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A characterization of two-weighted inequalities for singular operators and their commutators in generalized weighted Morrey spaces on spaces of homogeneous type

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Abstract. In this paper we give a characterization of two-weighted inequalities for singular operators and their commutators in generalized weighted Morrey spaces on spaces of homogeneous type $\mathcal{M}^{p,\varphi}_{\omega}(X)$. We prove the boundedness of the Calderón-Zygmund singular operators T and their commutators $[b, T]$ from the spaces $\mathcal{M}^{p,\varphi_1}_{\omega_1^{\delta}}(X)$ to the spaces $\mathcal{M}^{p,\varphi_2}_{\omega_2^{\delta}}(X)$, where $1 < p < \infty$, $0 < \delta < 1$ and $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$. Finally we give some examples for singular integral operators on $\mathcal{M}^{p,\varphi}_{\omega}(X)$ as applications of our results.

1. Introduction

As a generalization of Lebesgue spaces, the classical Morrey spaces were introduced by Charles Morrey [33] in 1938 to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [17, 32, 35] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [18, 19, 40]).

Recently, Komori and Shirai [29] defined the weighted Morrey spaces $L^{p,\kappa}_w(\mathbb{R}^n)$ and studied the boundedness of some classical operators such as the Hardy- Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Also, Guliyev in [20] first introduced the generalized weighted Morrey spaces $M^{p,\varphi}_w(\mathbb{R}^n)$ and studied the boundedness of the sublinear operators and their higher order commutators generated by Calderón-Zygmund operators and Riesz potentials in these spaces (see, also [23], [27]). Note that, Guliyev [20] gave the concept of generalized weighted Morrey space which could be viewed as an extension of both $M^{p,\varphi}_w(\mathbb{R}^n)$ and $L^{p,\kappa}_w(\mathbb{R}^n)$.

R. Coifman and G. Weiss introduced certain topological measure spaces which are equipped with a metric which is compatible with the given measure in a sense in the 1970s. These spaces are called spaces of homogeneous type. In this work, we consider $\mathcal{M}^{p,\varphi}_{\omega}(X)$ generalized weighted Morrey spaces on spaces

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of homogeneous type and give a characterization of two-weighted inequalities for singular operators and their commutators in these spaces. Moreover, we give some examples for singular integral operators on $\mathcal{M}_{\omega}^{p,\varphi}(X)$.

2. Preliminaries

We say that $X = (X, d, \mu)$ is a space of homogeneous type in the sense of Coifman and Weiss [9] if d is a quasi-metric on X and μ is a positive measure satisfying the doubling condition, i.e. X is a topological space endowed with a quasi-metric d and a positive measure μ such that

$$\begin{aligned} d(x,y) &= d(y,x) \geq 0 \text{ for all } x, y \in X, \\ d(x,y) &= 0 \text{ if and only if } x = y\,, \\ d(x,y) &\leq C_k\left[d(x,z) + d(z,y)\right] \text{ for all } x, y, z \in X, \end{aligned}$$

the balls $B(x,r) = \{y \in X : d(x,y) < r\}$, $r > 0$, form a basis of neighborhoods of point x, μ is defined on a σ-algebra of subsets of X which contains the balls, and

$$0 < \mu(B(x,2r)) < C_{\mu}\, \mu(B(x,r)) < \infty, \tag{1}$$

where $C_k, C_{\mu} \geq 1$ are constants independent of $x, y, z \in X$ and $r > 0$. As usual, the dilation of a ball $B = B(x,r)$ will be denoted by $\lambda B = B(x, \lambda r)$ for every $\lambda > 0$.

Note that, we say (X, d, μ) satisfies a reverse doubling condition if there exist $C'_{\mu} > 1, M > 1$ such that for all $x \in X$, $r > r_x = \sup\{r > 0 : B(x,r) = \{x\}\}$,

$$\mu(B(x, Mr)) \geq C'_{\mu}\, \mu(B(x,r)). \tag{2}$$

Let (X, d, μ) be a homogeneous space, $1 \leq p < \infty$, φ be a positive measurable function on $(0, \infty)$ and ω be a non-negative measurable function on X. We denote by $\mathcal{M}_{\omega}^{p,\varphi}$ the generalized weighted Morrey space on spaces of homogeneous type, the space of all functions $f \in L_{p,\omega}^{loc}(X)$ with finite norm

$$\|f\|_{\mathcal{M}_{\omega}^{p,\varphi}} = \sup_{x \in X, r > 0} \frac{1}{\varphi(x,r)\|\omega\|_{L_p(B(x,r))}} \|f\|_{L_{p,\omega}(B(x,r))},$$

where the supremum is taken over all balls $B(x,r)$ in X and $L_{p,\omega}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,\omega}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,\omega}(X)} = \left(\int_{B(x,r)} |f(y)|^p \omega(y) d\mu(y) \right)^{\frac{1}{p}}.$$

Moreover, by $W\mathcal{M}_{\omega}^{p,\varphi}$ we denote the weak generalized weighted Morrey space on spaces of homogeneous type of all functions $f \in WL_{p,\omega}^{loc}(X)$ with finite norm

$$\|f\|_{W\mathcal{M}_{\omega}^{p,\varphi}} = \sup_{x \in X, r > 0} \frac{1}{\varphi(x,r)\|\omega\|_{L_p(B(x,r))}} \|f\|_{WL_{p,\omega}(B(x,r))},$$

where $WL_{p,\omega}(B(x,r))$ denotes the weak weighted L_p-space of measurable functions f for which

$$\|f\|_{WL_{p,\omega}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,\omega}(X)} = \sup_{t>0} t \left(\int_{\{y \in B(x,r) : |f(y)| > t\}} |f(y)|^p \omega(y) d\mu(y) \right)^{\frac{1}{p}}.$$

Note that if $\omega(x) = \chi_{B(x,r)}$, then $\mathcal{M}^{p,\varphi}_{\omega}(X) = \mathcal{M}^{p,\varphi}(X)$ is the generalized Morrey space and if $\varphi(x,r) = \left(\frac{r^{\lambda}}{\mu(B(x,r))}\right)^{\frac{1}{p}}$, then $\mathcal{M}^{p,\varphi}_{\omega}(X) = L_{p,\lambda}(X)$ is the classic Morrey space.

