Filomat 39:1 (2025), 239–247
https://doi.org/10.2298/FIL2501239M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fractional Navier-Stokes equations regularity criteria in terms of deformation tensor

Dandan Ma[a], Fan Wu[a,*]

[a] *Key Laboratory of Engineering Mathematics and Advanced Computing of Nanchang Institute of Technology, College of Science, Nanchang Institute of Technology, Nanchang, Jiangxi 330099, China*

Abstract. We consider the singularity formation of strong solutions to the three-dimensional incompressible fractional Navier-Stokes equations in the whole space. By making use of the Bony decomposition technique, we prove that a unique local strong solution does not blow-up at time T if deformation tensor belongs to nonhomogeneous Besov spaces. As a bi-product, the result improves some well-known results on regularity for the particular case of classical Navier-Stokes equations.

1. Introduction

In this paper, we consider the Cauchy problem for the 3D fractional Navier-Stokes equations (FNSE for short)

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + (-\Delta)^\alpha u + \nabla p = 0, & x\in\mathbb{R}^3, t>0 \\ \nabla\cdot u = 0, & x\in\mathbb{R}^3, t>0 \\ u|_{t=0} = u_0(x), & x\in\mathbb{R}^3. \end{cases} \tag{1}$$

Here $u = u(x,t) : \mathbb{R}^3\times[0,\infty)\to\mathbb{R}^3$ is the velocity and the scalar function $p = p(x,t) : \mathbb{R}^3\times[0,\infty)\to\mathbb{R}$ is the total kinetic pressure. The constant α is a positive parameter to measure the dissipations. Moreover, the operator $(-\Delta)^\alpha$ is defined as follows:

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\hat{f}(\xi),$$

where $\hat{f}$ denotes the fourier transform of the function f. More details on $(-\Delta)^\alpha$ can be found in [21], as a notation, we take Λ as $(-\Delta)^{\frac{1}{2}}$.

It is well-known that the local and global-in time existence of strong solutions to the FNSE (1) were established by Lions [16] for $\alpha\geq\frac{5}{4}$ (also see [29]). In [29], Wu also showed that when $\alpha>0$, equations (1) with $u_0\in L^2$ possess a global weak solution and local in time strong solution for given initial value $u_0\in H^1$. Unfortunately, while such weak solutions are well suited to study in the sense that global-in-time existence is guaranteed for all finite energy initial data, they are not known to be either smooth or unique,

2020 *Mathematics Subject Classification*. Primary 35Q35 Secondary 76D05.
Keywords. Fractional Navier-Stokes equations, Regularity, Singularity formation, Bony decomposition.
Received: 02 October 2023; Revised: 11 August 2024; Accepted: 09 November 2024
Communicated by Maria Alessandra Ragusa
* Corresponding author: Fan Wu
Email address: wufan0319@yeah.net (Fan Wu)
ORCID iD: https://orcid.org/0000-0003-2367-4916 (Fan Wu)

leaving major problems for the well-posedness theory. On one hand, Katz-Pavlović [13] first showed that if $1 < \alpha < \frac{5}{4}$, the Hausdorff dimension of the singular set at the time of first possible blow-up is at most $(5-4\alpha)$. Recently, Tang-Yu [23] study the partial regularity of fractional Navier-Stokes equations (1) with $\frac{3}{4} < \alpha < 1$, more precisely, they show that the suitable weak solution is regular away from a relatively closed singular set whose $(5-4\alpha)$-dimentional Hausdorff measure is zero. Further partial regularity results of equations (1), we refer readers to [5, 20]. On the other hand, Zhou in [30] proved the following regularity condition

$$u \in L^p\left(0,T;L^q\left(\mathbb{R}^3\right)\right); \frac{2\alpha}{p}+\frac{3}{q} \leqslant 2\alpha-1, \frac{3}{2\alpha-1} < q \leqslant \infty, 1 \leq \alpha \leq \frac{3}{2}$$

or

$$\Lambda^\alpha u \in L^p\left(0,T;L^q\left(\mathbb{R}^3\right)\right); \frac{2\alpha}{p}+\frac{3}{q} \leqslant 3\alpha-1, \frac{3}{3\alpha-1} < q \leqslant \frac{3}{\alpha-1}, 1 \leq \alpha \leq \frac{5}{4}.$$

