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Existence results for nonlinear hybrid fractional differential equations with generalized $[\Psi, \Phi]-$ Caputo-Fabrizio derivative

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Abstract. The aim of this paper is to develop a theory of fractional hybrid differential equations with perturbations of second type involving $[\Psi, \Phi]$ Caputo-Fabrizio fractional derivative of an arbitrary order $\nu \in (0, 1)$. We demonstrate the existence and uniqueness of solutions for a particular class of nonlinear fractional hybrid differential equations with initial conditions by applying Banach's fixed point theorem and some fundamental concepts on fractional analysis. As an example, a significant case is given to illustrate the utility of our theoretical findings. We also, give a simulation of the solution by applying the Adams Bashford with three steps method .

1. Introduction

The fractional calculus has been used in many fields, including finance, physics, and engineering, to model non-local and non-linear phenomena that are not adequately described by classical calculus. The major concepts and definitions of fractional calculus are introduced by Diethelm and Ford [8], Dalir and Bashour [10] and Osler [6].
In recent decades, many classes of fractional differential equations (FDEs) have been thoroughly investigated and studied. Intensive research has been done on the existence and uniqueness by using Riemann-Liouville [11], Caputo [12],and Hilfer [4]. Generalized fractional deferentials and integrals are investigated by sevral authors. For example Ψ-Riemann-Liouville introduced by Kilbas [13],TheΨ-Caputo has been defined by Almedia [18].
There is a singular kernel in the aforementioned derivatives. Therefore, recently, some authors presented some new types of (FDs) in which they replaced a singular kernel with a non singular kernel [7, 19]. These types of (FDEs) have shown its major usefulness in modeling real problems in several fields of science.
Regarding this, Al-Refai and Jarrah [14] introduced the notion of $[\Psi, \Phi]$-Caputo- Fabrizio fractional derivative, where $\Psi(x)$ is a monotone function and Φ is a weight function.

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Motiveted by studies [1, 5, 14, 16, 22], we consider the flowing wheithed Hybrid fractional deferential model:

$$\begin{cases} {}^{CF}\mathbb{D}^{\nu,\Phi}_{a,\Psi}\Big(z(t) - f(t, z(t))\Big) = g(t, z(t)), \qquad t \in \overline{\Omega} = [0, T], \\ \\ z(0) - f(0, z(0)) = 0. \end{cases} \tag{1}$$

Where $T > 0$, ${}^{CF}\mathbb{D}^{\nu,\Phi}_{a,\Psi}$ is the $[\Psi, \Phi]$ Caputo-Fabrizio derivative and f, g in $C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.
The topic of the innovative weighted operators with another function is something we pay close attention to, as far as we are aware, no studies addressing the qualitative aspects of the aforementioned problems using the $[\Psi, \Phi]$Caputo-Fabrizio Fractional derivative have been published. As a result, we develop and expand the existence and uniqueness of solution for problem (1).
Our manuscript is organized as follows. In Section 2, we give some basic definitions and properties of $[\Psi, \Phi]$fractional integral and $[\Psi, \Phi]$Caputo-fractional derivative which will be used in the rest of our paper. In Section 3, we establish the existence and uniqness of solution of the $[\Psi, \Phi]$Caputo-Fabrizio fractional with initial value problem (1) by using Banach fixed point theorem. As application, an illustrative example is presented in Section 4, in section 5, we will simulate the solution of this example with different fractional order followed by conclusion.

2. Preliminaries

In this section, we begin by introducing some notations and fundamental vocabulary.
Let $\overline{\Omega} := [t_0, t_0 + \delta]$, $\delta > 0$, and $\mathbb{R}$ be the set of real numbers. $C(\overline{\Omega}, \mathbb{R})$ and $AC(\overline{\Omega}, \mathbb{R})$ denote the set of continuous and absolutely continuous functions, respectively on $\overline{\Omega}$ equipped with the usual supremum norm

$$\| z \| = \sup_{t \in \overline{\Omega}} \Big| z(t) \Big|.$$

Let $\Psi(x)$ and $\Phi(x)$ be the monotone and weight function, respectively, with $\Psi, \Phi' \in C^1(\overline{\Omega})$ and $\Psi, \Psi', \Phi' > 0$ on $\overline{\Omega}$. The weighted Caputo-Fabrizio fractional derivative is defined as:

