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Fuzzy $\mathcal{Z}$-proximal contraction in strong fuzzy metric spaces

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Abstract. In this note, we define two distinct categories of fuzzy $\mathcal{Z}$-proximal contractions and we use these two fuzzy $\mathcal{Z}$-proximal contractive inequalities as a tool to obtain best proximity point for a non-self mapping which is defined between two distinct non-empty subsets of a strong fuzzy metric space. Further, prove some proximity theorems by using these categories of fuzzy $\mathcal{Z}$-proximal in a complete strong fuzzy metric space. For the support of these innovative results we produce a few validation of examples. At last, we provide a solution of a non-linear second-order ordinary differential equation with the help of fuzzy $\mathcal{Z}$-proximal contractive inequality provided that assumed space is strong fuzzy metric space.

1. Introduction and Preliminaries

The mathematician Lotfi A. Zadeh [18] made significant contributions to the field of fuzzy set theory in his seminal paper in 1965. He introduced the concept of a membership function, which enhanced the original crisp set theory. In theory of crisp set, a member either belongs or not belongs to a set. However, with the introduction of membership functions, fuzzy set theory permits for partial membership, where a member can have a degree of membership between 0 and 1. This concept of partial membership is a generalization of the characteristic function used in crisp set theory.

Building upon the ideas of fuzzy set theory, Kramosil and Michálek [8] introduced the notion of a fuzzy metric space. In a fuzzy metric space, the distance between two points is represented by a fuzzy number or a membership function that assigns a degree of nearness between the points. This concept expanded the traditional notion of metric spaces, where distances are typically crisp values. George and Veeramani further improved upon the concept by defining a Hausdorff topology for fuzzy metric spaces which provides a framework for studying the properties of fuzzy metric spaces and their associated distances. Another significant development in the field of fuzzy set theory was the exploration of fixed point theory. Grabic was the first mathematicians to experiment with fuzzy concepts in fixed point theory. He demonstrated fuzzy versions of the Banach and Edelstein contraction theorems. These theorems establish conditions

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under which mappings on a set have unique fixed points. By extending these theorems to the realm of fuzzy sets, Grabic laid the foundation for further research in fuzzy fixed point theory.

Later, researchers began to question whether fixed point equations could still have solutions when considering non-self mappings. In the traditional context of fixed point theory, self-mappings were primarily studied, where the mapping operated on elements within the same set. However, this limitation raised the question of whether fixed points could be found when the mapping was non-self, meaning it operated between different sets.

To address this doubt, mathematicians extended the scope of fixed point theory to non-self mappings. They explored various conditions and properties under which best proximity points could still exist. This expansion led to the development of best proximity point theory. Now we need to recognize the notion of space introduced by George and Veeramani [3].

Definition 1.1. *[3]. A fuzzy metric space is an ordered triple $(\mathcal{E}, \mathcal{F}_z, \diamond)$ such that $\mathcal{E}$ is a non-empty set $\diamond$ is a continuous triangular-norm and $\mathcal{F}_z$ is a fuzzy set on $\mathcal{E} \times \mathcal{E} \times (0, +\infty)$ satisfying the following conditions, for all $\alpha, \beta, \gamma \in \mathcal{E}$ and $r, s > 0$;*

1. $\mathcal{F}_z(\beta, \gamma, r) > 0$;
2. $\mathcal{F}_z(\beta, \gamma, r) = 1$ *if and only if* $\gamma = \beta$;
3. $\mathcal{F}_z(\beta, \gamma, r) = \mathcal{F}_z(\gamma, \beta, r)$;
4. $\mathcal{F}_z(\beta, \alpha, r + s) \geq \mathcal{F}_z(\beta, \gamma, r) \diamond \mathcal{F}_z(\gamma, \alpha, s)$;
5. $\mathcal{F}_z(\beta, \gamma, .) : (0, +\infty) \to (0, 1]$ *is continuous.*

*If we replace axiom (4) by (4)′ $\mathcal{F}_z(\beta, \alpha, max\{r, s\}) \geq \mathcal{F}_z(\beta, \gamma, r) \diamond \mathcal{F}_z(\gamma, \alpha, s)$, then $(\mathcal{E}, \mathcal{F}_z, \diamond)$ is known as **non-Archimedean or strong fuzzy metric space**. Since (4) implies (4)′, then each non-Archimedean fuzzy metric space is a fuzzy metric space.*

In this context, the existence of a fixed point for a non-self mapping $\mathcal{H} : \mathcal{L} \to \mathcal{M}$ is not guaranteed where $\mathcal{L}$ and $\mathcal{M}$ are two non-empty subsets of a strong fuzzy metric space. Therefore, it becomes essential to investigate the existence of an element η that is as close to $\mathcal{H}(\eta)$ as possible. In other words, when the fixed point equation $\mathcal{H}(\eta) = \eta$ has no solution, the goal is to find an approximate solution x that minimizes the error in terms of the degree of nearness between η and $\mathcal{H}\eta$. Best proximity point theorems are aimed at establishing the existence of such optimal approximate solutions, known as best proximity point, for the fixed point equation $\mathcal{H}(\eta) = \eta$ when an exact solution does not exist. In this context, it is required that the degree of membership $\mathcal{F}_z(\eta, \mathcal{H}\eta, r)$ should be at least $\mathcal{F}_z(\mathcal{L}, \mathcal{M}, r)$, where $\mathcal{F}_z(\mathcal{L}, \mathcal{M}, r)$ represents the degree of nearness between these two sets $\mathcal{L}$ and $\mathcal{M}$ with respect to parameter r. A best proximity point theorem ensures the absolute minimum error $\mathcal{F}_z(\eta, \mathcal{H}(\eta), r)$ by providing an approximate solution η that satisfies the condition $\mathcal{F}_z(\eta, \mathcal{H}\eta, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r)$.

Suppose $\mathcal{L}$ and $\mathcal{M}$ are the two non-empty subsets of a strong fuzzy metric spaces $(\mathcal{E}, \mathcal{F}_z, \diamond)$. We denote $\mathcal{L}_0(r)$ and $\mathcal{M}_0(r)$ by

$$\left.\begin{aligned} \mathcal{L}_0(r) &= \{\beta \in \mathcal{L} : \mathcal{F}_z(\beta, \gamma, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r) \text{ for some } \gamma \in \mathcal{M}\} \\ \mathcal{M}_0(r) &= \{\gamma \in \mathcal{M} : \mathcal{F}_z(\beta, \gamma, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r) \text{ for some } \beta \in \mathcal{L}\} \end{aligned}\right\} \tag{1.1}$$

where $\mathcal{F}_z(\mathcal{L}, \mathcal{M}, r) = \sup\{\mathcal{F}_z(\beta, \gamma, r) : \beta \in \mathcal{L}, \gamma \in \mathcal{M}\}$.

Definition 1.2. *[14] The pair $(\mathcal{L}, \mathcal{M})$ with $\mathcal{L}_0 \neq 0$ is said to have the fuzzy weak p-property if*

$$\left.\begin{aligned} \mathcal{F}_z(\beta_1, \gamma_1, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r) \\ \mathcal{F}_z(\beta_2, \gamma_2, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r) \end{aligned}\right\} \text{ implies } \ \mathcal{F}_z(\beta_1, \beta_2, r) \geq \mathcal{F}_z(\gamma_1, \gamma_2, r)$$

where $\beta_1, \beta_2 \in \mathcal{L}_0$ and $\gamma_1, \gamma_2 \in \mathcal{M}_0$.

Shukla et al. [17] define group of function $\zeta : (0, 1] \times (0, 1] \to \mathbb{R}$ that satisfies

1. $\zeta(s_1, s_2) > s_2$ for every $s_1, s_2 \in (0, 1)$;

2. $\zeta(1,1)=1$.

By using this group they introduced below class of fuzzy $\mathcal{Z}$-contractive mapping with respect to $\zeta \in \mathcal{Z}$ in fuzzy metric space.
An operator $\mathcal{H}:\mathcal{E}\to\mathcal{E}$ is said to be a $\mathcal{Z}$-fuzzy contractive mapping if there exists $\zeta\in\mathcal{Z}$ such that

$$\mathcal{F}_z(\mathcal{H}\alpha,\mathcal{H}\beta,r)\geq \zeta(\mathcal{F}_z(\mathcal{H}\alpha,\mathcal{H}\beta,r),\mathcal{F}_z(\alpha,\beta,r)) \tag{1.2}$$

for every $\alpha,\beta\in\mathcal{E}$, $\mathcal{H}\alpha\neq\mathcal{H}\beta$ and $r>0$.
By introducing this class, they were able to unify various groups of fuzzy contractive mappings, such as Gregori and Sapena [6], Tirado, Mihet [9] based on the notion of George and Veeramani [3].

2. Main results

With this group of mappings $\zeta\in\mathcal{Z}$, we define a fuzzy $\mathcal{Z}$-proximal contractive mapping and before that require to define the following characteristic-$\mathcal{S}$.