When $\alpha = 1$, equations (1) reduces to the classical incompressible NSE. In 1934, Leray showed that global existence of weak solutions to the 3D NSE, but are not known to be either smooth or unique. Ladyzhenskaya-Prodi-Serrin [24, 25] showed that if

$$u \in L^q\left(0,T;L^p\left(\mathbb{R}^3\right)\right) \quad \text{with} \quad \frac{2}{q}+\frac{3}{p}=1 \quad \text{and} \quad 3 < p \leq \infty, \tag{2}$$

then the weak solution u is regular on $(0,T]$. Escauriaza-Seregin-Sverak in [8] extended Serrin type criteria to the endpoint case, $p=3, q=\infty$. Recently, Tao [22] further extended this regularity criterion giving a quantitative lower bound on the rate of the blowup of the L^3-norm. This result is very slightly supercritical- in fact triple logarithmic- with respect to scaling, and is the first supercritical regularity criterion for the NSE.

Two crucially important objects for the study of the NSE are the strain, which is the symmetric gradient of the velocity, $S = \nabla_{sym} u$, with $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, and the vorticity, which is a vector that represents the antisymmetric part of the velocity and is given by $\omega = \nabla \times u$. Physically, the strain describes how a parcel of the fluid is deformed, while the vorticity describes how a parcel of the fluid is rotated [18].

The vorticity has been studied fairly exhaustively for its role in the dynamics of the NSE. For example, the Beale-Kato-Majda (BKM) regularity criterion [2], which holds for smooth solutions of both the Euler and NSE, states that if $T_{\max} < +\infty$, then

$$\int_0^{T_{\max}} \|\omega(\cdot,t)\|_{L^\infty} \mathrm{d}t = +\infty. \tag{3}$$

Beirão da Veiga [3] showed that any Leray-Hopf weak solution of NSE satisfies

$$\omega \in L^q\left(0,T;L^p\left(\mathbb{R}^3\right)\right) \quad \text{with} \quad \frac{2}{q}+\frac{3}{p}=2 \quad \text{and} \quad \frac{3}{2} < p < \infty \tag{4}$$

is actually smooth. Skalák used the nonhomogeneous Besov spaces and proved in [26] that if

$$\omega \in L^q\left(0,T;B^{-\frac{3}{p}}_{\infty,\infty}\right), \tag{5}$$

where $q \in [1,\infty)$ and $\frac{2}{q}+\frac{3}{p}=2$, then u is regular on $(0,T]$. There are many refined regularity criteria in terms of partial components of vorticity appeared over years. The readers can be referred to, for example [4, 6, 9–11] and the related references therein.

The relationship between strain (deformation tensor) and the singularity formation of NSE has been investigated much less thoroughly, but strain can provide some insights that do not follow as clearly from the vorticity. Motivated by [18, 26], in this paper we will consider the singularity formation of FNSE (1) via strain (deformation tensor $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ in nonhomogeneous Besov spaces. The main result of this paper state as follow:

Theorem 1.1. *Let $0 < \alpha < \frac{5}{4}$, $u_0 \in H^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, and assume that u be a unique local strong solution to FNSE* (1). *If $S = \left(S_{ij}\right) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$ satisfies the following condition*

$$S \in L^{\frac{2r\alpha}{2r\alpha-3}}\left(0, T; B^{-\frac{3}{r}}_{\infty,\infty}\right), \quad \frac{3}{2\alpha} < r \le \infty, \quad 0 < T < \infty. \tag{6}$$

Then the solution u can be extended beyond $T > 0$.

