, 22:1–24, 2020. Paper No. 81.
[32] N. Shahzad and O. Valero. A Nemytskii-Edelstein type fixed point theorem for partial metric spaces. *Fixed Point Theory Appl.*, 2015:1–15, 2015. Article number 26.
[33] T. Suzuki. A new type of fixed point theorem in metric spaces. *Nonlinear Anal.*, 71(11):5313–5317, 2009.
[34] T. Suzuki. Another generalization of Edelstein's fixed point theorem in generalized metric spaces. *Linear Nonlinear Anal.*, 2(2):271–279, 2016.
[35] T. Suzuki. The weakest contractive conditions for Edelstein's mappings to have a fixed point in complete metric spaces. *J. Fixed Point Theory Appl.*, 19(4):2361–2368, 2017.
[36] T. Suzuki. Some comments on Edelstein's fixed point theorems in v-generalized metric spaces. *Bull. Kyushu Inst. Technol. Pure Appl. Math.*, (65):23–42, 2018.
[37] T. Suzuki. Generalizations of Edelstein's fixed point theorem in compact metric spaces. *Fixed Point Theory*, 20(2):703–713, 2019.
[38] T. Suzuki, B. Alamri, and M. Kikkawa. Edelstein's fixed point theorem in generalized metric spaces. *J. Nonlinear Convex Anal.*, 16(11):2301–2309, 2015.
[39] M. Turinici. Functional contractions in local Branciari metric spaces. *ROMAI J.*, 8(2):189–199, 2012.
[40] D. Wardowski. Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.*, 2012:1–6, 2012. Paper No. 94.