Definition 2.1. *Suppose that $\zeta\in\mathcal{Z}$ and a non-self map $\mathcal{H}:\mathcal{L}\to\mathcal{M}$ where $\mathcal{L}$ and $\mathcal{M}$ are two non empty subsets of a strong fuzzy metric space $(\mathcal{E},\mathcal{F}_z,\diamond)$. Then the quadruplet $(\mathcal{E},\mathcal{F}_z,\mathcal{H},\zeta)$ has characteristic-$\mathcal{S}$, if for any sequence $\{\xi_n\}$ in $\mathcal{L}_0$ satisfying $\mathcal{F}_z(\xi_{n+1},\mathcal{H}\xi_n,r)=\mathcal{F}_z(\mathcal{L},\mathcal{M},r)$ such that*

$$\mathcal{F}_z(\xi_{n+1},\xi_{n+2},r)\geq\mathcal{F}_z(\xi_n,\xi_{n+1},r)$$

for every $n\in\mathbb{N}$ and $r>0$ implies

$$\lim_{n\to+\infty}\zeta(\mathcal{F}_z(\xi_{n+1},\xi_{n+2},r),\mathcal{F}_z(\xi_n,\xi_{n+1},r))=1.$$

Theorem 2.2. *Assume that two non-empty closed subsets $\mathcal{L}$ and $\mathcal{M}$ of a complete strong fuzzy metric space $(\mathcal{E},\mathcal{F}_z,\diamond)$ with $\mathcal{L}_0(r)\neq\phi$ and $\mathcal{H}:\mathcal{L}\to\mathcal{M}$. If the following assertions satisfying:*

1. *$\mathcal{H}(\mathcal{L}_0)\subseteq\mathcal{M}_0$;*
2. *The mapping $\mathcal{H}$ is a fuzzy $\mathcal{Z}$-contractive;*
3. *The pair $(\mathcal{L},\mathcal{M})$ has a fuzzy weak p-property;*
4. *The quadruplet $(\mathcal{E},\mathcal{F}_z,\mathcal{H},\zeta)$ has the characteristic-$\mathcal{S}$.*

Then there exists a member $u\in\mathcal{L}$ such that $\mathcal{F}_z(u,\mathcal{H}u,r)=\mathcal{F}_z(\mathcal{L},\mathcal{M},r)$.

Proof. Consider ξ_0 is any arbitrary member of $\mathcal{L}_0(r)$. Since $\mathcal{H}(\xi_0)\in\mathcal{H}(\mathcal{L}_0(r))\subseteq\mathcal{M}_0(r)$ there exists a member ξ_1 in $\mathcal{L}_0(r)$ such that $\mathcal{F}_z(\xi_1,\mathcal{H}\xi_0,r)=\mathcal{F}_z(\mathcal{L},\mathcal{M},r)$. Since $\mathcal{H}\xi_1\in\mathcal{H}(\mathcal{L}_0(r))\subseteq\mathcal{M}_0(r)$, it will follow that there is a member ξ_2 in $\mathcal{L}_0(r)$ such that $\mathcal{F}_z(\xi_2,\mathcal{H}\xi_1,r)=\mathcal{F}_z(\mathcal{L},\mathcal{M},r)$.
Recursively, we can get a sequence $\{\xi_n\}$ in $\mathcal{L}_0(r)$ satisfying

$$\mathcal{F}_z(\xi_{n+1},\mathcal{H}\xi_n,r)=\mathcal{F}_z(\mathcal{L},\mathcal{M},r) \tag{2.1}$$

for every $n\in\mathbb{N}$ and $r>0$. Clearly, if for some $n\in\mathbb{N}$, $\xi_{n+1}=\xi_n$ then from (2.1), nothing will remains to prove. Therefore, $\xi_{n+1}\neq\xi_n$ for every $n\in\mathbb{N}$. Consider $c_n(r)=\mathcal{F}_z(\xi_n,\xi_{n+1},r)$ for all $n\in\mathbb{N}$ and $r>0$. From (1.2) and using definition of ζ,

$$c_n(r)=\mathcal{F}_z(\xi_n,\xi_{n+1},r)\geq\zeta(\mathcal{F}_z(\xi_n,\xi_{n+1},r),\mathcal{F}_z(\xi_{n-1},\xi_n,r))>\mathcal{F}_z(\xi_{n-1},\xi_n,t)=c_{n-1}(r)$$

for each $r>0$. Therefore, the sequence is $\{c_n(r)\}$ is a non-decreasing which is converges to $c(r)$ that is $\lim_{n\to+\infty}c_n(r)=c(r)$.
We claim that $c(r)=1$, for each $r>0$. Take a contradiction such that $0<c(s)<1$ for some $s>0$, again by (1.2)

$$c_n(s)=\mathcal{F}_z(\xi_n,\xi_{n+1},s)\geq\zeta(\mathcal{F}_z(\xi_n,\xi_{n+1},s),\mathcal{F}_z(\xi_{n-1},\xi_n,s))>\mathcal{F}_z(\xi_{n-1},\xi_n,s)=c_{n-1}(s)$$

applying limit $n \to +\infty$ and applying characteristic-$\mathcal{S}$,

$$\lim_{n\to+\infty} \mathcal{F}_z(\xi_n, \xi_{n+1}, s) = 1.$$

This contradiction verifies the claim. Thus for every $r > 0$

$$\lim_{n\to+\infty} \mathcal{F}_z(\xi_n, \xi_{n+1}, r) = 1. \tag{2.2}$$

Next we claim that $\{\xi_n\}$ is a Cauchy sequence. Assume that $\{\xi_n\}$ is not a Cauchy sequence, then for each $\epsilon \in (0, 1)$ and $r > 0$, there is $k \in \mathbb{N}$ such that

$$\mathcal{F}_z(\xi_{m_k}, \xi_{n_k}, r) \leq 1 - \epsilon \tag{2.3}$$

for all $m_k > n_k \geq k$. Assume that m_k is the smallest integer greater than n_k, satisfying (2.3)

$$\mathcal{F}_z(\xi_{m_k - 1}, \xi_{n_k}, r) > 1 - \epsilon. \tag{2.4}$$

Using (2.3),

$$1 - \epsilon \geq \mathcal{F}_z(\xi_{m_k}, \xi_{n_k}, r) \geq \mathcal{F}_z(\xi_{m_k}, \xi_{m_{k-1}}, r) \diamond \mathcal{F}_z(\xi_{m_{k-1}}, \xi_{n_k}, r) > \mathcal{F}_z(\xi_{m_{k-1}}, \xi_{m_k}, r) \diamond 1 - \epsilon.$$

applying limit $k \to +\infty$, we deduce

$$\lim_{k\to+\infty} \mathcal{F}_z(\xi_{m_k}, \xi_{n_k}, r) = 1 - \epsilon.$$

Further

$$\mathcal{F}_z(\xi_{m_{k+1}}, \xi_{n_{k+1}}, r) \geq \mathcal{F}_z(\xi_{m_{k+1}}, \xi_{m_k}, r) \diamond \mathcal{F}_z(\xi_{m_k}, \xi_{n_k}, r) \diamond \mathcal{F}_z(\xi_{n_k}, \xi_{n_k+1}, r).$$

Taking limit as $k \to +\infty$,

$$\lim_{k\to+\infty} \mathcal{F}_z(\xi_{m_k+1}, \xi_{n_k+1}, r) = 1 - \epsilon. \tag{2.5}$$

Now by using the (1.2),

$$\mathcal{F}_z(\xi_{m_k}, \xi_{n_k}, r) \geq \zeta(\mathcal{F}_z(\xi_{m_k}, \xi_{n_k}, r), \mathcal{F}_z(\xi_{m_{k-1}}, \xi_{n_{k-1}}, r)) > \mathcal{F}_z(\xi_{m_{k-1}}, \xi_{n_{k-1}}, r).$$

applying limit $k \to +\infty$, we get

$$1 - \epsilon \geq \lim_{k\to+\infty} \zeta(\mathcal{F}_z(\xi_{m_k}, \xi_{n_k}, r), \mathcal{F}_z(\xi_{m_{k-1}}, \xi_{n_{k-1}}, r)) > 1 - \epsilon,$$

with the use of characteristic-$\mathcal{S}$ we get a contradiction. Therefore $\{\xi_n\}$ is a Cauchy sequence. Completeness property of the space $(\mathcal{E}, \mathcal{F}_z, \diamond)$ ensure that sequence $\{\xi_n\}$ converges to some $u \in \mathcal{E}$, it means

$$\lim_{n\to+\infty} \mathcal{F}_z(\xi_n, u, r) = 1 \tag{2.6}$$

for all $r > 0$. Since $\mathcal{H}$ is continuous, $\mathcal{H}\xi_n \to \mathcal{H}u$ and by the continuity of $\mathcal{F}_z$ implies

$$\lim_{n\to+\infty} \mathcal{F}_z(\xi_{n+1}, \mathcal{H}\xi_n, r) \to \mathcal{F}_z(u, \mathcal{H}u, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r). \tag{2.7}$$