We now recall the definition of Hardy-Littlewood and Calderón-Zygmund operators on space of homogeneous type.

Let f be a locally integrable function on X. The so-called of Hardy-Littlewood maximal function is defined by the formula

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| d\mu(y),$$

where $\mu(B(x,r))$ is measure of the ball $B(x,r)$.

The study of maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators.

Calderón-Zygmund type singular operator is defined as

$$Tf(x) = \int_X K(x,y) f(y) d\mu(y) \text{ for a.e. } x \notin supp f,$$

where $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y) \in X \times X : x \neq y\}$ and satisfying the estimates: for all $x \neq y$,

$$|K(x,y)| \leq \frac{C}{\mu(B(x, 2d(x,y)))}, \tag{3}$$

and for all $M > 1$, $r > 0$, $x_0 \in X$, $x \in B(x_0, r)$, $y \notin B(x_0, Mr)$

$$|K(x_0,y) - K(x,y)| \leq \frac{C}{\mu(B(x_0, 2d(x_0,y)))} \frac{d(x_0,x)^{\beta}}{d(x_0,y)^{\beta}}, \ \beta > 0. \tag{4}$$

Let

$$T^* f(x) = \sup_{\varepsilon>0} |T_{\varepsilon} f(x)|$$

be the maximal singular operator, where $T_{\varepsilon} f(x)$ is the usual truncation

$$T_{\varepsilon} f(x) = \int_{\{y \in X : d(x,y) \geq \varepsilon\}} K(x,y) f(y) d\mu(y).$$

It is well known that $T^* f$ exists almost everywhere whenever f is a step function. The almost everywhere existence of the limit (of certain integral averages) was known for dense subset of L_1 and the result was extended to all of L_1 by establishing control over the corresponding maximal operator.

Theorem 2.1. *[4, 31] Let d be a quasi-metric on a set X. Then there exists a quasi-metric ϑ on X such that:*
1) d and ϑ are equivalent, that is there exist constants C_1, C_2 such that for every $x, y \in X$

$$C_1 \vartheta(x,y) \leq d(x,y) \leq C_2 \vartheta(x,y),$$

2) ϑ is "locally Hölder continous", more precisely there exist $\alpha \in (0,1]$ and $C_3 > 0$ such that for every $x, y, z \in X$

$$\left| \vartheta(x,z) - \vartheta(y,z) \right| \leq C_3 \vartheta(x,y)^{\alpha} \left[\vartheta(x,z) + \vartheta(y,z) \right]^{1-\alpha}.$$

By Theorem 2.1, we can endow X with a quasi-metric ϑ equivalent to d and locally Hölder of exponent α. Consider the functions $\Psi_j : [0, \infty) \longrightarrow [0, 1]$ $(j = 1, 2)$ defined by

$$\Psi_1(t) = \begin{cases} 0 & , \quad t \le 1 \\ t-1 & , \quad 1 < t < 2 \\ 1 & , \quad t \ge 2. \end{cases}$$

$$\Psi_2(t) = \begin{cases} 1 & , \quad t \le 2 \\ 3-t & , \quad 2 < t < 3 \\ 0 & , \quad t \ge 3. \end{cases}$$

For any $\epsilon \in (0, 1)$ set

$$\Psi_\epsilon(t) = \begin{cases} \Psi_1\left(\frac{t}{\epsilon}\right) & , \quad 0 \le t \le 2 \\ \\ \Psi_2(\epsilon t) & , \quad t \ge 2 \end{cases}$$

$$K_\epsilon(x, y) = K(x, y)\Psi_\epsilon(\vartheta(x, y)).$$

Theorem 2.2. *[4] Let (X, d, μ) be a homogeneous space, μ is a regular measure. Let $K : X \times X \backslash \{x \neq y\} \longrightarrow \mathbb{R}$ a kernel satisfying the following conditions:*

1) The condition (3).

2) The condition (4).

3) There exists $C_r > 0$ such that for every r, R, $0 < r < R < \infty$, a.e. x

$$\left| \int_{r<d(x,y)<R} K(x, y)d\mu(y) \right| \le C_r. \tag{5}$$

Theorem 2.3. *[4] Let (X, d, μ) be a homogeneous space such that μ is a regular measure and, if X is unbounded, the reverse doubling condition* (2) *holds. Let K be a kernel satisfying all the assumptions of Theorem 2.2. Moreover assume that for a.e. $x \in X$ there exists*

$$\lim_{\epsilon \longrightarrow 0} \int_{d(x,y)<1} K_\epsilon(x, y)d\mu(y) \equiv b(x). \tag{6}$$

Throughout this paper we always assume that $\mu(X) = \infty$, the space of compactly supported continuous function is dense in $L_1(X, \mu)$ and that X is N-homogeneous ($N > 0$), i.e.

$$C_1 r^N \le \mu(B(x, r)) \le C_2 r^N, \tag{7}$$

where $C_i \ge 1$ $(i = 1, 2)$ are constants independent of x and r.

Now, conditions 3 and 4 can be rewritten respectively as: for all $x \neq y$

$$|K(x, y)| \le \frac{C}{d(x, y)^N},$$

and for all $M > 1$, $x_0 \in X$, $x \in B(x_0, r)$ and $y \notin B(x_0, Mr)$

$$|K(x_0, y) - K(x, y)| \leq C \frac{d(x_0, x)^{\beta}}{d(x_0, y)^{\beta+N}}, \ \beta > 0. \tag{8}$$

In this paper we aim to give a characterization of two-weighted inequalities for singular operators and their commutators in generalized weighted Morrey spaces on spaces of homogeneous type. Two-weight norm inequalities for fractional maximal operators and singular integrals on Lebesgue spaces were widely studied (see, for example [10–12, 28, 30]). The weighted norm inequalities with different types of weights on Morrey spaces were also studied (see, for example [24, 36, 39]). The two-weight norm inequality for the Hardy-Littlewood maximal function on Morrey spaces was obtained in [42]. Two-weight norm inequalities on weighted Morrey spaces for fractional maximal operators and fractional integral operators were obtained in [38]. Also, two-weighted inequalities for maximal operator and its commutators in generalized weighted Morrey spaces on spaces of homogeneous type were studied in [2] and [3], respectively.

