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Unified approach to Carlitz and Cigler q-analogue for the bi-periodic Fibonacci and Lucas sequences

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Abstract. In this study, our aim is to establish and generalize a q-analogue for the bi-periodic Fibonacci and Lucas polynomials. We introduce two types of q-analogue for the bi-periodic Lucas polynomials, namely the q-bi-periodic Lucas polynomials of the first and second kinds. We extend and unify various aspects including explicit forms, recurrence relations, generating functions, and other combinatorial properties. Moreover, we give a q-analogue of the relationship between bi-periodic Fibonacci sequence and bi-periodic second-order recurrences.

1. Introduction

The bi-periodic Fibonacci and Lucas sequences are defined recursively, for $n \geq 2$, by

$$t_n = \begin{cases} at_{n-1} + t_{n-2}, & \text{if } n \text{ is even,} \\ bt_{n-1} + t_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \quad \text{and} \quad l_n = \begin{cases} bl_{n-1} + l_{n-2}, & \text{if } n \text{ is even,} \\ al_{n-1} + l_{n-2}, & \text{if } n \text{ is odd,} \end{cases}$$

with initial values $t_0 = 0$, $t_1 = 1$, $l_0 = 2$, and $l_1 = a$, where a and b are nonzero real numbers (see [7, 13, 18]). Note that if $a = b = 1$, then t_n and l_n correspond to the n-th Fibonacci and Lucas numbers, respectively.

Let $q \in \mathbb{C}$ be an indeterminate. The q-integer and q-factorial of the number n are defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \neq 1, \\ n, & \text{if } q = 1 \end{cases} \quad \text{and} \quad [n]_q! = \begin{cases} [n]_q[n-1]_q[n-2]_q \cdots [1]_q, & \text{if } n \neq 0, \\ 1, & \text{if } n = 0. \end{cases}$$

The Gaussian or q-binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad (0 \leq k \leq n),$$

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with $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ for $n < k$, where $(x;q)_n$ is the q-shifted factorial, defined as $(x;q)_0 = 1$ and $(x;q)_n = \prod_{j=0}^{n-1}(1-xq^j)$.

The q-difference operator D_q is defined as follows

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

There exist several different q-analogues of the Fibonacci and Lucas polynomials, as well as extensive research on the subject; see, for example, [1, 4, 6, 8–12, 15]. In particular, Cigler [12] proposed a unified approach for the q-Fibonacci and q-Lucas polynomials as follows

$$\Phi_{n+1}(x,y,m,q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q x^{n-2k} y^k, \quad n \geq 0, \tag{1}$$

$$\Lambda_n(x,y,m,q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{(m+1)\binom{k}{2}} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q x^{n-2k} y^k, \quad n \geq 1. \tag{2}$$

When $m = 1$, these expressions lead to the well-known Carlitz-type q-Fibonacci and q-Lucas polynomials (see [8, 12]), which are given by

$$F_{n+1}(x,y,q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q x^{n-2k} y^k, \quad n \geq 0,$$

$$L_n(x,y,q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k^2-k} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q x^{n-2k} y^k, \quad n \geq 1.$$

For $m = 0$, we recover Cigler-type q-Fibonacci and q-Lucas polynomials (see [9–12]), expressed as

$$\mathbf{F}_{n+1}(x,y,q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q x^{n-2k} y^k, \quad n \geq 0,$$

$$Luc_n(x,y,q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q x^{n-2k} y^k, \quad n \geq 1.$$

Recently, in [14] Ramirez and Sirvent defined a Carlitz-type q-analogue of the bi-periodic Fibonacci sequence, namely q-bi-periodic Fibonacci sequence, for any integer $n \geq 2$, as follows

$$F_n^{(a,b)}(y,q) = \begin{cases} aF_{n-1}^{(a,b)}(y,q) + q^{n-2} y F_{n-2}^{(a,b)}(y,q), & \text{if } n \text{ is even}, \\ bF_{n-1}^{(a,b)}(y,q) + q^{n-2} y F_{n-2}^{(a,b)}(y,q), & \text{if } n \text{ is odd}, \end{cases}$$

with $F_0^{(a,b)}(y,q) = 0$ and $F_1^{(a,b)}(y,q) = 1$. They also obtained the explicit formula for the q-bi-periodic Fibonacci sequence

$$F_{n+1}^{(a,b)}(y,q) = a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q (ab)^{\lfloor n/2 \rfloor - k} y^k,$$

where $\xi(n) = n - 2\lfloor n/2 \rfloor$, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd. Motivated by the results in [14], Tan in [16] introduced a Carlitz-type q-analogue of the bi-periodic Lucas sequence as

$$l_n^{(a,b)}(y,q) = a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k^2-k} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q (ab)^{\lfloor n/2 \rfloor - k} y^k, \quad n \geq 1,$$

with $l_0^{(a,b)}(y,q) = 2$. Additionally, she provided a matrix representation of the q-bi-periodic Fibonacci sequence, which can be expressed as follows

$$C\left(\chi_n, q^{n-1}y\right)C\left(\chi_{n-1}, q^{n-2}y\right)\cdots C\left(\chi_1, y\right) = \begin{pmatrix} yF_{n-1}^{(a,b)}(qy,q) & \left(\frac{b}{a}\right)^{\xi(n+1)} F_n^{(a,b)}(y,q) \\ yF_n^{(a,b)}(qy,q) & \left(\frac{b}{a}\right)^{\xi(n)} F_{n+1}^{(a,b)}(y,q) \end{pmatrix},$$

where $C\left(\chi_n, y\right) = \begin{pmatrix} 0 & 1 \\ y & \chi_n \end{pmatrix}$ and $\chi_n := a^{\xi(n+1)}b^{\xi(n)}$.

The authors in [3], defined a Cigler-type q-analogue of the bi-periodic Fibonacci and Lucas polynomials as

$$\mathbf{F}_n^{(a,b)}(x,y,q) = U_n^{(a,b)}\left(x + \frac{(q-1)\,y}{ab}D_q, y\right)\mathbf{1} \quad \text{and} \quad Luc_n^{(a,b)}(x,y,q) = V_n^{(a,b)}\left(x + \frac{q-1}{ab}yD_q, y\right)\mathbf{1},$$

where $\mathbf{1} = \mathbf{1}(x,y) = 1$ is a constant polynomial, and $U_n^{(a,b)}(x,y)$ and $V_n^{(a,b)}(x,y)$ represent the bivariate bi-periodic Fibonacci and Lucas polynomials defined in [17] and [2], respectively, as follows

$$U_0^{(a,b)}(x,y) = 0,\ U_1^{(a,b)}(x,y) = 1,\ \text{and}\ U_n^{(a,b)}(x,y) = \begin{cases} axU_{n-1}^{(a,b)}(x,y) + yU_{n-2}^{(a,b)}(x,y), & \text{if } n \text{ is even}, \\ bxU_{n-1}^{(a,b)}(x,y) + yU_{n-2}^{(a,b)}(x,y), & \text{if } n \text{ is odd}, \end{cases} \quad (n \geq 2),$$

$$V_0^{(a,b)}(x,y) = 2,\ V_1^{(a,b)}(x,y) = ax,\ \text{and}\ V_n^{(a,b)}(x,y) = \begin{cases} bxV_{n-1}^{(a,b)}(x,y) + yV_{n-2}^{(a,b)}(x,y), & \text{if } n \text{ is even}, \\ axV_{n-1}^{(a,b)}(x,y) + yV_{n-2}^{(a,b)}(x,y), & \text{if } n \text{ is odd}, \end{cases} \quad (n \geq 2).$$

They satisfy the following recurrence relations for $n \geq 2$,

$$\mathbf{F}_n^{(a,b)}(x,y,q) = a^{\xi(n+1)}b^{\xi(n)}\left(x + \frac{q-1}{ab}yD_q\right)\mathbf{F}_{n-1}^{(a,b)}(x,y,q) + y\mathbf{F}_{n-2}^{(a,b)}(x,y,q), \tag{3}$$

$$Luc_n^{(a,b)}(x,y,q) = a^{\xi(n+1)}b^{\xi(n)}\left(x + \frac{q-1}{ab}yD_q\right)Luc_{n-1}^{(a,b)}(x,y,q) + yLuc_{n-2}^{(a,b)}(x,y,q), \tag{4}$$

with the initial values $\mathbf{F}_0^{(a,b)}(x,y,q) = 0$, $\mathbf{F}_1^{(a,b)}(x,y,q) = 1$ and $Luc_0^{(a,b)}(x,y,q) = 2$, $Luc_1^{(a,b)}(x,y,q) = ax$. The recurrence in which the operator $(q-1)yD_q$ appears is called a D-recurrence. We also provided the following explicit formulas

$$\mathbf{F}_{n+1}^{(a,b)}(x,y,q) = a^{\xi(n)}\sum_{k=0}^{\lfloor n/2\rfloor} q^{\binom{k+1}{2}}\begin{bmatrix} n-k \\ k \end{bmatrix}_q (ab)^{\lfloor n/2\rfloor - k}\, x^{n-2k}y^k,\ \ n \geq 0, \tag{5}$$

$$Luc_n^{(a,b)}(x,y,q) = a^{\xi(n)}\sum_{k=0}^{\lfloor n/2\rfloor} q^{\binom{k}{2}}\frac{[n]_q}{[n-k]_q}\begin{bmatrix} n-k \\ k \end{bmatrix}_q (ab)^{\lfloor n/2\rfloor - k}\, x^{n-2k}y^k,\ \ n \geq 1. \tag{6}$$

The Cigler-type q-bi-periodic Fibonacci polynomials satisfy the following recurrence relations for $n \geq 2$,

$$\mathbf{F}_n^{(a,b)}(x,y,q) = a^{\xi(n-1)}b^{\xi(n)}x\mathbf{F}_{n-1}^{(a,b)}(x,y,q) + q^{n-2}y\mathbf{F}_{n-2}^{(a,b)}(x,\frac{y}{q},q), \tag{7}$$

$$\mathbf{F}_n^{(a,b)}(x,y,q) = a^{\xi(n-1)}b^{\xi(n)}x\mathbf{F}_{n-1}^{(a,b)}(x,qy,q) + qy\mathbf{F}_{n-2}^{(a,b)}(x,qy,q). \tag{8}$$

Furthermore, in [3], two types of q-bi-periodic Lucas polynomials were introduced: the q-bi-periodic Lucas polynomials of the first kind and the second kind. These polynomials are defined for $n \geq 1$ as follows

$$\mathbf{L}_n^{(a,b)}(x,y,q) = a^{\xi(n)}\sum_{k=0}^{\lfloor n/2\rfloor} q^{\binom{k}{2}}\begin{bmatrix} n-k \\ k \end{bmatrix}_q (ab)^{\lfloor n/2\rfloor - k}\left(1 + \frac{[k]_q}{[n-k]_q}\right)x^{n-2k}y^k,$$

$$\mathbb{L}_n^{(a,b)}(x,y,q) = a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q (ab)^{\lfloor n/2 \rfloor - k} \left(1 + q^{n-2k} \frac{[k]_q}{[n-k]_q}\right) x^{n-2k} y^k,$$

with $\mathbf{L}_0^{(a,b)}(x,y,q) = \mathbb{L}_0^{(a,b)}(x,y,q) = 2$.