Therefore u is best proximity point for $\mathcal{H}$. Next, demonstrate that u is unique for $\mathcal{H}$. Assume that for any $r > 0$, $0 < \mathcal{F}_z(u, w, r) < 1$, and w is another best proximity point of $\mathcal{H}$, i.e. $u \neq w$, and it follows from the condition that $\mathcal{H}$ is fuzzy $\mathcal{Z}$-contraction and fuzzy weak p-property,

$$\mathcal{F}_z(u, \mathcal{H}u, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r) \quad and \quad \mathcal{F}_z(w, \mathcal{H}w, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r) \implies \mathcal{F}_z(u, w, r) \geq \mathcal{F}_z(\mathcal{H}u, \mathcal{H}w, r).$$

Now

$$\mathcal{F}_z(u, w, r) \geq \mathcal{F}_z(\mathcal{H}u, \mathcal{H}w, r) \geq \zeta(\mathcal{F}_z(\mathcal{H}u, \mathcal{H}w, r), \mathcal{F}_z(u, w, r)) > \mathcal{F}_z(u, w, r), \tag{2.8}$$

which is contradiction. Thus $\mathcal{F}_z(u, w, r) = 1$ for all $r > 0$, it means $u = w$. □

Corollary 2.3. *Suppose $(\mathcal{E},\mathcal{F}_z,\diamond)$ is a strong fuzzy metric space and $\mathcal{H}:\mathcal{E}\to\mathcal{E}$ be a continuous fuzzy $\mathcal{Z}$-contraction. The quadruplet $(\mathcal{E},\mathcal{F}_z,\mathcal{H},\zeta)$ has characteristic-$\mathcal{S}$. Then $\mathcal{H}$ has a unique fixed point in $\mathcal{E}$.*

The key finding of this study is to transform the results given by Shukla et al. [17] for a non-self mapping by defining two kinds of fuzzy $\mathcal{Z}$-proximal contractive inequalities to obtain the best proximity point.

Definition 2.4. *Suppose $(\mathcal{E},\mathcal{F}_z,\diamond)$ is a strong fuzzy metric space. A map $\mathcal{H}:\mathcal{L}\to\mathcal{M}$ is known as a fuzzy $\mathcal{Z}$-proximal contractive of first kind, if there exists $\zeta\in\mathcal{Z}$ such that*

$$\left.\begin{aligned}\mathcal{F}_z(u_1,\mathcal{H}x_1,r)=\mathcal{F}_z(\mathcal{L},\mathcal{M},r)\\ \mathcal{F}_z(u_2,\mathcal{H}x_2,r)=\mathcal{F}_z(\mathcal{L},\mathcal{M},r)\end{aligned}\right\}\implies \mathcal{F}_z(u_1,u_2,r)\geq\zeta(\mathcal{F}_z(u_1,u_2,r),\mathcal{F}_z(x_1,x_2,r)) \tag{2.9}$$

for every $u_1,u_2,x_1,x_2\in\mathcal{L}$ and $r>0$.

Definition 2.5. *Suppose $(\mathcal{E},\mathcal{F}_z,\diamond)$ is a strong fuzzy metric space. A map $\mathcal{H}:\mathcal{L}\to\mathcal{M}$ is known as a fuzzy $\mathcal{Z}$-proximal contractive of second kind, if there exists $\zeta\in\mathcal{Z}$ such that*

$$\left.\begin{aligned}\mathcal{F}_z(u_1,\mathcal{H}x_1,r)=\mathcal{F}_z(\mathcal{L},\mathcal{M},r)\\ \mathcal{F}_z(u_2,\mathcal{H}x_2,r)=\mathcal{F}_z(\mathcal{L},\mathcal{M},r)\end{aligned}\right\}\implies \mathcal{F}_z(\mathcal{H}u_1,\mathcal{H}u_2,r)\geq\zeta(\mathcal{F}_z(\mathcal{H}u_1,\mathcal{H}u_2,r),\mathcal{F}_z(\mathcal{H}x_1,\mathcal{H}x_2,r)) \tag{2.10}$$

for every $u_1,u_2,x_1,x_2\in\mathcal{L}$ and $r>0$.

Now prove fuzzy $\mathcal{Z}$-proximal contractive mapping for of first kind and prove best proximity point results without help of fuzzy weak p-property. Also it will be required to define characteristic-Q_1 to show constructed sequence is a Cauchy sequence.

Definition 2.6. *A mapping $\mathcal{H}:\mathcal{L}\to\mathcal{M}$ said to have characteristic-Q_1, if for a sequence $\{\xi_n\}$ defined as $\mathcal{F}_z(\xi_{n+1},\mathcal{H}\xi_n,r)=\mathcal{F}_z(\mathcal{L},\mathcal{M},r)$ such that*

$$\lim_{n\to+\infty}\mathcal{F}_z(\xi_n,\xi_{n+1},r)=1$$

for every $n\in\mathbb{N}$. For any two subsequences $\{\xi_{n_k}\}$ and $\{\xi_{m_k}\}$ of sequence $\{\xi_n\}$ where $n_k>m_k>k$ and $k\in\mathbb{N}$. Then

$$\mathcal{F}_z(\xi_{n_k},\mathcal{H}\xi_{n_{k-1}},r)=\mathcal{F}_z(\mathcal{L},\mathcal{M},r) \text{ and } \mathcal{F}_z(\xi_{m_k},\mathcal{H}\xi_{m_{k-1}},r)=\mathcal{F}_z(\mathcal{L},\mathcal{M},r)$$

holds.

Theorem 2.7. *Assume that two non-empty closed subsets $\mathcal{L}$ and $\mathcal{M}$ of a complete strong fuzzy metric space $(\mathcal{E},\mathcal{F}_z,\diamond)$ with $L_0(r)\neq\phi$ and $\mathcal{H}:\mathcal{L}\to\mathcal{M}$ satisfying:*

1. *$\mathcal{H}$ is a fuzzy $\mathcal{Z}$-proximal contractive mapping of first kind;*
2. *$\mathcal{H}(\mathcal{L}_0)\subseteq\mathcal{M}_0$;*
3. *The quadruplet $(\mathcal{E},\mathcal{F}_z,\mathcal{H},\zeta)$ has the characteristic-$\mathcal{S}$ and characteristic-Q_1;*
4. *for any sequence $\{y_n\}$ in $\mathcal{M}_0(r)$ and $x\in\mathcal{L}$ satisfying $\mathcal{F}_z(x,y_n,r)\to\mathcal{F}_z(\mathcal{L},\mathcal{M},r)$ as $n\to+\infty$ then $x\in\mathcal{L}_0(r)$.*

Then there exists a member u in $\mathcal{L}$ such that $\mathcal{F}_z(u,\mathcal{H}u,r)=\mathcal{F}_z(\mathcal{L},\mathcal{M},r)$ for all $r>0$.

Proof. Sequence construction is similar to the Theorem 2.2. Consider $b_n(r)=\mathcal{F}_z(\xi_n,\xi_{n+1},r)$ for every $n\in\mathbb{N}$ and $r>0$. From (2.9) and using definition of ζ,

$$b_n(r)=\mathcal{F}_z(\xi_n,\xi_{n+1},r)\geq\zeta(\mathcal{F}_z(\xi_n,\xi_{n+1},r),\mathcal{F}_z(\xi_{n-1},\xi_n,r))>\mathcal{F}_z(\xi_{n-1},\xi_n,r)=b_{n-1}(r)$$

for each $r>0$. Therefore, the sequence $\{b_n(r)\}$ is a increasing sequence. Let $\lim_{n\to+\infty}b_n(r)=b(r)$. We claim that $b(r)=1$, for each $r>0$. Take a contradiction that $0<b(s)<1$ for some $s>0$ and then by (2.9)

$$b_n(s)=\mathcal{F}_z(\xi_n,\xi_{n+1},s)\geq\zeta(\mathcal{F}_z(\xi_n,\xi_{n+1},s),\mathcal{F}_z(\xi_{n-1},\xi_n,s))>\mathcal{F}_z(\xi_{n-1},\xi_n,s)=b_{n-1}(s)$$

applying limit $n \to +\infty$ and characteristic-$\mathcal{S}$,

$$\lim_{n\to+\infty} \mathcal{F}_z(\xi_n, \xi_{n+1}, s) = b(s) = 1.$$

which contradict verifies the claim. Thus for each $r > 0$,

$$\lim_{n\to+\infty} \mathcal{F}_z(\xi_n, \xi_{n+1}, r) = 1. \tag{2.11}$$

Next, we claim that constructed sequence $\{\xi_n\}$ is a Cauchy sequence. Assuming again a contradiction that $\{\xi_n\}$ is not a Cauchy sequence, then for each $\epsilon \in (0, 1)$ and $r > 0$, there exists a $k \in \mathbb{N}$ such that

$$\mathcal{F}_z(\xi_{m_k}, \xi_{n_k}, r) \leq 1 - \epsilon \tag{2.12}$$

for each $m_k > n_k \geq k$. Consider that m_k is the smallest positive integer greater than n_k, satisfying (2.12), so

$$\mathcal{F}_z(\xi_{m_k-1}, \xi_{n_k}, r) > 1 - \epsilon \tag{2.13}$$

for all $k \in \mathbb{N}$. Using (2.3)

$$1 - \epsilon \geq \mathcal{F}_z(\xi_{m_k}, \xi_{n_k}, r) \geq \mathcal{F}_z(\xi_{m_k}, \xi_{m_{k-1}}, r) \diamond \mathcal{F}_z(\xi_{m_{k-1}}, \xi_{n_k}, r) > \mathcal{F}_z(\xi_{m_{k-1}}, \xi_{m_k}, r) \diamond 1 - \epsilon.$$

applying limit $k \to +\infty$,

$$\lim_{k\to+\infty} \mathcal{F}_z(\xi_{m_k}, \xi_{n_k}, r) = 1 - \epsilon. \tag{2.14}$$