In the sequel we use the letter C for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence. For every $p \in [1, \infty]$, we denote p' the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. $\mathfrak{M}(\mathbb{R}_+)$, $\mathfrak{M}^+(\mathbb{R}_+)$ and $\mathfrak{M}^+(\mathbb{R}_+;\uparrow)$
stand for the set of Lebesgue-measurable functions on $\mathbb{R}_+$, and its subspaces of nonnegative and nonnegative non-decreasing functions, respectively.

Let (X, d, μ) be space of N-homogeneous type as mentioned in Section 1. We now recall the definition of A_p weight functions.

Definition 2.4. *The weight function ω belongs to the class $A_p(X)$ for $1 \leq p < \infty$, if*

$$\sup_{x \in X, r>0} \left(\mu(B(x,r))^{-1} \int_{B(x,r)} \omega^p(y) d\mu(y) \right)^{\frac{1}{p}} \left(\mu(B(x,r))^{-1} \int_{B(x,r)} \omega^{-p'}(y) d\mu(y) \right)^{\frac{1}{p'}}$$

is finite and ω belongs to $A_1(X)$, if there exists a positive constant C such that for any $x \in X$ and $r > 0$

$$\mu(B(x,r))^{-1} \int_{B(x,r)} \omega(y) d\mu(y) \leq C \operatorname*{ess\,sup}_{y \in B(x,r)} \frac{1}{\omega(y)}.$$

The weight function (ω_1, ω_2) belongs to the class $\widetilde{A}_p(X)$ for $1 < p < \infty$, if

$$\sup_{x \in X, r>0} \left(\mu(B(x,r))^{-1} \int_{B(x,r)} \omega_2^p(y) d\mu(y) \right)^{\frac{1}{p}} \left(\mu(B(x,r))^{-1} \int_{B(x,r)} \omega_1^{-p'}(y) d\mu(y) \right)^{\frac{1}{p'}}$$

is finite.

The following theorem was proved in [34].

Theorem 2.5. *Let $1 \leq p < \infty$.*
1) Then the operator M is bounded in $L_{p,\omega}(X)$ if and only if $\omega \in A_p(X)$.
2) Then the operator M is bounded from $L_{1,\omega}(X)$ to $WL_{1,\omega}(X)$ if and only if $\omega \in A_1(X)$.

Lemma 2.6. *[37] Let $1 < p < \infty$ and $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$, then $(\omega_2^{-1}, \omega_1^{-1}) \in \widetilde{A}_{p'}(X)$, with $\frac{1}{p} + \frac{1}{p'} = 1$.*

Lemma 2.7. *[37] Let $1 < p < \infty$, $0 < \delta < 1$ and $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$. If $\frac{q-1}{p-1} = \delta$, then $(\omega_1, \omega_2) \in \widetilde{A}_{q'}(X)$, with $\frac{1}{p} + \frac{1}{p'} = 1$.*

Corollary 2.8. *[37] Let $1 < p < \infty$, $0 < \delta < 1$ and $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$, then the operator M is bounded from $L_{p,\omega_1^\delta}(X)$ to $L_{p,\omega_2^\delta}(X)$.*

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} \mu\,(B\,(x,r))^{-1} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}| d\mu\,(y)\,,$$

where $f_{B(x,r)}(x) = \mu\,(B\,(x,r))^{-1} \int\limits_{B(x,r)} f(y) d\mu\,(y)$.

Lemma 2.9. *Let $1 < p < \infty$ and $\omega \in A_p(X)$. Then*

$$\|f\omega\|_{L_p} \le C\|\omega M^\sharp f\|_{L_p}$$

with a constant $C > 0$ not depending on f.

Proposition A. [1] *Let T be a Calderon-Zygmund operator. Then for arbitrary s; $0 < s < 1$, there exists a constant $C_s > 0$ such that*

$$[(|Tf|^s)^\sharp]^{\frac{1}{s}}(x) \le C_s Mf(x)$$

for all $f \in C_0^\infty(X)$; $x \in X$.

Theorem 2.10. *Let $1 < p < \infty$, $0 < \delta < 1$ and $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$. Then the operators T and T^* are bounded from $L_{p,\omega_1^\delta}(X)$ to $L_{p,\omega_2^\delta}(X)$.*

Proof. By the Proposition A, Lemma 2.9 and Theorem 2.5, we derive the operator T is bounded from $L_{p,\omega_1^\delta}(X)$ to $L_{p,\omega_2^\delta}(X)$.

The boundedness of the operator T^* follows from the known estimate

$$T^* f(x) \le c[M(Tf)(x) + Mf(x)],$$

from Theorem 2.5 and Theorem 2.10.

