Filomat 39:1 (2025), 267–277
https://doi.org/10.2298/FIL2501267S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Total domination and minimal total domination polynomial of $H-$join graphs

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Abstract. Let H be a connected labeled graph. In this article, we characterize all the total dominating sets and the minimal total dominating sets of $H-$join graphs. Consequently, we compute the (multivariate) total domination polynomial and the (multivariate) minimal total domination polynomial of $H-$join graphs. We also compute the total domination number of $H-$join graphs. Finally, as an illustration, we calculate the total domination polynomial and the minimal total domination polynomial of the join of graphs, multipartite graphs, K_n-join graphs, $K_{n_1,\ldots,n_m}-$join graphs, the corona product of graphs and the windmill graphs.

1. Introduction

A *dominating set* D of a graph G (may be disconnected) is a vertex subset D of G such that for every $v \in V(G) \setminus D$ there exists $u \in D$ such that v is adjacent to u. The *domination number* $\gamma(G)$ of G is the cardinality of a smallest dominating set of G. A subset $D \subset V(G)$ is said to be a *total dominating set* if for every $v \in V(G)$ there exists $u \in D$ such that v is adjacent to u. The *total domination number* $\gamma_t(G)$ of G is the cardinality of a smallest total dominating set of G. A subset $D \subset V(G)$ is said to be a *minimal total dominating set* if it is a total dominating set and none of its proper subset is a total dominating set. For an overview and in-depth analysis of the literature on dominating sets of graphs, refer [13, 14] and on total dominating sets of graphs refer [15, 16].

In graph theory, various types of polynomials are associated with graphs, such as characteristic polynomials, clique polynomials, chromatic polynomials, Tutte polynomials, etc. We study two specific polynomials related to total dominating sets in graphs: total domination and minimal total domination polynomials of graphs. In recent decades, various authors extensively studied (total) domination polynomials. For instance, investigations on domination polynomials of graphs, refer [1, 6–9, 17, 21] and on total domination polynomials of graphs, refer [2–5, 11, 12]. In particular, for insights into the domination and total polynomial of certain products of graphs, refer [9, 12, 17, 18, 21].

2020 *Mathematics Subject Classification*. Primary 05C31; Secondary 05C69, 05C76.
Keywords. $H-$join graph, Total domination polynomial, Minimal total domination polynomial.
Received: 04 January 2024; Revised: 22 May 2024; Accepted: 20 November 2024
Communicated by Paola Bonacini
The second author is supported by the NBHM Post-doctoral fellowship [Order number 0204/16(19)/2022/R&D-II/11991] India.
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Let H be a connected graph with at least two vertices and let $\mathcal{G} = \{G_v : v \in V(H)\}$ be a family of pairwise disjoint graphs. The $H-$*join operation* of the family of graphs $\mathcal{G}$, denoted by $\bigvee_H \mathcal{G}$, is obtained by replacing each vertex v of H by the graph G_v and every vertex of G_v is made adjacent with every vertex of G_w, whenever v is adjacent to w in H. Precisely, $\bigvee_H \mathcal{G}$ is the graph with the vertex set $V(\bigvee_H \mathcal{G}) = \bigcup_{v\in V(H)} V(G_v)$ and the edge set $E(\bigvee_H \mathcal{G}) = \bigcup_{v\in V(H)} E(G_v) \cup \bigcup_{vw\in E(H)}\{xy : x \in V(G_v), y \in V(G_w)\}$. A $H-$*join graph* G is a graph obtained by $H-$join operation of a family of graphs $\mathcal{G} = \{G_v : v \in V(H)\}$. Given a $H-$join graph $G := \bigvee_H \mathcal{G}$, there is a canonical projection $\pi : V(G) \to V(H)$ defined by $\pi(x_v) = v$, where $v \in V(H)$ and $x_v \in V(G_v)$.

It is a fact that given a graph G, there exists a graph H and a family of graphs $\mathcal{G} = \{G_v : v \in V(H)\}$ such that $G = \bigvee_H \mathcal{G}$, refer [[22], Lemma 3.1]. A generalization of $H-$join operation, known as $H-$generalized join operation of family of graphs $\mathcal{G} = \{G_v : v \in V(H)\}$ constrained by family of vertex subsets $\mathcal{S} = \{S_v \subset V(G_v) : v \in V(H)\}$, denoted by $\bigvee_{H,\mathcal{S}} \mathcal{G}$, has been introduced in [10]. Recently, we studied graph invariants such as the (maximal) clique polynomials, the (maximal) independent polynomials of $\bigvee_{H,\mathcal{S}} \mathcal{G}$ in [19] and the (minimal) domination polynomial of $\bigvee_{H,\mathcal{S}} \mathcal{G}$ in [21].

In [20], the authors described all $(1,2)-$dominating sets of $H-$join graphs. For $k \geq 1$, a vertex subset S of a graph G is said to be $(1,k)-$dominating set if for every vertex $v \in V(G)$ there exist two distinct vertices $u,w \in S$ such that $uv \in E(G)$ and the distance between v and w in G is at most k.

In this article, we provide a formula for computing the (multivariate) total domination polynomial and the (multivariate) minimal total domination polynomial of a $H-$join graph by identifying all (minimal) total dominating sets of $H-$join graphs in terms of certain vertex subsets of H and G_v.

