Filomat 39:1 (2025), 279–294
https://doi.org/10.2298/FIL2501279H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some fixed point theorems under relational Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contractions with an application

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Abstract. In this article, we introduce the idea of relational Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contractions and utilize the same to prove some fixed point results in $\mathcal{R}$-complete metric spaces employing binary relation. Our newly proved results yield sharpened versions of several known results of the existing literature. An example substantiates the genuineness of our main result. Furthermore, we have demonstrated the applicability of our main result in establishing the solution of a fractional thermostat model.

1. Introduction

The strength of Banach Contraction Principle [5](abbreviated as BCP) lies in its wide applications, which fall into several domains, namely: Differential equation, Integral equation, Economics, fractal theory, aquatic problem, market equilibrium, etc., which leads us to consider the BCP as an epitome of classical result for overall existing fixed point theorems. The BCP can be naturally extended and generalized by
• relaxing the completeness of space,
• the underlying contractive condition,
• the number of mappings involved *etc*.

Following the above-indicated spirit, Branciari [6] introduced a new distance notion and utilized that to define a space known as Branciari distance space and proved an analog of BCP in such spaces.

With a view to relax the contractive condition on the whole space, Suzuki [17] introduced a new contractive condition, now known as Suzuki contraction, which runs as follows:

Definition 1.1. *[17] Let (X, d) be a compact metric space and $T : X \to X$ satisfying the condition*

$$\frac{1}{2}d(x, Tx) < d(x, y) \implies d(Tx, Ty) < d(x, y) \text{ for all } x, y \in X (x \neq y). \tag{1}$$

Then, T is said to be Suzuki contraction on X.

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25.
Keywords. Binary relation, metric space, Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contraction, fractional thermostat model.
Received: 09 March 2023; Revised: 28 May 2024; Accepted: 13 October 2024
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In 2015, Alam and Imdad [3] introduced the relation-theoretic variant of BCP that unifies transitive relation due to Turinici [18], order-theoretic relation by Ran and Reurings [15], Nieto and Rodríguez-López [14], and several others. In this regard, the technical details are available in Alam and Imdad [3] and Alam et al. [2].

In 2014, Jleli and Samet [11] introduced the θ-contraction and utilized such contractions to prove fixed point results without continuity of the involved controlled function. Later, Ahmad et al. [1] used the continuity condition at the expanse of one of the requirements on the involved controlled function to prove their results. In the recent past, Cho [7] introduced $\mathcal{L}$-contraction by defining a class of simulation functions and proved fixed point results in Branciari distance space. After that, Cho [8] again proved the fixed point results in a generalized sense by using the Suzuki condition on the Branciari distance space. Due to the advantages of several topological properties (*e.g.*, continuity, convergence and compatibility) of metric space over Branciari distance space, Hasanuzzaman et al. [9] proved the result of Cho [7] in metric spaces equipped with an arbitrary binary relation.

In this article, we have introduced the Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contraction under arbitrary binary relations and proved the existence and uniqueness of fixed point results utilizing binary relation. We have incorporated an example to validate our main result. Lastly, we have utilized our main result to establish the existence of a positive solution for the fractional thermostat model under suitable requirements.

2. Relation-theoretic notions

We keep in mind the following terminological and notational conventions to make our presentation potentially self-contained. The sets of natural numbers, rational numbers, and real numbers are denoted in the text below by the symbols $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Before proceeding further, we summarize some fundamental relation theoretical concepts, definitions, and relevant results as recorded in the following lines:
A binary relation $\mathcal{R}$ on a non-empty set X is defined as an arbitrary subset of $X \times X$. Trivially, the terms empty relation and universal relation on X are respectively $\emptyset$ and $X \times X$. From now on, a non-empty binary relation will be denoted by $\mathcal{R}$. If $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ imply $(x, z) \in \mathcal{R}$, for all $x, y, z \in X$ then $\mathcal{R}$ is said to be transitive relation on X. Furthermore, if T is a self mapping on X, then $\mathcal{R}$ is said to be T-transitive if it is transitive on $T(X)$.