Further

$$\mathcal{F}_z(\xi_{m_{k+1}}, \xi_{n_{k+1}}, r) \geq \mathcal{F}_z(\xi_{m_{k+1}}, \xi_{m_k}, r) \diamond \mathcal{F}_z(\xi_{m_k}, \xi_{n_k}, r) \diamond \mathcal{F}_z(\xi_{n_k}, \xi_{n_k}+1, r).$$

applying limit $k \to +\infty$,

$$\lim_{k\to+\infty} \mathcal{F}_z(\xi_{m_k+1}, \xi_{n_k+1}, r) = 1 - \epsilon. \tag{2.15}$$

Using the characteristic-$\mathcal{Q}_1$,

$$\mathcal{F}_z(\xi_{m_k}, \mathcal{H}\xi_{m_{k-1}}, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r) \text{ and } \mathcal{F}_z(\xi_{n_k}, \mathcal{H}\xi_{n_{k-1}}, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r)$$

implies

$$\mathcal{F}_z(\xi_{m_k}, \xi_{n_k}, r) \geq \zeta(\mathcal{F}_z(\xi_{m_k}, \xi_{n_k}, r), \mathcal{F}_z(\xi_{m_{k-1}}, \xi_{n_{k-1}}, r)) > \mathcal{F}_z(\xi_{m_{k-1}}, \xi_{n_{k-1}}, r)$$

applying limit $k \to +\infty$,

$$1 - \epsilon \geq \lim_{k\to+\infty} \zeta(\mathcal{F}_z(\xi_{m_k}, \xi_{n_k}, r), \mathcal{F}_z(\xi_{m_{k-1}}, \xi_{n_{k-1}}, r)) > 1 - \epsilon,$$

which contradict characteristic-$\mathcal{S}$. Thus $\{\xi_n\}$ is a Cauchy sequence. With the completeness of the space $(\mathcal{E}, \mathcal{F}_z, \diamond)$ and $\mathcal{L}$ is a closed subset of $\mathcal{E}$ ensuring sequence $\{\xi_n\}$ converges to some $u \in \mathcal{L}$ for every $r > 0$, that is

$$\lim_{n\to+\infty} \mathcal{F}_z(\xi_n, u, r) = 1.$$

Moreover,

$$\begin{aligned}
\mathcal{F}_z(\mathcal{L}, \mathcal{M}, r) = \mathcal{F}_z(\xi_{n+1}, \mathcal{H}\xi_n, r) &\geq \mathcal{F}_z(\xi_{n+1}, u, r) \diamond \mathcal{F}_z(u, \mathcal{H}\xi_n, r) \\
&\geq \mathcal{F}_z(\xi_{n+1}, u, r) \diamond \mathcal{F}_z(u, \xi_{n+1}, r) \diamond \mathcal{F}_z(\xi_{n+1}, \mathcal{H}\xi_n, r) \\
&\geq \mathcal{F}_z(\xi_{n+1}, u, r) \diamond \mathcal{F}_z(u, \mathcal{H}\xi_n, r) \\
&\geq \mathcal{F}_z(\xi_{n+1}, u, r) \diamond \mathcal{F}_z(u, \xi_{n+1}, r) \diamond \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r)
\end{aligned}$$

applying limit $n \to +\infty$,

$$\mathcal{F}_z(\mathcal{L},\mathcal{M},r) \geq 1 \diamond \lim_{n\to+\infty} \mathcal{F}_z(u,\mathcal{H}\xi_n,r) \geq 1 \diamond 1 \diamond \mathcal{F}_z(\mathcal{L},\mathcal{M},r) \implies \lim_{n\to+\infty} \mathcal{F}_z(u,\mathcal{H}\xi_n,r) \to \mathcal{F}_z(\mathcal{L},\mathcal{M},r)$$

with assertion-4, $u \in \mathcal{L}_0(r)$. Since $\mathcal{H}(\mathcal{L}_0(r)) \subseteq \mathcal{M}_0(r)$ then there exists $v \in \mathcal{L}_0(r)$ such that

$$\mathcal{F}_z(v,\mathcal{H}u,r) = \mathcal{F}_z(\mathcal{L},\mathcal{M},r). \tag{2.16}$$

Combining (2.16) with (2.1),

$$\left.\begin{aligned}\mathcal{F}_z(v,\mathcal{H}u,r) = \mathcal{F}_z(\mathcal{L},\mathcal{M},r)\\ \mathcal{F}_z(\xi_{n+1},\mathcal{H}\xi_n,r) = \mathcal{F}_z(\mathcal{L},\mathcal{M},r)\end{aligned}\right\} \implies \mathcal{F}_z(v,\xi_{n+1},r) \geq \zeta(\mathcal{F}_z(v,\xi_{n+1},r),\mathcal{F}_z(u,\xi_n,r)) > \mathcal{F}_z(u,\xi_n,r)$$

applying limit $n \to +\infty$ and by characteristic-$\mathcal{S}$, for every $r > 0$

$$\lim_{n\to+\infty} \mathcal{F}_z(v,\xi_{n+1},r) = 1.$$

Since limit should be unique for any sequence, so conclude $v = u$, that is

$$\mathcal{F}_z(u,\mathcal{H}u,r) = \mathcal{F}_z(v,\mathcal{H}u,r) = \mathcal{F}_z(\mathcal{L},\mathcal{M},r).$$

Take a contrary, for any $r > 0$, $0 < \mathcal{F}_z(u,w,r) < 1$, and $u \neq w$, w is another best proximity point of $\mathcal{H}$ so that $\mathcal{F}_z(u,\mathcal{H}u,r) = \mathcal{F}_z(\mathcal{L},\mathcal{M},r)$ and $\mathcal{F}_z(w,\mathcal{H}w,r) = \mathcal{F}_z(\mathcal{L},\mathcal{M},r)$ then $\mathcal{F}_z(u,w,r) \geq \zeta(\mathcal{F}_z(u,w,r),\mathcal{F}_z(u,w,r)) > \mathcal{F}_z(u,w,r)$, which is contradiction. Thus best proximity point for non-self map is unique. □

Example 2.8. *A fuzzy set $\mathcal{F}_z : \mathcal{E}\times\mathcal{E}\times(0,+\infty) \to [0,1]$, where $\mathcal{E} = \mathbb{R}$ is defined by*

$$\mathcal{F}_z(\beta,\eta,r) = \left(\frac{r}{r+1}\right)^{d(\beta,\eta)}$$

for every $r > 0$ where d is a usual metric $d(\beta,\eta) = |\beta_1 - \eta_1| + |\beta_2 - \eta_2|$ for all $\beta = (\beta_1,\beta_2)$, $\eta = (\eta_1,\eta_2) \in \mathcal{E}$. Thus $(\mathcal{E},\mathcal{F}_z,\diamond)$ is complete strong fuzzy metric where $\diamond$ is product triangular-norm.
Define

$$\mathcal{L} = \left\{(0,1-\frac{1}{n}) : n \in \mathbb{N}\right\} \cup \{(0,1)\}$$
$$\mathcal{M} = \left\{(1,1-\frac{1}{n}); n \in \mathbb{N}\right\} \cup \{(1,1)\}$$

where $d(\mathcal{L},\mathcal{M}) = 1$. Clearly $\mathcal{L}$ and $\mathcal{M}$ are non-empty closed subsets of $\mathcal{E}$. Define $\mathcal{H} : \mathcal{L} \to \mathcal{M}$ by

$$\mathcal{H}(l,m) = \begin{cases}\left(1,1-\dfrac{1}{n+1}\right); & \text{if } (l,m) = (0,1-\dfrac{1}{n})\\ (1,1); & \text{if } (l,m) = (0,1).\end{cases}$$

Clearly $\mathcal{H}(\mathcal{L}_0(r)) \subseteq \mathcal{M}_0(r)$, $\mathcal{L}_0(r) = \mathcal{L}$ and $\mathcal{M}_0(r) = \mathcal{M}$. Define a function $\zeta : (0,1]\times(0,1] \to \mathbb{R}$ by

$$\zeta(s_1,s_2) = \begin{cases}\dfrac{s_1+s_2}{2}; & \text{if } s_1 > s_2\\ 1 \quad ; & \text{otherwise}\end{cases}$$

Consider $\{\xi_n\} = (0,1-\frac{1}{n}), n \in \mathbb{N}$ such that $\mathcal{F}_z(\xi_{n+1},\mathcal{H}(\xi_n),r) = \mathcal{F}_z(\mathcal{L},\mathcal{M},r)$.