Corollary 2.11. *Let $1 < p < \infty$ and $\omega \in A_p(X)$, then the singular integral operator T is bounded in $L_{p,\omega}(X)$.*

Definition 2.12. *We define the $BMO(X)$ space as the set of all locally integrable functions f such that*

$$\|f\|_{BMO} = \sup_{x\in X,\, r>0} \mu\,(B\,(x,r))^{-1} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}| d\mu\,(y) < \infty$$

or

$$\|f\|_{BMO} = \inf_{C} \sup_{x\in X,\, r>0} \mu\,(B\,(x,r))^{-1} \int\limits_{B(x,r)} |f(y) - C| d\mu\,(y) < \infty.$$

Definition 2.13. *We define the $BMO_{p,\omega}(X)$ $(1 \le p < \infty)$ space as the set of all locally integrable functions f such that*

$$\|f\|_{BMO_{p,\omega}} = \sup_{x\in X,\, r>0} \frac{\|(f(\cdot) - f_{B(x,r)})\chi_{B(x,r)}\|_{L_{p,\omega}(X)}}{\|\omega\|_{L_p(B(x,r))}}$$

or

$$\|f\|_{BMO_{p,\omega}} = \sup_{x\in X,\, r>0} \frac{1}{\mu\,(B\,(x,r))} \|(f(\cdot) - f_{B(x,r)})\chi_{B(x,r)}\|_{L_{p,\omega}(X)} \|\omega^{-1}\|_{L_{p'}(B(x,r))} < \infty.$$

Theorem 2.14. *[26] Let $1 \le p < \infty$ and ω be a Lebesgue measurable function. If $\omega \in A_p(X)$, then the norms $\|\cdot\|_{BMO_{p,\omega}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

We will need the following lemma while proving our main theorems.

Lemma 2.15. *[25] Let $b \in BMO(X)$. Then there is a constant $C > 0$ such that*

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \le C\|b\|_{BMO} \ln \frac{t}{r} \quad \text{for} \quad 0 < 2r < t,$$

where C is independent of b, x, r, and t.

Let $L_{\infty,\omega}(\mathbb{R}_+)$ be the weighted L_∞-space with the norm

$$\|g\|_{L_{\infty,\omega}(\mathbb{R}_+)} = \operatorname*{ess\,sup}_{t>0} \omega(t)g(t).$$

We denote

$$\mathbb{A} = \left\{\varphi \in \mathfrak{M}^+(\mathbb{R}_+;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0\right\}.$$

Let u be a continuous and non-negative function on $\mathbb{R}_+$. We define the supremal operator $\overline{S}_u$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(0,t)}, \quad t \in (0,\infty).$$

The following theorem was proved in [5].

Theorem 2.16. *[5] Suppose that v_1 and v_2 are nonnegative measurable functions such that $0 < \|v_1\|_{L_\infty(0,t)} < \infty$ for every $t > 0$. Let u be a continuous nonnegative function on $\mathbb{R}$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(\mathbb{R}_+)$ to $L_{\infty,v_2}(\mathbb{R}_+)$ on the cone $\mathbb{A}$ if and only if*

$$\left\|v_2 \overline{S}_u\left(\|v_1\|^{-1}_{L_\infty(0,\cdot)}\right)\right\|_{L_\infty(\mathbb{R}_+)} < \infty.$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_0^t g(s)w(s)ds, \quad H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [21].

Theorem 2.17. *[21] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \le C \sup_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Theorem 2.18. *[21, 22] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)H_w g(t) \le C \sup_{t>0} v_1(t)g(t) \tag{9}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_0^t \frac{w(s)ds}{\sup_{0<\tau<s} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (9).

3. Two-weighted inequalities for singular integral operators and their commutators in $\mathcal{M}_{\omega}^{p,\varphi}(X)$

Let T be a Calderón-Zygmund singular integral operator and $b \in BMO(X)$. A well known result of Coifman, Rochberg and Weiss [8] states that the commutator operator $[b,T]f = T(bf) - b\,Tf$ is bounded on $L_p(X)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [6], [7], [14], [15], [16]).

In this section we prove two-weighted inequalities for singular integral operators and their commutators in generalized weighted Morrey spaces on spaces of homogeneous type. We start with the following lemma.

Lemma 3.1. *[14] Let $1 < s < \infty$, $b \in BMO(X)$, then there exists $C > 0$ such that for all $x \in X$, the following inequality holds*

$$|[b,T]f|(x) \le M(|[b,T]f|(x)) \le C\|b\|_{BMO}\Big(\big(M|Tf|^s\big)^{\frac{1}{s}}(x) + \big(M|f|^s\big)^{\frac{1}{s}}(x)\Big).$$

Theorem 3.2. *Let $1 < p < \infty$, $0 < \delta < 1$ and $(\omega_1,\omega_2) \in \widetilde{A}_p(X)$. Then*

$$\|Tf\|_{L_{p,\omega_2^\delta}(B(x,r))} \le C\|\omega_2^\delta\|_{L_p(B(x,r))} \int_r^\infty \frac{\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}} \frac{dt}{t} \tag{10}$$

for every $f \in L_{p,\omega_1^\delta}(X)$, where C does not depend on f, x and r.

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{B(x,2kr)}(y), \quad f_2(y) = f(y)\chi_{X\setminus B(x,2kr)}(y),\ r > 0, \tag{11}$$

where k is the constant from the quasi-triangle inequality and have

$$\|Tf\|_{L_{p,\omega_2^\delta}(B(x,r))} \le \|Tf_1\|_{L_{p,\omega_2^\delta}(B(x,r))} + \|Tf_2\|_{L_{p,\omega_2^\delta}(B(x,r))}.$$

From Theorem 2.10 we obtain

$$\|Tf_1\|_{L_{p,\omega_2^\delta}(B(x,r))} \le \|Tf_1\|_{L_{p,\omega_2^\delta}(X)} \le C\|f_1\|_{L_{p,\omega_1^\delta}(X)} = C\|f\|_{L_{p,\omega_1^\delta}(B(x,2kr))}, \tag{12}$$

where C does not depend on f. From (12) we get

$$\|Tf_1\|_{L_{p,\omega_2^\delta}(B(x,r))} \le C\|\omega_2^\delta\|_{L_p(B(x,r))} \int_r^\infty \frac{\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}} \frac{dt}{t}, \tag{13}$$

which is easily obtained from the fact that $\|f\|_{L_{p,\omega_1^\delta}(B(x,2kr))}$ is non-decreasing in r, therefore $\|f\|_{L_{p,\omega_1^\delta}(B(x,2kr))}$ on the right-hand side of (12) is dominated by the right-hand side of (13). To estimate $\|Tf_2\|_{L_{p,\omega_2^\delta}(B(x,r))}$, we observe that

$$|Tf_2(y)| \le C \int\limits_{X\setminus B(x,2kr)} \frac{|f(z)|}{d(y,z)^N} d\mu(z),$$

where $y \in B(x,r)$ and the inequalities $d(x,y) \le r$, $d(y,z) \ge 2kr$ imply

$$\frac{1}{2k} d(y,z) \le d(x,z) \le \left(k + \frac{1}{2}\right) d(y,z),$$

finally we get

$$|Tf_2(y)| \le C \int\limits_{X\setminus B(x,2kr)} \frac{|f(z)|}{d(x,z)^N} d\mu(z).$$