Now, let us recall some terminologies that are needed for the article. Let H be a labeled graph. For each vertex v of H, we associate a variable $\mathbf{x_v}$. Define $X_J := \prod_{v\in J} \mathbf{x_v}$ for a vertex subset J of H. If $J = \emptyset$ then assume that $X_\emptyset := 1$. Let $\mathcal{B}$ be a collection of some special vertex subsets of H. (Some examples are the collection of cliques of H, the collection of dominating sets of H, the collection of total dominating sets of H, etc). Then, the multivariate polynomial for $\mathcal{B}$ of H is defined as

$$\mathcal{MV}_{\mathcal{B}}(H;X) = \sum_{J\subset V(H)} [J]X_J,$$

where $[J] = 1$ if $J \in \mathcal{B}$ and $[J] = 0$ if $J \notin \mathcal{B}$. In other words,

$$\mathcal{MV}_{\mathcal{B}}(H;X) = \sum_{J\in\mathcal{B}} X_J.$$

The polynomial for $\mathcal{B}$ of H is defined as

$$\mathcal{P}_{\mathcal{B}}(H;x) = \sum_{k=0}^{|V(H)|} c_k x^k,$$

where c_k is the number of elements of $\mathcal{B}$ of cardinality k.

Let $\mathcal{TD} = \mathcal{TD}om(H)$ be the set of all total dominating sets of H. Then, the polynomial $\mathcal{P}_{\mathcal{TD}}(H;x)$ (resp., the multivariate polynomial $\mathcal{MV}_{\mathcal{TD}}(H;X)$) for $\mathcal{TD}$ of H is called the *total domination polynomial* (resp. the *multivariate total domination polynomial*) of H. Similarly, let $\mathcal{MT}$ be the set of all minimal total dominating sets of H. Then the polynomial $\mathcal{P}_{\mathcal{MT}}(H;x)$ (resp. the multivariate polynomial $\mathcal{MV}_{\mathcal{MT}}(H;X)$) for $\mathcal{MT}$ of H is called the *minimal total domination polynomial* (resp. *multivariate minimal total domination polynomial*) of H.

We associate multivariables X, X^v and $Z = (X^v)_{v\in V(H)}$ to the vertex sets $V(H)$, $V(G_v)$ and $V(G)$ respectively for the purpose of writing their multivariate polynomials in subsequent sections.

2. Total Dominating sets of $H-$join graphs

In this section, we provide a formula for the total domination polynomial of a $H-$join graph by characterizing total dominating sets of $H-$join graphs.

Let M be a vertex subset of a graph H. We denote the degree of a vertex $v \in M$ in the induced subgraph $H[M]$ of H by $deg^M(v)$. Let $M^i = \{v \in M : deg^M(v) = i\}$ and $M^+ = \{v \in M : deg^M(v) \geq 1\}$. A vertex v in $M \subset V(H)$ is said to be $M-$isolated if v is isolated in $H[M]$, i.e., $v \in M^0$. For a vertex $v \in V(G)$, the *open neighborhood of* v in G is the set $N(v)$ consists of all vertices $u \in V(G)$ such that u is adjacent to v. For a set $D \subset V(G)$, the *open neighborhood of* D is the set $N(D) = \cup_{v \in D} N(v)$.

Now, we characterize all total dominating sets of $H-$join graphs.

Proposition 2.1. *Let $G = \bigvee_H \mathcal{G}$ be a $H-$join graph. Let D be a vertex subset of G. Then, D is a total dominating set in G if and only if $M := \pi(D)$ is a dominating set in H and there exists a family of nonempty sets $\{D_v \subset V(G_v)\}_{v \in M}$ such that $D = \cup_{v \in M} D_v$ and D_v is a total dominating set in G_v whenever v is $M-$isolated vertex in H, i.e., $v \in M^0$.*

Proof. For a total dominating set D of G, we first show that $M := \pi(D)$ is a dominating set in H. If $M = V(H)$, then nothing to prove. Assume that $M \neq V(H)$. Let $v \in V(H) \setminus M$. Since D is a total dominating set in G, for $x \in V(G_v)$ there exists $y \in D$ such that x is adjacent to y in G. Hence by the definition of $H-$join, $v = \pi(x)$ is adjacent to $\pi(y) \in M$ in H. Thus, M is a dominating set in H.

For $v \in M$, define $D_v = V(G_v) \cap D$. We show that for each $M-$isolated vertex v, the set D_v is a total dominating set in G_v. Suppose not, there exists a $M-$isolated vertex v such that D_v is not a total dominating set in G_v. Then, there exists a vertex $x \in V(G_v)$ such that x is adjacent to no element of D_v. Since D is a total dominating set in G, there exists $y \in D \setminus D_v$ such that y is adjacent to x. This implies that, by the definition of $H-$join, v is adjacent to $u = \pi(y) \in M \setminus \{v\}$ in H. Hence, $deg^M(v) > 0$, which is a contradiction. Thus, D_v is a total dominating set in G_v, for all $v \in M^0$.

Conversely, suppose that M is a dominating set in H and $\{D_v \subset V(G_v)\}_{v \in M}$ is a family of nonempty sets such that D_v is a total dominating set in G_v whenever v is $M-$isolated in H. We show that $D = \cup_{v \in M} D_v$ is a total dominating set in G. Let $x \in V(G)$. If $w := \pi(x) \notin M$ then w is adjacent to some vertex u in M, as M is a dominating set in H. Hence, x is adjacent to a vertex in D_u. If $w \in M^0$, then $V(G_w) \subset N(D_w)$ as D_w is a total dominating set in G_w. If $w \in M^+$, then there exists a vertex $u \in M$ such that w is adjacent to u. It follows that x is adjacent to a vertex in D_u. Hence, D is a total dominating set in G. □

As a consequence of the above proposition, we have the following theorem.