Remark 1.2. *Our Theorem 1.1 greatly improves the result of [26] by using the deformation tensor S instead of the Vorticity ω or the gradient of the velocity ∇u.*

Remark 1.3. *We remark here that the following continuous embeddings:*

$$L^p \hookrightarrow \dot{B}^0_{r,\infty} \hookrightarrow \dot{B}^{-\frac{3}{r}}_{\infty,\infty}, \quad r \in [1, \infty)$$

in the case of the homogeneous Besov spaces and

$$L^p \hookrightarrow B^0_{r,\infty} \hookrightarrow B^{-\frac{3}{r}}_{\infty,\infty}, \quad r \in [1, \infty]$$

in the case of the nonhomogeneous Besov spaces. In particular, the homogeneous version of the Besov space is smaller than the nonhomogeneous one, i.e.,

$$\dot{B}^r_{\infty,\infty} \subset B^r_{\infty,\infty}, \quad \textit{for all} \quad r < 0.$$

In view of these facts, Theorem 1 is the largest extension and extends/improves the corresponding result obtained recently in [7, 14, 26] and some references therein.

Remark 1.4. *By using the Littlewood-Paley decomposition technique, we can further extend our result to the general $\dot{B}^0_{p,\infty}$ space or the largest Vishik space (See [27, 28]). We will not go into details here and leave it to interested readers.*

Remark 1.5. *When $\alpha = 1$, the FNSE reduces to the classical NSE, thus our results improves/extendeds some of the classical and newly found results on NSE/FNSE regularity. In particular, Theorem 1.1 indicates that the singularity formation via strain of fluid to the FNSE* (1). *In addition, NSE regularity criteria for single components of the deformation tensor S is still an open problem, and we hope consider it in the near future.*

2. Preliminaries

In this section, we recall some definitions and give several lemmas, which will be used in the proof of theorem 1.1.

First we will introduce some notations. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x)e^{-ix\cdot\xi}dx.$$

Let (χ, φ) be a couple of smooth functions valued in $[0, 1]$ such that χ is supported in $B = \{\xi \in \mathbb{R}^3 : |\xi| \le \frac{4}{3}\}$, φ is supported in $C = \{\xi \in \mathbb{R}^3 : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{j\ge0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^3,$$

$$\sum_{j\in Z} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Denoting $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$, and then we define the homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operator $\dot{S}_j$ as follows:

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{3j}\int_{\mathbb{R}^3} h(2^j y)u(x-y)dy,$$

and

$$\dot{S}_j u = \chi(2^{-j}D) = 2^{3j}\int_{\mathbb{R}^3} \tilde{h}(2^j y)u(x-y)dy.$$

Informally, $\dot{\Delta}_j$ is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$, while $\dot{S}_j$ is a frequency projection to the ball $\{|\xi| \lesssim 2^j\}$. And one can easily verify that $\dot{\Delta}_j\dot{\Delta}_k f = 0$ if $|j-k| \geq 2$. We also define $\Delta_j f = \dot{\Delta}_j f$ if $j \geq 0$, $\Delta_{-1} f = \dot{S}_0 f$ and $\Delta_j f = 0$ if $j < -1$. Let now $p, r \in [1,\infty]$ and $s \in \mathbb{R}$.

Definition 2.1. *The nonhomogeneous Besov space $B^s_{p,r}$ is defined as*

$$B^s_{p,r} = \left\{ f \in S'; \|f\|_{B^s_{p,r}} < \infty \right\},$$

where

$$\begin{cases} \|f\|_{B^s_{p,r}} = \left(\sum_{j\geq -1} 2^{jsr} \left\|\Delta_j f\right\|_p^r \right)^{\frac{1}{r}}, \ r < \infty, \\ \|f\|_{B^s_{p,r}} = \sup_{j\geq -1} 2^{js} \left\|\Delta_j f\right\|_p, \ r = \infty. \end{cases}$$

Definition 2.2 (Homogeneous Besov spaces [1]). *Let*

$$S'_h = \left\{ f \in S'; \lim_{j\to-\infty} \left\|\dot{S}_j f\right\|_\infty = 0 \right\}.$$

