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Geometric Asian option pricing under the environment of a mixed fractional Brownian motion and discrete dividend payments

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Abstract. In this paper, the price process for underlying assets is modeled as a mixed fractional Brownian motion (MFBM) to account for the long memory in financial markets and remove arbitrage opportunities. Furthermore, the issue of the behaviors of price options with known discrete cash dividends on the underlying asset is taken into account. This study addresses the pricing issue of knowing dividends by rewriting the equation for the price movement of the underlying asset using the MFBM model. The pricing expression for geometric Asian options in the context of MFBM, with discrete dividend payouts, is proposed and derived in this work. Then numerical experiments have been conducted on the crucial parameters within the expression, wherein an analysis of the impacts of parameter variations on the prices of geometric Asian options has been also carried out. Subsequently, empirical analyses are undertaken to compare our model with option prices under continuous dividend payments, elucidating the effectiveness and practicality of this model.

1. Introduce

One knows that options are financial derivatives that provide the holder to call option an underlying asset with a strike price within a specified period. Options often can be divided into two categories, call options and put options, based on the direction in which the holder exercises them. Actually, there are various different options, such as European options and American options. We know that American options are exercise flexibility, early exercise, and higher premiums. While European options are exercisable only at expiration, fixed exercise date, and lower premium[1]. The exotic options, including lookback options, barrier options, and Asian options, which are distinct from standard American and European options, are typically tailored to meet specific investment or risk management objectives[2]. One should note that the characteristics of Asian options are averaging features, path-dependent nature, price smoothing, pricing complexity, and popularity in commodity markets[3]. Obviously, various options have different characteristics.

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It is worth noting that pricing options are a crucial problem in financial mathematics and hold significant importance for stock markets, companies, and investors. Over the years, numerous market practitioners and academic researchers have employed different models for option pricing. We know that the Black-Scholes (BS) model is a classic and elegant option pricing model[4], which primarily relies on several factors, such as the option's time to expiration, volatility and risk-free interest rate. However, extensive empirical studies have revealed significant limitations of the BS option pricing model. For instance, the BS model is based on the hypothesis that the market rapidly adjusts and reflects all available information. Yet, in real-world scenarios, the market is not perfectly efficient and can exhibit situations of asymmetric information. Additionally, the logarithmic returns distribution often possesses characteristics, including self-similarity, autocorrelation, and heavy tails[5]. The BS model assumes that asset price movements are continuous and smooth, following geometric Brownian motion (GBM). In actual markets, however, price changes can be discrete and discontinuous. Therefore, some scholars have proposed that stock price models should adhere to fractional Brownian motion (FBM) models[6]. However, due to the fact that FBM is neither a classical Markov process nor a semimartingale, the standard theory of stochastic integration on semimartingales cannot be applied to define stochastic integrals on FBM[7, 8]. Furthermore, some scholars have found that using FBM to simulate the volatility of stock price movements is unreasonable because it allows for arbitrage opportunities[9, 10]. To solve this issue and consider long memory, this work will use a mixed fractional Brownian motion (MFBM)[11] to describe the price movement process of the underlying asset.

In previous studies, Cheridito[12] demonstrated that, for Hurst index $H \in (\frac{3}{4}, 1)$, MFBM can be equal to Brownian motion (BM), thus allowing for no-arbitrage opportunities. Subsequently, Xiao and Zhang[13] established a pricing model for options under the environment of MFBM and demonstrated the superiority of the model through a genetic algorithm. Sun[14] provided pricing formulas for European call currency options and mixed-fractional partial differential equations and empirically demonstrated the rationality of the mixed-fractional Brownian pricing model. Recently, Ref.[15] investigated the pricing of geometric Asian options under the assumption of stock price dynamics based on the mixed-fractional diffusion BS model and derived pricing formulas for geometric average Asian options. Guo[16] addressed the Asian options under the mechanism of subdiffusion GBM. Moreover, numerous scholars have conducted research on the Asian options in uncertain financial market environments[17–19].

In addition, for classic option pricing, the BS model not only gives an analytical expression for option pricing but also discusses the impact of cash dividends upon option pricing. However, the BS model does not further explore how to derive the option pricing formula after the payment of cash dividends. After the BS model, the option pricing of the underlying asset to pay cash dividends has been a hot topic for scholars. For instance, Bos and Vandermark[20] proposed a linearly split dividend model, obtaining European option pricing formulas that consider both the stock price and exercise price. Liu and Guo[21] further developed the ideas of Bos and Vandermark and applied them to the pricing of American options under known dividend payments. Vellekoop and Nieuwenhuis[22] achieved a quasi-exact valuation of American and European options without altering the assumption of log-normality in the model. Joao[23] derived the pricing of European options with known dividends based on the convexity of the option price function. Veiga and coauthor[24] utilized Taylor series expansion to approximate the pricing formula for European options with discrete dividends. Recently, Zhu[25] also used the Taylor series to derive an approximate formula, which applies to European option pricing with single-share dividends and multi-share discrete dividends.

The option pricing research on dividend payment of the underlying asset is mainly divided into two categories: the first type is to study the continuous or random dividend payment of the underlying asset, and the second type is to study the discrete dividend payment of the underlying asset[26]. Many scholars have focused on the first category. However, in the actual financial market, almost all dividends are paid discretely rather than continuously, and it is generally difficult to accurately determine the amount of dividends using the dividend rate[27]. Discrete dividends could be divided into two cases, the first case is to adjust the amount of dividends to price options. The second scenario is to price options by adjusting the volatility of the underlying asset before and after the dividend payment[28].

Regarding the adjustment of dividend quantities for option pricing, several models and methods have been investigated and addressed. The Escrowed Model, introduced by Roll[29], separates the stock price into two parts: the first part can be treated as the difference between stock price and present value of future dividends, while the second part is the present value of dividends. Thus, the stock price can be then adjusted by subtracting the present value of dividends, allowing for the application of the BS model for option pricing. While the Forward Model, proposed by Heath and Jarrow[30], involves adjusting the exercise price by adding the future present value of cash dividends. They assume that the sum of the stock price and the future present dividends obey a geometric Brownian motion. Chiu[31] introduced a segmented lognormal model that dynamically captures the process of stock price fluctuation in response to dividend payments, which provides a relationship between stock prices and dividend payments.

In summary, this article proposes a new pricing model for geometric Asian options that takes into account both the long-memory characteristics of stock prices and the option pricing problem after known discrete dividends have been paid. The academic contributions of this work could be primarily summed up as follows.

* To capture the long-memory characteristics of financial markets, this work employs a MFBM to describe the dynamic changes in stock prices. This represents an innovative approach to modeling stock price movements and contributes to a better understanding of the underlying dynamics.
* Building upon the MFBM model, this work revises the expression for the underlying asset's price to account for the issue of paying discrete dividends. This modification is essential in addressing a gap in the existing literature, which often overlooks the impact of discrete dividend payments on option pricing.
* This work derives a pricing formula for geometric Asian options based on the mixed fractional Brownian motion model. This formula provides a comprehensive solution for valuing these options, incorporating both the long-memory nature of stock prices and the influence of discrete dividend payments.
* The article conducts an analysis of the key parameters and their impact on the pricing model.

It specifically compares the differences in option prices when dealing with continuous dividend payments versus discrete dividend payments. This analysis contributes to a deeper understanding of the factors influencing option prices in different scenarios. This empirical validation is crucial in demonstrating the model's practical applicability.

The remainder of this work is organized as follows. Section 2 makes a brief introduction of the definition and properties of MFBM. In section 2.2, we make a brief presentation of Asian options and understand their classification. In section 3, we derive the stock price model under the environment of MFBM and rewrite the stock price expression based on the discrete dividend payment. In section 4, this paper puts forward the geometric Asian option pricing model based on MFBM in a discrete dividend payment environment, and carries on the corresponding proof and theoretical derivation. In section 5, in order to apply the model, some numerical calculation are performed to observe the influence of parameters on the model. In section 6, this work makes an empirical analysis and compares the gap between the theoretical price calculated by our proposed model and the market price. Section 7 draws some conclusion.

2. Preliminaries

Before moving on, the background and preliminaries of MFBM and Asian options will be introduced in this section.

2.1. MFBM

One knows that MFBM is a stochastic process that combines the properties of FBM and standard BM[14, 32]. In fact, MFBM is a generalization of FBM. Therefore, MFBM provides a more flexible modeling framework for various applications, such as financial analysis, security analysis, network traffic, and asset pricing[14, 32]. The definition and properties of MFBM are given bellow.

Definition 2.1. *MFBM on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a linear combination of different FBM and BM,*

$$M_t^H = \alpha B_t + \beta B_t^H,$$

where B_t *represents BM, and* B_t^H *indicates an independent FBM with Hurst parameter* $H \in (0,1)$*, while* α *and* β *denotes two constants with* $(\alpha, \beta) \neq (0,0)$*.*

To exhibit non-linear and self-similar characteristics of MFBM M_t^H, some following properties will be presented.

(i) MFBM M_t^H is a Gaussian process but not a Markovian process for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.

(ii)The covariance function function of MFBM is defined by

$$\mathrm{Cov}\left(M_t^H, M_s^H\right) = \alpha^2 (t \wedge s) + \frac{\beta^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), t, s > 0,$$

where $\wedge$ means taking the minimum values of t and s.

(iii) MFBM $M_t^H(\alpha, \beta)$ is mixed-self-similar,

$$M_{ht}^H(\alpha, \beta) \triangleq M_t^H\left(\alpha h^{\frac{1}{2}}, \beta h^H\right), h > 0,$$

where $\triangleq$ means"to have the same law"

(iv) For $t > 0$,

$$E\left[\left(M_t^H(\alpha, \beta)\right)^n\right] = \begin{cases} 0, & n = 2l+1, \\ \frac{(2l)!}{2^l l!}\left(\alpha^2 t + \beta^2 t^{2H}\right)^l, & n = 2l. \end{cases}$$

MFBM exhibits long-term correlation for $H > \frac{1}{2}$, indicating that past values have a significant impact on future values.