Theorem 2.2. *Let $G = \bigvee_H \mathcal{G}$ be a $H-$join graph. The number of total dominating sets of size k in G is given by*

$$d_k(G) = \sum_{s=1}^{k} \sum_{\substack{M \in \mathcal{D}om(H) \\ |M|=s}} \sum_{\mathfrak{a}_M = k} \left(\prod_{v \in M^0} d_{a_v}(G_v) \prod_{u \in M^+} \binom{|V(G_u)|}{a_u} \right),$$

where $\mathcal{D}om(H)$ denotes the set of all dominating sets of H, the third sum is over all possible sums $k = \mathfrak{a}_M = \sum_{v \in M} a_v$ of positive integers, and $d_{a_v}(G_v)$ denotes the number of total dominating sets in G_v of size a_v.

Proof. By Proposition 2.1, D is a total dominating set in G of size k if and only if $M := \pi(D)$ is a dominating set in H and $\{D_v = D \cap V(G_v) : v \in M\}$ is a partition of D such that D_v is a total dominating set in G_v whenever $v \in M^0$ and $\sum_{v \in \pi(D)} |D_v| = k$. Hence, the number of total dominating sets D of size k in G such that $\pi(D) = M$ and $|D \cap V(G_v)| = a_v$ for each $v \in M$ is equal to the product $\prod_{v \in M^0} d_{a_v}(G_v) \prod_{u \in M^+} \binom{|V(G_u)|}{a_u}$. Hence, the result follows. □

For a subset M of a set N, we consider the characteristic function $\chi_M : N \to \{0, 1\}$ defined as $\chi_M(v) = 1$ if and only if $v \in M$.

Theorem 2.3. *Let $G = \bigvee_H \mathcal{G}$ be a $H-$join graph. Let $\mathcal{MV}_{\mathcal{D}}(H; X) = \sum_{M \subset V(H)} [M] X_M$ be the multivariate domination polynomial of H, where $\mathcal{D} = \mathcal{D}om(H)$. Then,*

1. *the total domination polynomial of G is given by*

$$\mathcal{P}_{\mathcal{TD}}(G; x) = \sum_{M \subset V(H)} [M] \left[\prod_{v \in M} \left(\chi_{M^0}(v) \mathcal{P}_{\mathcal{TD}}(G_v; x) + (1 - \chi_{M^0}(v)) \left((1 + x)^{|V(G_v)|} - 1 \right) \right) \right].$$

2. *the multivariate total domination polynomial of G is given by*

$$\mathcal{MV}_{\mathcal{TD}}(G;Z)=\sum_{M\subset V(H)}[M]\Big[\prod_{v\in M}\Big(\chi_{M^0}(v)\mathcal{MV}_{\mathcal{TD}}(G_v;X^v)+(1-\chi_{M^0}(v))\mathcal{MV}_{\mathcal{D}}(K_{|V(G_v)|};X^v)\Big)\Big],$$

where $\mathcal{MV}_{\mathcal{D}}(K_{|V(G_v)|};X^v)$ *is the multivariate domination polynomial of the complete graph* $K_{|V(G_v)|}$ *of order* $|V(G_v)|$.

Proof. Observe that the coefficient $d_k(G)$ of x^k in the domination polynomial of G is equal to the sum over all dominating sets M of H of all products $\prod_{v\in M^0} d_{a_v}(G_v)\prod_{v\in M^+}\binom{|V(G_v)|}{a_v}$ of coefficients $d_{a_v}(G_v)$ and $\binom{|V(G_v)|}{a_v}$ of x^{a_v} of polynomials $\mathcal{P}_{\mathcal{TD}}(G_v;x)$ (if $v\in M^0$) and $(1+x)^{|V(G_v)|}-1$ (if $v\in M^+$) respectively such that $\sum_{v\in M} a_v=k$. Hence, the first result immediately follows from Theorem 2.2. The proof of the statement (2) is also similar to that of (1). □

2.1. *The total domination number of a H−join graph*

Now, we discuss the total domination number of H−join graphs.

Theorem 2.4. *If* $G=\bigvee_H \mathcal{G}$ *is a H−join graph, then, the total domination number* $\gamma_t(G)$ *of G is given by*

$$\gamma_t(G)=min\Big\{|M|-|M^0|+\sum_{v\in M^0}\gamma_t(G_v): M\in \mathcal{D}om(H)\Big\}.$$

Proof. The proof follows from Theorem 2.2 and Theorem 2.3. □

Theorem 2.5. *Let* $G=\bigvee_H \mathcal{G}$ *be a H−join graph. Then,* $\gamma(H)\le\gamma_t(G)\le\gamma_t(H)$. *Moreover, if* $\gamma_t(H)=\gamma(H)$, *then* $\gamma(H)=\gamma_t(G)=\gamma_t(H)$. *The equality* $\gamma_t(G)=\gamma(H)$ *holds only when* $\gamma_t(H)=\gamma(H)$.

Proof. The proof follows from Theorem 2.4 and the following facts:

- If M is a total dominating set of a graph G' then $|M|\ge 2$.
- A dominating set M of a graph G' is a total dominating set in G' if and only if M has no M−isolated vertex, i.e., $M^0=\emptyset$.