The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as

$$\dot{B}^s_{p,r} = \left\{ f \in S'_h; \|f\|_{\dot{B}^s_{p,r}} < \infty \right\}$$

where

$$\begin{cases} \|f\|_{\dot{B}^s_{p,r}} = \left(\sum_{j\in Z} 2^{jsr} \left\|\dot{\Delta}_j f\right\|_p^r \right)^{\frac{1}{r}}, \ \text{if } r < \infty, \\ \|f\|_{\dot{B}^s_{p,r}} = \sup_{j\in Z} 2^{js} \left\|\dot{\Delta}_j f\right\|_p, \ r = \infty. \end{cases}$$

The following Bony decomposition will be used in the next section. Let two tempered distributions f and g, the Bony decomposition of the product fg can be formally written as

$$fg = T_f g + T_g f + R(f,g)$$

where

$$T_f g = \sum_{j\geq 1} \dot{S}_{j-1} f \Delta_j g$$

and

$$R(f,g) = \sum_{j,k\geq -1, |j-k|\leq 1} \Delta_j f \Delta_k g.$$

The main continuity properties of the operators T and R are described in the following lemmas [1].

Lemma 2.3. *Let $p, r, r_1, r_2 \in [1, \infty], 1/r = 1/r_1 + 1/r_2, s \in R, t < 0$ and $s + t < 3/p$. Then there exists a constant $c > 0$ such that*

$$\|T_u v\|_{B^{s+t}_{p,r}} \leq c\|u\|_{B^t_{\infty,r_1}} \|v\|_{B^s_{p,r_2}}$$

for every $u \in B^t_{\infty,r_1}$ and $v \in B^s_{p,r_2}$.

Lemma 2.4. *Let $p, p_1, p_2, r, r_1, r_2 \in [1, \infty], \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}, s_1, s_2 \in \mathbb{R}$ and $s_1 + s_2 > 0$. Then there exists a constant $C > 0$ such that*

$$\|R(f, g)\|_{B^{s_1+s_2}_{p,r}} \leq C\|f\|_{B^{s_1}_{p_1,r_1}} \|g\|_{B^{s_2}_{p_2,r_2}}$$

for every $f \in B^{s_1}_{p_1,r_1}$ and $g \in B^{s_2}_{p_2,r_2}$.

Lemma 2.5 (logarithmic Sobolev inequality in Besov spaces[15]). *Let $p, \varrho, \sigma \in [1, \infty], q \in [1, \infty)$ and $s > \frac{3}{q}$. Then there exists a constant c such that*

$$\|f\|_{L^\infty} \leq c\left(1 + \|f\|_{\dot{B}^{\frac{3}{p}}_{p,\varrho}} \left(\ln^+ \|f\|_{B^s_{q,\sigma}}\right)^{1-\frac{1}{\varrho}}\right) \quad \textit{for all } f \in \dot{B}^{\frac{3}{p}}_{p,\varrho} \cap B^s_{q,\sigma}.$$

Lemma 2.6 (Gagliardo-Nirenberg inequality[19]). *Let $0 \leq m, \alpha \leq l$, then we have*

$$\left\|\Lambda^\alpha f\right\|_{L^p(\mathbb{R}^3)} \leq C \left\|\Lambda^m f\right\|^{1-\theta}_{L^q(\mathbb{R}^3)} \left\|\Lambda^l f\right\|^\theta_{L^r(\mathbb{R}^3)},$$

where $\theta \in [0, 1]$ and α satisfies

$$\frac{\alpha}{3} - \frac{1}{p} = \left(\frac{m}{3} - \frac{1}{q}\right)(1 - \theta) + \left(\frac{l}{3} - \frac{1}{r}\right)\theta.$$

Here, when $p = \infty$, we require that $0 < \theta < 1$.

Lemma 2.7. *[17] For all $-\frac{3}{2} < \alpha < \frac{3}{2}$ and for all u divergence free in the sense that $\xi \cdot \hat{u}(\xi) = 0$ almost everywhere,*

$$\|S\|^2_{\dot{H}^\alpha} = \|A\|^2_{\dot{H}^\alpha} = \frac{1}{2}\|\omega\|^2_{\dot{H}^\alpha} = \frac{1}{2}\|\nabla \otimes u\|^2_{\dot{H}^\alpha}, \tag{7}$$

where symmetric part $S = S_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$, which we refer to as the strain tensor, anti-symmetric part $A = A_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)$, $\omega = \nabla \times u$.