Definition 2.1. *[3] Let $\mathcal{R}$ be a binary relation on X. Then for $x, y \in X$,*

(i) *inverse relation $\mathcal{R}^{-1} := \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}$ and symmetric closure $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$,*
(ii) *x and y are $\mathcal{R}$-comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. It is denoted by $[x, y] \in \mathcal{R}$.*
(iii) *if $(x, y) \in \mathcal{R}^s \iff [x, y] \in \mathcal{R}$.*
(iv) *a sequence $\{x_n\} \subset X$ is termed as $\mathcal{R}$-preserving if*

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall \ n \in \mathbb{N}_0.$$

Definition 2.2. *[3] For a self-mapping T on nonempty set X, any binary relation $\mathcal{R}$ on X is said to be T-closed if for all $x, y \in X$,*

$$(x, y) \in \mathcal{R} \implies (Tx, Ty) \in \mathcal{R}.$$

Definition 2.3. *[3, 4] Let (X, d) be a metric space and $\mathcal{R}$ a binary relation on X. Then,*

(i) *(X, d) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X converges.*

(ii) *$T : X \to X$ is called $\mathcal{R}$-continuous at $x \in X$ if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$, we have $T(x_n) \xrightarrow{d} T(x)$. Furthermore, T is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of X.*

(iii) *If* $\{x_n\}$ *is an* $\mathcal{R}$*-preserving sequence with* $x_n \xrightarrow{d} x$, *there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *with* $[x_{n_k}, x] \in \mathcal{R}$ *for all* $k \in \mathbb{N}_0$, *then* $\mathcal{R}$ *is said to be d-self-closed.*

Definition 2.4. *[12] For* $x, y \in X$, *a path (of length* n, $n \in \mathbb{N}$*) in* $\mathcal{R}$ *from* x *to* y *is a finite sequence* $\{x_0, x_1, x_2, \ldots, x_n\} \subseteq X$ *such that* $x_0 = x,\ x_n = y$ *with* $(x_i, x_{i+1}) \in \mathcal{R}$, *for each* $i \in \{0, 1, \ldots, n-1\}$.

It is worth mentioning here that a path of length n involves $n + 1$ elements of X (not necessarily distinct).

Definition 2.5. *[4] A subset* $D \subseteq X$ *is said to be* $\mathcal{R}$*-connected if for each* $x, y \in D$, *there exists a path from* x *to* y *in* $\mathcal{R}$.

3. Preliminaries on $\mathcal{L}$-contractions

Definition 3.1. *Following [11], let* Θ *be the set of all function* $\theta : (0, \infty) \to (1, \infty)$ *satisfying the following conditions:*

(θ_1) θ *is non decreasing,*
(θ_2) *for each sequence* $\{\beta_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(\beta_n) = 1 \Longleftrightarrow \lim_{n\to\infty} \beta_n = 0$,
(θ_3) *there exist* $\kappa \in (0, 1)$ *and* $\gamma \in (0, \infty]$ *such that* $\lim_{\beta\to 0^+} \frac{\theta(\beta)-1}{\beta^\kappa} = \gamma$.

After that, Ahmad et al. [1] replaced the condition (θ_3) *by the following:*

(θ_4) θ *is continuous.*

Let us denote Θ^* be the family of all functions satisfying $(\theta_1), (\theta_2)$ and (θ_4). Here, for the sake of convenience we provide some examples of such functions.

Example 3.2. *[10, 11] Define* $\theta : (0, \infty) \to (1, \infty)$ *by*

1. $\theta(\beta) = e^{e^{-\frac{1}{\sqrt{\beta}}}}$, *then* $\theta \in \Theta^*$,
2. $\theta(\beta) = e^{\sqrt{\beta}}$, *then* $\theta \in \Theta$ *as well as* $\theta \in \Theta^*$,
3. $$\theta(\beta) = \begin{cases} e^{\sqrt{\beta}} & \beta \leq k', \\ e^{2(k'+1)} & \beta > k', \end{cases}$$
   where $k' \geq 1$ *(a fixed real number). Then* $\theta \in \Theta$ *but* $\theta \notin \Theta^*$,
4. $\theta(\beta) = e^{e^{-\frac{1}{\beta}}}$, *then* $\theta \in \Theta^*$ *but* $\theta \notin \Theta$.

In recent past, Cho [7] initiated the idea of $\mathcal{L}$-simulation functions as follows:

Definition 3.3. *A mapping* $\zeta : [1, \infty) \times [1, \infty) \to \mathbb{R}$ *is said to be a* $\mathcal{L}$*-simulation function if the following conditions are satisfied:*

(ζ_1) $\zeta(1, 1) = 1$;
(ζ_2) $\zeta(x, y) < \frac{y}{x}$ *for all* $x, y > 1$;
(ζ_3) *if* $\{x_n\}, \{y_n\}$ *are sequences in* $(1, \infty)$ *such that* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n > 1$, *then* $\limsup_{n\to\infty} \zeta(x_n, y_n) < 1$.