□

3. Minimal Total Dominating sets of *H*−join graphs

In this section, we describe all minimal total dominating sets of H−join graphs. Consequently, we compute the minimal total domination polynomial of H−join graphs.

Let H be a graph. Consider a vertex subset S of H. For $v\in S$, *the* S*−private neighborhood* $pn(v,S)$ *of* v in H is defined by $pn(v,S)=\{u\in V(H): N(u)\cap S=\{v\}\}$. A vertex in $pn(v,S)$ is called a S*−private neighbor* of v. A vertex $v\in S$ is said to be S*−private* if the S−private neighborhood $pn(v,S)$ of $v\in S$ is nonempty. A vertex subset S of H is said to be *completely irredundant* if for each vertex $v\in S$ either v is S−private or S−isolated.

Now, we describe the minimal total dominating sets of H−join graphs.

Proposition 3.1. *Let* $G=\bigvee_H \mathcal{G}$ *be a H−join graph. Let D be a vertex subset of G. Then, D is a minimal total dominating set in G if and only if* $M:=\pi(D)$ *is a completely irredundant dominating set in H and there exists a family of nonempty sets* $\{D_v\subset V(G_v)\}_{v\in M}$ *such that* $D=\cup_{v\in M}D_v$ *and satisfies the following properties:*

1. *If* $v\in M^0$ *then* D_v *is a minimal total dominating set in* G_v.
2. *If* $v\in M^+$ *then* D_v *is a singleton set in* $V(G_v)$.

Proof. For a minimal dominating set D of G, let $M := \pi(D)$. Set $D_v = D \cap V(G_v)$, for all $v \in V(H)$. First, we show that M is a completely irredundant dominating set in H. It follows from Proposition 2.1 that M is a dominating set in H. It is enough to show that v is a $M-$private vertex if $v \in M^+$. Suppose that $v \in M^+$ is not a $M-$private vertex, i.e., the $M-$private neighborhood $pn(v, M)$ of v is empty. Then, every element of $N(v)$ is adjacent to an element of $M \setminus \{v\}$. Hence, $M \setminus \{v\}$ is a dominating set in H as M is a dominating set in H. Now, to get a contradiction, we claim that $D \setminus V(G_v)$ is a total dominating set in G. By Proposition 2.1, it is enough to prove that for all $u \in (M \setminus \{v\})^0$, the set D_u is a total dominating set in G_u. Since $v \in M^+$, we see that $(M \setminus \{v\})^0 = M^0 \cup (N(v) \cap M^1)$. But $pn(v, S) = \emptyset$ implies that $N(v) \cap M^1 = \emptyset$. Hence, $(M \setminus \{v\})^0 = M^0$. Then, the claim follows from Proposition 2.1. This is a contradiction to the minimality of D in G. Hence, M is a completely irredundant dominating set in H.

Now, we show that for each $v \in M^0$, the set D_v is a minimal total dominating set in G_v. It follows from Proposition 2.1 that for each $v \in M^0$, the set D_v is a total dominating set in G_v. Suppose that $D_v \setminus \{x\}$ is a total dominating set in G_v for some $x \in D_v$. Then by Proposition 2.1, we see that $D \setminus \{x\} = (D_v \setminus \{x\}) \cup_{w \in M \setminus \{v\}} D_w)$ is a total dominating set in G as each D_w is a total dominating set in G_w for $w \in M^0 \setminus \{v\}$. This is a contradiction to the minimality of D in G. Hence, D_v is a minimal total dominating set in G_v.

Finally, the set D_u is singleton, for each $u \in M^+$. Because if $\{x, y\} \subset D_u$ for some $u \in M^+$, then by a similar argument using Proposition 2.1, one can show that $D \setminus \{y\}$ is a total dominating set of G.

Conversely, suppose that M is a completely irredundant dominating set in H and there exists a family of vertex subsets $\{D_v \subset G_v\}_{v \in M}$ that satisfies (1) and (2). We show that $D = \cup_{v \in M} D_v$ is a minimal total dominating set in G. It follows from Proposition 2.1 that D is a total dominating set in G. Suppose that D is not a minimal total dominating set in G. Then, there exists a vertex x in D such that $D \setminus \{x\}$ is a total dominating set in G. Let $u := \pi(x)$. Suppose that u is $M-$isolated, i.e., $u \in M^0$. Then by hypothesis (1), D_u is a minimal total dominating set in G_u. Hence, $D_u \setminus \{x\} = (D \setminus \{x\}) \cap V(G_v)$ is not a total dominating set in G_u. But by Proposition 2.1, $D \setminus \{x\}$ is a total dominating set in G implies that the set $D_u \setminus \{x\}$ is a total dominating set of G_u if $D_u \setminus \{x\} \neq \emptyset$, or $\pi(D \setminus \{x\}) = M \setminus \{u\}$ is a dominating set of H if $D_u \setminus \{x\} = \emptyset$. This is a contradiction. Hence, $deg^M(u) > 0$.

If $u \in M^+$, then by hypothesis u is a $M-$private vertex and $D_u = \{x\}$. If there is a $M-$private neighbor w of u that is not in M, then $\pi(D \setminus \{x\}) = M \setminus \{u\}$ is not a dominating set in H. If there is a $M-$private neighbor w of u that belongs to M, then $w \in (M \setminus \{u\})^0$ and D_w is not a total domination set in G_w as it is a singleton set. In both cases, we get a contradiction to the hypothesis. Hence, D is a minimal total dominating set in G. □

As a consequence of Proposition 3.1, we have the following theorem.