$$\begin{aligned}\mathcal{F}_z(\xi_n,\xi_{n+1},r) &= \mathcal{F}_z\left(\left(0,1-\frac{1}{n}\right),\left(0,1-\frac{1}{n+1}\right),r\right)\\ &= \left(\frac{r}{r+1}\right)^{|1-\frac{1}{n}-1+\frac{1}{n+1}|}\end{aligned}$$

$$= \left(\frac{r}{r+1}\right)^{|\frac{1}{n+1}-\frac{1}{n}|}$$
$$\leq \left(\frac{r}{r+1}\right)^{|\frac{1}{n+2}-\frac{1}{n+1}|}$$
$$= \mathcal{F}_z\left(\left(0, 1-\frac{1}{n+1}\right), \left(0, 1-\frac{1}{n+2}\right), r\right) = \mathcal{F}_z(\xi_{n+1}, \xi_{n+2}, r)$$

implies

$$\lim_{n\to+\infty} \zeta\left(\mathcal{F}_z(\xi_{n+1}, \xi_{n+2}, r), \mathcal{F}_z(\xi_n, \xi_{n+1}, r)\right) = \lim_{n\to+\infty} \zeta\left(\left(\frac{r}{r+1}\right)^{|\frac{1}{n+2}-\frac{1}{n+1}|}, \left(\frac{r}{r+1}\right)^{|\frac{1}{n+1}-\frac{1}{n}|}\right)$$
$$= \lim_{n\to+\infty}\left[\frac{\left(\frac{r}{r+1}\right)^{|\frac{1}{n+2}-\frac{1}{n+1}|} + \left(\frac{r}{r+1}\right)^{|\frac{1}{n+1}+\frac{1}{n}|}}{2}\right] = 1.$$

Then quadruplet $(\mathcal{E}, \mathcal{F}_z, \mathcal{H}, \zeta)$ *where* $\zeta \in \mathcal{Z}$ *has the characteristic-*$\mathcal{S}$*. For any sequence* $\{\xi_n\} = (0, 1-\frac{1}{n})$ *where* $n \in \mathbb{N}$

$$\lim_{n\to\infty} \mathcal{F}_z(\xi_n, \xi_{n+1}, r) = \lim_{n\to\infty} \mathcal{F}_z\left(\left(0, 1-\frac{1}{n}\right), \left(0, 1-\frac{1}{n+1}\right), r\right)$$
$$= \lim_{n\to\infty}\left(\frac{r}{r+1}\right)^{|\frac{1}{n+1}-\frac{1}{n}|} = 1$$

then there exist two subsequences $\xi_{m_k} = \xi_{3k} = \left(0, 1-\frac{1}{3k}\right)$ *and* $\xi_{n_k} = \xi_{2k} = \left(0, 1-\frac{1}{2k}\right)$ *of* ξ_n *where* $3k > 2k > k$, $k \in \mathbb{N}$ *such that*

$$\mathcal{F}_z(\xi_{m_k}, \mathcal{H}\xi_{m_k-1}, r) = \mathcal{F}_z\left(\left(0, 1-\frac{1}{3k}\right), \mathcal{H}\left(0, 1-\frac{1}{3k-1}\right), r\right)$$
$$= \mathcal{F}_z\left(\left(0, 1-\frac{1}{3k}\right), \left(1, 1-\frac{1}{3k}\right), r\right)$$
$$= \frac{r}{r+1} = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r).$$

$$\mathcal{F}_z(\xi_{n_k}, \mathcal{H}\xi_{n_k-1}, r) = \mathcal{F}_z\left(\left(0, 1-\frac{1}{2k}\right), \mathcal{H}\left(0, 1-\frac{1}{2k-1}\right), r\right)$$
$$= \mathcal{F}_z\left(\left(0, 1-\frac{1}{2k}\right), \left(1, 1-\frac{1}{2k}\right), r\right)$$
$$= \frac{r}{r+1} = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r).$$

Then quadruplet $(\mathcal{E}, \mathcal{F}_z, \mathcal{H}, \zeta)$ *has the characteristic-*$\mathcal{Q}_1$*. Assume that,* $\mathcal{F}_z(l, \mathcal{H}m, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r)$ *for some* $l, m \in \mathcal{L}$*. Then*

$$(l, m) = \left\{((0,1),(0,1)), \left(\left(0, 1-\frac{1}{n+1}\right), \left(0, 1-\frac{1}{n}\right)\right)\right\}$$

Now, we may calculate the following cases:

1. If $(l_1, m_1) = \left(\left(0, 1-\frac{1}{n_1+1}\right), \left(0, 1-\frac{1}{n_1}\right)\right)$ *and* $(l_2, m_2) = \left(\left(0, 1-\frac{1}{n_2+1}\right), \left(0, 1-\frac{1}{n_2}\right)\right)$ *for all* $n_1, n_2 \in \mathbb{N}$.

$$\mathcal{F}_z(l_1, \mathcal{H}m_1, r) = \mathcal{F}_z\left(\left(0, 1-\frac{1}{n_1+1}\right), \mathcal{H}\left(0, 1-\frac{1}{n_1}\right), r\right)$$
$$= \mathcal{F}_z\left(\left(0, 1-\frac{1}{n_1+1}\right), \left(1, 1-\frac{1}{n_1+1}\right), r\right)$$

$$
\begin{aligned}
&= \frac{r}{r+1} = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r).\\
\mathcal{F}_z(l_2, \mathcal{H}m_2, r) &= \mathcal{F}_z\left(\left(0, 1-\frac{1}{n_2+1}\right), \mathcal{H}\left(0, 1-\frac{1}{n_2}\right), r\right)\\
&= \mathcal{F}_z\left(\left(0, 1-\frac{1}{n_2+1}\right), \left(1, 1-\frac{1}{n_2+1}\right), r\right)\\
&= \frac{r}{r+1} = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r).
\end{aligned}
$$

implies

$$
\begin{aligned}
&\zeta(\mathcal{F}_z(l_1, l_2, r), \mathcal{F}_z(m_1, m_2, r))\\
=&\left(\mathcal{F}_z\left(\left(0, 1-\frac{1}{n_1+1}\right), \left(0, 1-\frac{1}{n_2+1}\right), r\right), \mathcal{F}_z\left(\left(0, 1-\frac{1}{n_1}\right), \left(0, 1-\frac{1}{n_2}\right), r\right)\right)\\
=&\zeta\left(\left(\frac{r}{r+1}\right)^{\left|\frac{1}{n_2+1}-\frac{1}{n_1+1}\right|}, \left(\frac{r}{r+1}\right)^{\left|\frac{1}{n_2}-\frac{1}{n_1}\right|}\right)\\
=&\left[\frac{\left(\frac{r}{r+1}\right)^{\left|\frac{1}{n_2+1}-\frac{1}{n_1+1}\right|} + \left(\frac{r}{r+1}\right)^{\left|\frac{1}{n_2}-\frac{1}{n_1}\right|}}{2}\right]\\
\leq&\left(\frac{r}{r+1}\right)^{\left|\frac{1}{n_2+1}-\frac{1}{n_1+1}\right|}\\
=&\mathcal{F}_z\left(\left(0, 1-\frac{1}{n_1+1}\right), \left(0, 1-\frac{1}{n_2+1}\right), r\right) = \mathcal{F}_z(l_1, l_2, r)
\end{aligned}
$$

2. If $(l_1, m_1) = ((0,1), (0,1))$ and $(l_2, m_2) = \left(\left(0, 1-\frac{1}{n_2+1}\right), \left(0, 1-\frac{1}{n_2}\right)\right)$ for all $n_1 \in \mathbb{N}$, we have

$$
\begin{aligned}
\mathcal{F}_z(l_1, \mathcal{H}m_2, r) &= \mathcal{F}_z((0,1), \mathcal{H}(0,1), r)\\
&= \mathcal{F}_z((0,1), (1,1), r)\\
&= \left(\frac{r}{r+1}\right) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r).\\
\mathcal{F}_z(l_2, \mathcal{H}m_2, r) &= \mathcal{F}_z\left(\left(0, 1-\frac{1}{n_2+1}\right), \mathcal{H}\left(0, 1-\frac{1}{n_2}\right), r\right)\\
&= \mathcal{F}_z\left(\left(0, 1-\frac{1}{n_2+1}\right), \left(1, 1-\frac{1}{n_2+1}\right), r\right)\\
&= \left(\frac{r}{r+1}\right) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r).
\end{aligned}
$$

implies

$$
\begin{aligned}
\zeta(\mathcal{F}_z(l_1, l_2, r), \mathcal{F}_z(m_1, m_2, r)) = \;& \zeta\left(\left(\frac{r}{r+1}\right)^{\frac{1}{n_2+1}}, \left(\frac{r}{r+1}\right)^{\frac{1}{n_2}}\right)\\
=&\left[\frac{\left(\frac{r}{r+1}\right)^{\frac{1}{n_2+1}} + \left(\frac{r}{r+1}\right)^{\frac{1}{n_2}}}{2}\right]\\
\leq&\left(\frac{r}{r+1}\right)^{\frac{1}{n_2+1}} = \mathcal{F}_z\left((0,1), \left(0, 1-\frac{1}{n_2+1}\right), r\right) = \mathcal{F}_z(l_1, l_2, r).
\end{aligned}
$$

This shows $\mathcal{H}$ is a fuzzy $\mathcal{Z}$- proximal contraction of first kind and we may see graphical behaviour of this inequality by the figure 1 where $r = 2$ and $m, n \in \mathbb{N}$.