To estimate $Tf_2(y)$, for $y \in B(x,r)$

$$\begin{aligned}
&\int\limits_{X\setminus B(x,2kr)} \frac{|f(z)|}{d(x,z)^N} d\mu(z) \\
&= -N \int\limits_{X\setminus B(x,2kr)} |f(z)| \int\limits_{d(x,z)}^{\infty} t^{-N-1} dt d\mu(z) \\
&\le C \int\limits_{2kr}^{\infty} t^{-N-1} \int\limits_{2kr\le d(x,z)\le t} |f(z)| d\mu(z)\, dt \\
&\le C \int\limits_{2r}^{\infty} t^{-N-1} \int\limits_{B(x,t)} |f(z)| d\mu(z)\, dt \\
&\le C \int\limits_{r}^{\infty} t^{-N-1} \|\omega_1^{-\delta} \chi_{B(x,t)}\|_{L_{p'}(X)} \|f\|_{L_{p,\omega_1^\delta}(B(x,t))} dt.
\end{aligned}$$

We prove the following inequality

$$\int\limits_{X\setminus B(x,2kr)} \frac{|f(z)|}{d(x,z)^N} d\mu(z) \le C \int\limits_{r}^{\infty} t^{-N-1} \|\omega_1^{-\delta}\|_{L_{p'}(B(x,t))} \|f\|_{L_{p,\omega_1^\delta}(B(x,t))} dt. \tag{14}$$

Hence by inequality (14), we get

$$\begin{aligned}
\|Tf_2\|_{L_{p,\omega_2^\delta}(B(x,r))} &\le C \|\chi_{B(x,r)}\|_{L_{p,\omega_2^\delta}(X)} \int\limits_{r}^{\infty} t^{-N-1} \|\omega_1^{-\delta}\|_{L_{p'}(B(x,t))} \|f\|_{L_{p,\omega_1^\delta}(B(x,t))} dt \\
&\le C \|\omega_2^\delta\|_{L_p(B(x,r))} \int\limits_{r}^{\infty} \frac{\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}} \frac{dt}{t}. \qquad (15)
\end{aligned}$$

From (13) and (15) we arrive at (10).

Theorem 3.3. *Let* $1 < p < \infty$, $0 < \delta < 1$, $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$ *and let the functions* $\varphi_1(x, r)$ *and* $\varphi_2(x, r)$ *satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\|\omega_1^\delta\|_{L_p(B(x,s))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}} \frac{dt}{t} \le C\varphi_2(x,r), \tag{16}$$

where C *does not depend on* $x \in X$ *and* r.

Then the operator T *is bounded from the space* $\mathcal{M}^{p,\varphi_1}_{\omega_1^\delta}(X)$ *to the space* $\mathcal{M}^{p,\varphi_2}_{\omega_2^\delta}(X)$.

Proof. Let $f \in \mathcal{M}^{p,\varphi_1}_{\omega_1^\delta}(X)$. By (16), Theorems 2.17 and 3.2 with $v_2 = \frac{1}{\varphi_2(x,t)}$, $g = \|f\|_{L_{p,\omega_1^\delta}(B(x,t))}$, $w = t^{-1}\|\omega_2^\delta\|^{-1}_{L_p(B(x,t))}$ and $v_1 = \frac{1}{\varphi_1(x,t)\|\omega_1^\delta\|_{L_p(B(x,t))}}$ we get

$$\begin{aligned}
\|Tf\|_{\mathcal{M}^{p,\varphi_2}_{\omega_2^\delta}(X)} &\le C \sup_{x\in X,\, r>0} \frac{\|\omega_2^\delta\|_{L_p(B(x,r))}}{\varphi_2(x,r)\|\omega_2^\delta\|_{L_p(B(x,r))}} \int_r^\infty \frac{\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}} \frac{dt}{t} \\
&\le C \sup_{x\in X,\, r>0} \frac{1}{\varphi_1(x,r)\|\omega_1^\delta\|_{L_p(B(x,r))}} \|f\|_{L_{p,\omega_1^\delta}(B(x,t))} \\
&= C\|f\|_{\mathcal{M}^{p,\varphi_1}_{\omega_1^\delta}(X)},
\end{aligned}$$

which completes the proof.

Theorem 3.4. *Let* $1 < p < \infty$, $0 < \delta < 1$, $b \in BMO(X)$ *and* $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$, $\omega_1 \in A_p(X)$. *Then the operator* $[b, T]$ *is bounded from* $L_{p,\omega_1^\delta}(X)$ *to* $L_{p,\omega_2^\delta}(X)$.

Proof. Let $f \in L_{p,\omega_1^\delta}(X)$, $b \in BMO(X)$ and $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$, $\omega_1 \in A_p(X)$. From Lemma 3.1, Corollary 2.8 and Corollary 2.11 we have

$$\begin{aligned}
&\left\|[b,T]f\right\|_{L_{p,\omega_2^\delta}(X)} \le \left\|M([b,T]f)\right\|_{L_{p,\omega_2^\delta}(X)} \le C\|b\|_{BMO} \left\|(M|Tf|^s)^{\frac{1}{s}} + (M|f|^s)^{\frac{1}{s}}\right\|_{L_{p,\omega_2^\delta}(X)} \\
&\le C\|b\|_{BMO} \left[\left\|(M|Tf|^s)^{\frac{1}{s}}\right\|_{L_{p,\omega_2^\delta}(X)} + \left\|(M|f|^s)^{\frac{1}{s}}\right\|_{L_{p,\omega_2^\delta}(X)}\right] \\
&\le C\|b\|_{BMO} \left[\left\|(|Tf|^s)^{\frac{1}{s}}\right\|_{L_{p,\omega_1^\delta}(X)} + \left\|(|f|^s)^{\frac{1}{s}}\right\|_{L_{p,\omega_1^\delta}(X)}\right] \le C\|b\|_{BMO} \left\|f\right\|_{L_{p,\omega_1^\delta}(X)}.
\end{aligned}$$

We can easily get the following.