3. The proof of Theorem 1.1

The proof is based on the establishment of a priori estimate for u that allows us to extend the smooth solution beyond time T. We will establish corresponding prior estimates in two cases.

Testing the first equation of (1) by u, we see that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} |\Lambda^\alpha u|^2\, dx = 0.$$

(I) Case 1: $S \in L^{\frac{2r\alpha}{2r\alpha-3}}\left(0, T; B^{-\frac{3}{r}}_{\infty,\infty}\right)$, $\frac{3}{2\alpha} < r < \infty$.

Taking $\nabla\times$ on the first equation of (1), one has

$$\partial_t\omega + (u \cdot \nabla)\omega + (-\Delta)^\alpha\omega - S\omega = 0, \tag{8}$$

taking the operator ∇_{sym} $\left(i.e., S = \nabla_{sym}(u)_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\right)$ to the first equation of (1) to obtain

$$\partial_t S + (u \cdot \nabla)S + (-\Delta)^\alpha S + S^2 + \frac{1}{4}\omega \otimes \omega - \frac{1}{4}|\omega|^2 I_3 + Hess(p) = 0, \tag{9}$$

where more details can refer to [17]. Multiplying (8) by ω and integrating over $\mathbb{R}^3$, we have

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\Lambda^\alpha \omega\|_{L^2}^2 = \int_{\mathbb{R}^3} S\omega \cdot \omega dx. \tag{10}$$

Testing (9) by S and integrating by parts in $\mathbb{R}^3$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|S\|_{L^2}^2 + \|\Lambda^\alpha S\|_{L^2}^2 &= -\int_{\mathbb{R}^3} S^2 \cdot Sdx - \frac{1}{4}\int_{\mathbb{R}^3} \omega\otimes\omega \cdot Sdx - \int_{\mathbb{R}^3} \mathrm{Hess}(p)\cdot Sdx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} |\omega|^2 I_3 \cdot Sdx \\ &= -\int_{\mathbb{R}^3} S^2 \cdot Sdx - \frac{1}{4}\int_{\mathbb{R}^3} \omega\otimes\omega \cdot Sdx,\end{aligned} \tag{11}$$

where we used the facts that

$$\left\langle |\omega|^2 I_3, S\right\rangle_{L^2} = 0, \quad \langle \mathrm{Hess}(p), S\rangle_{L^2} = 0.$$

Thanks to Lemma 2.5, it follows from (10) that

$$\frac{d}{dt}\|S\|_{L^2}^2 + 2\|\Lambda^\alpha S\|_{L^2}^2 = \int_{\mathbb{R}^3} S\omega\otimes\omega dx. \tag{12}$$

Combining (11) and (12), it yields that

$$\frac{d}{dt}\|S\|_{L^2}^2 + 2\|\Lambda^\alpha S\|_{L^2}^2 = -\frac{4}{3}\int_{\mathbb{R}^3} S^2 \cdot Sdx. \tag{13}$$

To estimate the integral term $-\frac{4}{3}\int_{\mathbb{R}^3} S^2 \cdot Sdx$, we will applying the Bony decomposition to

$$S^2 = R(S,S) + 2T_S S,$$

one concludes that

$$\int_{\mathbb{R}^3} S^2 \cdot Sdx = \int_{\mathbb{R}^3} SR(S,S) + 2\int_{\mathbb{R}^3} ST_S S.$$