Definition 2.1. *([14]) Let $0 < \nu < 1$, and $\rho \in AC[\overline{\Omega}, \mathbb{R}]$. The left $[\Psi, \Phi]$ Caputo-Fabrizio Fractional Derivitave is defined as:*

$$ {}^{CF}\mathbb{D}^{\nu,\Phi}_{a,\Psi}\rho(t) = \frac{\aleph(\nu)}{1-\nu} \frac{1}{\Phi(t)} \int_a^t e^{-\ell_\nu(\Psi(t) - \Psi(s))} \frac{\mathrm{d}}{\mathrm{d}s}(\Phi\rho)(s) ds. \tag{2}$$

Where ${}^{CF}\mathbb{D}^{\nu,\Phi}_{a,\Psi}$ is a Caputo-Fabrizio Fractional Derivitave, $\ell_\nu = \frac{\nu}{1-\nu}$, and $\aleph(\nu)$ is a normalization function satisfying $\aleph(0) = \aleph(1) = 1$.

We can write the preceding operator as:

$$ {}^{CF}\mathbb{D}^{\nu,\Phi}_{a,\Psi}\rho(t) = \frac{\aleph(\nu)}{1-\nu} \frac{e^{-\ell_\nu \Psi((t)}}{\Phi(t)} \int_a^t e^{\ell_\nu \Psi(s)} \frac{\mathrm{d}}{\mathrm{d}s}(\Phi\rho)(s) ds. \tag{3}$$

Definition 2.2. *([14]). Let $0 < \nu < 1$, and $f \in AC[\overline{\Omega}, \mathbb{R}]$. The left $[\Psi, \Phi]$ Caputo-Fabrizio Fractional integral is defined as:*

$$ {}^{CF}\mathcal{I}^{\nu,\Phi}_{a,\Psi}\rho(x) = \frac{1-\nu}{\aleph(\nu)} f(x) + \frac{\nu}{\aleph(\nu)} \frac{1}{\Phi(x)} \int_a^x \Psi'(s)\Phi(s)\rho(s) \mathrm{d}s. \tag{4}$$

Remark 2.3. *For $\Psi(t) = 1$ and $\Phi(t) = 1$, the left $[\Psi, \Phi]$ Caputo-Fabrizio Fractional integral coincides with Caputo-fabrizio fractional integral defined in [15]*

Proposition 2.4. *[14] Let $\rho \in AC[\overline{\Omega}, \mathbb{R}]$, then we have*

1) ${}^{CF}\mathbb{D}^{\nu,\Phi}_{a,\Psi}({}^{CF}\mathcal{I}^{\nu,\Phi}_{a,\Psi})\rho(t) = \rho(t) - \frac{\Phi(a)\rho(a)e^{-\ell_\nu \Psi(a)}}{\Phi(t)}$.
2) ${}^{CF}\mathcal{I}^{\nu,\Phi}_{a,\Psi}({}^{CF}\mathbb{D}^{\nu,\Phi}_{a,\Psi})\rho(t) = \rho(t) - \rho(a)$.
3) ${}^{CF}\mathcal{I}^{\nu,\Phi}_{a,\Psi}$ *is linear and bounded from* $AC[\overline{\Omega}, \mathbb{R}]$ *to* $AC[\overline{\Omega}, \mathbb{R}]$.

Remark 2.5. *for $\rho \in AC[\overline{\Omega}, \mathbb{R}]$ such as $\rho(a) = 0$ we have:*

$${}^{CF}\mathbb{D}^{\nu,\Phi}_{a,\Psi}({}^{CF}\mathcal{I}^{\nu,\Phi}_{a,\Psi})\rho(t) = \rho(t)$$

and

$${}^{CF}\mathcal{I}^{\nu,\Phi}_{a,\Psi}({}^{CF}\mathbb{D}^{\nu,\Phi}_{a,\Psi})\rho(t) = \rho(t)$$

Theorem 2.6. *([17]) Let $\mathcal{X}$ be a non-empty, closed convex subset of a Banach space and Let $\mathcal{M}_1 : \mathcal{X} \to \mathcal{X}$ be an operator such that:*
$\mathcal{M}_1$ is a contraction, then the equation $\mathcal{M}_1 x = x$ admits a unique solution in $\mathcal{X}$.

Theorem 2.7. *([3]).(Gronwall) Let z, φ and Φ be real continuous functions defined in $[a, b]$, $\Phi(t) \geq 0$ for $t \in [a, b]$. If we have the inequality:*

$$z(t) \leq \varphi(t) + \int_a^t \Phi(s)z(s)ds.$$

for all $t \in [a, b]$,Then

$$z(t) \leq \varphi(t) + \int_a^t \Phi(s)\varphi(s) \exp\left[\int_a^t \Phi(u)du\right] ds$$

for all $t \in [a, b]$.