2.2. Asian Options

Asian options, different from European or American options, are a typical type of derivative contract. The defining characteristic of Asian options is their reliance on the average or accumulated value of the underlying asset's price during the observation period to determine whether the option would be settled. Asian options, in practical applications, can offer more flexible investment strategies and risk management approaches. They are suitable for investors who have a certain understanding of market volatility and mean reversion effects.

Based on the different observation periods and exercise methods, Asian options can be categorized into average fixed strike Asian option, average floating Asian option, fixed Asian option, fixed Asian option, and basket Asian option.

The profits of a fixed strike call option and put option are respectively given by $(A(T)-K)_+$ and $(K-A(T))_+$, in which K represents the strike price, T denotes the exercise time, and $A(T)$ means the average price during the time interval, $[0, T]$.

While the profits of a floating strike call option and put option are respectively given by $(S(T) - A(T))_+$ and $(A(T) - S(T))_+$, here, $S(T)$ stands for the price of a stock at time T.

In addition, the average price of Asian options can be calculated by arithmetic average or geometric average. The arithmetic average $A(T)$ could be defined by

$$A(T) = \frac{1}{T}\int_0^T S(\tau)\mathrm{d}\tau,$$

while the geometric average $A(T)$ can be given by

$$A(T) = \exp\{\frac{1}{T}\int_0^T S(\tau)\mathrm{d}\tau\}.$$

During the predetermined interval $[0, T]$, in this work, based on a fixed strike price, the average price of Asian options will be by geometric average.

3. Pricing model

3.1. Assumptions

For the convenience of description and the establishment of the pricing model, the following hypothesises are presented.

(i)The financial market is frictionless, without transaction costs when buying or selling stocks and options.

(ii)Options are only exercised at expiration.

(iii)The risk-free interest rate r_t is a constant.

According to the results in Ref[34], one should be pointed that M_t^H is not a semimartingale for $H \in (0, 0.5) \cup (0.5, 0.75)$. And the filtration generated by M_t^H is a semi-martingale equivalent in law to BM for $H \in (0.75, 1]$. In order to ensure M_t^H is a semimartingale, we also give the following assumption.

(iv)The underlying stock price follows a MFBM with a Hurst exponent $H > \frac{3}{4}$.

3.2. Stock price model

On account of the assumptions in the BS model and together with the above assumptions, the process of stock price $\{S_t\}_{t\geq 0}$ with MFBM, in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is modeled as

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}M_t^H,\ S_0 = S(0) > 0,\ t \in [0, T],$$

in which r denotes the risk-free interest rate, and T represents the expiration date. Especially, taking MFBM $\alpha = \beta = 1$, stock price process $\{S_t, t \geq 0\}$ could also be written as

$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma \mathrm{d}B_t + \sigma \mathrm{d}B_t^H,\ S_0 = S(0) > 0,\ t \in [0, T], \tag{1}$$

here, σ indicates the stock price volatility.

Considering the fact that the risk-neutral measure is a martingale measure of the dynamics of $\{S_t\}_{t\geq 0}$, utilizing fractional Ito's lemma[41, 42], if we square a stochastic differential equation (1), then one can obtain

$$(\mathrm{d}S_t)^2 = \sigma^2 S_t^2 t + \sigma^2 S_t^2 t^{2H}.$$

Taking the natural logarithm of S_t and substituting it into the fractional Ito's lemma or Wick–Itô formula[41, 42], one has

$$\begin{aligned}
\mathrm{d}\ln S_t =& \frac{\partial \ln S_t}{\partial S_t}\mathrm{d}S_t + \frac{1}{2}\frac{\partial^2 \ln S_t}{\partial S_t^2}(\mathrm{d}S_t)^2,\\
=& r\mathrm{d}t + \sigma(\mathrm{d}B_t + \mathrm{d}B_t^H) - \frac{1}{2}\sigma^2(\mathrm{d}B_t + \mathrm{d}B_t^H)^2.
\end{aligned}$$

We can derive the solution of Eq.(1) given by

$$S_t = S_0 \exp\{rt + \sigma(B_t + B_t^H) - \frac{1}{2}\sigma^2 t - \frac{1}{2}\sigma^2 t^{2H}\},\ t \in [0, T]. \tag{2}$$

This shows that stock prices S_t follows a log-normal distribution,

$$\log S(t) \simeq N(\log S_0 + rt - \frac{1}{2}\sigma^2 t - \frac{1}{2}\sigma^2 t^{2H}, \sigma^2 t + \sigma^2 t^{2H})$$

where $N(\mu, \nu^2)$ indicates the Gaussian distribution with means μ and variance ν^2. For strike price K, the call option price[14], $C(S(0), T)$ at time 0, is written as

$$C(S(0), T) = S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where

$$d_1 = \frac{\ln(S(0)/K) + rT - \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}},$$

$$d_2 = d_1 - \sqrt{\sigma^2 T + \sigma^2 T^{2H}},$$

and $\Phi(.)$ stands for the standard normal distribution function. Utilizing the put-call option evaluation formula, one could yield the put option price,

$$P(S(0), T) = Ke^{-rT}\Phi(-d_2) - S(0)\Phi(-d_1).$$

4. Pricing of discrete dividend Asian options

The majority of recent research on stock dividend payments focuses on incorporating continuous dividend yields into the pricing process[44, 45]. Nonetheless, discrete dividend distributions are more typical in real financial markets. As a result, this study tackles the issue of discrete dividend payments and presents a novel model that is more in line with actual financial markets. For the case of adjusting the amount of dividends to price options, previous scholars have studied many methods. Ref.[29] proposed the stock price could be divided into two parts. One of two parts is the present dividend, and the other one is the stock price minus the future dividends. We can suppose that the first part grows at the rate of the risk-free rate, and the second part follows the MFBM. Therefore, the stock price can be adjusted to the stock price minus the dividend, so that the BS formula can continue to be used for pricing. One should point out that both discrete dividend and continuous dividend options have their own advantages and disadvantages. Due to flexibility, accurate pricing, and enhanced hedging strategies, for a single stock, it is much more suitable to introduce discrete dividends. Considering the impact of discrete dividend, instead of the entire stock price, we take the stock price minus the present value of the dividend, let $Y_t = S_t - \sum_j D_j e^{-rt}$, follows the lognormal diffusion,

$$\frac{\mathrm{d}Y_t}{Y_t} = r\mathrm{d}t + \sigma_d \mathrm{d}B_t + \sigma_d \mathrm{d}B_t^H.$$

Solved by fractional Ito's lemma [41, 42], the stock price S_t in Eq.(2) could be written as

$$S_t = S_0 \exp\{rt + \sigma_d(B_t + B_t^H) - \frac{1}{2}\sigma_d^2 t - \frac{1}{2}\sigma_d^2 t^{2H}\} - \sum_i D_i,\ 0 \le t \le T, \tag{3}$$

where D_i denotes the present dividend. After adjusting stock prices to pay dividends, we need to adjust the impact of volatility on options. Considering the fact that the current volatility is different from the volatility of the overall stock price, only the volatility of the risky portion needs to be taken into account. Given that the stock price volatility σ of the stock price process is smaller than the volatility σ_d of the difference between the stock price and the present dividends, after multiplying σ by a scaling factor k_d, the corresponding stock price volatility σ_d should also be rewritten as[35],

$$\sigma_d = \sigma \cdot k_d = \frac{\sigma S_0}{S_0 - \sum_i D_i}. \tag{4}$$

Before going on, the following two lemmas will be introduced to obtain our main results. One should note that Eq.(1) is under martingale measure. Therefore, the following risk-neutral option evaluation holds.

Lemma 4.1. *[6] The price $F(t)$ at time t ($t \in [0, T]$) of a bounded F_T^H-measurable claim with payoff function $G(\{S_u, 0 \leqslant u \leqslant T\})$ can be written as*

$$F(t) = \exp(-(T-t)r)\mathbb{E}\left[G(\{S_u, 0 \leqslant u \leqslant T\}) \mid \mathcal{F}_t\right],\ t \in [0, T],$$

here, $\{\mathcal{F}_t\}_{t \geq 0}$ denotes the natural filtration.

According to Ref.[25, 36, 37], the expectation of neutral risk is given by

$$\mathbb{E}[max(S_T - K, 0)].$$

From Ref.[25, 36, 37], the call option price C at neutral risk with discount value could be written as

$$C = \mathrm{e}^{-rT}\mathbb{E}[max(S_T - K, 0)]. \tag{5}$$

Based on the assumptions addressed in subsection 3.1, we can derive our following main results.