Theorem 3.5. *Let* $1 < p < \infty$, $0 < \delta < 1$, $b \in BMO(X)$ *and* $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$, $\omega_1, \omega_2 \in A_p(X)$. *Then*

$$\|[b,T]f\|_{L_{p,\omega_2^\delta}(B(x,r))} \le C\|b\|_{BMO}\|\omega_2^\delta\|_{L_p(B(x,r))} \int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}} \frac{dt}{t} \tag{17}$$

for every $f \in L_{p,\omega_1^\delta}(X)$, *where* C *does not depend on* f, x *and* r.

Proof. We represent function f as in (11) and have

$$\|[b,T]f\|_{L_{p,\omega_2^\delta}(B(x,r))} \le \|[b,T]f_1\|_{L_{p,\omega_2^\delta}(B(x,r))} + \|[b,T]f_2\|_{L_{p,\omega_2^\delta}(B(x,r))}.$$

By Theorem 3.4 we obtain

$$\begin{aligned} &\|[b,T]f_1\|_{L_{p,\omega_2^\delta}(B(x,r))} \le \|[b,T]f_1\|_{L_{p,\omega_2^\delta}(X)} \\ &\le C\|b\|_{BMO}\|f_1\|_{L_{p,\omega_1^\delta}(X)} = C\|b\|_{BMO}\|f\|_{L_{p,\omega_1^\delta}(B(x,2kr))}, \end{aligned} \tag{18}$$

where C does not depend on f. From (18) we obtain

$$\|[b,T]f_1\|_{L_{p,\omega_2^\delta}(B(x,r))} \le C\|b\|_{BMO}\|\omega_2^\delta\|_{L_p(B(x,r))} \int_r^\infty \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}} \frac{dt}{t} \tag{19}$$

which is easily obtained from the fact that $\|f\|_{L_{p,\omega_1^\delta}(B(x,2kr))}$ is non-decreasing in r, therefore $\|f\|_{L_{p,\omega_1^\delta}(B(x,2kr))}$ on the right-hand side of (18) is dominated by the right-hand side of (19). To estimate $\|[b,T]f_2\|_{L_{p,\omega_2^\delta}(B(x,r))}$, we observe that

$$|[b,T]f_2(y)| \le C \int_{X\setminus B(x,2kr)} \frac{|b(y)-b(z)|}{d(y,z)^N} |f(z)|\, d\mu(z),$$

where $y \in B(x,r)$ and the inequalities $d(x,y) \le r$, $d(y,z) \ge 2kr$ imply

$$\frac{1}{2k} d(y,z) \le d(x,z) \le \left(k+\frac{1}{2}\right) d(y,z),$$

and therefore

$$|[b,T]f_2(y)| \le C \int_{X\setminus B(x,2kr)} \frac{|b(y)-b(z)|}{d(x,z)^N} |f(z)|\, d\mu(z).$$

To estimate $[b,T]f_2$, we first prove the following auxiliary inequality

$$\int_{X\setminus B(x,2kr)} \frac{|b(y)-b(z)|}{d(x,z)^N} |f(z)|\, d\mu(z) \le C\|b\|_{BMO} \int_r^\infty t^{-N-1} \left(1+\ln\frac{t}{r}\right) \|\omega_1^{-\delta}\|_{L_{p'}(B(x,t))} \|f\|_{L_{p,\omega_1^\delta}(B(x,t))} dt. \tag{20}$$

To estimate $[b,T]f_2(y)$, we observe that for $y \in B(x,r)$ we have

$$\begin{aligned} &\int_{X\setminus B(x,2kr)} \frac{|b(y)-b(z)|}{d(x,z)^N} |f(z)|\, d\mu(z) \\ &\le \int_{X\setminus B(x,2kr)} \frac{|b(z)-b_{B(x,r)}|}{d(x,z)^N} |f(z)|\, d\mu(z) + \int_{X\setminus B(x,2kr)} \frac{|b(y)-b_{B(x,r)}|}{d(x,z)^N} |f(z)|\, d\mu(z) = J_1 + J_2. \end{aligned}$$

By Lemma 2.15 ,we obtain

$$J_1 = \int\limits_{X\setminus B(x,2kr)} \frac{|b(z)-b_{B(x,r)}|}{d(x,z)^N}|f(z)|\,d\mu(z)$$

$$= -N \int\limits_{X\setminus B(x,2kr)} |b(z)-b_{B(x,r)}||f(z)|d\mu(z) \int\limits_{d(x,z)}^{\infty} t^{-N-1}dt$$

$$\le C \int\limits_{2kr}^{\infty} t^{-N-1} \int_{2kr\le d(x,z)\le t} |b(z)-b_{B(x,r)}||f(z)|d\mu(z)\,dt$$

$$\le C \int\limits_{2r}^{\infty} t^{-N-1}\|b(\cdot)-b_{B(x,t)}\|_{L_{p',\omega_1^{-\delta}}(B(x,t))}\|f\|_{L_{p,\omega_1^{\delta}}(B(x,t))}dt + C\int\limits_{2r}^{\infty} t^{-N-1}|b_{B(x,r)}-b_{B(x,t)}| \int\limits_{B(x,t)} |f(z)|d\mu(z)\,dt$$

$$\le C\|b\|_{BMO}\int\limits_{r}^{\infty} t^{-N-1}\|\omega_1^{-\delta}\|_{L_{p'}(B(x,t))}\|f\|_{L_{p,\omega_1^{\delta}}(B(x,t))}dt + C\|b\|_{BMO}\int\limits_{r}^{\infty} t^{-N-1}\ln\frac{t}{r}\|\omega_1^{-\delta}\|_{L_{p'}(B(x,t))}\|f\|_{L_{p,\omega_1^{\delta}}(B(x,t))}dt.$$