Lemma 2.8. *([3]). If φ is constant, then from*

$$z(t) \leq \varphi + \int_a^t \Phi(u)z(u)du$$

it follows that

$$z(t) \leq \varphi \exp\left(\int_a^t \Phi(u)du\right).$$

We assume the following assumptions throughout the rest of our paper.

($\mathcal{S}_1$) The functions $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are absolutely continuous.

($\mathcal{S}_2$) $g(0, z(0)) = 0$.

($\mathcal{S}_3$) There exists a constant $0 < \ell_1 < 1$ such that:
$\left|f(t, u) - f(t, v)\right| \leq \ell_1 \left|u - v\right| \quad for \quad all \quad t \in \overline{\Omega} \quad and \quad u, v \in \mathbb{R}.$

($\mathcal{S}_4$) There exists a constant $0 < \ell_2 < 1$ such that :
$\|g(t, u) - g(t, v)\| \leq \ell_2 \|u - v\| \quad for \quad all \quad t \in \overline{\Omega} \quad and \quad u, v \in \mathbb{R}.$

3. Main results

In this section, we begin by establishing what we signify by solution of the problem (1), after we will show the existence and the unicity of the solution of this problem using Banach's fixed point theorem.

Definition 3.1. *The function $z \in AC[\overline{\Omega}, \mathbb{R}]$ is said to be a solution of the problem (1) if $z(t)$ verifies the equation*

$$^{CF}\mathbb{D}^{\nu,\Phi}_{0,\Psi}\Big(z(t) - f(t, z(t))\Big) = g(t, z(t)), \quad t \in \overline{\Omega} = [0, T],$$

and $\quad z(0) - f(0, z(0)) = 0$ *on* $\overline{\Omega}$

Before we can confirm the existence and uniqueness of solution of the fractional problem (1), we must first establish the following basic lemma.

Proposition 3.2. *Consider that hypothesis $(\mathcal{S}_1) - (\mathcal{S}_2)$ is true, then the function $z(t) \in AC(\overline{\Omega}, \mathbb{R})$ is a solution of the fractional problem (1) if and only if z verifies the following fractional hybrid integral equation*

$$z(t) = f(t, z(t)) + a_\nu g(t, z(t)) + b_\nu(t) \int_0^t \Psi'(s)\Phi(s)g(s, z(s))\mathrm{d}s. \tag{5}$$

Where $\; a_\nu = \frac{1-\nu}{\aleph(\nu)} = cte$ *, and* $\; b_\nu(t) = \frac{\nu}{\aleph(\nu)} \frac{1}{\Phi(t)}$.

Proof. Let z be a solution of the problem (1), then we apply the $[\Psi, \Phi]$ fractional integral $^{CF}\mathcal{I}^{\nu,\Phi}_{0,\Psi}$ on both sides of (1) we have

$$^{CF}\mathcal{I}^{\nu,\Phi}_{0,\Psi}(^{CF}\mathbb{D}^{\nu,\Phi}_{0,\Psi})\Big(z(t) - f(t, z(t))\Big) = {}^{CF}\mathcal{I}^{\nu,\Phi}_{0,\Psi} g(t, z(t)),$$

from Proposition 2.4 we obtain

$$z(t) - f(t, z(t)) = {}^{CF}\mathcal{I}^{\nu,\Phi}_{0,\Psi} g(t, z(t)),$$

which implies

$$z(t) = f(t, z(t)) + {}^{CF}\mathcal{I}^{\nu,\Phi}_{0,\Psi} g(t, z(t)),$$

thus

$$z(t) = f(t, z(t)) + \frac{1-\nu}{\aleph(\nu)} g(t, z(t)) + \frac{\nu}{\aleph(\nu)} \frac{1}{\Phi(t)} \int_0^t \Psi'(s)\Phi(s)g(s, z(s))\mathrm{d}s.$$

Involves

$$z(t) = f(t, z(t)) + a_\nu g(t, z(t)) + b_\nu(t) \int_0^t \Psi'(s)\Phi(s)g(s, z(s))\mathrm{d}s.$$

Hence equation (5) holds.