In this paper, we present a unified and generalized approach that combines Carlitz's and Cigler's approaches for the q-analogue of the bi-periodic Fibonacci and Lucas polynomials. We introduce two types of q-analogue for the bi-periodic Lucas polynomials, namely the q-bi-periodic Lucas polynomials of the first and second kinds. We extend and unify explicit forms, recurrence relations, generating functions, and other combinatorial properties. In addition, we expand the results in [6] to the generalized q-bi-periodic Fibonacci polynomials. We establish the q-analogue of the relationship between bi-periodic Fibonacci sequence and bi-periodic second-order recurrences.

2. Generalized q-bi-periodic Fibonacci and Lucas polynomials

In this section, we will begin by defining the generalized q-bi-periodic Fibonacci and Lucas polynomials.

Definition 2.1. *The generalized q-bi-periodic Fibonacci and Lucas polynomials are defined, respectively, as*

$$\mathcal{F}_n(x,y,m,q) := \mathfrak{U}_m\left(\mathbf{F}_n^{(a,b)}(x,y,q)\right) \quad and \quad \mathcal{L}_n(x,y,m,q) := \mathfrak{U}_m\left(Luc_n^{(a,b)}(x,y,q)\right),$$

where $\mathfrak{U}_m$ be the linear operator on the polynomials in y defined by

$$\mathfrak{U}_m y^k = q^{m\binom{k}{2}} y^k.$$

This definition leads to the following theorem.

Theorem 2.2. *For all $m \in \mathbb{Z}$, we have*

$$\mathcal{F}_{n+1}(x,y,m,q) = a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2} + m\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q (ab)^{\lfloor n/2 \rfloor - k} x^{n-2k} y^k, \quad n \geqslant 0, \tag{9}$$

$$\mathcal{L}_n(x,y,m,q) = a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{(1+m)\binom{k}{2}} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q (ab)^{\lfloor n/2 \rfloor - k} x^{n-2k} y^k, \quad n \geqslant 1. \tag{10}$$

Proof. We apply the operator $\mathfrak{U}_m$ to (5) and (6), yielding the results. □

Remark 2.3. *It is clear that $\mathfrak{U}_1\left(\mathbf{F}_n^{(a,b)}(x,y,q)\right) = F_n^{(a,b)}(x,y,q)$ and $\mathfrak{U}_1\left(Luc_n^{(a,b)}(x,y,q)\right) = l_n^{(a,b)}(x,y,q)$ provides the Carlitz approach for the q-bi-periodic Fibonacci and Lucas polynomials.*

Lemma 2.4. *For all $i, j \in \mathbb{Z}$, we have*

$$\mathfrak{U}_m\left(y^j \mathbf{F}_n^{(a,b)}(x, q^i y, q)\right) = q^{m\binom{j}{2}} y^j \mathcal{F}_n(x, q^{i+mj} y, m, q), \tag{11}$$

$$\mathfrak{U}_m\left(y^j Luc_n^{(a,b)}(x, q^i y, q)\right) = q^{m\binom{j}{2}} y^j \mathcal{L}_n(x, q^{i+mj} y, m, q). \tag{12}$$

Proof. From Definition 2.1, we obtain

$$\begin{aligned}
\mathfrak{U}_m\left(y^j \mathbf{F}_n^{(a,b)}(x, q^i y, q)\right) &= \mathfrak{U}_m\left(a^{\xi(n-1)} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} q^{\binom{k+1}{2}} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q (ab)^{\lfloor (n-1)/2 \rfloor - k} q^{ik} x^{n-2k-1} y^{k+j}\right) \\
&= a^{\xi(n-1)} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} q^{\binom{k+1}{2}} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q (ab)^{\lfloor (n-1)/2 \rfloor - k} q^{ik} x^{n-2k-1} q^{m\binom{k+j}{2}} y^{k+j} \\
&= a^{\xi(n-1)} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} q^{\binom{k+1}{2} + m\binom{k}{2}} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q (ab)^{\lfloor (n-1)/2 \rfloor - k} q^{m\binom{j}{2} + (i+mj)k} x^{n-2k-1} y^{k+j} \\
&= q^{m\binom{j}{2}} y^j \mathcal{F}_n(x, q^{i+mj} y, m, q).
\end{aligned}$$

In a similar way, we find (12). □

The polynomials $\mathcal{F}_n(x,y,m,q)$ and $\mathcal{L}_n(x,y,m,q)$ satisfy the following D-recurrence relations.

Theorem 2.5. *For $n \geq 2$, we have*

$$\mathcal{F}_n(x,y,m,q) = a^{\xi(n+1)} b^{\xi(n)} x\mathcal{F}_{n-1}(x,y,m,q) + \frac{q-1}{ab} yD_q\mathcal{F}_{n-1}^{(a,b)}(x,q^m y,m,q) + y\mathcal{F}_{n-2}(x,q^m y,m,q),$$

$$\mathcal{L}_n(x,y,m,q) = a^{\xi(n+1)} b^{\xi(n)} x\mathcal{L}_{n-1}(x,y,m,q) + \frac{q-1}{ab} yD_q\mathcal{L}_{n-1}(x,q^m y,m,q) + y\mathcal{L}_{n-2}(x,q^m y,m,q),$$

with the initial values $\mathcal{F}_0(x,y,m,q) = 0$, $\mathcal{F}_1(x,y,m,q) = 1$, $\mathcal{L}_0(x,y,m,q) = 2$, and $\mathcal{L}_1(x,y,m,q) = ax$.

Proof. Applying $\mathfrak{U}_m$ to (3) and (4) and using Lemma 2.4, we obtain results. □

The polynomials $\mathcal{F}_n(x,y,m,q)$ satisfy further recurrence relations, as shown by the following theorem.

Theorem 2.6. *For $n \geqslant 2$, we have*

$$\mathcal{F}_n(x,y,m,q) = a^{\xi(n-1)} b^{\xi(n)} x\mathcal{F}_{n-1}(x,y,m,q) + q^{n-2} y\mathcal{F}_{n-2}(x,q^{m-1} y,m,q), \tag{13}$$

$$\mathcal{F}_n(x,y,m,q) = a^{\xi(n-1)} b^{\xi(n)} x\mathcal{F}_{n-1}(x,qy,m,q) + qy\mathcal{F}_{n-2}(x,q^{m+1} y,m,q). \tag{14}$$

Proof. Applying $\mathfrak{U}_m$ to equation (7), and using (11), we obtain

$$\begin{aligned}\mathfrak{U}_m\left(\mathbf{F}_n^{(a,b)}(x,y,q)\right) &= \mathfrak{U}_m\left(a^{\xi(n-1)} b^{\xi(n)} x\mathbf{F}_{n-1}^{(a,b)}(x,y,q) + q^{n-2} y\mathbf{F}_{n-2}^{(a,b)}(x,\frac{y}{q},q)\right)\\ &= a^{\xi(n-1)} b^{\xi(n)} x\mathcal{F}_{n-1}(x,y,m,q) + q^{n-2} y\mathcal{F}_{n-2}(x,q^{m-1} y,m,q).\end{aligned}$$

Applying $\mathfrak{U}_m$ to equation (8), and using (11), we obtain

$$\begin{aligned}\mathfrak{U}_m\left(\mathbf{F}_n^{(a,b)}(x,y,q)\right) &= \mathfrak{U}_m\left(a^{\xi(n-1)} b^{\xi(n)} x\mathbf{F}_{n-1}^{(a,b)}(x,qy,q) + qy\mathbf{F}_{n-2}^{(a,b)}(x,qy,q)\right)\\ &= a^{\xi(n-1)} b^{\xi(n)} x\mathcal{F}_{n-1}(x,qy,m,q) + qy\mathcal{F}_{n-2}(x,q^{m+1} y,m,q).\end{aligned}$$

□

In the following proposition, we express the generalized q-bi-periodic Lucas polynomials in terms of the generalized q-bi-periodic Fibonacci polynomials.

Proposition 2.7. *For $n \geq 1$, we have*

$$\mathcal{L}_n(x,y,m,q) = \mathcal{F}_{n+1}(x,y,m,q) + y\mathcal{F}_{n-1}(x,q^m y,m,q), \tag{15}$$

$$\mathcal{L}_n(x,qy,m,q) = \mathcal{F}_{n+1}(x,y,m,q) + q^n y\mathcal{F}_{n-1}(x,q^m y,m,q). \tag{16}$$

Proof. Using the explicit formulas for the generalized q-bi-periodic Fibonacci and Lucas polynomials, we obtain

$$\begin{aligned}\mathcal{L}_n(x,y,m,q) &= a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{(1+m)\binom{k}{2}} \left(q^k \begin{bmatrix} n-k \\ k \end{bmatrix}_q + \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q\right) (ab)^{\lfloor n/2 \rfloor - k} x^{n-2k} y^k\\ &= a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2} + m\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q (ab)^{\lfloor n/2 \rfloor - k} x^{n-2k} y^k\\ &\quad + a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2} + m\binom{k}{2}} \begin{bmatrix} n-k-2 \\ k \end{bmatrix}_q (ab)^{\lfloor n/2 \rfloor - k-1} x^{n-2k-2} q^{mk} y^{k+1}\\ &= \mathcal{F}_{n+1}(x,y,m,q) + y\mathcal{F}_{n-1}(x,q^m y,m,q).\end{aligned}$$

In similar ways, we prove the second relation. □

3. Generalized q-bi-periodic Lucas polynomials of the first and second kinds

The polynomials $\mathcal{L}_n(x, y, m, q)$ do not follow simple recurrences like those in (13) and (14). Therefore, we introduce new types of q-analogues of bi-periodic Lucas polynomials, known as the generalized q-bi-periodic Lucas polynomials of the first kind and the generalized q-bi-periodic Lucas polynomials of the second kind.