To estimate J_2, we have

$$J_2 = |b(y)-b_{B(x,r)}| \int\limits_{X\setminus B(x,2kr)} \frac{|f(z)|}{d(x,z)^N}d\mu(z)$$

$$\le C\mu(B(x,r))^{-1}\int\limits_{B(x,r)} |b(y)-b(z)|d\mu(z)\int\limits_{2r}^{\infty} t^{-N-1}\|\omega_1^{-\delta}\|_{L_{p'}(B(x,t))}\|f\|_{L_{p,\omega_1^{\delta}}(B(x,t))}dt$$

$$\le CM_b\chi_{B(x,r)}(y)\int\limits_{r}^{\infty} t^{-N-1}\|\omega_1^{-\delta}\|_{L_{p'}(B(x,t))}\|f\|_{L_{p,\omega_1^{\delta}}(B(x,t))}dt,$$

where C does not depend on x, t.

Hence by inequality (20), we get

$$\|[b,T]f_2\|_{L_{p,\omega_2^{\delta}}(B(x,r))} \le C\|\chi_{B(x,r)}\|_{L_{p,\omega_2^{\delta}}(X)}\|b\|_{BMO}\int\limits_{r}^{\infty} t^{-N-1}\left(1+\ln\frac{t}{r}\right)\|\omega_1^{-\delta}\|_{L_{p'}(B(x,t))}\|f\|_{L_{p,\omega_1^{\delta}}(B(x,t))}dt$$

$$\le C\|b\|_{BMO}\|\omega_2^{\delta}\|_{L_p(B(x,r))}\int\limits_{r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{p,\omega_1^{\delta}}(B(x,t))}}{\|\omega_2^{\delta}\|_{L_p(B(x,t))}}\frac{dt}{t}. \tag{21}$$

From (19) and (21) we arrive at (17).

Theorem 3.6. *Let $1 < p < \infty$, $0 < \delta < 1$, $b \in BMO(X)$ and $(\omega_1,\omega_2) \in \widetilde{A}_p(X)$, $\omega_1,\omega_2 \in A_p(X)$. Let the functions $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the condition*

$$\int\limits_{r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)\|\omega_1^{\delta}\|_{L_p(B(x,s))}}{\|\omega_2^{\delta}\|_{L_p(B(x,t))}}\frac{dt}{t} \le C\varphi_2(x,r), \tag{22}$$

where C does not depend on $x \in X$ and r.

Then the operator $[b,T]$ is bounded from the space $\mathcal{M}^{p,\varphi_1}_{\omega_1^{\delta}}(X)$ to the space $\mathcal{M}^{p,\varphi_2}_{\omega_2^{\delta}}(X)$.

Proof. Let $f \in \mathcal{M}^{p,\varphi_1}_{\omega_1^\delta}(X)$. By (22), Theorems 2.17 and 3.5 we obtain

$$\begin{aligned}&\|[b,T]f\|_{\mathcal{M}^{p,\varphi_2}_{\omega_2^\delta}(X)}\\&\le C\|b\|_{BMO}\sup_{x\in X,\,r>0}\frac{\|\omega_2^\delta\|_{L_p(B(x,r))}}{\varphi_2(x,r)\|\omega_2^\delta\|_{L_p(B(x,r))}}\int_r^\infty\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}}\frac{dt}{t}\\&\le C\|b\|_{BMO}\sup_{x\in X,\,r>0}\frac{1}{\varphi_1(x,r)\|\omega_1^\delta\|_{L_p(B(x,r))}}\|f\|_{L_{p,\omega_1^\delta}(B(x,r))}=C\|b\|_{BMO}\|f\|_{\mathcal{M}^{p,\varphi_1}_{\omega_1^\delta}(X)},\end{aligned}$$

which completes the proof.

4. Some examples for singular integral operators on $\mathcal{M}^{p,\varphi}_{\omega}(X)$

In this section we give some applications of our main results. We apply the theorems of Section 3 to the operators which are estimated from above by singular integral operators. Now we give some examples.

Example 1. [4, 13]

Let $X=\mathbb{R}^N$, μ is the Lebesgue measure, d the distance defined as follows. Let $\alpha_1,\alpha_2,...,\alpha_N$ be $N\in\mathbb{R}$, with $1\le\alpha_1\le\alpha_2\le...\le\alpha_N$, for all $x\in\mathbb{R}^N\setminus\{0\}$ a unique positive solution $\rho(x)$, the following equation holds

$$\sum_{k=1}^N\frac{x_k^2}{\rho^{2\alpha_k}}=1.$$

Set $\rho(0)=0$ and define $d(x,y)=\rho(x-y)$. Introducing the polar type change of variables

$$\begin{cases}x_1=\rho^{\alpha_1}\cos\varphi_1...\cos\varphi_{N-2}\cos\varphi_{N-1}\\x_2=\rho^{\alpha_2}\cos\varphi_1...\sin\varphi_{N-1}\\...\\x_N=\rho^{\alpha_N}\sin\varphi_1\end{cases}$$

we find $dx=\rho^{\alpha-1}d\rho d\sigma$, with $\alpha=\sum_{k=1}^N\alpha_k$ and $d\sigma$ the surface measure on $\sum_{N-1}=\{|x|=1\}$. Hence for all $r>0$, we can estimate

$$\mu(B(x,r))=C_N\,r^\alpha. \tag{23}$$

Particularly (7) is satisfied and $\left(\mathbb{R}^N,d,dx\right)$ is a homogeneous space. The unit ball related to d is the Euclidean unit ball. Let

$$K(x,y)=\frac{\Omega(x-y)}{\rho(x-y)^\alpha}, \tag{24}$$

where the function Ω has the following mixed homogeneity of degree zero, for all $t>0$, $x\in\mathbb{R}^N$

$$\Omega(t^{\alpha_1}x_1,\ t^{\alpha_2}x_2,...,t^{\alpha_N}x_N)=\Omega(x_1,\ x_2,...,x_N) \tag{25}$$

and the Hölder condition holds i.e. there exist $C>0$, $\beta\in(0,1]$ such that for all $x,y\in\sum_{N-1}$

$$|\Omega(x)-\Omega(x)|\le C\left|x-y\right|^\beta. \tag{26}$$

It is easy to check that the following condition (8) is fullfilled:

If α, β are as above, there exists a constant C such that for all $x_0 \in \mathbb{R}^N$, $x \in B(x_0, r)$, $y \notin B(x_0, 2r)$

$$|K(x_0, y) - K(x, y)| \leq C \frac{d(x_0, x)^{\beta}}{d(x_0, y)^{\beta+\alpha}}. \tag{27}$$

The condition (3) follows the definition of K and from (23). Furthermore, if we suppose that Ω holds the vanishing property

$$\int_{\sum_{N-1}} \Omega(x)\, d\sigma(x) = 0, \tag{28}$$

then conditions (5) and (6) are trivial. Furthermore, by Theorem 2.2, 2.3, for all $p \in (1, \infty)$ it is well defined

$$Tf(x) = \lim_{\epsilon \longrightarrow 0} \int K_{\epsilon}(x, y) f(y)\, d(y),$$

where the limit exists in L_p sense and T is continuous.

Kernels of the above kind arise for instance considering parabolic operators with constants coefficients

$$L = \sum_{j,k=1}^{n} a_{jk} \partial^2_{\xi_j \xi_k} - \partial_t,$$

where a_{jk} is a positive and symmetric matrix. Here x$\equiv (\xi, t) \in \mathbb{R}^{n+1} \equiv \mathbb{R}^N$ (we set $N = n + 1$). Let Γ^0 be the fundamental solution for L, with pole at the origin, and let

$$K(x, y) = \left(\partial^2_{\xi_j \xi_k} \Gamma^0\right)(x - y), \text{ for any } j, k = 1,\ 2, ..., n. \tag{29}$$

Then K is a kernel as in (24)- (25), with $\alpha_1 = \alpha_2 = ... = \alpha_n = 1$, $\alpha_{n+1} = 2$. For any test function u we can write

$$u(x) = \int_{\mathbb{R}^{n+1}} \Gamma^0(x - y)\, Lu(y)\, dy$$

and, differentiating twice the above formula with respect to ξ we find

$$\partial^2_{\xi_j \xi_k} u(x) = P.V. \int_{\mathbb{R}^{n+1}} \partial^2_{\xi_j \xi_k} \Gamma^0(x - y)\, Lu(y)\, dy + \ CLu(x).$$

Furthermore, the L_p-continuity of the singular integral implies L_p -estimates for the second derivatives of u, in terms of Lu and u.

Corollary 4.1. *Let* $1 < p < \infty$, $0 < \delta < 1$, $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$, $\omega_1, \omega_2 \in A_p(X)$ *and the functions* $\varphi_1(x, r)$, $\varphi_2(x, r)$ *satisfy the condition* (16) *. Then the singular integral operator T given with the kernel* $K(x, y)$ *given with equation* (29) *is bounded from the space* $\mathcal{M}^{p,\varphi_1}_{\omega_1^{\delta}}(X)$ *to the space* $\mathcal{M}^{p,\varphi_2}_{\omega_2^{\delta}}(X)$.

Example 2. [4, 41]

Let $0 \leq \gamma < n$, $X = \mathbb{R}^n$, $d\mu = |x|^{-\gamma} dx$ and d the eucledean distance. It is well known that for $-n < \alpha < n(p-1)$, $|x|^{\alpha} \in A_p(\mathbb{R}^n)$, hence for $1 \leq p < \infty$, $0 \leq \gamma < n$, $|x|^{-\gamma} \in A_p(\mathbb{R}^n)$. Particularly $d\mu$ is doubling, so that X is a homogeneous space, one can prove that $d\mu$ satisfies also a reverse doubling condition. Note that $d\mu$ is not translation invariant. Now, let

$$K(x, y) = \frac{\Omega(x - y)}{|x - y|^n} \left(|x|^{\gamma} - |y|^{\gamma}\right) \tag{30}$$

where Ω is a homogeneous function of degree zero satisfying (26) - (27). In this case (7) does not hold, but the following inequalities can be easily verified

$$\mu\left(B\left(x,2d\left(x,y\right)\right)\right)\leq\begin{cases} C\left|x-y\right|^{n}|x|^{-\gamma} & , \quad \left|x-y\right|<\frac{|x|}{2} \\ C\left|x-y\right|^{n-\gamma} & , \quad \left|x-y\right|>\frac{|x|}{2}. \end{cases} \tag{31}$$

By (30) and (31), we can see that K satisfies condition (3). Moreover, one can prove that K satisfies condition (4) with exponent $\beta' = \min(\beta,\gamma)$ (β is the same number appearing in (27)). Finally, also (5) and (6) can be proved, the last one following from analogous result for classical Calderón-Zygmund integrals on $\mathbb{R}^n$. So also in this case, by Theorems 2.2, 2.3 the kernel K defines a unique Calderón-Zygmund operator, for which the commutator estimate holds.

Corollary 4.2. *Let $1 < p < \infty$, $0 < \delta < 1$, $(\omega_1,\omega_2) \in \widetilde{A}_p(X)$, $\omega_1,\omega_2 \in A_p(X)$ and the functions $\varphi_1(x,r)$, $\varphi_2(x,r)$ satisfy the condition* (16) *. Then the singular integral operator T given with the kernel $K(x,y)$ given with equation* (30) *is bounded from the space $\mathcal{M}^{p,\varphi_1}_{\omega_1^\delta}(X)$ to the space $\mathcal{M}^{p,\varphi_2}_{\omega_2^\delta}(X)$.*

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