Theorem 4.2. *Under the risk-neutral probability measure, supposed the stock price $S(t)$ is governed by Eq.(3), the geometric Asian call option price $C(S(0), T)$ is written as*

$$\begin{aligned} C(S(0), T) = (S_0 - \sum_i D_i)\exp\{-rT + \frac{\frac{1}{2}rT - \frac{1}{2}(\frac{\sigma_d^2 T}{2} + \frac{\sigma_d^2 T^{2H}}{2H+1})}{\ln\sum_i D_i} \\ + \frac{\sigma_d^2}{2(\ln\sum_i D_i)^2}\cdot(\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)})\}\Phi(d_1) - Ke^{-rT}\Phi(d_2), \end{aligned} \tag{6}$$

where

$$d_2 = \frac{\ln S_0 - \ln K \ln\sum_i D_i + \frac{1}{2}rT - \frac{1}{2}(\frac{\sigma_d^2 T}{2} + \frac{\sigma_d^2 T^{2H}}{2H+1})}{\sigma_d\sqrt{\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}}},$$

$$d_1 = d_2 + \frac{\sigma_d}{\ln\sum_i D_i}\sqrt{\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}}.$$

Proof. To begin with, one can let

$$G(T) = \frac{1}{T}\int_0^T \ln S(t)\mathrm{d}t,$$

and

$$A(T) = \exp G(T).$$

According to the stock price model mentioned in Eq.(3), one can know that $G(T)$ meets a Gaussian distribution. For the convenience of description, we denote $\mathbb{E}(\cdot)$ as the expected value, μ and ν^2 respectively indicate the mean and variance of $G(T)$. Therefore, one can calculate the mean and variance of $G(T)$ as follows.

$$\begin{aligned} \mu = \mathbb{E}[G(T)] &= \frac{1}{T}\int_0^T \mathbb{E}[\ln S(t)]\mathrm{d}t \\ &= \frac{1}{T}\int_0^T \mathbb{E}[\ln(S_0\exp\{rt + \sigma_d(B_t + B_t^H) - \frac{1}{2}\sigma_d^2 t - \frac{1}{2}\sigma_d^2 t^{2H}\} - \sum_i D_i)]\mathrm{d}t \\ &= \frac{1}{T}\int_0^T \ln(S_0 - \sum_i D_i)\mathrm{d}t + \frac{1}{T}\int_0^T \frac{rt}{\ln\sum_i D_i}\mathrm{d}t - \frac{1}{2T}\int_0^T \frac{\sigma_d^2 t + \sigma_d^2 t^{2H}}{\ln\sum_i D_i}\mathrm{d}t \\ &= \frac{1}{\ln\sum_i D_i}\{\ln S_0 + \frac{1}{2}rT - \frac{1}{2}(\frac{\sigma_d^2 T}{2} + \frac{\sigma_d^2 T^{2H}}{2H+1})\}. \end{aligned} \tag{7}$$

Due to the fact that BM B_t and FBM B_t^H are are independent, we can also calculate the variance given by

$$\begin{aligned}
\nu^2 &= Var[G(T)] = \mathbb{E}[G(T)-\mu]^2 \\
&= \frac{1}{T^2}\int_0^T\int_0^T \frac{\sigma_d^2}{(\ln\sum_i D_i)^2}(\mathbb{E}[B(t)B(\tau)] + \mathbb{E}[B(t)^H B(\tau)^H])\mathrm{d}t\mathrm{d}\tau \\
&= \frac{\sigma_d^2}{(\ln\sum_i D_i)^2}(\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}),
\end{aligned} \tag{8}$$

where σ_d is specified in Eq.(4). According to the expressions of μ and ν^2 for the random variable $G(T)$, we can see that $A(T)$ follows a log-normal distribution, and the random variable $\ln A(T)$ follows a Gaussian distribution. Therefore, from Eq.(5), we can derive the geometric Asian call option price,

$$\begin{aligned}
C(S(0),T) &= \mathrm{e}^{-rT}\mathbb{E}[(A(T)-K)_+] \\
&= \mathrm{e}^{-rT}\int_l(\mathrm{e}^\lambda - K)\frac{1}{\sqrt{2\pi}\nu}\exp -\frac{(\lambda-\mu)^2}{2\nu^2}\mathrm{d}\lambda,
\end{aligned}$$

where l is the set $[\lambda : \mathrm{e}^\lambda > K]$. The following calculation we get the geometric Asian call option price explicit formula. Let $\phi(\cdot)$ denote the standard normal distribution. Make substitutions for the above definite integral $\lambda = \mu + \nu y$, and we can yield

$$\begin{aligned}
C(S(0),T) &= \mathrm{e}^{-rT}\int_l(\mathrm{e}^{\mu+\nu y} - K)\phi(y)\mathrm{d}y \\
&= \mathrm{e}^{-rT+\mu}\int_{-d_2}^{\infty}\mathrm{e}^{\nu y}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y - K\mathrm{e}^{-rT}\int_{-d_2}^{\infty}\phi(y)\mathrm{d}y \\
&= \mathrm{e}^{-rT+\mu+\frac{1}{2}\nu^2}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(y-\nu)^2}\mathrm{d}y - K\mathrm{e}^{-rT}\phi(d_2) \\
&= \mathrm{e}^{-rT+\mu+\frac{1}{2}\nu^2}\int_{-d_2-\nu}^{\infty}\phi(y)\mathrm{d}y - K\mathrm{e}^{-rT}\phi(d_2) \\
&= \mathrm{e}^{-rT+\mu+\frac{1}{2}\nu^2}\phi(d_1) - K\mathrm{e}^{-rT}\phi(d_2).
\end{aligned}$$

Substituting μ, ν^2 in Eqs.(7) and (8), one can yield the geometric Asian call option price,

$$\begin{aligned}
C(S(0),T) = (S_0 - \sum_i D_i)\exp\{-rT + \frac{\frac{1}{2}rT - \frac{1}{2}(\frac{\sigma_d^2 T}{2} + \frac{\sigma_d^2 T^{2H}}{2H+1})}{\ln\sum_i D_i} \\
+ \frac{\sigma_d^2}{2(\ln\sum_i D_i)^2}\cdot(\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)})\}\Phi(d_1) - K\mathrm{e}^{-rT}\Phi(d_2),
\end{aligned}$$

where

$$l = \{\lambda : A(T) > K\} = \{y : \mathrm{e}^{\mu+\nu y} > K\} = \{y : \mu + \nu y > \ln K\} = \{y : y > -d_2\},$$

and d_1 and d_2 are specified by

$$d_2 = \frac{\ln S_0 - \ln K \ln\sum_i D_i + \frac{1}{2}rT - \frac{1}{2}(\frac{\sigma_d^2 T}{2} + \frac{\sigma_d^2 T^{2H}}{2H+1})}{\sigma_d\sqrt{\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}}},$$

$$d_1 = d_2 + \frac{\sigma_d}{\ln\sum_i D_i}\sqrt{\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}}.$$

This completes the proof of *Theorem*1. □

Remark 4.3. *By using a similar method and the put-call option parity formula, it can be shown that the geometric Asian put option is based on the price in MFBM with discrete dividend payment written as*

$$P(S(0),T) = K\mathrm{e}^{-rT}\Phi(-d_2) - (S_0 - \sum_i D_i)\exp\{-rT + \frac{\frac{1}{2}rT - \frac{1}{2}(\frac{\sigma_d^2 T}{2} + \frac{\sigma_d^2 T^{2H}}{2H+1})}{\ln\sum_i D_i} + \frac{\sigma_d^2}{2(\ln\sum_i D_i)^2}\cdot(\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)})\}\Phi(-d_1).$$

5. Numerical experiment

To address the effectiveness of the geometric Asian option pricing formula proposed in our work, we use the corresponding data to do numerical calculation, and then analyze the impacts of different parameters in the option pricing formula. First, based on historical data, the values of our given basic parameters are addressed in Table 1.

Table 1: **The chosen parameter values for calculation.**

S	K	T	H	r	σ	ΣD_i
50	50	1	0.85	0.03	0.35	3

To begin with, we analyze the influence of the Hurst index and volatility changes on geometric Asian option prices. We begin by calculating the sensitivity of the option's price to changes in the volatility of the underlying asset, v, which is the ratio of the change in the option's value to the change in volatility. If a trading portfolio has a high absolute value of v, it is extremely sensitive to modest fluctuations in volatility. When the absolute value of v approaches zero, changes in asset volatility have no effect on the value of the trading portfolio. For convenient calculation, we use the following notation,

$$A = -rT + \frac{\frac{1}{2}rT - \frac{1}{2}(\frac{\sigma_d^2 T}{2} + \frac{\sigma_d^2 T^{2H}}{2H+1})}{\ln\sum_i D_i} + \frac{\sigma_d^2}{2(\ln\sum_i D_i)^2}\cdot(\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}),$$

And then we can compute v,

$$\begin{aligned} v = \frac{\partial C}{\partial \sigma_d} &= (S_0 - \sum_i D_i)e^A\Phi(d_1)\frac{\partial A}{\partial \sigma_d} + (S_0 - \sum_i D_i)e^A\Phi'(d_1)\ln\sum_i D_i\sqrt{\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}} \\ &+ [(S_0 - \sum_i D_i)e^A\Phi'(d_1) - Ke^{-rT}\Phi'(d_2)]\frac{\partial d_2}{\partial \sigma_d}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \frac{\partial A}{\partial \sigma_d} &= -\frac{\frac{\sigma_d T^{2H}}{2H+1} + \frac{\sigma_d T}{2}}{ln\sum_i D_i} + \frac{\sigma_d}{ln\sum_i D_i}(\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}), \\ \frac{\partial d_2}{\partial \sigma_d} &= \{(-\frac{\sigma_d T^{2H}}{2H+1} - \frac{\sigma_d T}{2})\sigma_d\sqrt{\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}} - [\ln S_0 - \ln K\ln\sum_i D_i \\ &+ \frac{1}{2}rT - \frac{1}{2}(\frac{\sigma_d^2 T}{2} + \frac{\sigma_d^2 T^{2H}}{2H+1})]\sqrt{\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}}\}/\sigma_d^2(\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}). \end{aligned} \tag{10}$$

Similar to literature[43], we can obtain $\frac{\partial C}{\partial H}$, refers to the ratio of the change in option value to the change in the Hurst H,

$$\frac{\partial C}{\partial H} = (S_0 - \sum_i D_i)e^A \Phi(d_1)\frac{\partial A}{\partial H} + (S_0 - \sum_i D_i)e^A \Phi'(d_1)\frac{\partial d_1}{\partial H} - Ke^{-rT}\Phi'(d_2)\frac{\partial d_2}{\partial H}, \tag{11}$$

where

$$\begin{aligned}
\frac{\partial A}{\partial H} &= -\frac{2\sigma_d^2 T^{2H}[\ln T(2H+1)-1]}{\ln\sum\limits_i D_i(2H+1)^2} + \frac{\sigma_d^2 T^{2H}[\ln T(2H+2)-1]}{4(\ln\sum\limits_i D_i)^2(H+1)^2},\\
\frac{\partial d_1}{\partial H} &= \frac{\partial d_2}{\partial H} + \frac{\sigma_d}{2\ln\sum\limits_i D_i}(\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)})^{-\frac{1}{2}}\frac{T^{2H}[\ln T(2H+2)-1]}{2(H+1)^2},\\
\frac{\partial d_2}{\partial H} &= \{-\frac{2\sigma_d^2 T^{2H}[\ln T(2H+1)-1]}{(2H+1)^2}(\sigma_d\sqrt{\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}}) - [\ln S_0 - \ln K \ln\sum_i D_i + \frac{1}{2}rT\\
&\quad - \frac{1}{2}(\frac{\sigma_d^2 T}{2} + \frac{\sigma_d^2 T^{2H}}{2H+1})]\frac{\sigma_d}{2}(\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)})^{-\frac{1}{2}}\frac{T^{2H}[\ln T(2H+2)-1]}{2(H+1)^2}\}/\sigma_d^2(\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}).
\end{aligned} \tag{12}$$

Since the Hurst index is a special kind of long memory process when $H > \frac{1}{2}$, and is arbitrageless when H belongs to $(\frac{3}{4}, 1)$, it is more in line with today's financial market. The price changes are observed for $\sigma = 0.35$ and $\sigma = 0.25$. The calculation results are exhibited in Table 2 and Fig. 1.