It follows from Lemma 2.4 and Lemma 2.6 (Gagliardo-Nirenberg's inequality)that

$$\begin{aligned}\int_{\mathbb{R}^3} SR(S,S) &\le \|S\|_{B_{\infty,\infty}^{-\frac{3}{r}}} \|R(S,S)\|_{B_{1,1}^{\frac{3}{r}}} \\ &\le C\|S\|_{B_{\infty}^{-\frac{3}{r}}} \|S\|^2_{B_{2,2}^{\frac{3}{2r}}} \\ &\le C\|S\|_{B_{\infty,\infty}^{-\frac{3}{r}}} \|S\|^2_{H^{\frac{3}{2r}}} \\ &\le C\|S\|_{B_{\infty,\infty}^{-\frac{3}{r}}} \|S\|_{L^2}^{2-\frac{3}{\alpha r}} \|\Lambda^\alpha S\|_{L^2}^{\frac{3}{\alpha r}} \\ &\le C\|S\|^{\frac{2\alpha r}{2\alpha r-3}}_{B_{\infty,\infty}^{-\frac{3}{r}}} \|S\|_{L^2}^2 + \epsilon\|\Lambda^\alpha S\|_{L^2}^2.\end{aligned} \tag{14}$$

Applying Lemma 2.3 and the same tricks as (14) yields that

$$\begin{aligned}2\int_{\mathbb{R}^3} ST_S S &\le C\|S\|_{B_{2,2}^{\frac{3}{2r}}} \|T_S S\|_{B_{2,2}^{-\frac{3}{2r}}} \\ &\le C\|S\|_{B_{2,2}^{\frac{3}{2r}}} \|S\|_{B_{\infty,\infty}^{-\frac{3}{r}}} \|S\|_{B_{2,2}^{\frac{3}{2r}}} \\ &\le C\|S\|^{\frac{2\alpha r}{2\alpha r-3}}_{B_{\infty,\infty}^{-\frac{3}{r}}} \|S\|_{L^2}^2 + \epsilon\|\Lambda^\alpha S\|_{L^2}^2.\end{aligned} \tag{15}$$

Putting (14) and (15) into (13), we get

$$\frac{d}{dt}\|S\|_{L^2}^2+\|\Lambda^\alpha S\|_{L^2}^2\le C\|S\|_{B_{\infty,\infty}^{-\frac{3}{r}}}^{\frac{2\alpha r}{2\alpha r-3}}\|S\|_{L^2}^2. \tag{16}$$

Therefore Gronwall's inequality implies that for all $0<t\le T$ and $\frac{3}{2\alpha}<r<\infty$, one has

$$\sup_{0<t\le T}\|S\|_{L^2}^2+\int_0^T\|\Lambda^\alpha S\|_{L^2}^2dt\le\|S_0\|_{L^2}^2\exp C\int_0^T\|S\|_{B_{\infty,\infty}^{-\frac{3}{r}}}^{\frac{2\alpha r}{2\alpha r-3}}dt. \tag{17}$$

(II) Case 2: $S\in L^1\left(0,T;B_{\infty,\infty}^0\right)$.

Applying $\Lambda^3\left(\Lambda:=(-\Delta)^{\frac12}\right)$ to equations (1) and multiplying the resulting equations by Λ^3u, it yields that

$$\frac12\frac{d}{dt}\left\|\Lambda^3u(t)\right\|_{L^2}^2+\left\|\Lambda^{3+\alpha}u\right\|_{L^2}^2=-\int_{\mathbb{R}^3}\left[\Lambda^3,u\cdot\nabla\right]u\cdot\Lambda^3u\,dx, \tag{18}$$

here and in what follows $[\Lambda^s,f]\,g:=\Lambda^s(fg)-f\Lambda^sg$ stands for the standard commutator notation. Moreover, we have used the following identitiy

$$\int_{\mathbb{R}^2}u\cdot\nabla\Lambda^3u\cdot\Lambda^3udx=0.$$

We now resort to the following bilinear commutator estimate [12]

$$\left\|[\Lambda^s,f]\,g\right\|_{L^p}\le C\left(\|\nabla f\|_{L^{p_1}}\left\|\Lambda^{s-1}g\right\|_{L^{p_2}}+\left\|\Lambda^sf\right\|_{L^{p_3}}\|g\|_{L^{p_4}}\right), \tag{19}$$

with $s>0$ and $p_2,p_3\in(1,\infty)$ such that

$$\frac1p=\frac1{p_1}+\frac1{p_2}=\frac1{p_3}+\frac1{p_4}$$

According to inequality (19), we infer that

$$\begin{aligned}-\int_{\mathbb{R}^3}\left[\Lambda^3,u\cdot\nabla\right]u\cdot\Lambda^3u\,dx&\le\left\|\left[\Lambda^3,u\cdot\nabla\right]u\right\|_{L^2}\left\|\Lambda^3u\right\|_{L^2}\\&\le C\|\nabla u\|_{L^\infty}\left\|\Lambda^3u\right\|_{L^2}^2\end{aligned} \tag{20}$$