Theorem 3.2. *Let $G = \bigvee_H \mathcal{G}$ be a $H-$join graph. Then, the number of minimal total dominating sets of size k in G is given by*

$$m_k(G) = \sum_{s=1}^{k} \sum_{\substack{M \in \mathcal{D}om^*(H) \\ |M|=s}} \sum_{\mathfrak{a}_M = k} \left(\prod_{v \in M^0} m_{a_v}(G_v) \prod_{v \in M^+} |V(G_v)| \right),$$

where $\mathcal{D}om^*(H)$ is the set of all completely irredundant dominating sets in H, the third sum is over all possible sums $k = \mathfrak{a}_M = |M^+| + \sum_{v \in M^0} a_v$ of positive integers and $m_{a_v}(G_v)$ denotes the number of minimal total dominating sets in G_v of size a_v.

Proof. The proof follows from Proposition 3.1 and by a similar approach of the proof of Theorem 2.2. □

Now, we have an immediate theorem.

Theorem 3.3. *Let $G = \bigvee_H \mathcal{G}$ be a $H-$join graph. Let $\mathcal{MV}_{\mathcal{D}^*}(H; X) = \sum_{M \subset V(H)} [M] X_M$ be the multivariate polynomial of H for $\mathcal{D}^* := \mathcal{D}om^*(H)$. Then,*

1. *the minimal total domination polynomial of G is given by*

$$\mathcal{P}_{\mathcal{MT}}(G;x) = \sum_{M\subset V(H)} [M]\left[\prod_{v\in M}\left(\chi_{M^0}(v)\mathcal{P}_{\mathcal{MT}}(G_v;x) + (1-\chi_{M^0}(v))|V(G_v)|x\right)\right].$$

2. *the multivariate minimal total domination polynomial of G is given by*

$$\mathcal{MV}_{\mathcal{MT}}(G;Z) = \sum_{M\subset V(H)} [M]\left[\prod_{v\in M}\left(\chi_{M^0}(v)\mathcal{MV}_{\mathcal{MT}}(G_v;X^v) + (1-\chi_{M^0}(v))\sum_{w\in V(G_v)} X_w^v\right)\right].$$

4. Illustration of results using examples

Now, we illustrate our theorems using some examples. The (minimal) total domination polynomials of some of the following examples may be well-known in the literature. For example, the total domination polynomial of the join of graphs (Subsection 4.1) can be found in [3] and the total domination polynomial of Corona product (Subsection 4.5) of a certain class of graphs have been discussed in [4]. But, here we obtain the polynomials of the examples using our theorems.

4.1. Join of graphs

The join $G_1 + G_2$ of two graphs G_1 and G_2 is a K_2−join of graphs G_1 and G_2, i.e., $G_1 + G_2 = \bigvee_{K_2}\{G_1, G_2\}$. Let $|G_i| = n_i$ for $i = 1, 2$.

By Theorem 2.2, we get

$$d_k(G_1 + G_2) = d_k(G_1) + d_k(G_2) + \sum_{\substack{a_1+a_2=k\\ a_i\geq 1}} \binom{n_1}{a_1}\binom{n_2}{a_2}.$$

Hence, the total domination polynomial of $G_1 + G_2$ is given by

$$\mathcal{P}_{\mathcal{TD}}(G_1 + G_2;x) = \mathcal{P}_{\mathcal{TD}}(G_1;x) + \mathcal{P}_{\mathcal{TD}}(G_2;x) + ((1+x)^{n_1} - 1)((1+x)^{n_2} - 1).$$

Similarly by Theorem 3.3, the minimal total domination polynomial of $G_1 + G_2$ is given by

$$\mathcal{P}_{\mathcal{MT}}(G_1 + G_2;x) = \mathcal{P}_{\mathcal{MT}}(G_1;x) + \mathcal{P}_{\mathcal{MT}}(G_2;x) + n_1n_2x^2.$$

4.2. K_m−join graphs

Suppose that H is a complete graph. Let $G := \bigvee_{K_m} \mathcal{G}$ be a K_m−join graph with a family $\mathcal{G} = \{G_i : i = 1, 2, ..., m\}$, where $V(K_m) = \{v_1, v_2, ..., v_m\}$ and $|V(G_i)| = n_i$, for some integers $m, n_1, ..., n_m$. One can see that every vertex subset of K_m is a dominating set. Hence by Theorem 2.3, the total domination polynomial of G is given by

$$\mathcal{P}_{\mathcal{TD}}(G;x) = \sum_{i=1}^{n} \mathcal{P}_{\mathcal{TD}}(G_i;x) + \sum_{\substack{J\subset V(K_m)\\ |J|\geq 2}} \prod_{v_j\in J} ((1+x)^{n_j} - 1).$$

Now, we calculate the minimal total domination polynomial of $G = \bigvee_{K_m} \mathcal{G}$. Note that the set $\mathcal{D}om^*(K_m)$ of all completely irredundant dominating sets M in K_m is given by $\{M \subset V(K_m) : 0 < |M| \leq 2\}$. Hence by Theorem 3.3, the minimal total domination polynomial of G is

$$\begin{aligned}\mathcal{P}_{\mathcal{MT}}(G;x) &= \sum_{i=1}^{n} \mathcal{P}_{\mathcal{MT}}(G_i;x) + \sum_{\{v_i,v_j\}\subset V(K_m)} |V(G_i)||V(G_j)|x^2\\ &= \sum_{i=1}^{n} \mathcal{P}_{\mathcal{MT}}(G_i;x) + \sum_{\{v_i,v_j\}\subset V(K_m)} n_in_jx^2.\end{aligned}$$