Conversely, if $z(t)$ satisfies the equation (5), then we have:

$$z(t) - f(t, z(t)) = a_\nu g(t, z(t)) + b_\nu(t) \int_0^t \Psi'(s)\Phi(s)g(s, z(s))\mathrm{d}s, \tag{6}$$

we employ the $[\Psi, \Phi]$ Caputo-Fabrizio fractional derivative $^{CF}\mathbb{D}^{\nu,\Phi}_{0,\Psi}$ to both sides of equation (6), we obtain

$$^{CF}\mathbb{D}^{\nu,\Phi}_{0,\Psi}\left(z(t) - f(t, z(t))\right) = {}^{CF}\mathbb{D}^{\nu,\Phi}_{0,\Psi}\left(a_\nu g(t, z(t)) + b_\nu(t) \int_0^t \Psi'(s)\Phi(s)g(s, z(s))\mathrm{d}s\right),$$

$$^{CF}\mathbb{D}^{\nu,\Phi}_{0,\Psi}\left(z(t) - f(t, z(t))\right) = {}^{CF}\mathbb{D}^{\nu,\Phi}_{0,\Psi}(^{CF}\mathcal{I}^{\nu,\Phi}_{0,\Psi}) g(t, z(t)),$$

and we practice Proposition 2.4,we find

$$ {}^{CF}\mathbb{D}^{\nu,\Phi}_{0,\Psi}(z(t)-f(t,z(t)))=g(t,z(t))-\nu e^{\ell_\nu\Psi(0)}\Phi(0)g(0,z(0)). $$

Since $z(0)=f(0,z(0))$ and $g(0,z(0))=0$ then we have

$$ {}^{CF}\mathbb{D}^{\nu,\Phi}_{a,\Psi}(z(t)-f(t,z(t)))=g(t,z(t)). $$

Finally, we need to make sure that $z(0)=f(0,z(0))$ in the equation (1) is likewise true. To accomplish this, we substitute $t=0$ in (5), we obtain:

$$ z(0)=f(0,z(0))+\frac{1-\nu}{\aleph(\nu)}g(0,z(0))+\frac{\nu}{\aleph(\nu)}\frac{1}{\Phi(0)}\int_0^0\Psi'(s)\Phi(s)g(s,z(s))\mathrm{d}s, $$

and from $(\mathcal{S}_2)$, it follows that

$$ z(0)=f(0,z(0)), $$

Hence, z is a solution to the problem (1). This completes the proof. □

Theorem 3.3. *Assume that hypotheses* $(\mathcal{S}_1)-(\mathcal{S}_4)$ *hold, and*

$$ \ell_1+\Big(a_\nu+(\Psi(T)-\Psi(0))\parallel\Phi\parallel.\parallel b_\nu\parallel\Big)\ell_2<1 $$

then fractional initial value problem (1) has a unique solution defined on $\overline{\Omega}$.

Proof. By using Proposition 3.2, the fractional hybrid differential equation (1) is equivalent to the following nonlinear fractional hybrid integral equation

$$ z(t)=f(t,z(t))+a_\nu g(t,z(t))+b_\nu(t)\int_a^t\Psi'(s)\Phi(s)g(s,z(s))\mathrm{d}s. \tag{7} $$

Let $\mathcal{M}_1:\mathcal{X}\to\mathcal{X}$ be an operator defined by

$$ \mathcal{M}_1z(t)=f(t,z(t))+a_\nu g(t,z(t))+b_\nu(t)\int_a^t\Psi'(s)\Phi(s)g(s,z(s))\mathrm{d}s. $$

where $\mathcal{X}=AC[\overline{\Omega},\mathbb{R}]$.

We can transforme the fractional integral equation (7) into the operator equation as follows

$$ \mathcal{M}_1z(t)=z(t),\quad t\in\overline{\Omega}. \tag{8} $$

Now, we will verify that the operator $\mathcal{M}_1$ satisfies the conditions of theorem 2.6.
Let us show that the operator $\mathcal{M}_1$ is a contraction on $AC[\overline{\Omega},\mathbb{R}]$.

$$ \begin{aligned} \Big|\mathcal{M}_1x(t)-\mathcal{M}_1y(t)\Big| &\le \ell_1\Big|x(t)-y(t)\Big|+a_\nu\ell_2\Big|x(t)-y(t)\Big|+\ell_2\Big|b_\nu(t)\int_a^t\Psi'(s)\Phi(s)\Big(x(t)-y(t)\Big)ds\Big| \\ \parallel\mathcal{M}_1x-\mathcal{M}_1y\parallel &\le \ell_1\parallel x-y\parallel+a_\nu\ell_2\parallel x-y\parallel+\ell_2\parallel b_\nu\parallel.\parallel\Phi\parallel.\parallel x-y\parallel\int_a^t\Psi'(s)ds \\ \parallel\mathcal{M}_1x-\mathcal{M}_1y\parallel &\le \Big(\ell_1+\Big(a_\nu+(\Psi(T)-\Psi(0))\parallel\Phi\parallel.\parallel b_\nu\parallel\Big)\ell_2\Big)\parallel x-y\parallel \end{aligned} $$