Table 2: **The option prices for** $H \in (\frac{1}{2}, 1)$ **and** $\sigma = 0.35, \sigma = 0.25$

H	$\sigma = 0.35$	$\sigma = 0.25$
0.85	3.1385	1.647
0.8	3.1436	1.645
0.75	3.1482	1.6426
0.7	3.1522	1.6395
0.65	3.1555	1.6358
0.6	3.1577	1.6313

It can be seen from Table 2 that, for the volatility of stock price $\sigma = 0.35$, the price of options is significantly higher than that for $\sigma = 0.25$. Due to the fact that the high volatility makes the stock price more likely to fluctuate greatly in the future period of time, thus, increasing the probability of realizing profit of options, the time value of options increases. The time value of options tends to increase, if the volatility of stock prices increases. Meanwhile, it increases the opportunity for options to realize benefits. Therefore, options have higher potential profits, resulting in an increase in the total value of options. Thus, under the same other conditions, high volatility will increase the option price. Although the volatility σ_d of the difference between the stock price and the dividend is larger than the volatility σ of the stock price, it has the same effect on the option price as the volatility of the stock price. As can be seen from Fig. 1, for $\sigma = 0.35$, option prices gradually decrease with the Hurst index increasing. It can be seen from Fig. 1, for $\sigma = 0.25$, with the Hurst index increasing, the price of the rights increases. The Hurst Index itself does not directly affect the price of options, however, the Hurst Index can provide investors with some reference and insight into the market. Specifically, if a market exhibits high self-similarity and long-term dependence (the Hurst index is close to 1), this may mean that market trends are more persistent and price movements may be more predictable. In this case, investors may be more inclined to use options strategies to profit because the market trend is relatively stable.

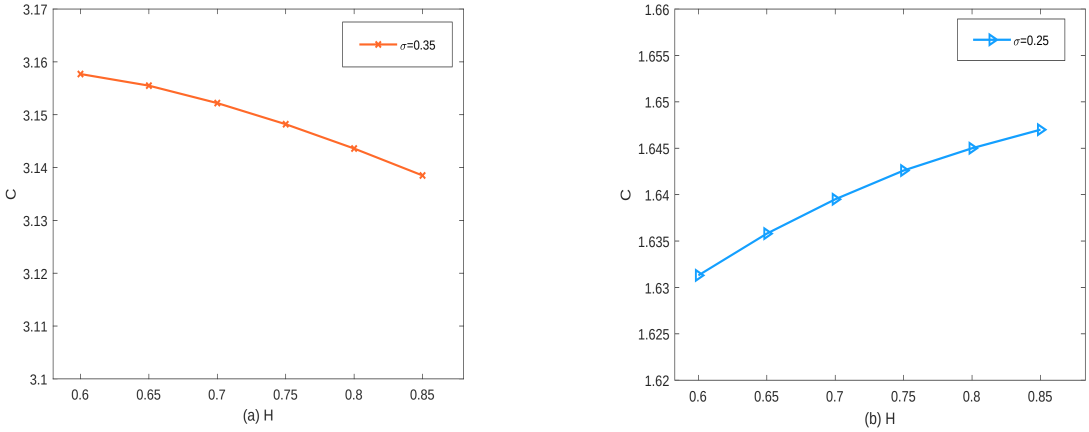


Fig. 1: The behaviors of option price versus H for (a) $\sigma = 0.35$,(b) $\sigma = 0.25$.

Furthermore, in reverse, for further addressing the behaviors of option price, we select the range of stock price volatility σ as $[0.2, 0.4]$, and select Hurst index $H = 0.8, H = 0.75$. The result is the same as above, option prices increase with the increase of stock price volatility. When the Hurst index $H \geq \frac{3}{4}$, the change of option price with the Hurst index is not significant. The behaviors of the option price versus σ are demonstrated in Fig. 2 and Table 3.

Table 3: **The change in option prices in** $\sigma \in [0.2, 0.4]$ **and** $H = 0.8, H = 0.75$

σ	$H = 0.8$	$H = 0.75$
0.2	0.9287	0.9238
0.25	1.645	1.6426
0.3	2.394	2.3954
0.35	3.1436	3.1482
0.4	3.8827	3.8906

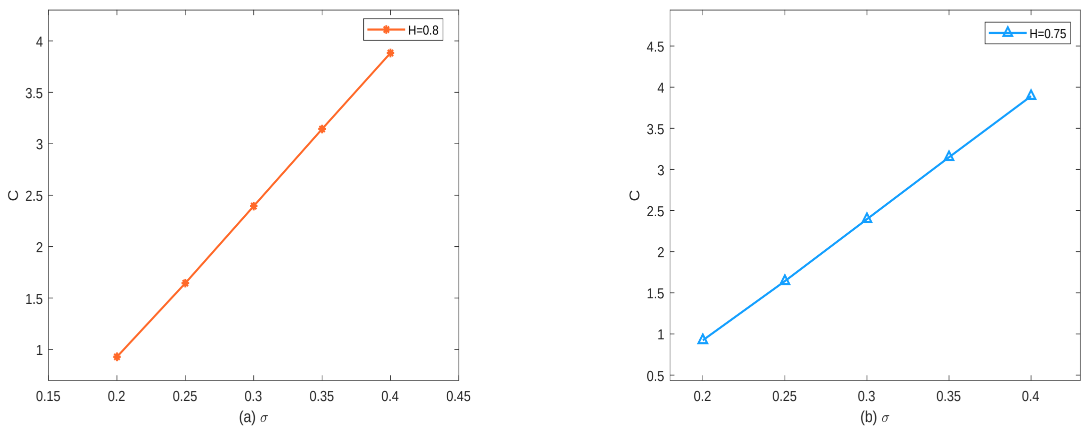


Fig. 2: Behaviors of the option price versus σ for (a) $H = 0.8$,(b) $H = 0.75$.

In the second part, this paper mainly takes Hurst index H and option expiration time T as variables to analyze the impacts of their changes on option prices. Similarly, we can work out the rate Θ at which the value of an options portfolio is predicted to fall over time. This rate is defined as the ratio of the change in the portfolio's value to the change in time, granted that all other conditions remain constant. It is sometimes known as time decay. An option's Θ is typically negative as the time to expiration decreases, assuming all other factors remain constant. Therefore, one has

$$\Theta = \frac{\partial C}{\partial T} = (S_0 - \sum_i D_i)e^A[\Phi(d_1)\frac{\partial A}{\partial T} + \Phi'(d_1)\frac{\partial d_1}{\partial T}] - Ke^{-rT}[\Phi'(d_2)\frac{\partial d_2}{\partial T} - r\Phi(d_2)], \tag{13}$$

where

$$\begin{aligned}
\frac{\partial A}{\partial T} &= -r + \frac{\frac{1}{2}r - \frac{1}{2}(\frac{\sigma_d^2}{2} + \frac{2H\sigma_d^2T^{2H-1}}{2H+1})}{\ln\sum_i D_i} + \frac{\sigma_d^2}{2(\ln\sum_i D_i)^2}(\frac{1}{6}T + \frac{HT^{2H-1}}{H+1}),\\
\frac{\partial d_1}{\partial T} &= \frac{\partial d_2}{\partial T} + \frac{\sigma_d}{\ln\sum_i D_i}[\frac{1}{2}(\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)})^{-\frac{1}{2}}(\frac{1}{6}T + \frac{HT^{2H-1}}{H+1})],\\
\frac{\partial d_2}{\partial T} &= \{[\frac{1}{2}r - \frac{1}{2}(\frac{\sigma_d^2}{2} + \frac{2H\sigma_d^2T^{2H-1}}{2H+1})]\sigma_d\sqrt{\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}} - [\ln S_0 - \ln K \ln\sum_i D_i + \frac{1}{2}rT\\
&\quad - \frac{1}{2}(\frac{\sigma_d^2T}{2} + \frac{\sigma_d^2T^{2H}}{2H+1})]\frac{\sigma_d}{2}(\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)})^{-\frac{1}{2}}(\frac{1}{6}T + \frac{HT^{2H-1}}{H+1})\}/\sigma_d^2(\frac{1}{3}T^2 + \frac{T^{2H}}{2(H+1)}).
\end{aligned} \tag{14}$$

For better comparing the impact of the option expiration date on option price, in this work, we selects the expiration time with a large time span and take the expiration time of option as $T = 0.25$ and $T = 2$ (unit: year). The behaviors of the option price versus H are demonstrated in Table 4 and Fig. 3.