The family of $\mathcal{L}$-simulation functions will be denoted by $\mathcal{L}$. Some examples of $\mathcal{L}$-simulation functions are as under:

Example 3.4. *[7] We define the mappings* $\zeta_i : [1, \infty) \times [1, \infty) \to \mathbb{R}$ *for* $i = 1, 2, 3$, *as follows:*

- $$\zeta_1(x,y) = \begin{cases} 1 & \text{if } (x,y) = (1,1); \\ \frac{x}{2y} & \text{if } x < y; \\ \frac{y^k}{x} & \text{elsewhere}, \end{cases}$$

  for all $x, y \in [1, \infty)$ and $k \in (0, 1)$.
- $\zeta_2(x,y) = \frac{y}{x\varphi(y)}$ *for all $x, y \in [1,\infty)$, where $\varphi : [1,\infty) \to [1,\infty)$ is a lower semi continuous and non-decreasing function such that $\varphi^{-1}(\{1\}) = \{1\}$.*
- $\zeta_3(x,y) = \frac{y^k}{x}$ *for all $x, y \in [1,\infty)$, where $k \in (0,1)$.*

Then ζ_i are $\mathcal{L}$-simulation functions for $i = 1, 2, 3$.

Example 3.5. *[8] Let $\zeta_k : [1,\infty) \times [1,\infty) \to \mathbb{R}$, $k = 4, 5$ be functions defined as follows:*

(1) $\zeta_4(x,y) = \frac{\psi(y)}{\phi(x)}$, $\forall x, y \geq 1$ *where $\psi, \phi : [1,\infty) \to [1,\infty)$ are continuous functions such that $\psi(x) = \phi(x) = 1$ if and only if $x = 1$, $\psi(x) < x \leq \phi(x)$, $x \geq 1$ and ϕ is an increasing function.*

(2) $\zeta_5(x,y) = \frac{\eta(y)}{x}$, $\forall x, y \geq 1$, *where $\eta : [1,\infty) \to [1,\infty)$ is upper semi-continuous with $\eta(x) < x$, $\forall x \geq 1$ and $\eta(x) = x$ if and only if $x = 1$.*

Then $\zeta_4, \zeta_5 \in \mathcal{L}$.

Utilizing $\mathcal{L}$-simulation functions, Cho [7] introduced $\mathcal{L}$-contraction in generalized metric space(often referred as Branciari distance space) without using (θ_4). However, Hasanuzzaman et al. [9] defined the $\mathcal{L}$-contraction for $\theta \in \Theta^*$ in the setting of metric space as follows:

Definition 3.6. *[9] Let (X, d) be a metric space and $T : X \to X$. Then T is said to be $\mathcal{L}$-contraction w.r.t. ζ if there exist $\zeta \in \mathcal{L}$ and $\theta \in \Theta^*$ such that*

$$\zeta\Big(\theta(d(Tx,Ty)), \theta(d(x,y))\Big) \geq 1 \tag{2}$$

for all $x, y \in X$ with $d(Tx,Ty) > 0$.

If we take $\zeta(x,y) = \frac{y^k}{x}$ for all $x, y \in [1,\infty)$ with $k \in (0,1)$, then $\mathcal{L}$-contraction reduces to θ-contraction.

Remark 3.7. *[9] Due to the condition (ζ_2), we have $\zeta(x,x) < 1$, for all $x > 1$. Therefore, if a mapping T is a $\mathcal{L}$-contraction then it cannot be an isometry (i.e., distance does not preserve under such mappings).*

Again, Hasanuzzaman et al. [9] introduced the relation-theoretic variant of $\mathcal{L}$-contraction known as $\mathcal{L}_{\mathcal{R}}$-contraction which is stated as follows:

Definition 3.8. *Let $\mathcal{R}$ be a binary relation on metric space (X, d) and $T : X \to X$. We say that T is $\mathcal{L}_{\mathcal{R}}$-contraction w.r.t. $\zeta \in \mathcal{L}$, if there exist $\zeta \in \mathcal{L}$ and $\theta \in \Theta^*$ such that the following condition holds:*

$$\zeta(\theta(d(Tx,Ty)), \theta(d(x,y))) \geq 1, \quad \forall\, x, y \in X \text{ with } (x,y) \in \mathcal{R}^* \tag{3}$$

where $(x,y) \in \mathcal{R}^* := \{(x,y) \in \mathcal{R} : Tx \neq Ty\}$.