Definition 3.1. *The generalized q-bi-periodic Lucas polynomials of the first kind, denoted as $\mathbf{P}_n(x, y, m, q)$, and the second kind, denoted as $\mathbb{P}_n(x, y, m, q)$, are defined by*

$$\mathbf{P}_n(x, y, m, q) = \mathfrak{U}_m\left(\mathbf{L}_n^{(a,b)}(x, y, q)\right), \tag{17}$$

$$\mathbb{P}_n(x, y, m, q) = \mathfrak{U}_m\left(\mathbb{L}_n^{(a,b)}(x, y, q)\right). \tag{18}$$

This definition leads to the following theorem.

Theorem 3.2. *For $n \geq 1$ and $m \in \mathbb{Z}$, we have*

$$\mathbf{P}_n(x, y, m, q) = a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{(m+1)\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \left(1 + \frac{[k]_q}{[n-k]_q}\right) (ab)^{\lfloor n/2 \rfloor - k} x^{n-2k} y^k,$$

$$\mathbb{P}_n(x, y, m, q) = a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2} + m\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \left(1 + q^{n-2k} \frac{[k]_q}{[n-k]_q}\right) (ab)^{\lfloor n/2 \rfloor - k} x^{n-2k} y^k,$$

with $\mathbf{P}_0(x, y, m, q) = \mathbb{P}_0(x, y, m, q) = 2$.

Note that for $m = 0$, we obtain the Cigler-type q-bi-periodic Lucas sequence of the first and second kinds as defined in [3], and for $m = 1$, we obtain the Carlitz-type q-bi-periodic Lucas sequence of the first and second kinds, defined for $n \geq 1$, as

$$\mathbf{P}_n(x, y, 1, q) = a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k^2-k} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \left(1 + \frac{[k]_q}{[n-k]_q}\right) (ab)^{\lfloor n/2 \rfloor - k} x^{n-2k} y^k,$$

$$\mathbb{P}_n(x, y, 1, q) = a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \left(1 + q^{n-2k} \frac{[k]_q}{[n-k]_q}\right) (ab)^{\lfloor n/2 \rfloor - k} x^{n-2k} y^k,$$

with $\mathbf{P}_0(x, y, 1, q) = \mathbb{P}_0(x, y, 1, q) = 2$.

In the following result, we state the generalized q-bi-periodic Lucas polynomials of both kinds in terms of the generalized q-bi-periodic Fibonacci polynomials.

Theorem 3.3. *For $n \geq 1$, we have*

$$\mathbf{P}_n(x, y, m, q) = \mathcal{F}_{n+1}(x, y/q, m, q) + y\mathcal{F}_{n-1}(x, q^m y, m, q), \tag{19}$$

$$\mathbb{P}_n(x, y, m, q) = \mathcal{F}_{n+1}(x, y, m, q) + q^{n-1} y\mathcal{F}_{n-1}(x, q^{m-1} y, m, q). \tag{20}$$

Proof. From Theorem 3.2, we have

$$\begin{aligned}
\mathbf{P}_n(x, y, m, q) &= a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{(m+1)\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q (ab)^{\lfloor n/2 \rfloor} x^{n-2k} y^k + a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{(m+1)\binom{k}{2}} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q (ab)^{\lfloor n/2 \rfloor - k} x^{n-2k} y^k \\
&= a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2} + m\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q (ab)^{\lfloor n/2 \rfloor} x^{n-2k} (y/q)^k \\
&\quad + a^{\xi(n)} y \sum_{k=0}^{\lfloor n/2 \rfloor - 1} q^{\binom{k+1}{2} + m\binom{k}{2}} \begin{bmatrix} n-k-2 \\ k \end{bmatrix}_q (ab)^{\lfloor n/2 \rfloor - k - 1} x^{n-2k-2} (q^m y)^k \\
&= \mathcal{F}_{n+1}(x, y/q, m, q) + y\mathcal{F}_{n-1}(x, q^m y, m, q).
\end{aligned}$$

In a similar way, we obtain the second identity. □

The recurrence relations satisfied by the q-bi-periodic Lucas polynomials of the first and second kinds are as follows.

Theorem 3.4. *For $n \geq 2$, we have*

$$\mathbf{P}_n(x,y,m,q) = a^{\xi(n)}b^{\xi(n-1)}x\mathbf{P}_{n-1}(x,y,m,q) + q^{n-2}y\mathbf{P}_{n-2}(x,q^{m-1}y,m,q), \tag{21}$$

$$\mathbb{P}_n(x,y,m,q) = a^{\xi(n)}b^{\xi(n-1)}x\mathbb{P}_{n-1}(x,qy,m,q) + qy\mathbb{P}_{n-2}(x,q^{m+1}y,m,q), \tag{22}$$

with the initial values $\mathbf{P}_0(x,y,m,q) = \mathbb{P}_0(x,y,m,q) = 2$ and $\mathbf{P}_1(x,y,m,q) = \mathbb{P}_1(x,y,m,q) = ax$.

Proof. Using the recurrence relations (13) and (14), along with the relations (19) and (20), we obtain the results. □

The Carlitz-type q-bi-periodic Lucas polynomials of the first and second kinds satisfy the following recurrence relations.

Corollary 3.5. *For $n \geq 2$, we obtain*

$$\mathbf{P}_n(x,y,1,q) = a^{\xi(n)}b^{\xi(n-1)}x\mathbf{P}_{n-1}(x,y,1,q) + q^{n-2}y\mathbf{P}_{n-2}(x,y,1,q),$$

$$\mathbb{P}_n(x,y,1,q) = a^{\xi(n)}b^{\xi(n-1)}x\mathbb{P}_{n-1}(x,qy,1,q) + qy\mathbb{P}_{n-2}(x,q^2y,1,q),$$

with the initial values $\mathbf{P}_0(x,y,1,q) = \mathbb{P}_0(x,y,1,q) = 2$ and $\mathbf{P}_1(x,y,1,q) = \mathbb{P}_1(x,y,1,q) = ax$.

Proposition 3.6. *For $n \geq 1$, we obtain*

$$\mathbf{P}_n(x,y,m,q) = 2\mathcal{F}_{n+1}(x,\frac{y}{q},m,q) - a^{\xi(n)}b^{\xi(n-1)}x\mathcal{F}_n(x,y,m,q),$$

$$\mathbb{P}_n(x,y,m,q) = 2\mathcal{F}_{n+1}(x,y,m,q) - a^{\xi(n)}b^{\xi(n-1)}x\mathcal{F}_n(x,y,m,q).$$

Proof. Using the recurrence relations (13), (14), (19), and (20), we arrive at the results obtained. □

Remark 3.7. *For $n \geq 1$, we deduce the following identities*

$$\mathbf{P}_n(x,y,m,q) = \mathcal{F}_n(x,y,m,q) + 2y\mathcal{F}_{n-1}(x,q^my,m,q),$$

$$\mathbb{P}_n(x,y,m,q) = \mathcal{F}_n(x,y,m,q) + 2q^{n-1}y\mathcal{F}_{n-1}(x,q^{m-1}y,m,q).$$

The generalized q-bi-periodic Fibonacci and Lucas polynomials satisfy the following properties.

Proposition 3.8. *For $n \geq 0$ and $m \in \mathbb{Z}$, we have*

$$\mathcal{L}_n(x,y,m,q) = \frac{1}{2}\left(\mathbf{P}_n(x,y,m,q) + \mathbb{P}_n(x,y,m,q)\right) \tag{23}$$

and for $y \neq 0$, we obtain

$$\mathcal{F}_{-n}(x,y,m,q) = (-1)^{n+1}\frac{q^{m\binom{n+1}{2}}}{y^n}\mathcal{F}_n(x,q^{-mn}y,m,q), \tag{24}$$

$$\mathcal{L}_{-n}(x,y,m,q) = (-1)^{n}\frac{q^{m\binom{n+1}{2}}}{y^n}\mathcal{L}_n(x,q^{-mn}y,m,q), \tag{25}$$

$$\mathbf{P}_{-n}(x,y,m,q) = (-1)^{n}\frac{q^{m\binom{n+1}{2}}}{y^n}\mathbb{P}_n(x,q^{-mn}y,m,q), \tag{26}$$

$$\mathbb{P}_{-n}(x,y,m,q) = (-1)^{n}\frac{q^{m\binom{n+1}{2}}}{y^n}\mathbf{P}_n(x,q^{-mn}y,m,q). \tag{27}$$

Remark 3.9. *From* (24), (25), (26), *and* (27), *the identities* (13), (14), (15), (16), (21), *and* (22) *holds for all $n \in \mathbb{Z}$.*

4. Generating Function

In this section, we derive the generating functions for the generalized q-bi-periodic Fibonacci and Lucas polynomials. We begin by stating the relationship between the generalized q-bi-periodic Fibonacci polynomials $\mathcal{F}_n(x, y, m, q)$ and the classical q-Fibonacci polynomials $\Phi_n(x, y, m, q)$.

Lemma 4.1. *For $n \geq 0$, we have*

$$\mathcal{F}_n(x, y, m, q) = \frac{\tau}{2}\Phi_n(\sqrt{ab}x, y, m, q) - (-1)^n \frac{\bar{\tau}}{2}\Phi_n(\sqrt{ab}x, y, m, q),$$

where $\tau = 1 + \sqrt{\frac{a}{b}}$ and $\bar{\tau} = 1 - \sqrt{\frac{a}{b}}$.