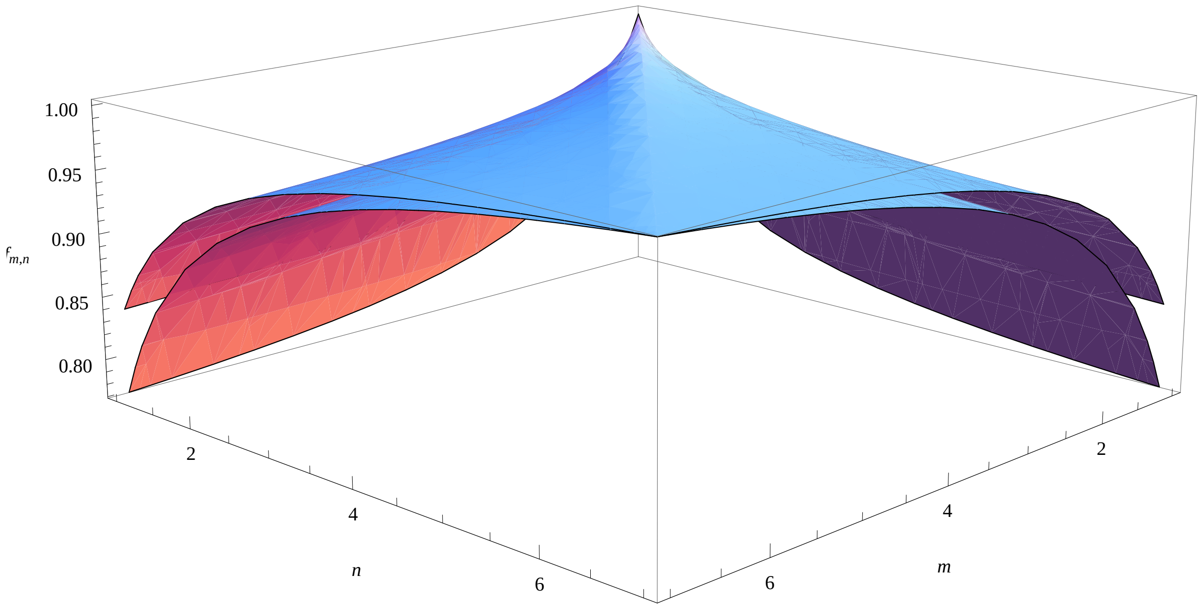

Figure 1:

3. If $(l_1, l_2) = (m_1, m_2) = ((0,1),(0,1))$*, we have*

$$\begin{aligned}\mathcal{F}_z(l_1, \mathcal{H}m_1, r) &= \mathcal{F}_z((0,1), \mathcal{H}(0,1), r)\\ &= \mathcal{F}_z((0,1),(1,1), r)\\ &= \left(\frac{r}{r+1}\right) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r).\\ \mathcal{F}_z(l_2, \mathcal{H}m_2, r) &= \mathcal{F}_z((0,1), \mathcal{H}(0,1), r)\\ &= \mathcal{F}_z((0,1),(1,1), r)\\ &= \left(\frac{r}{r+1}\right) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r).\end{aligned}$$

implies

$$\begin{aligned}\zeta(\mathcal{F}_z(l_1, l_2, r), \mathcal{F}_z(m_1, m_2, r)) = \ & \zeta\left(\left(\frac{r}{r+1}\right)^0, \left(\frac{r}{r+1}\right)^0\right)\\ &= 1 = \mathcal{F}_z((0,1),(0,1), t) = \mathcal{F}_z(l_1, l_2, r).\end{aligned}$$

Thus $\mathcal{H}$ *is fuzzy* $\mathcal{Z}$*-Proximal contraction of first kind. Thus, we conclude that all the assertions of Theorem (2.7) are hold so, there exists unique* $(0,1) \in \mathcal{L}$ *such that* $\mathcal{F}_z((0,1), \mathcal{H}(0,1), r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r)$ *for all* $r > 0$*.*

Theorem 2.9. *Assume that two non-empty closed subsets* $\mathcal{L}$ *and* $\mathcal{M}$ *of a complete strong fuzzy metric space* $(\mathcal{E}, \mathcal{F}_z, \diamond)$ *with* $\mathcal{L}_0(r) \neq \phi$ *and* $\mathcal{H} : \mathcal{L} \to \mathcal{M}$ *satisfying:*

1. a continuous map $\mathcal{H}$ *is a fuzzy* $\mathcal{Z}$*- proximal contractive of kind first with* $\mathcal{H}(\mathcal{L}_0(r)) \subseteq \mathcal{M}_0(r)$*;*

2. The quadruplet $(\mathcal{E}, \mathcal{F}_z, \mathcal{H}, \zeta)$ *has characteristic-*$\mathcal{S}$*;*
Then there exists a unique member u in $\mathcal{L}$ *such that* $\mathcal{F}_z(u, \mathcal{H}u, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r)$ *for all* $r > 0$*.*

Proof. Sequence construction is $\{\xi_n\}$ is similar to that in Theorem (2.2). We must show the sequence is Cauchy sequence, proof of that part is similar to the Theorem (2.7). The completeness property of the space $(\mathcal{E}, \mathcal{F}_z, \diamond)$ and $\mathcal{L}$ being a closed subset of $\mathcal{E}$, then there exists $u \in \mathcal{L}$ such that $\lim_{n\to+\infty} \xi_n = u$. Since $\mathcal{H}$ is continuous, $\mathcal{H}\xi_n \to \mathcal{H}u$ and the continuity of $\mathcal{F}_z$ implies $\mathcal{F}_z(\xi_{n+1}, \mathcal{H}\xi_n, r) \to \mathcal{F}_z(u, \mathcal{H}u, r)$. From (2.1)

$$\mathcal{F}_z(u, \mathcal{H}u, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r).$$

Uniqueness part is similar to Theorem (2.7). □

Now, we shall define characteristic-$\mathcal{Q}_2$ which is essential to prove the result for fuzzy $\mathcal{Z}$-proximal contraction of the second kind.

Definition 2.10. *The quadruplet $(\mathcal{E}, \mathcal{F}_z, \mathcal{H}, \zeta)$ has characteristic-$\mathcal{Q}_2$, if for any sequence $\{\xi_n\}$ in $\mathcal{L}_0$ satisfying $\mathcal{F}_z(\xi_{n+1}, \mathcal{H}\xi_n, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r)$ for every $r > 0$ such that*

$$\mathcal{F}_z(\mathcal{H}\xi_{n+1}, \mathcal{H}\xi_{n+2}, r) \geq \mathcal{F}_z(\mathcal{H}\xi_n, \mathcal{H}\xi_{n+1}, r)$$

implies

$$\lim_{n\to+\infty} \zeta(\mathcal{F}_z(\mathcal{H}\xi_{n+1}, \mathcal{H}\xi_{n+2}, r), \mathcal{F}_z(\mathcal{H}\xi_n, \mathcal{H}\xi_{n+1}, r)) = 1.$$

Theorem 2.11. *Assume that two non-empty closed subsets $\mathcal{L}$ and $\mathcal{M}$ of a complete strong fuzzy metric space $(\mathcal{E}, \mathcal{F}_z, \diamond)$ with $\mathcal{L}_0(r) \neq \phi$. Assume $\mathcal{L}$ is approximately compact with respect to $\mathcal{M}$ and $\mathcal{H} : \mathcal{L} \to \mathcal{M}$ satisfying:*

1. *a continuous map $\mathcal{H}$ is a fuzzy $\mathcal{Z}$- proximal contractive of kind second with $\mathcal{H}(\mathcal{L}_0(r)) \subseteq \mathcal{M}_0(r)$;*
2. *The quadruplet $(\mathcal{E}, \mathcal{F}_z, \mathcal{H}, \zeta)$ has the characteristic-$\mathcal{Q}_2$.*

Then there exists a unique member u in $\mathcal{L}$ such that $\mathcal{F}_z(u, \mathcal{H}u, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r)$ for every $r > 0$.