Since $\ell_1+\Big(a_\nu+(\Psi(T)-\Psi(0))\parallel\Phi\parallel\parallel b_\nu\parallel\Big)\ell_2<1$, the operator $\mathcal{M}_1$ is a contraction on $\mathcal{X}$.
From Theorem 2.6 we conclude that the problem (1) admits a unique solution. □

Lemma 3.4. .
The solution of fractional model with initial value (1) is defined in

$$\mathcal{M}_{\kappa,\omega} = \{y \in AC[\overline{\Omega}, \mathbb{R}] : \parallel y \parallel \leq \kappa e^{\omega T}\},$$

for hypothesis $(\mathcal{S}_1) - (\mathcal{S}_4)$ *, and* $\ell_1 + (1-\varrho)\ell_2 < 1$ *where*

$$\kappa = \frac{f_0 + \Big(a_\nu + \parallel b_\nu \parallel . \parallel \Psi' \parallel . \parallel \Phi \parallel \Big) g_0}{1-(\ell_1 + a_\nu \ell_2)}, \quad \omega = \frac{\parallel b_\nu \parallel . \parallel \Psi' \parallel . \parallel \Phi \parallel \ell_2}{1-(\ell_1 + a_\nu \ell_2)}$$

and

$$f_0 = \sup_{t\in\overline{\Omega}} f(t,0), \quad g_0 = \sup_{t\in\overline{\Omega}} g(t,0).$$

Proof. From equality (8) we have

$$\Big|z(t)\Big| = |\mathcal{M}_1 z(t)|$$

which means

$$\begin{aligned}
\Big|z(t)\Big| &= \Big|f(t,z(t)) + (a_\nu) g(t,z(t)) + b_\nu(t) \int_0^t \Psi'(s)\Phi(s) g(s,z(s)) ds\Big|, \\
\Big|z(t)\Big| &= \Big|f(t,z(t)) - f(t,0) + f(t,0) + (a_\nu)(g(t,z(t)) - g(t,0)) \\
&+ (a_\nu) g(t,0) + b_\nu(t) \int_0^t \Psi'(s)\Phi(s)\Big(g(s,z(s)) - g(s,0)\Big) ds + b_\nu(t) \int_0^t \Psi'(s)\Phi(s) g(s,0) ds\Big| \\
\Big|z(t)\Big| &\leq \Big|f(t,z(t)) - f(t,0)\Big| + \Big|f(t,0)\Big| + a_\nu \Big|g(t,0)\Big| + a_\nu \Big|(g(t,z(t)) - g(t,0))\Big| \\
&+ \Big|b_\nu(t)\Big| \|\Psi'\| \parallel \Phi \parallel \int_0^t \Big|g(s,z(s)) - g(s,0)\Big| ds + \Big|b_\nu(t)\Big| \parallel \Psi' \parallel . \parallel \Phi \parallel \int_0^t \Big|g(s,0)\Big| ds
\end{aligned}$$

By using the assymtions $(\mathcal{S}_1) - (\mathcal{S}_4)$,we find

$$\Big|z(t)\Big| \leq \ell_1 \Big|z(t)\Big| + f_0 + a_\nu g_0 + a_\nu \ell_2 \Big|z(t)\Big| + \parallel b_\nu \parallel . \parallel \Psi' \parallel . \parallel \Phi \parallel \ell_2 \int_0^t \Big|z(s)\Big| ds + \parallel b_\nu \parallel . \parallel \Psi' \parallel . \parallel \Phi \parallel T g_0,$$

$$(1 - \ell_1 - a_\nu \ell_2) |z(t)| \leq f_0 + a_\nu g_0 + \parallel b_\nu \parallel . \parallel \Psi' \parallel . \parallel \Phi \parallel T g_0 + \parallel b_\nu \parallel . \parallel \Psi' \parallel . \parallel \Phi \parallel \ell_2 \int_0^t \Big|z(s)\Big| ds,$$

From lemma 2.8, and the fact that $(\ell_1 + a_\nu \ell_2) < 1$, We have $|z(t)| \leq \kappa e^{\omega T}$
where

$$\kappa = \frac{f_0 + a_\nu g_0 + \parallel b_\nu \parallel . \parallel \Psi' \parallel . \parallel \Phi \parallel T g_0}{1 - \ell_1 - a_\nu \ell_2}$$

$$\omega = \frac{\parallel b_\nu \parallel . \parallel \Psi' \parallel . \parallel \Phi \parallel \ell_2}{1 - \ell_1 - a_\nu \ell_2}$$