Proof. According to (1) and (9), we obtain

$$\begin{aligned}
\mathcal{F}_n(x, y, m, q) &= a^{\xi(n+1)} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-1-k \\ k \end{bmatrix}_q (ab)^{(n-1-\xi(n+1))/2-k} x^{n-1-2k} y^k \\
&= \left(\sqrt{\frac{a}{b}}\right)^{\xi(n+1)} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q \left(\sqrt{ab}x\right)^{n-1-2k} y^k \\
&= \frac{\left(1+\sqrt{\frac{a}{b}}\right) - (-1)^n \left(1-\sqrt{\frac{a}{b}}\right)}{2} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q \left(\sqrt{ab}x\right)^{n-2k-1} y^k \\
&= \frac{1}{2}\left(1+\sqrt{\frac{a}{b}}\right)\Phi_n(\sqrt{ab}x, y, m, q) - \frac{(-1)^n}{2}\left(1-\sqrt{\frac{a}{b}}\right)\Phi_n(\sqrt{ab}x, y, m, q).
\end{aligned}$$

□

Lemma 4.2. *The generating function for the classical q-Fibonacci polynomials $\Phi_n(x, y, m, q)$ is given by*

$$\Psi_m(x, y, z) = \sum_{k \geq 0} q^{\binom{k+1}{2}+m\binom{k}{2}} \frac{y^k z^{2k+1}}{(xz; q)_{k+1}}.$$

Proof. Let $\Psi_m(x, y, z)$ be the generating function of the polynomials $\Phi_n(x, y, m, q)$, according to the well-known formula

$$\sum_{n \geq 0} \begin{bmatrix} n+k \\ k \end{bmatrix}_q z^n = \frac{1}{(z; q)_{k+1}},$$

we obtain

$$\begin{aligned}
\Psi_m(x, y, z) &= \sum_{n \geq 0} \Phi_n(x, y, m, q) z^n \\
&= \sum_{n \geq 0} z^n \sum_{n-1-k \geq 0} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q x^{n-2k-1} y^k \\
&= \sum_{k \geq 0} q^{\binom{k+1}{2}+m\binom{k}{2}} y^k \sum_{n-1 \geq k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q x^{n-2k-1} z^n \\
&= \sum_{k \geq 0} q^{\binom{k+1}{2}+m\binom{k}{2}} y^k \sum_{l \geq 0} \begin{bmatrix} l \\ k \end{bmatrix}_q x^{l-k} z^{l+k+1} \\
&= \sum_{k \geq 0} q^{\binom{k+1}{2}+m\binom{k}{2}} y^k z^{2k+1} \sum_{l \geq 0} \begin{bmatrix} l+k \\ k \end{bmatrix}_q x^l z^l \\
&= \sum_{k \geq 0} q^{\binom{k+1}{2}+m\binom{k}{2}} \frac{y^k z^{2k+1}}{(xz; q)_{k+1}}.
\end{aligned}$$

□

Theorem 4.3. *The generating function of the polynomials $\mathcal{F}_n(x,y,m,q)$ is given by*

$$G_m(z)=\sum_{k\geq 0} q^{\binom{k+1}{2}+m\binom{k}{2}} y^k \frac{\tau(-\sqrt{ab}xz;q)_{k+1}+\bar{\tau}(\sqrt{ab}xz;q)_{k+1}}{2(abx^2z^2;q^2)_{k+1}} z^{2k+1}. \tag{28}$$

Proof. From Lemma 4.1 and Lemma 4.2, we get

$$\begin{aligned} G_m(z) &= \sum_{n\geq 0} \mathcal{F}_n(x,y,m,q) z^n \\ &= \frac{\tau}{2}\sum_{n\geq 0} \Phi_n\left(\sqrt{ab}x,y,m,q\right) z^n - \frac{\bar{\tau}}{2}\sum_{n\geq 0} \Phi_n\left(\sqrt{ab}x,y,m,q\right)(-z)^n \\ &= \frac{\tau}{2}\Psi_m\left(\sqrt{ab}x,y,z\right) - \frac{\bar{\tau}}{2}\Psi_m\left(\sqrt{ab}x,y,-z\right) \\ &= \sum_{k\geq 0} q^{\binom{k+1}{2}+m\binom{k}{2}} y^k \frac{\tau(-\sqrt{ab}xz;q)_{k+1}+\bar{\tau}(\sqrt{ab}xz;q)_{k+1}}{2(abx^2z^2;q^2)_{k+1}} z^{2k+1}. \end{aligned}$$

□

Corollary 4.4. *The generating functions for the Carlitz-type and Cigler-type q-analogues of the bi-periodic Fibonacci polynomials are provided below, respectively, as*

$$G_1(z)=\sum_{k\geq 0} F_n^{(a,b)}(x,y,q) z^n = \sum_{k\geq 0} q^{k^2} y^k \frac{\tau(-\sqrt{ab}xz;q)_{k+1}+\bar{\tau}(\sqrt{ab}xz;q)_{k+1}}{2(abx^2z^2;q^2)_{k+1}} z^{2k+1},$$

$$G_0(z)=\sum_{k\geq 0} \mathbf{F}_n^{(a,b)}(x,y,q) z^n = \sum_{k\geq 0} q^{\binom{k+1}{2}} y^k \frac{\tau(-\sqrt{ab}xz;q)_{k+1}+\bar{\tau}(\sqrt{ab}xz;q)_{k+1}}{2(abx^2z^2;q^2)_{k+1}} z^{2k+1}.$$

Theorem 4.5. *The generating function of the polynomials $\mathcal{L}_n(x,y,m,q)$ is given by*

$$S_m(z)=\sum_{k\geq 0} q^{\binom{k+1}{2}+m\binom{k}{2}} y^k \frac{\tau(-\sqrt{ab}xz;q)_{k+1}+\bar{\tau}(\sqrt{ab}xz;q)_{k+1}}{2(abx^2z^2;q^2)_{k+1}} \left(\frac{1}{z}+q^{mk}yz\right) z^{2k+1}. \tag{29}$$

Proof. From (15), we have

$$\begin{aligned} S_m(z) &= \sum_{n\geq 0} \mathcal{L}_n(x,y,m,q) z^n \\ &= \sum_{n\geq 0} \left(\mathcal{F}_{n+1}(x,y,m,q) + y\mathcal{F}_{n-1}(x,q^m y,m,q)\right) z^n \\ &= \frac{1}{z}\sum_{n\geq 0} \mathcal{F}_n(x,y,m,q) z^n + yz\sum_{n\geq 0} \mathcal{F}_n(x,q^m y,m,q) z^n \\ &= \frac{1}{2z}\left(\tau\Psi_m(\sqrt{ab}x,y,z) - \bar{\tau}\Psi_m(\sqrt{ab}x,y,-z)\right) \\ &\quad + \frac{yz}{2}\left(\tau\Psi_m(\sqrt{ab}x,q^m y,z) - \bar{\tau}\Psi_m(\sqrt{ab}x,q^m y,-z)\right). \end{aligned}$$

According to the Lemma 4.2, we have

$$\Psi_m(x,q^m y,z)=\sum_{k\geq 0} q^{(m+1)\binom{k+1}{2}} \frac{y^k z^{2k+1}}{(xz;q)_{k+1}}. \tag{30}$$

Thus, we find the result. □

Corollary 4.6. *The generating functions of the Carlitz-type and Cigler-type for the q-bi-periodic Lucas polynomials are given, respectively, by*

$$S_1(z)=\sum_{n\geq 0} l_n^{(a,b)}(x,y,q)z^n=\sum_{k\geq 0} q^{k^2}y^k\frac{\tau(-\sqrt{ab}xz;q)_{k+1}+\bar{\tau}(\sqrt{ab}xz;q)_{k+1}}{2(abx^2z^2;q^2)_{k+1}}\left(\frac{1}{z}+q^kyz\right)z^{2k+1},$$

$$S_0(z)=\sum_{n\geq 0} Luc_n(x,y,q)z^n=\sum_{k\geq 0} q^{\binom{k+1}{2}}y^k\frac{\tau(-\sqrt{ab}xz;q)_{k+1}+\bar{\tau}(\sqrt{ab}xz;q)_{k+1}}{2(abx^2z^2;q^2)_{k+1}}\left(\frac{1}{z}+yz\right)z^{2k+1}.$$

Theorem 4.7. *The generating functions of the polynomials $\mathbf{P}_n(x,y,m,q)$ and $\mathbb{P}_n(x,y,m,q)$ are given by:*

$$\mathbf{L}_m(z)=\sum_{k\geq 0} q^{(m+1)\binom{k}{2}}y^k\frac{\tau(-\sqrt{ab}xz;q)_{k+1}+\bar{\tau}(\sqrt{ab}xz;q)_{k+1}}{2(abx^2z^2;q^2)_{k+1}}\left(\frac{1}{z}+q^{(m+1)k}yz\right)z^{2k+1}, \tag{31}$$

$$\mathbb{L}_m(z)=\sum_{k\geq 0} q^{(m+1)\binom{k}{2}}y^k\frac{\tau(-\sqrt{ab}xz;q)_{k+1}+\bar{\tau}(\sqrt{ab}xz;q)_{k+1}}{2(abx^2z^2;q^2)_{k+1}}\left(\frac{1}{z}(2q^k-1)+q^{(m+1)k}yz\right)z^{2k+1}. \tag{32}$$

Proof. From (19), we have

$$\begin{aligned}\sum_{n\geq 0}\mathbf{P}_n(x,y,m,q)z^n &= \frac{1}{z}\sum_{n\geq 0}\mathcal{F}_n(x,y/q,m,q)z^n+yz\sum_{n\geq 0}\mathcal{F}_n(x,q^my,m,q)z^n\\ &= \frac{1}{2z}\left(\tau\Psi_m(\sqrt{ab}x,y/q,z)-\bar{\tau}\Psi_m(\sqrt{ab}x,y/q,-z)\right)\\ &\quad+\frac{yz}{2}\left(\tau\Psi_m(\sqrt{ab}x,q^my,z)-\bar{\tau}\Psi_m(\sqrt{ab}x,q^my,-z)\right).\end{aligned}$$

According to the Lemma 4.2, we have

$$\Psi_m(x,y/q,z)=\sum_{k\geq 0} q^{(m+1)\binom{k}{2}}\frac{y^kz^{2k+1}}{(xz;q)_{k+1}}.$$

Using the above identity and (30), we find the result.
The generating function of the polynomials $\mathbb{P}_n(x,y,m,q)$ is obtained by using the relations (23), (29), and (31). □

Corollary 4.8. *The generating functions of the Carlitz-type and Cigler-type for the q-bi-periodic Lucas polynomials of the first and second kinds are given, respectively, by*

$$\mathbf{L}_1(z)=\sum_{n\geq 0}\mathbf{L}_n^{(a,b)}(x,y,1,q)z^n=\sum_{k\geq 0} q^{k^2-k}y^k\frac{\tau(-\sqrt{ab}xz;q)_{k+1}+\bar{\tau}(\sqrt{ab}xz;q)_{k+1}}{2(abx^2z^2;q^2)_{k+1}}\left(\frac{1}{z}+q^{2k}yz\right)z^{2k+1},$$

$$\mathbb{L}_1(z)=\sum_{n\geq 0}\mathbb{L}_n^{(a,b)}(x,y,1,q)z^n=\sum_{k\geq 0} q^{k^2-k}y^k\frac{\tau(-\sqrt{ab}xz;q)_{k+1}+\bar{\tau}(\sqrt{ab}xz;q)_{k+1}}{2(abx^2z^2;q^2)_{k+1}}\left(\frac{1}{z}(2q^k-1)+q^{2k}yz\right)z^{2k+1}$$

and

$$\mathbf{L}_0(z)=\sum_{n\geq 0}\mathbf{L}_n^{(a,b)}(x,y,0,q)z^n=\sum_{k\geq 0} q^{\binom{k}{2}}y^k\frac{\tau(-\sqrt{ab}xz;q)_{k+1}+\bar{\tau}(\sqrt{ab}xz;q)_{k+1}}{2(abx^2z^2;q^2)_{k+1}}\left(\frac{1}{z}+q^kyz\right)z^{2k+1},$$

$$\mathbb{L}_0(z)=\sum_{n\geq 0}\mathbb{L}_n^{(a,b)}(x,y,0,q)z^n=\sum_{k\geq 0} q^{\binom{k}{2}}y^k\frac{\tau(-\sqrt{ab}xz;q)_{k+1}+\bar{\tau}(\sqrt{ab}xz;q)_{k+1}}{2(abx^2z^2;q^2)_{k+1}}\left(\frac{1}{z}(2q^k-1)+q^kyz\right)z^{2k+1}.$$

5. Sum of the terms of the generalized q-bi-periodic Fibonacci and Lucas polynomials

In the following results, we provide identities concerning the sum of the terms of the generalized q-bi-periodic Fibonacci and Lucas polynomials.