4.3. Complete multipartite graphs

Let G be a complete multipartite graph. We can write G as a K_m−join graph of empty graphs $\overline{K_{n_i}}$'s, i.e., $G = \bigvee_{K_m} \mathcal{G}$, where $\mathcal{G} = \{\overline{K_{n_i}} : i = 1, 2, ..., m\}$ and $V(K_m) = \{v_1, v_2, ..., v_m\}$ for some integers $m, n_1, ..., n_m$. Note that no vertex subset of $\overline{K_{n_i}}$ is a total dominating set. Hence by the above example, the total domination polynomial of G is

$$\mathcal{P}_{\mathcal{TD}}(G; x) = \sum_{\substack{J \subset V(K_m) \\ |J| \geq 2}} \prod_{v_j \in J} ((1 + x)^{n_j} - 1),$$

and the minimal total domination polynomial of G is

$$\mathcal{P}_{\mathcal{MT}}(G; x) = \sum_{\{v_i, v_j\} \subset V(K_m)} n_i n_j x^2.$$

4.4. $K_{n_1,...,n_m}$−join graphs

Suppose that H is a complete multipartite graph $K_{n_1,...,n_m}$ with $n_1 \leq n_2 \leq ... \leq n_m$. Let $G = \bigvee_{K_{n_1,...,n_m}} \mathcal{G}$ be a $K_{n_1,...,n_m}$−join graph of a family $\mathcal{G}$. Write $V(H) = \bigsqcup_{i=1}^{m} V_i$ with $|V_i| = n_i$. Let $\rho_{V_1,...,V_m}(H) = \rho(V(H)) \setminus \bigcup_{i=1}^{m} \rho(V_i)$, where $\rho(A)$ is the power set of a set A.

The set of all dominating sets of H is given by

$$\mathcal{D}om(H) = \{V_i : i = 1, ..., m\} \bigcup \rho_{V_1,...,V_m}(H).$$

By Theorem 2.3, the total domination polynomial of G is

$$\mathcal{P}_{\mathcal{TD}}(G; x) = \sum_{i=1}^{m} \prod_{v \in V_i} \mathcal{P}_{\mathcal{TD}}(G_v; x) + \sum_{M \in \rho_{V_1,...,V_m}(H)} \prod_{v \in M} \left((1 + x)^{|V(G_v)|} - 1\right).$$

Let us calculate the minimal total domination polynomial of G. One can see that a vertex subset M of H is a completely irredundant dominating set if and only if either $M = V_i$ for some $i = 1, ..., m$ or $M \in \rho_{V_1,...,V_m}(H)$ with $|M| = 2$.

Hence by Theorem 3.3, the minimal total domination polynomial of G is

$$\mathcal{P}_{\mathcal{MT}}(G; x) = \sum_{i=1}^{m} \prod_{v \in V_i} \mathcal{P}_{\mathcal{MT}}(G_v; x) + \sum_{\{v_i, v_j\} \in \rho_{V_1,...,V_m}(H)} |V(G_{v_i})||V(G_{v_j})| x^2.$$

4.5. Corona Product of graphs

Let H' be a connected graph with vertex set $\{v_1, ..., v_n\}$ and G' be a graph. The corona product $H' \circ G'$ of H' and G' is obtained by taking a disjoint union of a copy of H' with $n = |V(H')|$ copies of G' and joining the vertex v_i of H' to all the vertices of the i^{th} copy of G', for each i. We can view the corona product $H' \circ G'$ of graphs H' and G' as a H−join graph $\bigvee_H \mathcal{G}'$, where $H = H' \circ K_1$ and $\mathcal{G}' = \{G_{v_1} = K_1 = \{u_1'\}, ..., G_{v_n} = K_1 = \{u_n'\}, G_{u_1} = G', ..., G_{u_n} = G'\}$, where u_i is the vertex of the i^{th} copy of K_1. Let $\pi : V(H' \circ G') \to V(H' \circ K_1)$ be the canonical map.

The total domination polynomial of $H' \circ G'$:

Let us describe dominating sets of $H = H' \circ K_1$. We write $V(H' \circ K_1) = \{v_1, ..., v_n, u_1, ..., u_n\}$, where the vertex set of i^{th} copy of K_1 is $\{u_i\}$ and v_i is adjacent to u_i for each $i = 1, ..., n$. Note that any dominating set of $H' \circ K_1$ contains at least one of u_i or v_i for all $i = 1, .., n$. Let $n \leq m \leq 2n$ and let M be a dominating set of $H' \circ K_1$ of size m. Then, M must contains a subset M_p which consists of $m - n$ pair of vertices u_i, v_i and contains a subset M_s which consists of $2n - m$ vertices either u_j or v_j but not both such that $M = M_p \sqcup M_s$.

One can easily see that the graph $G_{v_i} = \pi^{-1}(v_i) = \{u'_i\}$ has no total dominating set. Hence, if v_i is a $M-$isolated vertex of a dominating set M in H, by Proposition 2.1, there is no vertex subset D of $H' \circ G'$ such that $M = \pi(D)$ and D is a total dominating set of $H' \circ G'$. Thus, if D is a total dominating set of $H' \circ G'$, then $M = \pi(D)$ is a dominating set of H such that no vertex v_i of H' (in H) is an element of $M^0 \subset M_s$. Given a such M, let $L = M \cap V(H')$. Then, the set L^0 of degree zero vertices of $H'[L]$ should contained in M_p and for each $v_i \in \overline{L} = V(H') \setminus L$, the vertex $u_i \in M_s$.