Proof. Define $b_n(r) = \mathcal{F}_z(\mathcal{H}\xi_n, \mathcal{H}\xi_{n+1}, r)$ for all $n \in \mathbb{N} \cup \{0\}$ and $r > 0$. From (2.10), definition of ζ and characteristic-$\mathcal{Q}_2$, we may

$$\begin{aligned} b_n(r) = \mathcal{F}_z(\mathcal{H}\xi_n, \mathcal{H}\xi_{n+1}, r) &\geq \zeta(\mathcal{F}_z(\mathcal{H}\xi_n, \mathcal{H}\xi_{n+1}, r), \mathcal{F}_z(\mathcal{H}\xi_{n-1}, \mathcal{H}\xi_n, r) \\ &> \mathcal{F}_z(\mathcal{H}\xi_{n-1}, \mathcal{H}\xi_n, r) = b_{n-1}(r) \end{aligned} \tag{2.17}$$

for every $r > 0$. This implies $\{b_n(r)\}$ is increasing sequence must be convergences to some $b(r)$ as $n \to +\infty$ for every $r > 0$, that is

$$\lim_{n\to+\infty} b_n(r) = b(r).$$

To show $b(r) = 1$ for every $r > 0$. Instead, let us take $0 < b(r_0) < 1$ for some $r_0 > 0$ and then by (2.10)

$$\begin{aligned} b_n(r_0) = \mathcal{F}_z(\mathcal{H}\xi_n, \mathcal{H}\xi_{n+1}, r_0) &\geq \zeta(\mathcal{F}_z(\mathcal{H}\xi_n, \mathcal{H}\xi_{n+1}, r_0), \mathcal{F}_z(\mathcal{H}\xi_{n-1}, \mathcal{H}\xi_n, r_0)) \\ &> \mathcal{F}_z(\mathcal{H}\xi_{n-1}, \mathcal{H}\xi_n, r_0) = b_{n-1}(r_0) \end{aligned} \tag{2.18}$$

applying limit $n \to +\infty$ in equation (2.17),

$$\lim_{n\to+\infty} \mathcal{F}_z(\mathcal{H}\xi_n, \mathcal{H}\xi_{n+1}, r_0) = 1,$$

which contradict our assumption, so $b(r_0) = 1$ for every $r_0 > 0$. For next, with the similar process of Theorem (2.7), we can receive that $\{\mathcal{H}\xi_n\}$ is a Cauchy sequence in $\mathcal{M}$. Since the space $(\mathcal{E}, \mathcal{F}_z, \diamond)$ is complete and $\mathcal{M}$ is a closed subset of $\mathcal{E}$, there exists $z \in \mathcal{M}$ such that $\lim_{n\to+\infty} \mathcal{H}\xi_n = z$.
Furthermore,

$$\begin{aligned} \mathcal{F}_z(z, \mathcal{L}, r) &\geq \mathcal{F}_z(z, \xi_{n+1}, r) \\ &\geq \mathcal{F}_z(z, \mathcal{H}\xi_n, r) \diamond \mathcal{F}_z(\mathcal{H}\xi_n, \xi_{n+1}, r) \\ &= \mathcal{F}_z(z, \mathcal{H}\xi_n, r) \diamond \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r) \\ &\geq \mathcal{F}_z(z, \mathcal{H}\xi_n, r) \diamond \mathcal{F}_z(z, \mathcal{L}, r) \end{aligned}$$

applying limit as $n \to +\infty$,

$$\lim_{n\to+\infty} \mathcal{F}_z(z, \mathcal{H}\xi_n, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r)$$

Since $\mathcal{L}$ is approximately compact with respect to $\mathcal{M}$, there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ converging to element u in $\mathcal{L}$. Therefore

$$\mathcal{F}_z(u, z, r) = \lim_{k\to+\infty} \mathcal{F}_z(\xi_{n_k}, \mathcal{H}\xi_{n_k-1}, r) = \mathcal{F}_z(z, \mathcal{L}, r).$$

This implies $u \in \mathcal{L}_0(r)$, since $\lim_{k\to+\infty} \xi_{n_k} = u$. By continuity of $\mathcal{H}$ and z is convergent point of $\{\mathcal{H}\xi_n\}$, It means

$$\lim_{k\to+\infty} \mathcal{H}\xi_{n_k} = \mathcal{H}u = z.$$

Therefore

$$\mathcal{F}_z(u, \mathcal{H}u, r) = \lim_{k\to+\infty} \mathcal{F}_z(\xi_{n_k}, \mathcal{H}\xi_{n_k}, r) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r).$$

□

Example 2.12. *Consider a fuzzy set* $\mathcal{F}_z : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to (0, 1]$ *defines by*

$$\mathcal{F}_z(\beta, \gamma, r) = \frac{r}{r + d(\beta, \gamma)}$$

where d is usual metric on $\mathbb{R}$. *Then* $(\mathbb{R}, \mathcal{F}_z, \diamond)$ *is an complete strong fuzzy metric space where* $\diamond$ *is product triangular-norm. A function* $\zeta : (0, 1] \times (0, 1] \to \mathbb{R}$ *defined by* $\zeta(s_1, s_2) = \psi(s_2)$ *where* $\psi : (0, 1] \to (0, 1]$ *such that* $s < \psi(s)$ *for all* $s \in (0, 1)$ *and* $\psi(1) = 1$. *Consider*

$$\mathcal{L} = \{(\mu, -1) : \mu \in \mathbb{R}^+\} \ \text{and} \ \mathcal{M} = \{(\nu, 1) : \nu \in \mathbb{R}^+\}$$

A map $\mathcal{H} : \mathcal{L} \to \mathcal{M}$ *defined by, for each* $(\mu, -1) \in \mathcal{L}$

$$\mathcal{H}(\mu, -1) = \left(\frac{\mu}{\mu + 1}, 1\right),$$

where $\mathcal{H}(\mathcal{L}_0) \subseteq \mathcal{M}_0$ *and* $\mathcal{L}_0(r) = \mathcal{L}$ *and* $\mathcal{M}_0(r) = \mathcal{M}$. *Consider* $\{\xi_n\} = (\frac{1}{n_1}, -1)$ *for every* $n_1 \in \mathbb{N}$,

$$\mathcal{H}(\xi_n) = \mathcal{H}(\frac{1}{n_1}, -1) = \left(\frac{\frac{1}{n_1}}{\frac{1}{n_1} + 1}, 1\right) = \left(\frac{1}{n_1 + 1}, 1\right).$$

Now,

$$\begin{aligned}
\mathcal{F}_z(\xi_{n+1}, \mathcal{H}\xi_n, r) &= \mathcal{F}_z\left(\left(\frac{1}{n_1 + 1}, -1\right), \mathcal{H}\left(\frac{1}{n_1}, -1\right), r\right) \\
&= \mathcal{F}_z\left(\left(\frac{1}{n_1 + 1}, -1\right), \left(\frac{1}{n_1 + 1}, 1\right), r\right) \\
&= \frac{r}{r + |\frac{1}{n_1+1} - \frac{1}{n_1+1}| + 2} \\
&= \frac{r}{r + 2} \\
&= \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r).
\end{aligned}$$

Now, $\{\xi_n\}$ and $\{\xi_{n+1}\}$ be any sequences such that

$$\begin{aligned}\mathcal{F}_z(\mathcal{H}\xi_n, \mathcal{H}\xi_{n+1}, r) &= \mathcal{F}_z\left(\mathcal{H}\left(\frac{1}{n_1}, -1\right), \mathcal{H}\left(\frac{1}{n_1+1}, -1\right), r\right)\\ &= \mathcal{F}_z\left(\left(\frac{1}{n_1+1}, 1\right), \left(\frac{1}{n_1+2}, 1\right), r\right)\\ &= \frac{r}{r + |\frac{1}{n_1+1} - \frac{1}{n_1+2}|}\\ &\leq \frac{r}{r + |\frac{1}{n_1+2} - \frac{1}{n_1+3}|} = \mathcal{F}_z(\mathcal{H}\xi_{n+1}, \mathcal{H}\xi_{n+2}, r)\end{aligned}$$

implies

$$\lim_{n\to+\infty} \zeta\left(\frac{r}{r + |\frac{1}{n_1+1} - \frac{1}{n_1+2}|}, \frac{r}{r + |\frac{1}{n_1} - \frac{1}{n_1+1}|}\right) = \lim_{n\to+\infty} \psi\left(\frac{r}{r + |\frac{1}{n_1} - \frac{1}{n_1+1}|}\right) = 1.$$

Thus characteristic-$\mathcal{Q}_2$ holds. Consider for some $u_1, u_2, x_1, x_2 \in \mathcal{L}$ such that $u_1 = \left(\frac{\mu_1}{\mu_1+1}, -1\right), u_2 = \left(\frac{\mu_2}{\mu_2+1}, -1\right), x_1 = (\mu_1, -1), x_2 = (\mu_2, -1)$, we get

$$\begin{aligned}\mathcal{F}_z(u_1, \mathcal{H}x_1, r) &= \mathcal{F}_z\left(\left(\frac{\mu_1}{\mu_1+1}, -1\right), \mathcal{H}(\mu_1, -1), r\right)\\ &= \mathcal{F}_z\left(\left(\frac{\mu_1}{\mu_1+1}, -1\right), \left(\frac{\mu_1}{\mu_1+1}, 1\right), r\right)\\ &= \frac{r}{r+2} = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r).\\ \mathcal{F}_z(u_2, \mathcal{H}x_2, r) &= \mathcal{F}_z\left(\left(\frac{\mu_2}{\mu_2+1}, -1\right), \mathcal{H}(\mu_2, -1), r\right)\\ &= \mathcal{F}_z\left(\left(\frac{\mu_2}{\mu_2+1}, -1\right), \left(\frac{\mu_2}{\mu_2+1}, 1\right), r\right)\\ &= \frac{r}{r+2} = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r).\end{aligned}$$