Theorem 5.1. *For any integers $n \geq 0$ and $l \in \mathbb{Z}$, we have*

$$\sum_{k=0}^{n} (-1)^k a^{\xi(l-1)\xi(k)} b^{\xi(l)\xi(k)} (ab)^{\lfloor k/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \mathcal{F}_{2n+l-k}(x,y,m,q) = q^{nl+(m+1)\binom{n}{2}} y^n \mathcal{F}_l(x, q^{n(m-1)}y, m, q),$$

$$\sum_{k=0}^{n} (-1)^k a^{\xi(l)\xi(k)} b^{\xi(l+1)\xi(k)} (ab)^{\lfloor k/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \mathbf{P}_{2n+l-k}(x,y,m,q) = q^{nl+(m+1)\binom{n}{2}} y^n \mathbf{P}_l(x, q^{n(m-1)}y, m, q),$$

$$\sum_{k=0}^{n} \frac{(-1)^k}{q^{nk-\binom{k+1}{2}}} a^{\xi(l)\xi(k)} b^{\xi(l+1)\xi(k)} (ab)^{\lfloor k/2 \rfloor} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \mathbb{P}_{2n+l-k}(x, q^k y, m, q) = q^{\binom{n+1}{2}+m\binom{n}{2}} y^n \mathbb{P}_l(x, q^{n(m+1)}y, m, q),$$

$$\sum_{k=0}^{n} (-1)^k a^{\xi(l-1)\xi(k)} b^{\xi(l)\xi(k)} (ab)^{\lfloor k/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \mathcal{L}^{(a,b)}_{2n+l-k-1}(x,y,m,q) = q^{nl+(m+1)\binom{n}{2}} y^n \mathcal{F}_l(x, q^{n(m-1)}y, q)$$
$$+ q^{n(m+l-2)+(m+1)\binom{n}{2}} y^{n+1} \mathcal{F}_{l-2}(x, q^{m+n(m-1)}y, m, q).$$

Proof. The first formula can be proved by induction on n. It is trivially true when $n = 0$ for all $l \in \mathbb{Z}$. For $n = 1$, the result reduces to

$$\mathcal{F}_{l+2}(x,y,m,q) - a^{\xi(l-1)} b^{\xi(l)} x \mathcal{F}_{l+1}(x,y,m,q) = q^l y \mathcal{F}_l(x, q^{m-1}y, m, q),$$

which also holds for $l \in \mathbb{Z}$ by (14) and Remark 3.9. Assume that the identity holds for $i < n$ and all $l \in \mathbb{Z}$. Then, we get

$$\begin{aligned}
&\sum_{k} (-1)^k a^{\xi(l-1)\xi(k)} b^{\xi(l)\xi(k)} (ab)^{\lfloor k/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \mathcal{F}^{(a,b)}_{2n+l-k}(x,y,m,q) \\
&= \sum_{k} (-1)^k a^{\xi(l-1)\xi(k)} b^{\xi(l)\xi(k)} (ab)^{\lfloor k/2 \rfloor} q^{k+\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q x^k \mathcal{F}^{(a,b)}_{2n+l-k}(x,y,m,q) \\
&\quad + \sum_{k} (-1)^k a^{\xi(l-1)\xi(k)} b^{\xi(l)\xi(k)} (ab)^{\lfloor k/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q x^k \mathcal{F}^{(a,b)}_{2n+l-k}(x,y,m,q) \\
&= \sum_{k} (-1)^{k-1} a^{\xi(l-1)\xi(k-1)} b^{\xi(l)\xi(k-1)} (ab)^{\lfloor (k-1)/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q x^{k-1} \mathcal{F}^{(a,b)}_{2n+l-k+1}(x,y,m,q) \\
&\quad + \sum_{k} (-1)^k a^{\xi(l-1)\xi(k)} b^{\xi(l)\xi(k)} (ab)^{\lfloor k/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q x^k \mathcal{F}^{(a,b)}_{2n+l-k}(x,y,m,q) \\
&= \sum_{k} (-1)^{k-1} a^{\xi(l-1)\xi(k-1)} b^{\xi(l)\xi(k-1)} (ab)^{\lfloor (k-1)/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q x^{k-1} \\
&\quad \times \left(\mathcal{F}^{(a,b)}_{2n+l-k+1}(x,y,m,q) - a^{\xi(l+k)} b^{\xi(l+k-1)} x \mathcal{F}^{(a,b)}_{2n+l-k}(x,y,m,q) \right) \\
&= \sum_{k} (-1)^k a^{\xi(l-1)\xi(k)} b^{\xi(l)\xi(k)} (ab)^{\lfloor k/2 \rfloor} q^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q x^k q^{2n+l-k-2} y \mathcal{F}^{(a,b)}_{2n+l-k-2}(x, q^{m-1}y, m, q) \\
&= q^{2n+l-2} y \sum_{k} (-1)^k a^{\xi(l-1)\xi(k)} b^{\xi(l)\xi(k)} (ab)^{\lfloor k/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q x^k \mathcal{F}^{(a,b)}_{2n+l-k-2}(x, q^{m-1}y, m, q) \\
&= q^{2n+l-2} y q^{(n-1)l+(m+1)\binom{n-1}{2}} (q^{m-1}y)^{n-1} \mathcal{F}^{(a,b)}_l(x, q^{n(m-1)}y, m, q) \\
&= q^{nl+(m+1)\binom{n}{2}} y^n \mathcal{F}^{(a,b)}_l(x, q^{n(m-1)}y, m, q).
\end{aligned}$$

In the same way, we use the recurrences (21) and (22) to prove the second and third identities. Moreover, by using (15), we obtain the last identity. □

Note that if we take $m = 0$, we get the results given in [3] and for $m = 1$, we get the following results.

Corollary 5.2. *For any integers $n \geq 0$ and $l \in \mathbb{Z}$, we have*

$$\sum_{k=0}^{n} (-1)^k a^{\xi(l-1)\xi(k)} b^{\xi(l)\xi(k)} (ab)^{\lfloor k/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k F^{(a,b)}_{2n+l-k}(x,y,q) = q^{nl+2\binom{n}{2}} y^n F^{(a,b)}_l(x,y,q),$$

$$\sum_{k=0}^{n} (-1)^k a^{\xi(l)\xi(k)} b^{\xi(l+1)\xi(k)} (ab)^{\lfloor k/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \mathbf{P}_{2n+l-k}(x,y,1,q) = q^{nl+2\binom{n}{2}} y^n \mathbf{P}_l(x,y,1,q),$$

$$\sum_{k=0}^{n} \frac{(-1)^k}{q^{nk-\binom{k+1}{2}}} a^{\xi(l)\xi(k)} b^{\xi(l+1)\xi(k)} (ab)^{\lfloor k/2 \rfloor} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \mathbb{P}_{2n+l-k}(x,q^k y,1,q) = q^{\binom{n+1}{2}+\binom{n}{2}} y^n \mathbb{P}_l(x,q^{2n}y,1,q),$$

$$\sum_{k=0}^{n} (-1)^k a^{\xi(l-1)\xi(k)} b^{\xi(l)\xi(k)} (ab)^{\lfloor k/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k l^{(a,b)}_{2n+l-k-1}(x,y,q) = q^{nl+2\binom{n}{2}} y^n F^{(a,b)}_l(x,y,q) + q^{2\binom{n}{2}} y^{n+1} F^{(a,b)}_{l-2}(x,qy,q).$$

Theorem 5.3. *For all $n \geq 0$ and $l \in \mathbb{Z}$, we have*

$$\sum_{j=0}^{n} (-1)^j q^{lj-j^2+m\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q y^j \mathbf{P}_{l+n-2j}(x,q^{j(m-1)}y,m,q) = a^{\xi(n)} \left(\frac{b}{a}\right)^{\xi(l)\xi(n)} (ab)^{\lfloor n/2 \rfloor} x^n \mathbf{P}^{(a,b)}_l(x,y,m,q),$$

$$\sum_{j=0}^{n} (-1)^j q^{j^2+m\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q y^j \mathbb{P}_{l+n-2j}(x,q^{j(m+1)}y,m,q) = a^{\xi(n)} \left(\frac{b}{a}\right)^{\xi(l)\xi(n)} (ab)^{\lfloor n/2 \rfloor} x^n \mathbb{P}^{(a,b)}_l(x,q^n y,m,q),$$

$$\sum_{j=0}^{n} (-1)^j q^{j^2+m\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q y^j \mathcal{F}_{l+n-2j}(x,q^{j(m+1)}y,m,q) = b^{\xi(n)} \left(\frac{a}{b}\right)^{\xi(l)\xi(n)} (ab)^{\lfloor n/2 \rfloor} x^n \mathcal{F}^{(a,b)}_l(x,q^n y,m,q).$$

Proof. We prove the first identity by induction on n. For fixed l, the formula holds for $n = 1$. We assume that it is true for $i < n$. Then, using $\xi(n+l) = \xi(n) + \xi(l) - 2\xi(n)\xi(l)$ and $2\lfloor n/2 \rfloor = n - \xi(n)$, we get

$$\begin{aligned}
&\sum_{j\geq 0} (-1)^j q^{lj-j^2+m\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q y^j \mathbf{P}_{l+n-2j}(x,q^{j(m-1)}y,m,q) \\
&= \sum_{j\geq 0} (-1)^j q^{lj-j^2+m\binom{j}{2}} \left(\begin{bmatrix} n-1 \\ j \end{bmatrix}_q + q^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q \right) y^j \mathbf{P}_{l+n-2j}(x,q^{j(m-1)}y,m,q) \\
&= \sum_{j\geq 0} (-1)^j q^{lj-j^2+m\binom{j}{2}} \begin{bmatrix} n-1 \\ j \end{bmatrix}_q y^j \left(\mathbf{P}_{l+n-2j}(x,q^{j(m-1)}y,q) - q^{l+n-2-2j+j(m-1)} y \mathbf{P}_{l+n-2-2j}(x,q^{(j+1)(m-1)}y,m,q) \right) \\
&= \sum_{j\geq 0} (-1)^j q^{lj-j^2+m\binom{j}{2}} \begin{bmatrix} n-1 \\ j \end{bmatrix}_q a^{\xi(l+n)} b^{\xi(l+n-1)} x y^j \mathbf{P}^{(a,b)}_{l+n-2j-1}(x,q^{j(m-1)}y,m,q) \\
&= a^{\xi(l+n)} b^{\xi(l+n-1)} x a^{\xi(n-1)} \left(\frac{b}{a}\right)^{\xi(l)\xi(n-1)} (ab)^{\lfloor (n-1)/2 \rfloor} x^{n-1} \mathbf{P}^{(a,b)}_l(x,y,m,q) \\
&= a^{\xi(n)} \left(\frac{b}{a}\right)^{\xi(l)\xi(n)} (ab)^{\lfloor n/2 \rfloor} x^n \mathbf{P}_l(x,y,m,q).
\end{aligned}$$

In the same way, we prove the remaining formulas. □

Note that if we take $m = 0$, we get the results given in [3].