As a result $y \in C_{\kappa,\omega}$ This completes the proof of Lemma 3.4. □

4. An illustrative example

In this section, we provide a nontrivial example to illustrate the main results obtained above. For this purpose, we consider the fractional with initial value problem below :

$$\begin{cases} {}^{CF}\mathbb{D}^{\nu,\Phi}_{0,\Psi} (z(t) - f(t,z(t))) = g(t,z(t)), & t \in \overline{\Omega} = [0,T] \\ z(0) - f(0,z(0)) = 0 \end{cases} \tag{9}$$

Where $\nu = \frac{1}{3}$, $\aleph(\nu) = 1$, $\Psi(t) = t^3$, $\Phi(t) = t + 1$, $T = 1$, $f(t, z(t)) = \dfrac{2t}{15} sin(z(t))$
and $g(t, z(t)) = \dfrac{t}{5} \sqrt{z^2(t) + 3}$.
It is clear that the assumptions $(\mathcal{S}_1)$ and $(\mathcal{S}_2)$ are satisfied . Indeed, let $t \in \overline{\Omega}$ and $z, z' \in \mathbb{R}$, then we have

$$\begin{aligned} \left|f(t,z) - f(t,z')\right| &= \left|\frac{2t}{15}(sin(z) - sin(z')\right|, \\ \left|f(t,z) - f(t,z')\right| &\leq \left|\frac{2t}{15}\right|.\left|sin(z) - sin(z')\right|, \end{aligned}$$

By using the mean value theorem, we have

$$\left|f(t,z) - f(t,z')\right| \leq \frac{2}{15}\left|z - z'\right|,$$

thus, the hypothesis $(\mathcal{S}_3)$ in holds true with $\ell_1 = \frac{2}{15}$.

It remains to verify the supposition $(\mathcal{S}_4)$. Let $t \in \overline{\Omega}$ and $z, z' \in \mathbb{R}$, then we have

$$\begin{aligned} \left|g(t,z(t)) - g(t,z'(t))\right| &= \left|\frac{t}{5}\sqrt{z^2(t)+3} - \frac{t}{5}\sqrt{z'^2(t)+3}\right|, \\ \left|g(t,z(t)) - g(t,z'(t))\right| &\leq \frac{t}{5}\left|z(t) - z'(t)\right| \frac{\left|z(t) + z'(t)\right|}{\sqrt{z^2(t)+3} + \sqrt{z'^2(t)+3}}, \\ \left|g(t,z(t)) - g(t,z'(t))\right| &\leq \frac{1}{5}\left|z(t) - z'(t)\right|, \end{aligned}$$

thus, the assumption $(\mathcal{S}_4)$ in holds true with $\ell_2 = \frac{1}{5}$.
Moreover,$\ell_1 + \Big(a_\nu + (\Psi(T) - \Psi(0)) \parallel \Phi \parallel . \parallel b_\nu \parallel \Big)\ell_2 = \frac{2}{15} + \frac{4}{15} = \frac{2}{5} \leq 1$ Where $a_\nu = \frac{2}{3}$, $\parallel b_\nu \parallel = \frac{1}{3}$, $(\Psi(T) - \Psi(0)) = 1$ and $\parallel \Phi \parallel = 2$. Thus,all of the conditions of theorem 3.3 are satisfied.
Finally, the $[\Psi, \Phi]$ Caputo-Fabrizio fractional hybrid problem (9) has a unique solution defined on $\overline{\Omega}$.

5. Numerical simulations

Several numerical methods for obtaining approximate solutions to fractional differential equations have been proposed. These methods are primarily based on discretization of the independent variable and include adaptations of the integerorder methods such as the adams bashforth moulton type predictor corrector methods [9], finite difference methods [21], and finite element methods [20].In this paper,to approximate the solution of the wheithed Hybrid fractional deferential model (9),we will use a three-step fractional adams bashforth scheme.
We will first describe the three-step fractional adams bashforth model and then apply it to obtain numerical solutions for the $[\Psi, \Phi]$ Caputo–fabrizio fractional model (9).