Now, By the fuzzy $\mathcal{Z}$-proximal contraction of second kind,

$$\begin{aligned}\mathcal{F}_z(\mathcal{H}u_1, \mathcal{H}u_2, r) &= \mathcal{F}_z\left(\mathcal{H}\left(\frac{\mu_1}{\mu_1+1}, -1\right), \mathcal{H}\left(\frac{\mu_2}{\mu_2+1}, -1\right), r\right)\\ &= \mathcal{F}_z\left(\left(\frac{\mu_1}{2\mu_1+1}, 1\right), \left(\frac{\mu_2}{2\mu_2+1}, 1\right), r\right)\\ &= \frac{r}{r + |\frac{\mu_1}{2\mu_1+1} - \frac{\mu_2}{2\mu_2+1}|}\\ &\geq \zeta\left(\frac{r}{r + |\frac{\mu_1}{2\mu_1+1} - \frac{\mu_2}{2\mu_2+1}|}, \frac{r}{r + |\frac{\mu_1}{\mu_1+1} - \frac{\mu_2}{\mu_2+1}|}\right)\\ &= \psi\left(\frac{r}{r + |\frac{\mu_1}{\mu_1+1} - \frac{\mu_2}{\mu_2+1}|}\right)\\ &> \frac{r}{r + |\frac{\mu_1}{\mu_1+1} - \frac{\mu_2}{\mu_2+1}|}\\ &= \mathcal{F}_z(\mathcal{H}x_1, \mathcal{H}x_2, r).\end{aligned}$$

Hence $\mathcal{H}$ is fuzzy $\mathcal{Z}$-proximal contraction of second kind, so there exists a unique $(0, -1) \in \mathcal{L}$ such that $\mathcal{F}_z((0, -1), \mathcal{H}(0, -1), r)) = \mathcal{F}_z(\mathcal{L}, \mathcal{M}, r)$ for all $r > 0$.

3. Application

In this section, we solve the following boundary value problem. Consider a problem

$$-\frac{d^2w}{dv^2} = h(v, w(v)), v \in [0,1] \tag{3.1}$$

with $w(0) = w(1) = 0$, boundary values, where function $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous. The problem (3.1) is equivalent to the following integral equation

$$w(v) = \int_0^1 \mathcal{G}(v,s) \cdot h(s, w(s))ds, \tag{3.2}$$

where $v \in [0,1]$, $\mathcal{G}(v,s)$ is a green function defined as

$$\mathcal{G}(v,s) = \begin{cases} s(1-v); 0 \leq s < v \\ v(1-s); v < s \leq 1. \end{cases}$$

Consider $\mathcal{E} = C([0,1], \mathbb{R})$ is a set of all positive continuous real-valued functions defined in $[0,1]$. Define a fuzzy set $\mathcal{F}_z : \mathcal{E}^2 \times (0,+\infty) \to [0,1]$ such as $\mathcal{F}_z(w_1(v), w_2(v), r) = \dfrac{r}{r + d(w_1,w_2)}$ for every $r > 0$, where $d(w_1(v), w_2(v)) = \max_{v\in[0,1]} |w_1(v) - w_2(v)|$ for every $w_1, w_2 \in \mathcal{E}$. Then $(\mathcal{E}, \mathcal{F}_z, \diamond)$ is a complete strong fuzzy metric spaces where $\diamond$ is a continuous triangular-norm.
Consider $\zeta \in \mathcal{Z}$ such that

$$\zeta(s_1, s_2) = \begin{cases} \dfrac{s_1 + s_2}{2}, & if \ s_1 > s_2, \\ 1, & otherwise. \end{cases}$$

Now we insert an existence solution to find a unique solution of a non-linear ordinary differential equation of second order.

Theorem 3.1. *Suppose that*

1. *for every* $w_1(v), w_2(v) \in [0,1]$, $\lambda > 1$ *and* $v \in [0,1]$

$$|h(v, w_1(v)) - h(v, w_2(v))| \leq \frac{1}{\lambda}|w_1(v) - w_2(v)|$$

2. *a sequence* $\{w_n(v)\}$ *in* $\mathcal{E}$ *is non-increasing and convergent in* $(\mathcal{E}, d)$, *for each* $v \in [0,1]$.

Then the integral equation (3.2) has a unique solution in $\mathcal{E}$.

Proof. Define a mapping $\mathcal{H} : \mathcal{E} \to \mathcal{E}$ as

$$\mathcal{H}(w(v)) = \int_0^1 G(v,s) \cdot h(s, w(s))ds \text{ where } v, s \in [0,1].$$

Take

$$\begin{aligned} d(\mathcal{H}w_1, \mathcal{H}w_2) &= \max_{v\in[0,1]} \left|\mathcal{H}w_1(v) - \mathcal{H}w_2(v)\right| \\ &= \max_{v\in[0,1]} \left|\int_0^1 G(v,s) \cdot h(s, w_1(s))ds - \int_0^1 G(v,s) \cdot h(s, w_2(s))ds\right| \\ &\leq \max_{v\in[0,1]} \int_0^1 G(v,s)ds \cdot |h(s, w_1(s)) - h(s, w_2(s))| \\ &\leq \max_{v\in[0,1]} \int_0^1 G(v,s)ds \cdot \max_{s\in[0,1]} \left|\frac{w_1(s) - w_2(s)}{\lambda}\right| \end{aligned}$$

$$= \frac{1}{\lambda} d(w_1, w_2) \cdot \max_{v \in [0,1]} \int_0^1 G(v,s) ds.$$

We may verify $\int_0^1 G(v,s)ds = \frac{v}{2} - \frac{v^2}{2}$ and $\max_{v \in [0,1]} \int_0^1 G(v,s)ds = \frac{1}{8}$. This implies

$$d(\mathcal{H}w_1, \mathcal{H}w_2) \leq \frac{1}{8\lambda} d(w_1, w_2).$$

Now

$$\begin{aligned}
\zeta(\mathcal{F}_z(\mathcal{H}w_1(v), \mathcal{H}w_2(v), r), \mathcal{F}_z(w_1(v), w_2(v), r)) &= \zeta\left(\frac{8r}{8r + \frac{d(w_1(v), w_2(v))}{\lambda}}, \frac{r}{r + d(w_1(v), w_2(v))}\right) \\
&= \left[\frac{\frac{8r}{8r + \frac{d(w_1(v), w_2(v))}{\lambda}} + \frac{r}{r + d(w_1(v), w_2(v))}}{2}\right] \\
&\leq \frac{8r}{8r + \frac{d(w_1(v), w_2(v))}{\lambda}} = \mathcal{F}_z(\mathcal{H}w_1(v), \mathcal{H}w_2(v), r).
\end{aligned}$$

Thus $\mathcal{H}$ is fuzzy $\mathcal{Z}$-proximal contraction. If $\{w_n(v)\}$ be any sequence in $\mathcal{E}$ and using assumption (2), we have

$$\begin{aligned}
\mathcal{F}_z(w_{n+1}(v), w_{n+2}(v), r) &= \left(\frac{r}{r + d(w_{n+1}(v), w_{n+2}(v))}\right) \\
&\geq \left(\frac{r}{r + d(w_n(v), w_{n+1}(v))}\right) = \mathcal{F}_z(w_n(v), w_{n+1}(v), r)
\end{aligned}$$

implies

$$\begin{aligned}
&\lim_{n \to +\infty} \zeta\left(\mathcal{F}_z(w_{n+1}(v), w_{n+2}(v), r), \mathcal{F}_z(w_n(v), w_{n+1}(v), r)\right) \\
&= \lim_{n \to +\infty} \zeta\left(\frac{r}{r + d(w_{n+1}(v), w_{n+2}(v))}, \frac{r}{r + d(w_n(v), w_{n+1}(v))}\right) \\
&= \lim_{n \to +\infty} \left[\frac{\frac{8r}{8r + \frac{d(w_n(v), w_{n+1}(v))}{\lambda}} + \frac{r}{r + d(w_n(v), w_{n+1}(v))}}{2}\right] = 1.
\end{aligned}$$

The quadruplet $(\mathcal{E}, \mathcal{F}_z, \mathcal{H}, \zeta)$ satisfies the characteristic-$\mathcal{S}$. Hence, the mapping $\mathcal{H}$ satisfy all the conditions of Corollary (2.3). Thus integral equation (3.2) has a unique solution in $\mathcal{E}$. □

4. Conclusion

This note introduces two distinct categories of fuzzy $\mathcal{Z}$-proximal contractions which serve as a tool for finding the best proximity point for a non-self mapping defined between two non-empty subsets of a strong fuzzy metric space. To demonstrate the validity of the proposed results, a few validation examples are provided. Additionally, the note includes a solution to a non-linear second-order ordinary differential equation by employing the fuzzy $\mathcal{Z}$-proximal contractive inequality, assuming the space is a strong fuzzy metric space. We will also able to unify the various fuzzy proximal contractive mappings with the help of our results with regards to best proximity. This idea can be extend in different ways for more than one non-self mappings with weaker relativity in the context of generalized fuzzy spaces, b-fuzzy metric spaces, fuzzy metric-like spaces, PM spaces, etc.

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