In particular, if we take $l = 0$ in the previous theorem, we get the following result.

Corollary 5.4. *For $n \geq 0$, we get*

$$\sum_{j=0}^{n} (-1)^j q^{-j^2+m\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q y^j \mathbf{P}_{n-2j}(x, q^{j(m-1)}y, m, q) = 2a^{\xi(n)} (ab)^{\lfloor n/2 \rfloor} x^n,$$

$$\sum_{j=0}^{n} (-1)^j q^{j^2+m\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q y^j \mathbb{P}_{n-2j}(x, q^{j(m+1)}y, m, q) = 2a^{\xi(n)} (ab)^{\lfloor n/2 \rfloor} x^n,$$

$$\sum_{j=0}^{n} (-1)^j q^{m\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q y^j \mathcal{L}_{n-2j}(x, q^{mj}y, m, q) = 2a^{\xi(n)} (ab)^{\lfloor n/2 \rfloor} x^n,$$

$$\sum_{j=0}^{n} (-1)^j q^{j^2+m\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q y^j \mathcal{F}_{n-2j}(x, q^{j(m+1)}y, m, q) = 0.$$

Note that if we take $m = 1$ in Corollary 5.4, respectively, we get the new identities for the Carlitz-type q-bi-periodic Fibonacci and lucas polynomials.

Corollary 5.5. *For $n \geq 0$, we have*

$$\sum_{j=0}^{n} (-1)^j q^{-\binom{j+1}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q y^j \mathbf{P}_{n-2j}(x, y, 1, q) = 2a^{\xi(n)} (ab)^{\lfloor n/2 \rfloor} x^n,$$

$$\sum_{j=0}^{n} (-1)^j q^{j^2+\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q y^j \mathbb{P}_{n-2j}(x, q^{2j}y, 1, q) = 2a^{\xi(n)} (ab)^{\lfloor n/2 \rfloor} x^n,$$

$$\sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q y^j l_{n-2j}^{(a,b)}(x, q^j y, q) = 2a^{\xi(n)} (ab)^{\lfloor n/2 \rfloor} x^n,$$

$$\sum_{j=0}^{n} (-1)^j q^{j^2+\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q y^j F_{n-2j}^{(a,b)}(x, q^{2j}y, q) = 0.$$

6. Connection between q-bi-periodic Fibonacci sequence and bi-periodic second-order recurrences

Now, we provide the q-analogue of bi-periodic Fibonacci and Lucas identities $t_{2n+1} = \left(\frac{b}{a}\right)^{\xi(n+1)} t_{n+1}^2 + \left(\frac{b}{a}\right)^{\xi(n)} t_{n+1}^2$, $t_{2n} = t_{n+1}t_n + t_n t_{n-1}$ given in [18], and $l_{2n+1} = l_n t_n + l_{n+1}t_{n+1}$ and $l_{2n} = \left(\frac{b}{a}\right)^{\xi(n)} l_n t_{n-1} + \left(\frac{b}{a}\right)^{\xi(n+1)} l_{n+1} t_n$ given in [7].

Let $C(\varkappa_n, y) = \begin{pmatrix} 0 & 1 \\ y & \varkappa_n \end{pmatrix}$ and $\varkappa_n = a^{\xi(n+1)} b^{\xi(n)} x$. Then

$$C\left(\varkappa_n, q^{n-1}y\right) C\left(\varkappa_{n-1}, q^{n-2}y\right) \cdots C(\varkappa_1, y) = \begin{pmatrix} yF_{n-1}^{(a,b)}(x, qy, q) & \left(\frac{b}{a}\right)^{\xi(n+1)} F_n^{(a,b)}(x, y, q) \\ yF_n^{(a,b)}(x, qy, q) & \left(\frac{b}{a}\right)^{\xi(n)} F_{n+1}^{(a,b)}(x, y, q) \end{pmatrix}$$

and

$$C\left(\varkappa_{n+1}, q^{n-1}y\right) C\left(\varkappa_n, q^{n-2}y\right) \cdots C(\varkappa_2, y) = \begin{pmatrix} \left(\frac{b}{a}\right)^{\xi(n)} yF_{n-1}^{(a,b)}(x, qy, q) & F_n^{(a,b)}(x, y, q) \\ \left(\frac{b}{a}\right)^{\xi(n+1)} yF_n^{(a,b)}(x, qy, q) & F_{n+1}^{(a,b)}(x, y, q) \end{pmatrix}.$$

Thus

$$C\left(\varkappa_{n+k},q^{n-1}y\right)C\left(\varkappa_{n+k-1},q^{n-2}y\right)\cdots C\left(\varkappa_{k+1},y\right)=\begin{pmatrix}\left(\frac{b}{a}\right)^{\xi(n)\xi(k)}yF_{n-1}^{(a,b)}(x,qy,q) & \left(\frac{b}{a}\right)^{\xi(n+1)\xi(k+1)}F_{n}^{(a,b)}(x,y,q)\\ \left(\frac{b}{a}\right)^{\xi(n+1)\xi(k)}yF_{n}^{(a,b)}(x,qy,q) & \left(\frac{b}{a}\right)^{\xi(n)\xi(k+1)}F_{n+1}^{(a,b)}(x,y,q)\end{pmatrix}.$$

Theorem 6.1. *If the sequence $(G_n(x,y,q))_n$ satisfies the recurrence relation*

$$G_n(x,y,q)=\varkappa_nG_{n-1}(x,y,q)+q^{n-2}yG_{n-2}(x,y,q),$$

then we have

$$G_{n+k}(x,q^{-k}y,q)=\begin{cases}\left(\frac{b}{a}\right)^{\xi(n)}G_k\left(x,q^{-k}y,q\right)yF_{n-1}^{(a,b)}(x,qy,q)+G_{k+1}\left(x,q^{-k}y,q\right)F_n^{(a,b)}(x,y,q), & \text{if } k \text{ is even},\\ G_k\left(x,q^{-k}y,q\right)yF_{n-1}^{(a,b)}(x,qy,q)+\left(\frac{b}{a}\right)^{\xi(n+1)}G_{k+1}\left(x,q^{-k}y,q\right)F_n^{(a,b)}(x,y,q), & \text{if } k \text{ is odd}.\end{cases}$$

Proof. Let $(G_n(x,y,q))_n$ be a sequence satisfying the recurrence $G_n(x,y,q)=\varkappa_nG_{n-1}(x,y,q)+q^{n-2}yG_{n-2}(x,y,q)$. Then

$$G_{n+k+1}\left(x,q^{-k}y,q\right)=\varkappa_{n+k+1}G_{n+k}\left(x,q^{-k}y,q\right)+q^{n-1}yG_{n+k-1}\left(x,q^{-k}y,q\right).$$

Consequently,

$$\begin{pmatrix}G_{n+k}\left(x,q^{-k}y,q\right)\\ G_{n+k+1}\left(x,q^{-k}y,q\right)\end{pmatrix}=C\left(\varkappa_{n+k+1},q^{n-1}y\right)\begin{pmatrix}G_{n+k-1}\left(x,q^{-k}y,q\right)\\ G_{n+k}\left(x,q^{-k}y,q\right)\end{pmatrix}.$$

By induction, we get

$$\begin{pmatrix}G_{n+k}\left(x,q^{-k}y,q\right)\\ G_{n+k+1}\left(x,q^{-k}y,q\right)\end{pmatrix}=C\left(\varkappa_{n+k+1},q^{n-1}y\right)C\left(\varkappa_{n+k},q^{n-2}y\right)\cdots C\left(\varkappa_{k+2},y\right)\begin{pmatrix}G_{k}\left(x,q^{-k}y,q\right)\\ G_{k+1}\left(x,q^{-k}y,q\right)\end{pmatrix}.$$

Since

$$C\left(\varkappa_{n+k+1},q^{n-1}y\right)C\left(\varkappa_{n+k},q^{n-2}y\right)\cdots C\left(\varkappa_{k+2},y\right)=\begin{pmatrix}\left(\frac{b}{a}\right)^{\xi(n)\xi(k+1)}yF_{n-1}^{(a,b)}(x,qy,q) & \left(\frac{b}{a}\right)^{\xi(n+1)\xi(k)}F_{n}^{(a,b)}(x,y,q)\\ \left(\frac{b}{a}\right)^{\xi(n+1)\xi(k+1)}yF_{n}^{(a,b)}(x,qy,q) & \left(\frac{b}{a}\right)^{\xi(n)\xi(k)}F_{n+1}^{(a,b)}(x,y,q)\end{pmatrix},\quad(33)$$

we obtain the result. □

Consider $G_n(x,y,q)=F_n^{(a,b)}(x,y,q)$ in Theorem 6.1. First, we replace k with n; secondly, we replace k with n and n with $n+1$. Thus, we arrive at the following results.