Substituting (20) into (18), we obtain

$$\frac{d}{dt}\|\Lambda^3u\|_{L^2}^2+2\|\Lambda^{3+\alpha}u\|_{L^2}^2\le C\|\nabla u\|_{L^\infty}\left\|\Lambda^3u\right\|_{L^2}^2 \tag{21}$$

Taking into account (21), and using Lemma 2.5 with $p=\varrho=\infty$ and $q=\sigma=s=2$, we get

$$\begin{aligned}\|\nabla u\|_\infty\le&C\left(1+\|\nabla u\|_{\dot B_{\infty,\infty}^0}\ln^+\|\nabla u\|_{B_{2,2}^2}\right)\\\le&C\left(1+\|\nabla u\|_{\dot B_{\infty,\infty}^0}\ln^+C\left(1+\left\|\nabla^3u\right\|_2\right)\right).\end{aligned} \tag{22}$$

Since

$$S=\nabla_{\rm sym}u\quad\Longleftrightarrow\quad\nabla u=-2\nabla\,{\rm div}(-\Delta)^{-1}S,$$

we have $\left(I-\dot S_2\right)\nabla u=(-\Delta)^{-1}\nabla\,{\rm div}\left(I-\dot S_2\right)S$ and

$$\begin{aligned}\|\nabla u\|_{\dot B_{\infty,\infty}^0}&\le C+\left\|\left(I-\dot S_2\right)\nabla u\right\|_{\dot B_{\infty,\infty}^0}\\&\le C+\left\|\left(I-\dot S_2\right)S\right\|_{\dot B_{\infty,\infty}^0}\\&\le C+C\sup_{j\ge1}\left\|\dot\Delta_jS\right\|_{L^\infty}\le C+C\|S\|_{B_{\infty,\infty}^0},\end{aligned}$$

where we used some facts in Besov spaces (see [26]). The following H^3-bound is an easy consequence of Gronwall's inequality

$$\max_{0\leq t\leq T} \|u(t)\|_{H^3} < \infty.$$

Thus, the proof of Theorem 1.1 is immediately complete.

Acknowledgments

We wish to thank the reviewer for a careful reading of the paper and useful comments that helped to clarify and correct its original version.