Table 4: **The change in option prices in** $H \in [0.5, 0.8]$ **and** $T = 2, T = 0.25$

H	T=2	T=0.25
0.8	2.0933	0.1311
0.7	2.2187	0.3089
0.6	2.3264	0.5287
0.5	2.4159	0.7955

From Table 4 and Fig. 3, it could be found that the option price with expiration time $T = 2$ is significantly higher than that with expiration time $T = 0.25$. This is consistent with the actual situation, as longer maturities provide investors with greater flexibility and opportunities, as market conditions may change during the effective period. Longer expiration times mean that price movements may have more time to occur, and investors have a relatively low-risk perception of price movements and may be willing to pay higher option prices. Conversely, shorter expiration times limit investors' flexibility and may increase sensitivity to market fluctuations, leading to lower option prices. Besides, the option price will decrease with the increase of the Hurst index H during these two expiration times, but not significantly.

Finally, this paper analyzes the impacts of stock price S on option price. The sensitivity of an option's price to changes in the underlying asset, Δ, is defined as the ratio of the change in the option price to the change in the underlying asset's price. This ratio accurately represents the slope of the tangent line to the curve that describes the relationship between the option price and the price of the underlying asset. After

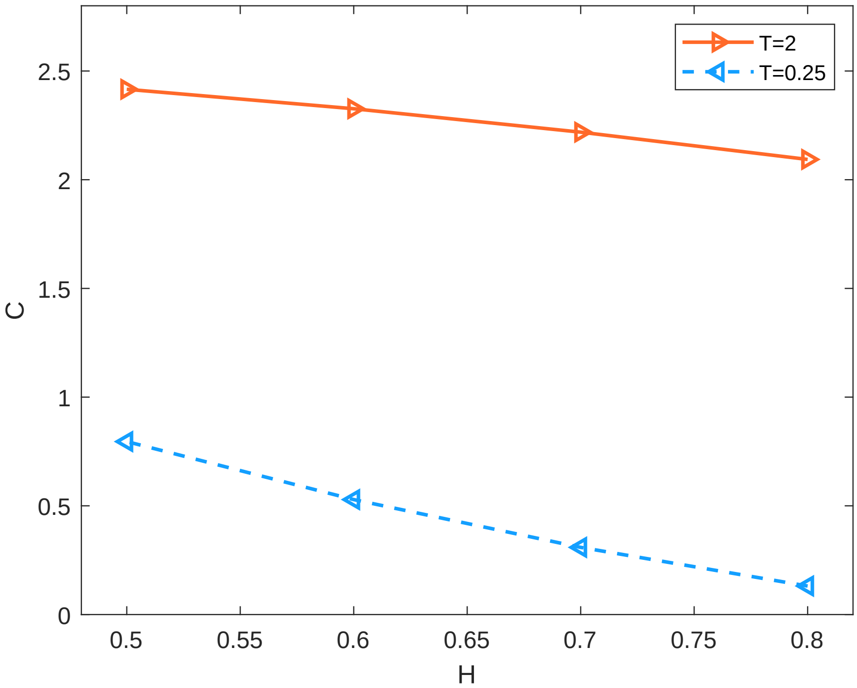


Fig. 3: Behaviors of the option price versus H for $T = 2, T = 0.25$.

calculation, we can derive

$$\begin{aligned}\Delta = \frac{\partial C}{\partial S_0} =& e^A\Phi(d_1) + [S_0e^A\Phi'(d_1) - \sum_i D_ie^A\Phi'(d_1) - Ke^{-rT}\Phi'(d_2)]\frac{\partial d_2}{\partial S_0}\\ &+[S_0e^A\Phi'(d_1) - \sum_i D_ie^A\Phi'(d_1)]\frac{\sqrt{\frac{1}{3}T^2+\frac{T^{2H}}{2(H+1)}}}{\ln\sum\limits_i D_i}\frac{\partial\sigma_d}{\partial S_0}\\ &+(S_0-\sum_i D_i)e^A\Phi(d_1)\frac{\partial A}{\partial\sigma_d}\frac{\partial\sigma_d}{\partial S_0},\end{aligned} \tag{15}$$

where

$$\begin{aligned}\frac{\partial\sigma_d}{\partial S_0} =& \frac{-\sigma\sum\limits_i D_i}{(S_0-\sum\limits_i D_i)^2},\\ \frac{\partial d_2}{\partial S_0} =& \{[\frac{1}{S_0}-\frac{1}{2}(\sigma_dT+\frac{2\sigma_dT^{2H}}{2H+1})\frac{\partial\sigma_d}{\partial S_0}]\sigma_d\sqrt{\frac{1}{3}T^2+\frac{T^{2H}}{2(H+1)}} - [\ln S_0 - \ln K\ln\sum_i D_i\\ &+\frac{1}{2}rT-\frac{1}{2}(\frac{\sigma_d^2T}{2}+\frac{\sigma_d^2T^{2H}}{2H+1})]\sqrt{\frac{1}{3}T^2+\frac{T^{2H}}{2(H+1)}}\frac{\partial\sigma_d}{\partial S_0}\}/\sigma_d^2(\frac{1}{3}T^2+\frac{T^{2H}}{2(H+1)}).\end{aligned} \tag{16}$$

and $\frac{\partial A}{\partial\sigma_d}$ is given by Eq.(10).

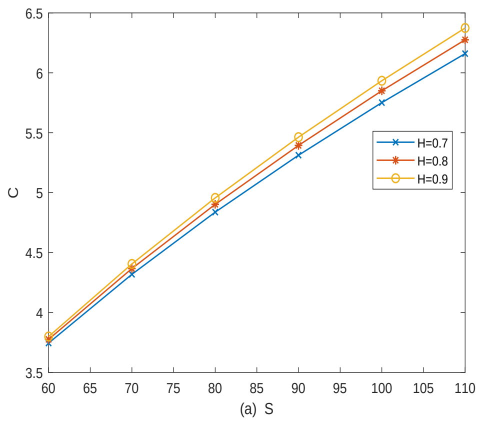
One can select the range of stock price S as $[60, 120]$, and calculate the Hurst index $H = 0.7, H = 0.8$, and $H = 0.9$ respectively. The behaviors of the option price versus the stock price S for different H are demonstrated in Table 5 and Fig. 4 (a).

Table 5: **The change in option prices in** $S \in [60, 120]$ **and** $H = 0.7, H = 0.8, H = 0.9$

S	H=0.7	H=0.8	H=0.9
60	3.1522	3.7526	3.7463
70	4.327	4.3297	4.3283
80	4.8719	4.881	4.8848
90	5.3948	5.4106	5.42
100	5.899	5.9218	5.937
110	6.3868	6.4167	6.438
120	6.8601	6.8975	6.9249

Table 6: **The change in option prices in** $S \in [50, 120]$ **and dividends are divided into discrete dividends and continuous dividends**

S	Continuous dividend	Discrete dividend
50	4.1704	3.1385
60	5.0045	3.7498
70	5.8385	4.3294
80	6.6726	4.8834
90	7.5067	5.416
100	8.3408	5.9302
110	9.1749	6.4283
120	10.0089	6.9123



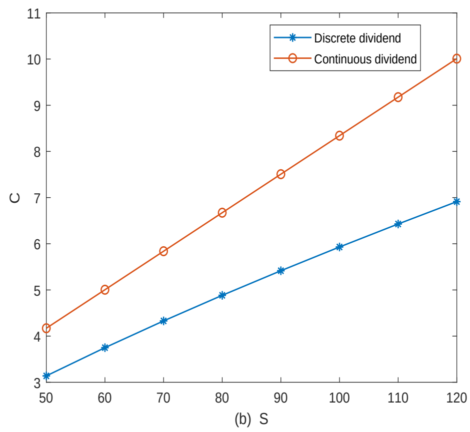


Fig. 4: The behaviors of the option price versus the stock price S.

By observing Table 5 and Fig. 4 (a), we can find that option prices increase significantly with the increase of stock prices. This is the same with reality because the price of the stock has an effect on the intrinsic value and time value of the option, and in the case of a call option when the price of the underlying stock rises, the intrinsic value of the option also gets increasing because the holder with the choice to buy the stock at a lower strike price and then deal it at a current higher market price. In addition, time value equals to the option premium minus the actual intrinsic value, which measures the possibility of realizing profit in the remaining time of the option, and the stock price has an indirect effect on time value. Higher stock prices usually increase the time value of the option, because the underlying asset has more room for fluctuation. Thus increasing the probability of the option can achieve profit, and improve the time value and the option price. On the contrary, a lower stock price will reduce the time value of the option, on account of the fact that the underlying asset has less room to fluctuate, reducing the probability that the option will realize a profit.