Corollary 6.2.

$$F_{2n}^{(a,b)}(x,\frac{y}{q^n},q)=F_n^{(a,b)}(x,\frac{y}{q^n},q)yF_{n-1}^{(a,b)}(x,qy,q)+F_{n+1}^{(a,b)}(x,\frac{y}{q^n},q)F_n^{(a,b)}(x,y,q),\quad(34)$$

$$F_{2n+1}^{(a,b)}(x,\frac{y}{q^n},q)=\left(\frac{b}{a}\right)^{\xi(n+1)}F_n^{(a,b)}(x,\frac{y}{q^n},q)yF_{n}^{(a,b)}(x,qy,q)+\left(\frac{b}{a}\right)^{\xi(n)}F_{n+1}^{(a,b)}(x,\frac{y}{q^n},q)F_{n+1}^{(a,b)}(x,y,q).\quad(35)$$

Remark 6.3. *By substituting y with q^ny in Corollary 6.2, we obtain the following results:*

$$F_{2n}^{(a,b)}(x,y,q)=q^nF_n^{(a,b)}(x,y,q)yF_{n-1}^{(a,b)}(x,q^{n+1}y,q)+F_{n+1}^{(a,b)}(x,y,q)F_n^{(a,b)}(x,q^ny,q),$$

$$F_{2n+1}^{(a,b)}(x,y,q)=\left(\frac{b}{a}\right)^{\xi(n+1)}q^nF_n^{(a,b)}(x,y,q)yF_{n}^{(a,b)}(x,q^{n+1}y,q)+\left(\frac{b}{a}\right)^{\xi(n)}F_{n+1}^{(a,b)}(x,y,q)F_{n+1}^{(a,b)}(x,q^ny,q).$$

By considering $G_n(x,y,q) = \left(\frac{a}{b}\right)^{\xi(n+1)} \mathbf{P}_n(x,y,1,q)$ in Theorem 6.1, we first replace k with n; secondly, we replace k with n and n with $n+1$. Thus, we arrive at the following results.

Corollary 6.4.

$$\mathbf{P}_{2n}(x,q^{-n}y,1,q) = \left(\frac{b}{a}\right)^{\xi(n)} \mathbf{P}_n\left(x,q^{-n}y,1,q\right) yF_{n-1}^{(a,b)}(x,qy,q) + \left(\frac{b}{a}\right)^{\xi(n+1)} \mathbf{P}_{n+1}\left(x,q^{-n}y,1,q\right) F_n^{(a,b)}(x,y,q), \tag{36}$$

$$\mathbf{P}_{2n+1}(x,q^{-n}y,1,q) = \mathbf{P}_n\left(x,q^{-n}y,1,q\right) yF_n^{(a,b)}(x,qy,q) + \mathbf{P}_{n+1}\left(x,q^{-n}y,1,q\right) F_{n+1}^{(a,b)}(x,y,q). \tag{37}$$

Theorem 6.5. *If the sequence $(G_n(x,y,q))_n$ satisfies the recurrence relation*

$$G_n(x,y,q) = \varkappa_n G_{n-1}(x,qy,q) + qyG_{n-2}(x,q^2y,q),$$

then we have

$$G_{n+k}(x,y,q) = \begin{cases} \left(\frac{b}{a}\right)^{\xi(n)} q^{n-1}G_k\left(x,q^ny,q\right) yF_{n-1}^{(a,b)}(x,y,q) + G_{k+1}\left(x,q^{n-1}y,q\right) F_n^{(a,b)}(x,y,q), & \textit{if } k \textit{ is even}, \\ q^{n-1}G_k\left(x,q^ny,q\right) yF_{n-1}^{(a,b)}(x,y,q) + \left(\frac{b}{a}\right)^{\xi(n+1)} G_{k+1}\left(x,q^{n-1}y,q\right) F_n^{(a,b)}(x,y,q), & \textit{if } k \textit{ is odd}. \end{cases}$$

Proof. Let $(G_n(x,y,q))_n$ be a sequence satisfying the recurrence $G_n(x,y,q) = \varkappa_n G_{n-1}(x,qy,q) + qyG_{n-2}(x,q^2y,q)$. Then

$$G_{n+k+1}\left(x,q^{-n}y,q\right) = \varkappa_{n+k+1}G_{n+k}\left(x,q^{1-n}y,q\right) + q^{1-n}yG_{n+k-1}\left(x,q^{2-n}y,q\right).$$

Consequently,

$$\begin{pmatrix} G_{n+k}\left(x,q^{1-n}y,q\right) \\ G_{n+k+1}\left(x,q^{-n}y,q\right) \end{pmatrix} = C\left(\varkappa_{n+k+1},q^{1-n}y\right) \begin{pmatrix} G_{n+k-1}\left(x,q^{2-n}y,q\right) \\ G_{n+k}\left(x,q^{1-n}y,q\right) \end{pmatrix}.$$

By induction, we get

$$\begin{pmatrix} G_{n+k}\left(x,q^{1-n}y,q\right) \\ G_{n+k+1}\left(x,q^{-n}y,q\right) \end{pmatrix} = C\left(\varkappa_{n+k+1},q^{1-n}y\right) C\left(\varkappa_{n+k},q^{2-n}y\right) \cdots C\left(\varkappa_{k+2},y\right) \begin{pmatrix} G_k\left(x,qy,q\right) \\ G_{k+1}\left(x,y,q\right) \end{pmatrix}.$$

Using (33), we obtain

$$\begin{pmatrix} G_{n+k}\left(x,q^{1-n}y,q\right) \\ G_{n+k+1}\left(x,q^{-n}y,q\right) \end{pmatrix} = \begin{pmatrix} \left(\frac{b}{a}\right)^{\xi(n)\xi(k+1)} yF_{n-1}^{(a,b)}\left(x,q^{-1}y,q^{-1}\right) & \left(\frac{b}{a}\right)^{\xi(n+1)\xi(k)} F_n^{(a,b)}(x,y,q^{-1}) \\ \left(\frac{b}{a}\right)^{\xi(n+1)\xi(k+1)} yF_n^{(a,b)}\left(x,q^{-1}y,q^{-1}\right) & \left(\frac{b}{a}\right)^{\xi(n)\xi(k)} F_{n+1}^{(a,b)}\left(x,y,q^{-1}\right) \end{pmatrix} \begin{pmatrix} G_k\left(x,qy,q\right) \\ G_{k+1}\left(x,y,q\right) \end{pmatrix},$$

therefore

$$G_{n+k}\left(x,q^{1-n}y,q\right) = \begin{cases} \left(\frac{b}{a}\right)^{\xi(n)} G_k(x,qy,q)yF_{n-1}^{(a,b)}\left(x,q^{-1}y,q^{-1}\right) + G_{k+1}(x,y,q)F_n^{(a,b)}\left(x,y,q^{-1}\right), & \text{if } k \text{ is even}, \\ G_k(x,qy,q)yF_{n-1}^{(a,b)}\left(x,q^{-1}y,q^{-1}\right) + \left(\frac{b}{a}\right)^{\xi(n+1)} G_{k+1}(x,y,q)F_n^{(a,b)}\left(x,y,q^{-1}\right), & \text{if } k \text{ is odd}. \end{cases}$$

To achieve the desired result, simply use $F_{n+1}^{(a,b)}\left(x,y,q^{-1}\right) = F_{n+1}^{(a,b)}\left(x,q^{-n}y,q\right)$. □

By considering $G_n(x,y,q) = F_n^{(a,b)}(x,y,q)$ in Theorem 6.5, if we replace k with n, and n with $n+1$, respectively, we arrive at the following results.

Corollary 6.6.

$$F_{2n}^{(a,b)}(x,y,q) = q^{n-1}F_n^{(a,b)}(x,q^n y,q)yF_{n-1}^{(a,b)}(x,y,q) + F_{n+1}^{(a,b)}(x,q^{n-1}y,q)F_n^{(a,b)}(x,y,q), \tag{38}$$

$$F_{2n+1}^{(a,b)}(x,y,q) = \left(\frac{b}{a}\right)^{\xi(n+1)} q^n F_n^{(a,b)}\left(x,q^{n+1}y,q\right)yF_n^{(a,b)}(x,y,q) + \left(\frac{b}{a}\right)^{\xi(n)} F_{n+1}^{(a,b)}\left(x,q^n y,q\right)F_{n+1}^{(a,b)}(x,y,q). \tag{39}$$

By considering $G_n(x,y,q) = \left(\frac{a}{b}\right)^{\xi(n+1)} \mathbb{P}_n(x,y,1,q)$ in Theorem 6.5, if we replace k with n, and n with $n+1$, respectively, we arrive at the following results.

Corollary 6.7.

$$\mathbb{P}_{2n}(x,y,1,q) = \left(\frac{b}{a}\right)^{\xi(n)} q^{n-1}\mathbb{P}_n(x,q^n y,1,q)\,yF_{n-1}^{(a,b)}(x,y,q) + \left(\frac{b}{a}\right)^{\xi(n+1)} \mathbb{P}_{n+1}\left(x,q^{n-1}y,1,q\right)F_n^{(a,b)}(x,y,q), \tag{40}$$

$$\mathbb{P}_{2n+1}(x,y,1,q) = q^n\mathbb{P}_n\left(x,q^{n+1}y,q\right)yF_n^{(a,b)}(x,y,q) + \mathbb{P}_{n+1}(x,q^n y,1,q)\,F_{n+1}^{(a,b)}(x,y,q). \tag{41}$$

To achieve a generalization of the corollaries mentioned above, it is necessary to employ new polynomial products.

Definition 6.8. *For $P(x,y)$ and $(Q_n(x,y))_{n\in\mathbb{Z}}$ in $\mathbb{R}\left[x,x^{-1},y,y^{-1}\right]$ with $P(x,y) = \sum_{i=c}^{d}\alpha_i(x)\,y^i$ where $\alpha_i(x) \in \mathbb{R}\left[x,x^{-1}\right]$ and $c,d \in \mathbb{Z}$, we note*

$$P(x,y) * Q_n(x,y) = \sum_{i=c}^{d}\alpha_i(x)\,y^i Q_n\left(x,q^{(m-1)i}y\right) \quad \text{and} \quad P(x,y)\,\Delta Q_n(x,y) = \sum_{i=c}^{d}\alpha_i(x)\,y^i q^{ni} Q_n\left(x,q^{(m-1)i}y\right).$$

Lemma 6.9. *We have*

$$\mathfrak{U}_{m-1}\left(W_l(x,y,q)F_n^{(a,b)}(x,y,q)\right) = \mathfrak{U}_{m-1}\left(W_l(x,y,q)\right) * \mathfrak{U}_{m-1}\left(F_n^{(a,b)}(x,y,q)\right), \tag{42}$$

where $W_l \in \left\{F_l^{(a,b)}(x,y,q), \mathbf{P}_l(x,y,1,q)\right\}$.

$$\mathfrak{U}_{m-1}\left(W_l(x,q^n y,q)F_n^{(a,b)}(x,y,q)\right) = \mathfrak{U}_{m-1}\left(W_l(x,y,q)\right)\Delta\mathfrak{U}_{m-1}\left(F_n^{(a,b)}(x,y,q)\right), \tag{43}$$

where $W_l \in \left\{F_l^{(a,b)}(x,y,q), \mathbb{P}_l(x,y,1,q)\right\}$.