On the other hand, for a vertex subset L of H', we construct a dominating set M of H with the above property as follows: Let L^0 be the set of degree zero vertices of $H'[L]$ and let M_L be a subset of L that contains L^0. Then, define M to be the union of the following three subsets of $V(H)$

$$M = \{u_i, v_i : v_i \in M_L\} \cup \{u_i : v_i \in \overline{L} = V(H') \setminus L\} \cup \{v_i : v_i \in L \setminus M_L\}.$$

Thus, we get a dominating set M of H with the property that no vertex v_i of H' is an element of M^0 and $M_p = \{u_i, v_i : v_i \in M_L\}$. In fact, any such M can be uniquely obtained in this way.

Now, let D be a total dominating set of $H' \circ G'$ and $M = \pi(D)$. Note that if a vertex $u_i \in M_s$ then it is an isolated vertex of $H[M]$ and if $u_j \in M_p$ then it is a non-isolated vertex of $H[M]$. Then, by using Proposition 2.1, we have the following:

1. $D_{v_i} = D \cap V(G_{v_i}) = \{u'_i\}$ and D_{u_i} is a nonempty vertex subset of G' if $\{u_i, v_i\} \subset M_p$.
2. $D_{v_j} = \{u'_j\} = G_{v_j}$ if $v_j \in M_s$ (as v_j is not $M-$isolated).
3. D_{u_j} is a total dominating set of G' if $u_j \in M_s$.

Now by Theorem 2.3, the total domination polynomial of Corona product $H' \circ G'$ is given by

$$\mathcal{P}_{\mathcal{TD}}(H' \circ G'; x) = \sum_{s=0}^{n} \sum_{\substack{L \subset V(H') \\ |L| = s}} \Bigg[\prod_{v_i \in L^0} \Big((1+x)^{|V(G_{v_i})|} - 1\Big)\Big((1+x)^{|V(G_{u_i})|} - 1\Big)$$
$$\times \sum_{J \subset L^+} \Bigg(\prod_{v_i \in J} \Big((1+x)^{|V(G_{v_i})|} - 1\Big)\Big((1+x)^{|V(G_{u_i})|} - 1\Big) \prod_{v_i \in J^c} x \Bigg) \times \prod_{v_i \in \overline{L}} \mathcal{P}_{\mathcal{TD}}(G_{u_i}; x) \Bigg]$$

$$\mathcal{P}_{\mathcal{TD}}(H' \circ G'; x) = \sum_{s=0}^{n} \sum_{\substack{L \subset V(H') \\ |L| = s}} \Bigg[\Bigg(P(x)^{|L^0|} \sum_{J \subset L^+} P(x)^{|J|} x^{|L^+| - |J|} \Bigg) \mathcal{P}_{\mathcal{TD}}(G'; x)^{n - |L|} \Bigg]$$
$$= \sum_{L \subset V(H')} \Bigg[P(x)^{|L^0|} (P(x) + x)^{|L| - |L^0|} \mathcal{P}_{\mathcal{TD}}(G'; x)^{n - |L|} \Bigg], \text{ where } P(x) = x((1+x)^{|V(G')|} - 1).$$

The minimal total domination polynomial of $H' \circ G'$:

Now, we describe the completely irredundant dominating sets of $H = H' \circ K_1$. Recall that a vertex subset M of H is completely irredundant if each element of M is either $M-$isolated or $M-$private. By the above case and by Proposition 3.1, if D is a minimal total dominating set of $H' \circ G'$, then $M = \pi(D)$ is a completely irredundant dominating set of H such that no vertex v_i of H' (in H) is an element of M^0.

Let M be a completely irredundant dominating set of H such that no vertex v_i of H' (in H) is an element of M^0. Note that each vertex $v_i \in M$ of H is $M-$private, because u_i is adjacent to the only vertex v_i in H. Also, each $u_i \in M_s$ is M−isolated as $v_i \notin M$. Now, one can see that $u_i \in M_p$ is M−private if and only if $v_i \in M$ is not adjacent with any other vertex $v_j \in M$, i.e., $v_i \in (M \cap V(H'))^0$. Hence, if D is a minimal total dominating set of $H' \circ G'$, then $M = \pi(D)$ is a dominating set of H such that no vertex v_i of H' (in H) is an element of M^0 and no vertex v_i with $deg^M(v_i) > 1$ of H' (in H) is an element of M_p.

It follows that if $L = M \cap V(H')$ then $M_p = L^0 \cup \{u_i : v_i \in L^0\}$ and $M_s = L^+ \cup \{u_i : v_i \in V(H') \setminus L\}$. Conversely, for a vertex subset L of H', we construct such a unique set M by defining M to be the union of the following subsets of $V(H)$

$$M = \{u_i, v_i : v_i \in L^0\} \cup \{u_i : v_i \in \overline{L} = V(H') \setminus L\} \cup \{v_i : v_i \in L^+\}.$$