References

[1] H. Bahouri, J. Y. Chemin and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Springer, 2011.
[2] J.T. Beale, T. Kato, A. Majda. *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Communications in Mathematical Physics. **94**(1)(1984), 61–66.
[3] H. Beirao da Viega, *A new regularity class for the Navier-Stokes equations in* $\mathbb{R}^n$, Chinese Annals of Mathematics Series B. **16** (1995), 407–412.
[4] C.S. Cao, E.S. Titi, *Global regularity criterion for the 3D Navier-Stokes equations involving one entry of the velocity gradient tensor*, Archive for Rational Mechanics and Analysis. **202** (3)(2011) 919-932.
[5] Y. K. Chen, C.H. Wei. *Partial regularity of solutions to the fractional Navier-Stokes equations*, Discrete & Continuous Dynamical Systems-A. **36** (10)(2016), 5309.
[6] J.Y. Chemin, P. Zhang, *On the critical one component regularity for 3D Navier-Stokes system*, Annales Scientifiques De L École Normale Supérieure. **49**(1)(2016), 131–167.
[7] C-J O. *Regularity criterion for weak solutions to the 3D Navier-Stokes equations via two vorticity components in* BMO^{-1}, Nonlinear Analysis: Real World Applications, 59: 103271.
[8] L. Escauriaza, G. Seregin, $L_{3,\infty}$*-solutions of the Navier-Stokes equations and backward uniqueness*, Nonlinear Problems in Mathematical Physics & Related Topics II. 18(2003), 353–366.
[9] Y. Fadila, C. Allaloua, M. Oukessoua, *Existence and uniqueness result for a Navier problem involving Leray-Lions type operators in weighted Sobolev spaces*, Filomat. **38**(6)(2024) 2143-2155.
[10] P. Jiang, J. Ni, L. Zhu, *Global well-posedness and large-time behavior for the equilibrium diffusion model in radiation hydrodynamics*, Journal of Function Spaces. **2023** (1)(2023), 2891000.
[11] A. Kassymov, M.A. Ragusa, M.Ruzhansky, et al. *Stein-Weiss-Adams inequality on Morrey spaces*, Journal of Functional Analysis. **285** (11)(2023), 110152.
[12] T. Kato, G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Communications on Pure and Applied Mathematics. **41**(7)(1988), 891–907.
[13] N. Katz, N. Pavlović, *A cheap Caffarelli-Kohn-Nirenberg inequality for the Navier-Stokes equation with hyper-dissipation*, Geometric & Functional Analysis GAFA. **12** (2)(2002), 355–379.
[14] J. Kim, *Some regularity criteria for the 3D generalized Navier-Stokes equations*, Zeitschrift für angewandte Mathematik und Physik. **72** (3)(2021), 118.
[15] H. Kozono, T. Ogawa, Y. Taniuchi, *The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations*, Mathematische Zeitschrift. **242** (2)(2002),251–278.
[16] J. Lions, *Quelques méthodes de résolution des problémes aux limites non linéaires*, Dunod; Gauthier-Villars, Paris, 1969.
[17] E. Miller, *A regularity criterion for the Navier-Stokes equation involving only the middle eigenvalue of the strain tensor*, Archive for Rational Mechanics and Analysis. **235**(1) (2020), 99–139.
[18] E. Miller, *Finite-time blowup for a Navier-Stokes model equation for the self-amplification of strain*, Analysis & PDE. **16** (4)(2023), 997–1032.
[19] L. Nirenberg, *On elliptic partial differential equations. principio di minimo e sue applicazioni alle equazioni funzionali*, Springer, (2011) 1–48.
[20] W. Ren, Y.Q. Wang, G. Wu, *Partial regularity of suitable weak solutions to the multi-dimensional generalized magnetohydrodynamics equations*, Communications in Contemporary Mathematics. **18** (06)(2016) 1650018.
[21] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, 1970.
[22] T. Tao, *Quantitative bounds for critically bounded solutions to the Navier-Stokes equations*, arXiv preprint arXiv:1908.04958, 2019.
[23] L. Tang, Y. Yu, *Partial regularity of suitable weak solutions to the fractional Navier-Stokes equations*, Communications in Mathematical Physics. **334** (3)(2015), 1455–1482.
[24] G. Prodi, *Un teorema di unicita per le equazioni di Navier-Stokes*, Annali di Matematica Pura ed Applicata. 1959, **48** (1)(1959), 173–182.
[25] J. Serrin, *On the interior regularity of weak solutions of the Navier-Stokes equations*, Archive for Rational Mechanics & Analysis. **9** (1)(1962), 187–195.
[26] Z. Skalák, *Regularity criteria for weak solutions to the Navier-Stokes equations using the Besov spaces*, Preprint. http://mat. fsv. cvut. cz/finales/preprints, 2014.
[27] F. Wu, *Navier-Stokes regularity criteria in Vishik spaces*, Applied Mathematics & Optimization. **84** (Suppl 1)(2021), 39–53.

[28] F. Wu, *On continuation criteria for the double-diffusive convection system in Vishik spaces*, Applicable Analysis. **103**(9)(2024), 1693–1703.
[29] J.H. Wu, *Generalized MHD equations*, Journal of Differential Equations. **195** (2)(2003), 284–312.
[30] Y. Zhou, *Regularity criteria for the generalized viscous MHD equations*, Annales de l'Institut Henri Poincare (C) Non Linear Analysis. **24** (3)(2007), 491–505.