In addition, this study focuses on the pricing problem of geometric Asian options with known discrete dividends, and we compare it with the pricing of geometric Asian options with continuous dividends. B.L.S. Prakasa Rao[38] investigated the pricing problem of geometric Asian options with continuous dividends and derived the pricing formula for the corresponding dividend rate q. To better observe the sensitivity of discrete dividends to option prices, similar to the previous discussion, one can derive the expression of $\frac{\partial C}{\partial \sum_i D_i}$,

$$\begin{aligned}\frac{\partial C}{\partial \sum_i D_i} =&(S_0-\sum_i D_i)e^A\Phi(d_1)\frac{\partial A}{\partial \sum_i D_i}+(S_0-\sum_i D_i)e^A\Phi'(d_1)\frac{\partial d_1}{\partial \sum_i D_i}\\ &-Ke^{-rT}\Phi'(d_2)\frac{\partial d_2}{\partial \sum_i D_i}-e^A\Phi(d_1),\end{aligned}$$

where

$$\begin{aligned}\frac{\partial \sigma_d}{\partial \sum_i D_i} =&\frac{\sigma S_0}{(S_0-\sum_i D_i)^2},\\ \frac{\partial A}{\partial \sum_i D_i} =&\{-\frac{1}{2}(\sigma_d T+\frac{2\sigma_d T^{2H}}{2H+1})\ln\sum_i D_i\frac{\partial \sigma_d}{\partial \sum_i D_i}-[\frac{1}{2}rT-\frac{1}{2}(\frac{\sigma_d^2 T}{2}+\frac{\sigma_d^2 T^{2H}}{2H+1})]\frac{1}{\sum_i D_i}\}/(\ln\sum_i D_i)^2\\ &+\{\sigma_d(\ln\sum_i D_i)^2\frac{\partial \sigma_d}{\partial \sum_i D_i}-\sigma_d^2\ln\sum_i D_i\frac{1}{\sum_i D_i}\}/(\ln\sum_i D_i)^4,\\ \frac{\partial d_1}{\partial \sum_i D_i} =&\frac{\partial d_2}{\partial \sum_i D_i}+\frac{\sigma S_0[\ln\sum_i D_i-(S_0-\sum_i D_i)\frac{1}{\sum_i D_i}]}{(S_0-\sum_i D_i)^2(\ln\sum_i D_i)^2},\\ \frac{\partial d_2}{\partial \sum_i D_i} =&\{[\frac{-\ln K}{\sum_i D_i}-(\frac{1}{2}\sigma_d T+\frac{\sigma_d T^{2H}}{2H+1})\frac{\partial \sigma_d}{\partial \sum_i D_i}]\sigma_d\sqrt{\frac{1}{3}T^2+\frac{T^{2H}}{2(H+1)}}-[\ln S_0-\ln K\ln\sum_i D_i\\ &+\frac{1}{2}rT-\frac{1}{2}(\frac{\sigma_d^2 T}{2}+\frac{\sigma_d^2 T^{2H}}{2H+1})]\sqrt{\frac{1}{3}T^2+\frac{T^{2H}}{2(H+1)}}\frac{\partial \sigma_d}{\partial \sum_i D_i}\}/\sigma_d^2(\frac{1}{3}T^2+\frac{T^{2H}}{2(H+1)}).\end{aligned}$$

In the case of varying stock prices, the comparison between option prices with discrete and continuous dividends is shown in Table 6 and Fig. 4 (b). Taking the dividend rate q as $q = 0.03$, Fig. 4 (b) reveals that the prices of geometric Asian options with continuous dividends are higher than those with discrete dividends. However, both prices increase with an increase in the stock price. The prices of options with continuous dividends exhibit a steeper growth compared to the relatively stable and smoother price growth of options with discrete dividends.

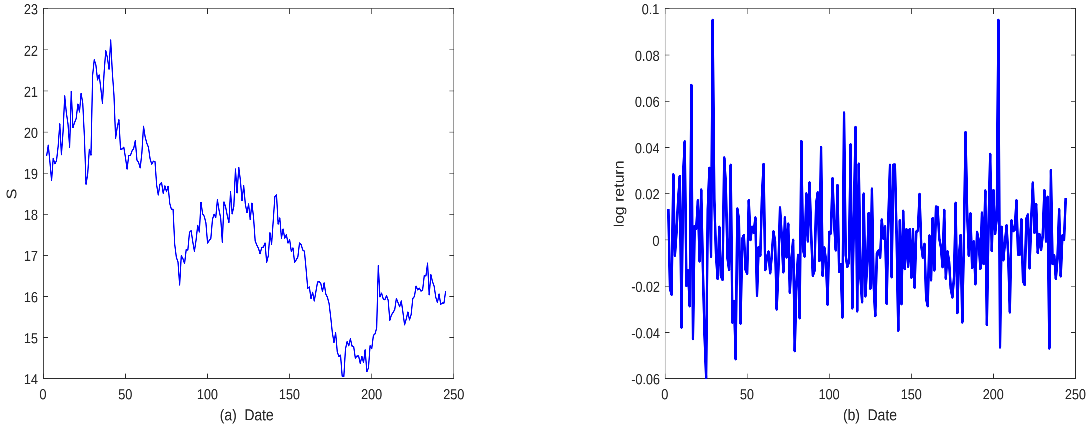


Fig. 5: (a)Daily closing prices, (b)Daily log return.

6. Empirical analysis

To further illustrate the effectiveness and practicality of this model, this study conducted an empirical analysis using data from Shenzhen Technology (stock code: 000021), a company in the deep technology sector. The closing price data for Shenzhen Technology was obtained from the Resset database (www.resset.com) and covered the period between January 4, 2021 and January 4, 2022. We now discuss the statistical characteristics of the sample returns of Shenzhen Technology, as depicted in Figs. 5 and 6.

Fig. 5 (a) displays the closing prices of Shenzhen Technology from January 4, 2021, to January 4, 2022. It is observed that the time series of Shenzhen Technology's prices exhibit distinct upward and downward jumps. Fig. 5 (b) illustrates the logarithmic return series of the Shenzhen Technology sample, where log return = $\ln(S_t/S_{t-1})$. Fig. 6 (a) presents the QQ plot of the corresponding logarithmic returns. The results indicate that the corresponding distribution of logarithmic returns reveals heavy-tailedness and diverges from normality. Finally, Fig. 6 (b) shows the distribution of the daily logarithmic returns of Shenzhen Technology. The solid red line represents the corresponding normal distribution. It can be observed that the distribution of logarithmic returns exhibits significantly higher kurtosis than the normal distribution and is not symmetric.

Table 7 summarizes the statistical interpretation of the Shenzhen Technology sample returns, with the first four columns containing basic descriptive statistics. Additionally, skewness and kurtosis statistics are included in the table. The skewness is 0.7103, which is larger than the skewness of $\phi(0)$. The kurtosis is 5.3257, which also exceeds the kurtosis of $\phi(3)$. This is consistent with the observed fat-tailedness and peakedness in the return distribution as depicted in the figures above. Thus, this is inconsistent with the assumptions of the classical BS model. In contrast, the MFBM model presented in this paper offers a flexible approach to simulating the complex phenomena in financial markets, including the features of fat tails and sharp peak, by combining stochastic processes with different characteristics.

Table 7: **Descriptive statistic of return rate**

Max	Min	Mean	Var	Skewness	Kurtosis
0.0951	-0.0596	-0.000763	0.00049	0.7103	5.3257

Next, we conducted a study on whether the logarithmic returns of this stock exhibit long memory. Fig. 7 presents the autocorrelation function (ACF) of the daily returns of the stock. From Fig. 7, we can observe that ACF of the sequence decays tardily over time, indicating that the logarithmic returns exhibit small yet significant autocorrelation over a prolonged time span. This verifies the long memory property of the

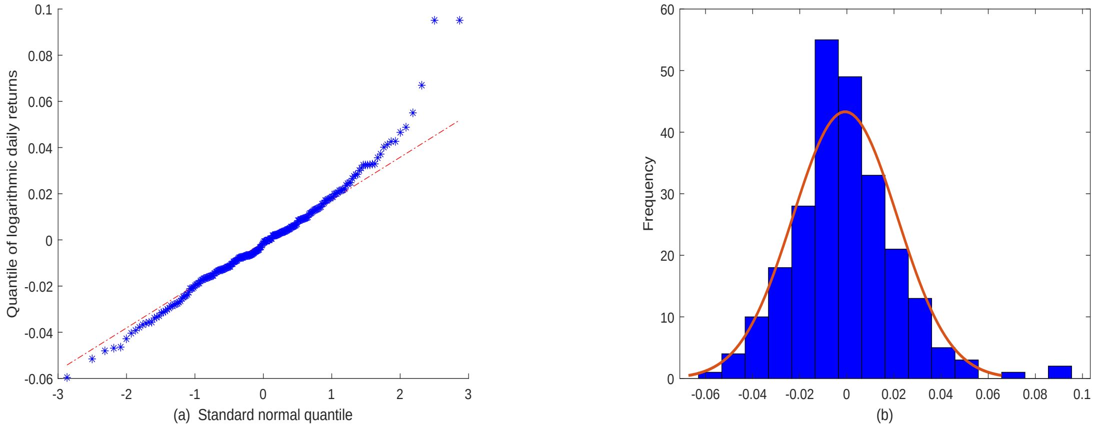


Fig. 6: (a)QQ plot of sample data, (b)Probability density of the return.

distribution of stock logarithmic returns. Hence, the stock price process in this model is described using mixed-fractional Brownian motion, which better captures long memory compared to previous geometric Brownian motion or fractional Brownian motion models.

Besides, we will use the model addressed in this work to calculate the price of geometric Asian call options and compare it with the results from existing valuation models. Using historical data, together with the R/S analysis method, one can evaluate the Hurst exponent H, resulting in $H = 0.825$. Additionally, other parameters are specified as follows: $r = 0.0135$, $\sigma = 0.35$, $\sum D = 0.6$, $q = 0.03$. We can input all the data into our pricing model and a known dividend yield q pricing model[38]. This involves comparing the option prices under known discrete dividend payments with those under continuous dividend payments, as illustrated in Figs. 8 and 9.

Examining Fig. 8 (a), it depicts the daily option prices under known discrete dividend payments versus those under continuous dividend payments. Overall, the option prices under discrete dividend payments are lower than those under continuous dividend payments. Observing Fig. 8 (b), it illustrates the variation in option prices with stock prices under the two models mentioned above. It can be observed that, after the stock price reaches a certain value, the option prices under discrete dividend payments are higher than those under continuous dividend payments. If the dividends are continuous, as the stock price rises, with more dividends being paid, the price of the call option decreases even more. Fig. 9 displays three-dimensional surface plots of option prices with respect to time T and stock price S under discrete and continuous dividend payments.