Proof. Let us first consider $W_l(x,y,q) = F_l^{(a,b)}(x,y,q)$. Then we have

$$F_l^{(a,b)}(x,y,q)\,F_n^{(a,b)}(x,y,q) = a^{\xi(l+1)}\sum_{k=0}^{\lfloor (l-1)/2\rfloor} q^{\binom{k+1}{2}+\binom{k}{2}}\begin{bmatrix} l-1-k \\ k\end{bmatrix}_q (ab)^{\lfloor \frac{l-1}{2}\rfloor-k}x^{l-1-2k}y^kF_n^{(a,b)}(x,y,q).$$

Therefore

$$\mathfrak{U}_{m-1}\left(F_l^{(a,b)}(x,y,q)F_n^{(a,b)}(x,y,q)\right) = \mathfrak{U}_{m-1}\left(a^{\xi(l+1)}\sum_{k=0}^{\lfloor (l-1)/2\rfloor} q^{\binom{k+1}{2}+\binom{k}{2}}\begin{bmatrix} l-1-k \\ k\end{bmatrix}_q (ab)^{\lfloor \frac{l-1}{2}\rfloor-k}x^{l-1-2k}y^kF_n^{(a,b)}(x,y,q)\right).$$

According Relation (11), we get

$$\begin{aligned}
&\mathfrak{U}_{m-1}\left(F_l^{(a,b)}(x,y,q)F_n^{(a,b)}(x,y,q)\right)\\
&= a^{\xi(l+1)}\sum_{k=0}^{\lfloor (l-1)/2\rfloor} q^{\binom{k+1}{2}+\binom{k}{2}}\begin{bmatrix} l-1-k \\ k\end{bmatrix}_q (ab)^{\lfloor \frac{l-1}{2}\rfloor-k}x^{l-1-2k}q^{(m-1)\binom{k}{2}}y^k\mathcal{F}_n\left(x,q^{(m-1)k}y,q\right)\\
&= a^{\xi(l+1)}\sum_{k=0}^{\lfloor (l-1)/2\rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}}\begin{bmatrix} l-1-k \\ k\end{bmatrix}_q (ab)^{\lfloor \frac{l-1}{2}\rfloor-k}x^{l-1-2k}y^k\mathfrak{U}_{m-1}\left(F_n^{(a,b)}\left(x,q^{(m-1)k}y,q\right)\right),\\
&= \mathfrak{U}_{m-1}\left(F_l^{(a,b)}(x,y,q)\right) * \mathfrak{U}_{m-1}\left(F_n^{(a,b)}(x,y,q)\right).
\end{aligned}$$

Using the same approach, we achieve the desired result for the polynomial $\mathbf{P}_l(x,y,1,q)$.
Next, consider

$$\begin{aligned}
&\mathfrak{U}_{m-1}\left(F_l^{(a,b)}(x,q^n y,q)F_n^{(a,b)}(x,y,q)\right)\\
&=a^{\xi(l+1)}\sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor}q^{\binom{k+1}{2}+\binom{k}{2}}\begin{bmatrix}l-1-k\\k\end{bmatrix}_q(ab)^{\lfloor\frac{l-1}{2}\rfloor-k}x^{l-1-2k}q^{(m-1)\binom{k}{2}}q^{nk}y^k\mathcal{F}_n\left(x,q^{(m-1)k}y,q\right)\\
&=a^{\xi(l+1)}\sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor}q^{\binom{k+1}{2}+m\binom{k}{2}}\begin{bmatrix}l-1-k\\k\end{bmatrix}_q(ab)^{\lfloor\frac{l-1}{2}\rfloor-k}x^{l-1-2k}y^kq^{nk}\mathfrak{U}_{m-1}\left(F_n^{(a,b)}\left(x,q^{(m-1)k}y,q\right)\right),\\
&=\mathfrak{U}_{m-1}\left(F_l^{(a,b)}(x,y,q)\right)\Delta\mathfrak{U}_{m-1}\left(F_n^{(a,b)}(x,y,q)\right).
\end{aligned}$$

Using the same approach, we obtain the desired result for the polynomial $\mathbb{P}_l(x,y,1,q)$. □

Theorem 6.10.

$$\mathcal{F}_{2n}(x,\frac{y}{q^n},m,q)=\mathcal{F}_n(x,\frac{y}{q^n},m,q)*y\mathcal{F}_{n-1}(x,q^m y,m,q)+\mathcal{F}_{n+1}(x,\frac{y}{q^n},m,q)*\mathcal{F}_n(x,y,m,q), \tag{44}$$

$$\mathcal{F}_{2n}(x,y,m,q)=\mathcal{F}_n(x,qy,m,q)\Delta q^{n-1}y\mathcal{F}_{n-1}(x,q^{m-1}y,m,q)+\mathcal{F}_{n+1}(x,\frac{y}{q},m,q)\Delta\mathcal{F}_n(x,y,m,q), \tag{45}$$

$$\mathcal{F}_{2n+1}(x,\frac{y}{q^n},m,q)=\left(\frac{b}{a}\right)^{\xi(n+1)}\mathcal{F}_n(x,\frac{y}{q^n},m,q)*y\mathcal{F}_n(x,q^m y,m,q)+\left(\frac{b}{a}\right)^{\xi(n)}\mathcal{F}_{n+1}(x,\frac{y}{q^n},m,q)*\mathcal{F}_{n+1}(x,y,m,q), \tag{46}$$

$$\mathcal{F}_{2n+1}(x,y,q)=\left(\frac{b}{a}\right)^{\xi(n+1)}q^n\mathcal{F}_n(x,qy,m,q)\Delta y\mathcal{F}_n\left(x,q^{m-1}y,m,q\right)+\left(\frac{b}{a}\right)^{\xi(n)}\mathcal{F}_{n+1}\left(x,\frac{y}{q},m,q\right)\Delta\mathcal{F}_{n+1}(x,y,m,q), \tag{47}$$

$$\mathbf{P}_{2n}(x,\frac{y}{q^n},m,q)=\left(\frac{b}{a}\right)^{\xi(n)}\mathbf{P}_n\left(x,\frac{y}{q^n},m,q\right)*y\mathcal{F}_{n-1}(x,q^m y,q)+\left(\frac{b}{a}\right)^{\xi(n+1)}\mathbf{P}_{n+1}\left(x,\frac{y}{q^n},m,q\right)*\mathcal{F}_n(x,y,q), \tag{48}$$

$$\mathbf{P}_{2n+1}(x,\frac{y}{q^n},m,q)=\mathbf{P}_n\left(x,\frac{y}{q^n},m,q\right)*y\mathcal{F}_n(x,q^m y,m,q)+\mathbf{P}_{n+1}\left(x,\frac{y}{q^n},m,q\right)*\mathcal{F}_{n+1}(x,y,m,q), \tag{49}$$

$$\mathbb{P}_{2n}(x,y,m,q)=\left(\frac{b}{a}\right)^{\xi(n)}\mathbb{P}_n(x,qy,m,q)\Delta q^{n-1}y\mathcal{F}_{n-1}\left(x,q^{m-1}y,m,q\right)+\left(\frac{b}{a}\right)^{\xi(n+1)}\mathbb{P}_{n+1}\left(x,\frac{y}{q},m,q\right)\Delta\mathcal{F}_n(x,y,m,q), \tag{50}$$

$$\mathbb{P}_{2n+1}(x,y,m,q)=\mathbb{P}_n(x,qy,m,q)\Delta q^n y\mathcal{F}_n\left(x,q^{m-1}y,m,q\right)+\mathbb{P}_{n+1}\left(x,\frac{y}{q},m,q\right)\Delta\mathcal{F}_{n+1}(x,y,m,q). \tag{51}$$

Proof. We applied the operator $\mathfrak{U}_{m-1}$ to (34) and (38), and using Lemma 6.9, we get

$$\begin{aligned}
\mathfrak{U}_{m-1}\left(F_{2n}^{(a,b)}(x,\frac{y}{q^n},q)\right)&=\mathfrak{U}_{m-1}\left(yF_n^{(a,b)}(x,\frac{y}{q^n},q)F_{n-1}^{(a,b)}(x,qy,q)\right)+\mathfrak{U}_{m-1}\left(F_{n+1}^{(a,b)}(x,\frac{y}{q^n},q)F_n^{(a,b)}(x,y,q)\right)\\
&=\mathfrak{U}_{m-1}\left(yF_n^{(a,b)}(x,\frac{y}{q^n},q)\right)*\mathfrak{U}_{m-1}\left(F_{n-1}^{(a,b)}(x,qy,q)\right)+\mathfrak{U}_{m-1}\left(F_{n+1}^{(a,b)}(x,\frac{y}{q^n},q)\right)*\mathfrak{U}_{m-1}\left(F_n^{(a,b)}(x,y,q)\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{U}_{m-1}\left(F_{2n}^{(a,b)}(x,y,q)\right)&=\mathfrak{U}_{m-1}\left(q^{n-1}yF_n^{(a,b)}(x,q^n y,q)F_{n-1}^{(a,b)}(x,y,q)\right)+\mathfrak{U}_{m-1}\left(F_{n+1}^{(a,b)}(x,q^{n-1}y,q)F_n^{(a,b)}(x,y,q)\right)\\
&=\mathfrak{U}_{m-1}\left(q^{n-1}yF_n^{(a,b)}(x,y,q)\right)\Delta\mathfrak{U}_{m-1}\left(F_{n-1}^{(a,b)}(x,y,q)\right)+\mathfrak{U}_{m-1}\left(F_{n+1}^{(a,b)}(x,\frac{y}{q^2},q)\right)\Delta\mathfrak{U}_{m-1}\left(F_n^{(a,b)}(x,y,q)\right).
\end{aligned}$$

Since $\mathcal{F}_n(x, y, m, q) = \mathfrak{U}_m\left(\mathbf{F}_n^{(a,b)}(x, y, q)\right) = \mathfrak{U}_{m-1}\left(F_n^{(a,b)}(x, y, q)\right)$, we get (44) and (45).

Similarly, we applied the operator $\mathfrak{U}_{m-1}$ to equations (35) and (39), and using Lemma 6.9, we obtain (46) and (47).

Using the same method, we applied the operator $\mathfrak{U}_{m-1}$ to (36) and (37). Consequently, using (42), we obtain (48) and (49).

Finally, we applied the operator $\mathfrak{U}_{m-1}$ to (40) and (41),which enables us to obtain (50) and (51) using (43). □

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