5.1. *Numerical scheme of solution*

Consider the following fractional differential equation:

$$^{CF}\mathbb{D}^{\nu,\Phi}_{a,\Psi}(z(t) - f(t, z(t))) = g(t, z(t)) \tag{10}$$

where $^{CF}\mathbb{D}^{\nu,\Phi}_{a,\Psi} z(t)$ is the $[\Psi, \Phi]$ Caputo-Fabrizio fractional differential equation defined in (2). By integrating Equation (10) using the $[\Psi, \Phi]$ Caputo-Fabrizio fractional integral, we obtain:

$$^{CF}\mathcal{I}^{\nu,\Phi}_{0,\Psi}\left(^{CF}\mathbb{D}^{\nu,\Phi}_{a,\Psi}(z(t) - f(t, y))\right) = {}^{CF}\mathcal{I}^{\nu,\Phi}_{0,\Psi}(g(t, y))$$

$$z(t) = f(t,y) + a_\nu g(t,y) + b_\nu(t)\int_0^t \Psi'(s)\Phi(s)g(s,z(s))ds \tag{11}$$

Where a_ν and $b_\nu(t)$ are defined above in (5).

The time interval was discretized into steps with an interval of $\hbar$; we thus have $t_0 = 0.\ t_{i+1} = t_i + \hbar.\ldots i = 0 : n-1$. Now, Equation (11) can be rewritten as

$$z(t_{i+1}) = f(t_i, z(t_i)) + \frac{2}{3}g(t_i, z(t_i)) + b_\nu(t_i)\int_0^{t_{i+1}} \Psi'(s)\Phi(s)g(s,z(s))ds \tag{12}$$

Also, we have

$$z(t_i) = f(t_{i-1}, z(t_{i-1})) + \frac{2}{3}g(t_{i-1}, z(t_{i-1})) + b_\nu(t_i)\int_0^{t_i} \Psi'(s)\Phi(s)g(s,z(s))ds \tag{13}$$

Subtracting Equation (13) from Equation (12) gives

$$y(t_{i+1}) - z(t_i) = f(t_i, z(t_i) - f(t_{i-1}, y(t_{i-1}) + \frac{2}{3}(g(t_i, z(t_i)) - g(t_{i-1}, y(t_{i-1})) + b_\nu(t_i)\int_{t_i}^{t_{i+1}} \Psi'(s)\Phi(s)g(s,z(s))ds \tag{14}$$

In order to calculate Equation (14), we approximated the integral $\int_{t_i}^{t_{i+1}} f(s,z(s))ds$ by $\int_{t_i}^{t_{i+1}} P_2(s)ds$, where $P(s)$ is a Lagrange interpolating polynomial of degree two that can be calculated using the following formula:

$$P_2(t) = \sum_{j=0}^{2} f(t_{i-j}, z\left(t_{i-j}\right))L_j(t)$$

Where the $L_j(t)$ terms are the Lagrange basis polynomials on the three points (t_{i-2}, t_{i-1}, t_i).Using the change of variable $s = \frac{t_{i+1} \breve{} t}{\hbar}$.

Substituting for the Lagrange basis polynomials and integrating (for more details see ([2])), we obtain

$$\int_{t_i}^{t_{i+1}} \Psi'(s)\Phi(s)g(s,z(s))ds = \hbar\left[\frac{23}{12}\overline{f}(t_i, z_{i-2}) - \frac{4}{3}\overline{f}(t_{i-1}, z(t_{i-1})) + \frac{5}{12}\overline{f}(t_{i-2}, z(t_{i-2}))\right] \tag{15}$$

Where $\overline{f}(t,z(t)) = \Psi'(t)\Phi(t)g(t,z(t))$.
Hence

$$\int_{t_i}^{t_{i+1}} \overline{f}(s,z(s))ds = \hbar\left[\frac{23}{12}\overline{f}(t_i, z_i) - \frac{4}{3}\overline{f}(t_{i-1}, z_{i-1}) + \frac{5}{12}\overline{f}(t_{i-2}, z_{i-2})\right] \tag{16}$$

Where $z_i = z(t_i)$,$z_{i-1} = z(t_{i-1})$ and $z_{i-2} = z(t_{i-2})$.
Inserting (15) in eqs (14) ,we obtain the iterative formula as follows

$$\begin{aligned} z(t_{i+1}) - z(t_i) =& f(t_i, z_i) - f(t_{i-1}, z_{i-1}) + \frac{2}{3}(g(t_i, z_i) - g(t_{i-1}, z_{i-1})) \\ &+ \hbar * b_\nu(t_i)\left[\frac{23}{12}\overline{f}(t_i, z_i) - \frac{4}{3}\overline{f}(t_{i-1}, z_{i-1}) + \frac{5}{12}\overline{f}(t_{i-2}, z_{i-2})\right] \end{aligned} \tag{17}$$