Finally, this paper compares the calculated results with the market price and conducts a certain error analysis. The following calculation formulas, including mean square error (MSE), mean absolute error (MAE), root mean square error, and mean absolute percentage error (MAPE), will be used to measure modeling error[35, 36],

$$\mathrm{MSE} = \frac{1}{n}\sum_{i=1}^{n}(S_i - S)^2, \ \mathrm{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(S_i - S)^2},$$

$$\mathrm{MAE} = \frac{1}{n}\sum_{i=1}^{n}|S_i - S|, \ \mathrm{MAPE} = \frac{1}{n}\sum_{i=1}^{n}\frac{|S_i - S|}{S},$$

where S_i and S respectively denote market price and model price, and n means the number of options.

The above four statistics are used to measure the degree of difference of estimators. MSE is the absolute deviation of the pricing error, and RMSE is the average deviation of estimators, which reflects the total error. ME measures the absolute deviation from the actual price, while MAPE measures the relative deviation

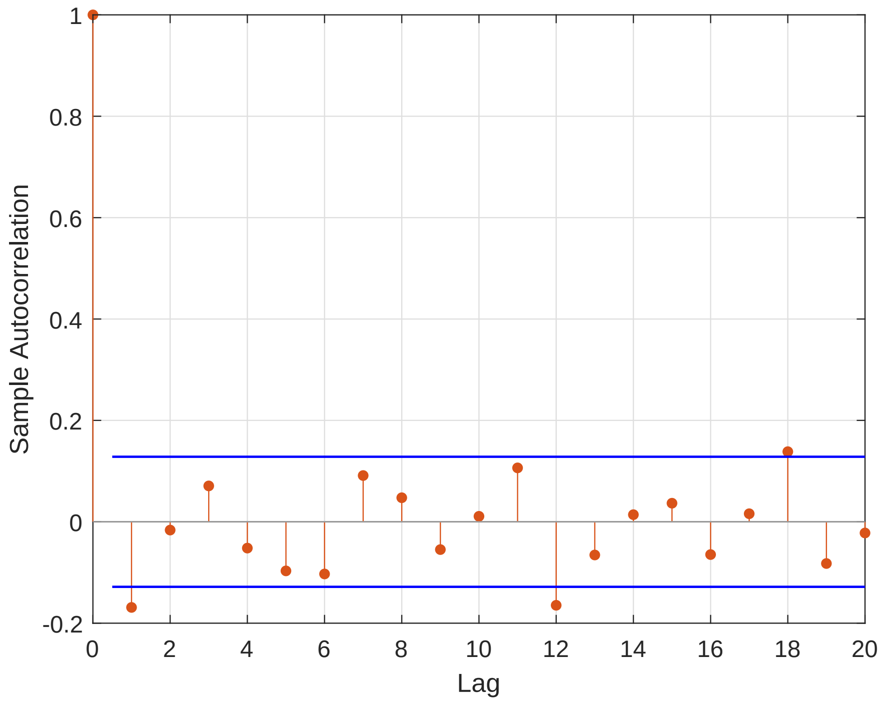


Fig. 7: The ACF plots of the log returns.

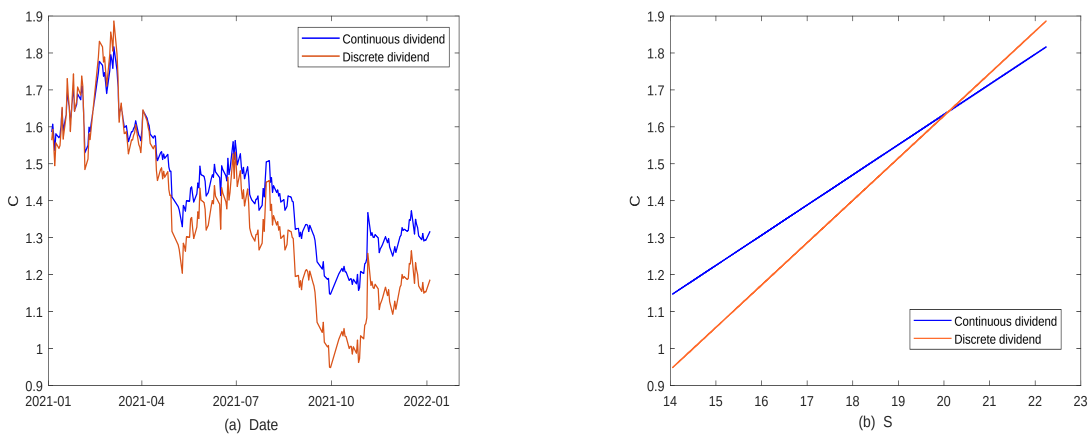


Fig. 8: (a)Pay the daily option price under discrete or continuous dividends, (b)The change of option price with stock price under different models.

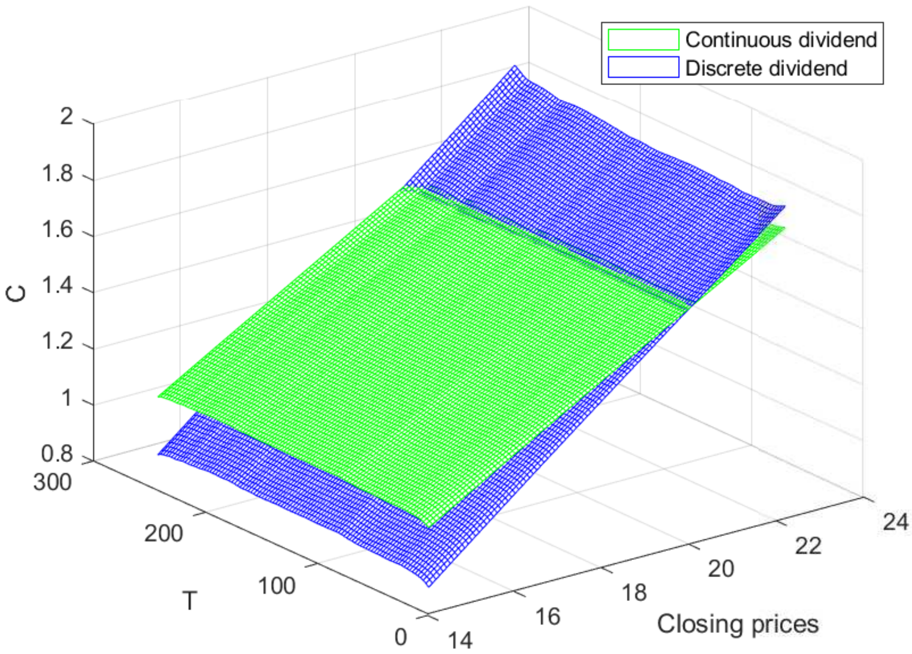


Fig. 9: Three-dimensional graph of stock prices and option prices under different models.

from the actual price. The larger the value is, the larger the error is and the worse the model effect is. The smaller the value, the smaller the error and the better the model effect. The calculation results are shown in Table 8.

Table 8 presents the error values between the theoretical option prices and the actual option prices under two models, where “discrete dividends” refers to the MFBM pricing model with discrete dividends, and “continuous dividends” indicates the pricing model with continuous dividends. The error values for the former are consistently smaller than those for the latter. This indicates that the overall error between the theoretical prices of our model and the market prices is smaller, both in relative and absolute terms. This further substantiates the effectiveness and accuracy of our model. To further highlight the superiority of the model presented in this study, Fig. 10 contrasts the theoretical option prices computed by both models with the actual option prices.

Table 8: **Each error analysis results**

	Continuous dividend	Discrete dividend
MSE	0.0503	0.0429
RMSE	0.2244	0.2071
MAE	0.1957	0.1638
MAPE	0.1543	0.1289

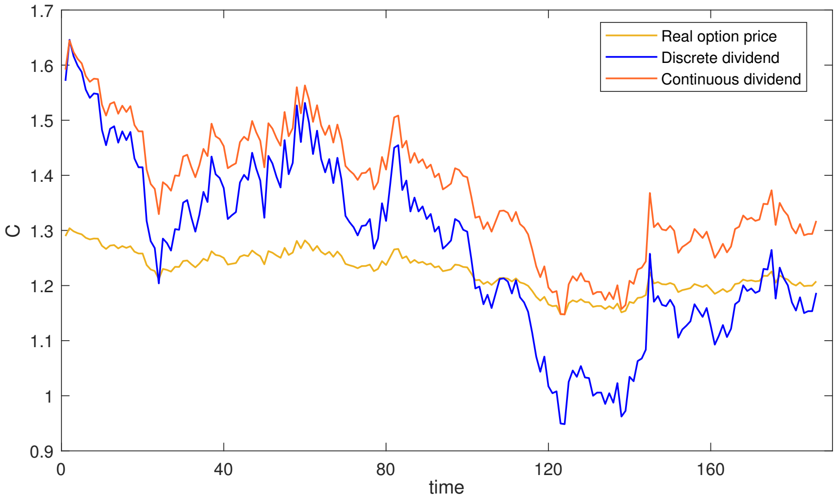


Fig. 10: Real option prices versus simulated prices.

7. Conclusion

Option is a kind of derivative instrument, which plays an important role in China's financial market. How to price them effectively and accurately is of great significance both in theory and practice. The Asian option studied in this work is a special exotic option, which is characterized by a smoother response to price fluctuations based on the average price over a specified period of time compared to European or American options. This means that it can mitigate the impact of short-term noise and occasional movements in the market on option prices, which is particularly beneficial in a high-volatility environment or a market with relatively smooth price movements. Moreover, Asian options can be used in market risk management and hedging strategies, and play a certain protective role in the investment portfolio[39]. It can reduce the impact of short-term price fluctuations on investors to a certain extent, and provide them with relatively stable returns. In addition, the flexibility of Asian options enables investors to invest according to their own needs and expectations. For bullish Asian options, investors can use the characteristics of average price to lock the rising trend of the underlying asset in a certain period of time and obtain a relatively low exercise price[40]. For Asian options, investors can use the average price to lock in a downtrend and get a relatively high strike price. The pricing of Asian options has been addressed in numerous studies. For instance, Ref.[46] builds a three-factor model based on Schwartz's framework and derives a closed-form solution by incorporating stochastic convenience yields, random interest rates, and jumps in the spot price. To study and price discretely observed Asian options, Ref.[47] uses a discontinuous Galerkin method that combines the ideas of finite volume (FV) and finite element (FE) methods by using piecewise polynomial approximations that allow for discontinuities between elements.