Hence by Theorem 3.3, the minimal total domination polynomial of Corona product $H' \circ G'$ is given by

$$\begin{aligned}\mathcal{P}_{\mathcal{MT}}(H' \circ G'; x) &= \sum_{s=0}^{n} \sum_{\substack{L \subset V(H') \\ |L|=s}} \left[\prod_{v_i \in L^0} \Big((1+x)^{|V(G_{v_i})|} - 1\Big)\Big((1+x)^{|V(G_{u_i})|} - 1\Big) \prod_{v \in L^+} x \prod_{v_i \in \overline{L}} \mathcal{P}_{\mathcal{MT}}(G_{u_i}; x) \right] \\ &= \sum_{L \subset V(H')} \left[P(x)^{|L^0|} x^{|L|-|L^0|} \mathcal{P}_{\mathcal{MT}}(G'; x)^{n-|L|} \right].\end{aligned}$$

4.6. *The windmill graph $Wd(m, n)$*

The (m, n)−windmill graph $Wd(m, n)$ is the graph constructed by taking m copies of the complete graph K_n of n vertices all sharing a single common vertex. Note that the $(m, 3)$−windmill graph is called a friendship graph. The total domination polynomial of the friendship graph is already studied in [4].

Note that the windmill graph $Wd(m, n)$ is the H−join of the family $\{G_u : u \in V(H)\}$, where

- H is a star graph St_m with vertex set $V(St_m) = \{v, v_1, \cdots, v_m\}$ and edge set $E(St_m) = \{vv_i : i \in [m]\}$, where $[m] = \{1, 2, ..., m\}$.
- G_v is a single vertex graph K_1.
- G_{v_i} is the complete graphs K_{n-1} of $n - 1$ vertices, for each $v_i \in V(St_m)$.

Total domination polynomial of $Wd(m, n)$:

One can see that the dominating sets of St_m are $\{v\}$, $\{v_1, \cdots, v_m\}$, and $\{v, v_{i_1}, \cdots, v_{i_k}\}$, for some $i_j \in [m]$.
For each dominating set M of St_m, let us analyze the set M^0.

1. If $M = \{v\}$ then $M^0 = \{v\}$. Since the cardinality of $V(G_v)$ is 1, there is no total dominating set in G_v. Hence, there is no total dominating set D of $Wd(m, n)$ such that $\pi(D) = \{v\}$.
2. If $v \in M$ and $M \neq \{v\}$ then $M^0 = \emptyset$ as $H[M]$ is connected.
3. If $v \notin M$, that is $M = \{v_1, \cdots, v_m\}$ then $M^0 = M$.

Now, by Theorem 2.3, the total domination polynomial of $Wd(m, n)$ is given by

$$\mathcal{P}_{\mathcal{TD}}(Wd(m, n); x) = \sum_{s=1}^{m} \left(\sum_{M=\{v, v_{i_1}, ..., v_{i_s}\}} \prod_{u \in M} ((1+x)^{|V(G_u)|} - 1) \right) + \prod_{u \in \{v_1, ..., v_m\}} \mathcal{P}_{\mathcal{TD}}(G_u; x).$$

Since each $G_u = K_{n-1}$, we have $\mathcal{P}_{\mathcal{TD}}(G_u; x) = (1 + x)^{n-1} - 1 - (n - 1)x$.
Hence,

$$\begin{aligned}\mathcal{P}_{\mathcal{TD}}(Wd(m, n); x) &= \sum_{s=1}^{m} \binom{m}{s} x\Big((1+x)^{n-1} - 1\Big)^s + \Big((1+x)^{n-1} - 1 - (n-1)x\Big)^m \\ &= x((1+x)^{(n-1)m} - 1) + ((1+x)^{n-1} - 1 - (n-1)x)^m.\end{aligned}$$

Minimal total domination polynomial of $Wd(m,n)$:

Now, we find the completely irredundant dominating sets of St_m. Since we already found the dominating sets of St_m, we have to describe which of them are completely irredundant.

1. If $M = \{v\}$ then v is $M-$isolated. But $|V(G_v)| = 1$, there is no total dominating set in G_v. Hence, there is no total dominating set D of $Wd(m,n)$ such that $\pi(D) = \{v\}$.
2. Assume that $v \in M$ and $M \neq \{v\}$.
   - Suppose that $\{v, v_i, v_j\} \subset M$, for some $i, j \in [m]$. Note that $H[M]$ is connected and $pn(v_i, M) = pn(v_j, M) = \emptyset$. Hence, v_i and v_j are not $M-$private in H. Thus, the set M is not completely redundant.
   - Suppose that $M = \{v, v_i\}$, for some $i \in [m]$. Note that v and v_i are $M-$privates as $pn(v_i, M) = \{v\}$ and $pn(v, M) = \{v_i\}$. Hence, $\{v, v_i\}$ is completely irredundant dominating set of St_m.
3. If $v \notin M$, that is $M = \{v_1, \cdots, v_m\}$, then $M^0 = M$. Hence, the set M is completely irredundant.

Now, by Theorem 3.3, the minimal domination polynomial of $Wd(m,n)$ is given by

$$\begin{aligned}\mathcal{P}_{\mathcal{MT}}(Wd(m,n);x) &= \sum_{i=1}^{m} \prod_{u\in\{v,v_i\}} |V(G_u)|x + \prod_{u\in\{v_1,\cdots,v_m\}} \mathcal{P}_{\mathcal{MT}}(G_u;x)\\ &= \sum_{i=1}^{m}(n-1)x^2 + \prod_{i=1}^{m}\binom{n-1}{2}x^2\\ &= m(n-1)x^2 + \left(\frac{(n-1)(n-2)}{2}x^2\right)^m.\end{aligned}$$

Conflict of Interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

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