5.2. *The truncation error*

The approximation by lagrange interpolating polynomial induces an error that we want to estimate,namely,

$$\overline{f}(t,z(t)) = P_2(t) + \overline{E}_2(t)$$

$$\overline{E}_2(t) = \frac{\overline{f}^{(3)}(\zeta_i, z(\zeta_i))}{3!}(t-t_i)(t-t_{i-1})(t-t_{i-2}), \quad \zeta_i \in (t_{i-2}, t_i)$$

One may calculate the truncation error for the three-step formula as follow.

$$\int_{t_i}^{t_{i+1}} \overline{E}_2(t)dt = \int_{t_i}^{t_{i+1}} \frac{\overline{f}^{(3)}\left(\zeta_i, z\left(\zeta_i\right)\right)}{3!} \left(t - t_i\right)\left(t - t_{i-1}\right)\left(t - t_{i-2}\right) dt$$

and by using the change of variable $\tau = \frac{t_{i+1} - t}{\hbar}$ to approximite the integral, we obtain

$$\int_{t_i}^{t_{i+1}} \overline{E}_2(t)dt = \frac{\hbar^3 \overline{f}^{(3)}\left(\zeta_i, z\left(\zeta_i\right)\right)}{6} \int_0^1 (\tau - 1)(\tau - 2)(\tau - 3)d\tau = \frac{3}{16}\hbar^3 \overline{f}^{(3)}\left(\zeta_i, z\left(\zeta_i\right)\right), \tag{18}$$

Where $\zeta_i \in (t_{i-2}, t_i)$. Denoting the entire right-hand side of Eq. (17) by $\tilde{z}_i$, then we have
$z_{i+1} = \tilde{z}_i + b_\nu(t)\frac{3}{16}\hbar^3 \overline{f}^{(3)}\left(\zeta_i, z\left(\zeta_i\right)\right)$.
Therefore, the local truncation error of the use of formula (17) is determined by

$$\frac{z_{i+1} - \tilde{z}_i}{\hbar} \leq \| b_\nu(t) \| \cdot \frac{\frac{3}{16}\hbar^3 \overline{f}^{(3)}\left(\zeta_i, z\left(\zeta_i\right)\right)}{\hbar}$$

This implies

$$\frac{z_{i+1} - \tilde{z}_i}{\hbar} \leq \frac{3}{32}\hbar^2 \overline{f}^{(3)}\left(\zeta_i, z\left(\zeta_i\right)\right) * \| b_\nu(t) \| \tag{19}$$

5.3. *Simulations*

The numerical simulation, which are based on a formula (17), are designed to simulate the behavior of the solutions $z(t)$ for different values of ν.

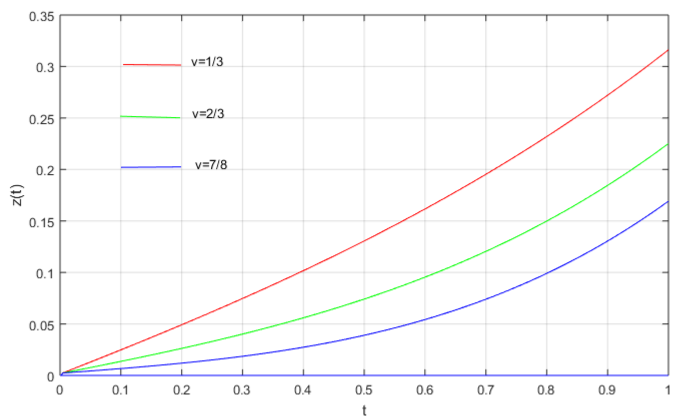

Figure 1: the time plots of the variable in model (10) when $\nu=\frac{1}{3}$, $\nu=\frac{2}{3}$, and $\nu=\frac{7}{8}$

For the fractional orders $\nu = \frac{1}{3}$, $\nu = \frac{2}{3}$, and $\nu = \frac{7}{8}$, figure (1) shows a time plot for the variable z(t) in model (9). We can see that when is increased, the curves of each state follow the same pattern. However, their values are somewhat different.

6. Conclusion

Using the $[\Psi, \Phi]$ Caputo-Fabrizio fractional derivative of order $\nu \in (0, 1)$, we defined solutions for the fractional hybrid with initial value problem in the current paper. Furthermore, the Banach fixed point theorem is used to demonstrate the existence and uniqueness of a solution to this problem. In addition, a nontrivial example is presented as an application to demonstrate our theoretical results.Finally, to obtain numerical solutions to the fractional equation of the exemple, the Three-step adams–bashforth techniques have been devised and implemented.

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