This work gives and deduces the pricing formula of geometric Asian options based on MFBM and discrete dividend payment. It should be pointed out that some dividend payments may be periodic cash distributions. Considering the fact that dividends can be paid at discrete time intervals, in this work, the dividend is modeled as discrete dividend payments. Additionally, it is a challenging problem to price options with known cash dividends. In addition, owing to the fact that MFBM has a unique advantage in predicting the underlying processes and future outcomes, and MFBM can better describe the nonlinearity and time variability in financial fluctuations, thus, in this work, the stock price process is described by a MFBM. Therefore, this work has a new description of the underlying asset price movement after dividend payment and tries to explore this problem in a useful and effective way. The underlying asset price also follows MFBM, which addresses the long memory of the financial market and excludes the arbitrage behavior and is a mathematical model of a strongly correlated stochastic process. Significantly, the analytic expression for geometric Asian call option price has been achieved. In contrast to Ref.[48], this study incorporates a jump process into the price dynamics but does not take dividend payments into account. In addition, numerical experiments have been conducted on different parameters in our proposed model, and the effects of important parameters on option prices are also addressed. Finally, the effectiveness and practicality of our model has been verified by empirical analysis. In a word, our results, incorporating MFBM and discrete dividend payments into the pricing of geometric Asian options, enrich the theory of the dynamics of the underlying asset.

References

[1] B. K. Sinem Kozpınar, Murat Uzunca, *Pricing european and american options under heston model using discontinuous galerkin finite elements*, Mathematics and Computers in Simulation **177** (2020), 568–587.
[2] J. Hussain, *Pricing of quanto power options and related exotic options*, Results in Applied Mathematics **18** (2023), 100371.
[3] S. F. AhmadianL D, Ballestra V, *A monte-carlo approach for pricing arithmetic asian rainbow options under the mixed fractional brownian motion, Chaos*, Solitons and Fractals **158** (2022), 112023.
[4] F. Black, M. Scholes, *The pricing of options and corporate liabilities*, Journal of political economy **81 (3)** (1973), 637–654.
[5] L. Meng, M. Wang, *Comparison of black-scholes formula with fractional black–scholes formula in the foreign exchange option market with changing volatility*, Asia-Pacific Financial Markets**17** (2010), 99–111.
[6] C. Necula, *Option pricing in a fractional brownian motion environment*, Advances in Economic and Financial Research**2 (3)** (2002), 259–273.
[7] S. Lin, *Stochastic analysis of fractional brownian motions*, Stochastics: An International Journal of Probability and Stochastic Processes**55 (1-2)** (1995), 121–140.
[8] F. Comte, E. Renault, *Long memory in continuous-time stochastic volatility models*, Mathematical finance**8 (4)** (1998), 291–323.
[9] Y. A. Kuznetsov, *The absence of arbitrage in a model with fractal brownian motion*, Russian Mathematical Surveys**54 (4)** (1999), 847.
[10] L. C. G. Rogers, *Arbitrage with fractional brownian motion*, Mathematical finance**7 (1)** (1997) 95–105.
[11] C. El-Nouty, *The fractional mixed fractional brownian motion*, Statistics & Probability Letters**65 (2)** (2003), 111–120.
[12] P. Cheridito, *Arbitrage in fractional brownian motion models*, Finance and stochastics**7 (4)** (2003), 533–553.
[13] W.-L. Xiao, W.-G. Zhang, X. Zhang, X. Zhang, *Pricing model for equity warrants in a mixed fractional brownian environment and its algorithm*, Physica A**391 (24)** (2012), 6418–6431.
[14] L. Sun, *Pricing currency options in the mixed fractional brownian motion*, Physica A **392 (16)** (2013), 3441–3458.
[15] F. Shokrollahi, *The evaluation of geometric asian power options under time changed mixed fractional brownian motion*, Journal of Computational and Applied Mathematics**344** (2018), 716–724.
[16] Z. Guo, X. Wang, Y. Zhang, *Option pricing of geometric asian options in a sub-diffusive brownian motion regime*, AIMS Mathematics**4** (2020), 5332–5343.
[17] Z. Zhang, W. Liu, *Geometric average asian option pricing for uncertain financial market*, Journal of Uncertain Systems**8 (4)** (2014), 317–320.
[18] J. Sun, X. Chen, *Asian option pricing formula for uncertain financial market*, Journal of Uncertainty Analysis and Applications**3 (1)** (2015), 1–11.
[19] W. Wang, P. Chen, *Pricing asian options in an uncertain stock model with floating interest rate*, International Journal for Uncertainty Quantification**8 (6)** (2018), 543-557.
[20] M. Bos, S. Vandermark, *Finessing fixed dividends*, Risk-London-Risk Maqazine Limited **15 (9)** (2002), 157–170.
[21] S. Guo, Q. Liu, *A simple accurate binomial tree for pricing options on stocks with known dollar dividend*s, The Journal of Derivatives**26 (4)** (2019), 54–70.
[22] M. H. Vellekoop, J. W. Nieuwenhuis, *Efficient pricing of derivatives on assets with discrete dividends*, Applied Mathematical Finance **13 (3)** (2006), 265–284.
[23] J. A. d. M. R. Dilão, B. Ferreira, *On the value of european options on a stock paying a discrete dividend*, Journal of Modelling in Management **4 (3)** (2009), 235–248.
[24] C. Veiga, U. Wystup, *Closed formula for options with discrete dividends and its derivatives*, Applied Mathematical Finance **16 (6)** (2009), 517–531.
[25] S.-P. Zhu, X.-J. He, *An accurate approximation formula for pricing european options with discrete dividend payments*, IMA Journal of Management Mathematics**29 (2)** (2018), 175–188.
[26] B. W. Muganda, I. Kyriakou, B. S. Kasamani, *Modelling asymmetric dependence in stochastic volatility and option pricing: A conditional copula approach*, Scientific African **21** (2023), e01765.
[27] Z. Zhou, Z. Jin, *Optimal equilibrium barrier strategies for time-inconsistent dividend problems in discrete time*, Insurance: Mathematics and Economics **94** (2020), 100–108.
[28] X.-J. He, S. Lin, *An accurate approximation to barrier option prices with discrete fixed-amount dividends: Nonlinear dynamics*, Expert Systems with Applications**204** (2022), 117543.
[29] R. Roll, *An analytic valuation formula for unprotected american call options on stocks with known dividends*, Journal of Financial Economics**5 (2)** (1977), 251–258.
[30] D. C. Heath, R. A. Jarrow, *Ex-dividend stock price behavior and arbitrage opportunities*, Journal of Business **61**(1988), 95–108.
[31] T.-S. Dai, C.-Y. Chiu, *Pricing barrier stock options with discrete dividends by approximating analytical formulae*, Quantitative Finance**14 (8)** (2014), 1367–1382.

[32] D. Ahmadian, L. Ballestra, *Pricing geometric asian rainbow options under the mixed fractional brownian motion*, Physica A**555** (2020), 124458.

[33] J. Garnier, K. Sølna,*Option pricing under fast varying long-memory stochastic volatility*, Mathematical finance **29(1)** (2019), 39–83.

[34] P. Cheridito, *Mixed fractional brownian motion*, Bernoull**7 (6)** (2001), 913–934.

[35] N. Chriss, *Black-scholes and beyond: option pricing models*, Sirirajmedj Com, 1997.

[36] J. C. Hull, *Options futures and other derivatives*, Pearson Education India, 2003.

[37] S. S. R T Vulandari, *Black-scholes model of european call option pricing in constant market condition*, International Journal of Computing Science and Applied Mathematics**6 (2)** (2020), 46–49.

[38] B. P. Rao, *Pricing geometric asian power options under mixed fractional brownian motion environment*, Physica A**446** (2016), 92–99.

[39] Z. Mao, Z. Liang, et al., *Evaluation of geometric asian power options under fractional brownian motion*, Journal of Mathematical Finance **4 (01)** (2013), 1.

[40] H. Geman, M. Yor, *Bessel processes, asian options, and perpetuities*, Mathematical finance**3** (4) (1993), 349–375.

[41] Bender C.*An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter[J]*. Stochastic Processes and their Applications, 2003, **104(1)**: 81-106.

[42] Kruk I, Russo F, Tudor C A. *Wiener integrals, Malliavin calculus and covariance measure structure[J]*. Journal of Functional Analysis, 2007, **249(1)**: 92-142.

[43] Araneda A. *The fractional and mixed-fractional CEV model[J]*. Journal of Computational and Applied Mathematics, 2020, **363**: 106-123.

[44] Merton R C. *Theory of rational option pricing[J]*. The Bell Journal of economics and management science **4(1)**: 141-183, 1973.

[45] Geske R. *The pricing of options with stochastic dividend yield[J]*. The Journal of Finance **33**(2): 617-625, 1978.

[46] Ewald C, Wu Y, Zhang A. *Pricing Asian options with stochastic convenience yield and jumps[R]*. Quantitative Finance **23**(4):677-692, 2023.

[47] Hozman J, Tichý T. *The discontinuous Galerkin method for discretely observed Asian options[J]*. Mathematical Methods in the Applied Sciences, 2020, **43**(13): 7726-7746.

[48] Shokrollahi F, Ahmadian D, Ballestra L V. *Pricing Asian options under the mixed fractional Brownian motion with jumps[J]*. Mathematics and Computers in Simulation, 2024, 226: 172-183.