Later on, Cho [8] introduced the Suzuki generalized $\mathcal{L}$-contraction in Branciari distance space. Below, we have stated the metrical version as follows:

Definition 3.9. *Let (X, d) be a metric space and $T : X \to X$. Then T is said to be a Suzuki generalized $\mathcal{L}$-contraction w.r.t. ζ if there exist $\zeta \in \mathcal{L}$ and $\theta \in \Theta^*$ such that*

$$\frac{1}{2}d(x,Tx) < d(x,y) \implies \zeta\Big(\theta(d(Tx,Ty)), \theta(\mathcal{M}(x,y))\Big) \geq 1 \tag{4}$$

for all $x, y \in X$ with $d(Tx,Ty) > 0$ and $\mathcal{M}(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\{d(x,Ty) + d(y,Tx)\}\}$.

The relation-theoretic version of Definition 3.9 is called Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contraction, which is stated as follows:

Definition 3.10. *Let $\mathcal{R}$ be a binary relation on metric space (X, d) and $T : X \to X$. We say that T is Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contraction w.r.t. $\zeta \in \mathcal{L}$, if there exist $\zeta \in \mathcal{L}$ and $\theta \in \Theta^*$ such that the following condition holds:*

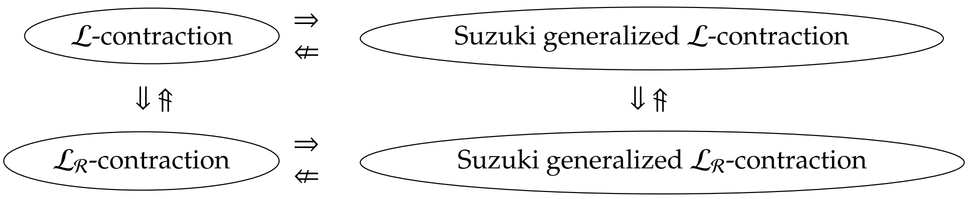
$$\frac{1}{2}d(x, Tx) < d(x, y) \implies \zeta(\theta(d(Tx, Ty)), \theta(\mathcal{M}(x, y))) \geq 1, \ \ \forall\, x, y \in X \text{ with } (x, y) \in \mathcal{R}^* \quad (5)$$

where $(x, y) \in \mathcal{R}^* := \{(x, y) \in \mathcal{R} : Tx \neq Ty\}$.

Remark 3.11. *The above definition remains true if we replace $\mathcal{M}(x, y)$ by $\mathcal{N}(x, y)$, where*

$$\mathcal{N}(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

In view of the Definitions 3.6, 3.8, 3.9 and 3.10, the following implications and non-implications hold good:



Notice that, Example 3.1 of [9] is an examples of $\mathcal{L}_{\mathcal{R}}$-contraction but fails to satisfy $\mathcal{L}$-contraction. Example 3.12 is an example of Suzuki generalized $\mathcal{L}$-contraction but not an $\mathcal{L}$-contraction. Also, Example 3.13 is an example of Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contraction but not an $\mathcal{L}_{\mathcal{R}}$-contraction. Whereas, Example 4.3 is an example of Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contraction but not an Suzuki generalized $\mathcal{L}$-contraction.

Example 3.12. *Consider $X = \{1, 3, 5\}$ with usual metric. Then X is a complete metric space. Define a self mapping T on X as $T(1) = 5, T(3) = 1$ and $T(5) = 3$. Then the pairs $(1, 5), (3, 5)$ satisfy the Suzuki condition and the Suzuki generalized $\mathcal{L}$-contraction for $\theta(t) = e^t$ and $\zeta(t, s) = \frac{s}{t\phi(s)}$ where the map $\phi : [1, \infty) \to [1, \infty)$ is defined by*

$$\phi(s) = \begin{cases} 1 & \text{if } s \leq e^2, \\ s^{\frac{1}{3}} & \text{if } s > e^2. \end{cases}$$

But it fails to satisfy the condition of $\mathcal{L}$-contraction, for $x = 1$ and $y = 3$.

Example 3.13. *Consider $X = [-2, 8]$ under usual metric. Define a self mapping T on X as*

$$Tx = \begin{cases} 7 & \text{if } x = 0, \\ 5 & \text{if } x = 2, \\ 6 & \text{otherwise.} \end{cases}$$

Let $\mathcal{R} = \{(0, 2), (3, 4), (5, 2), (5, 3), (6, 1), (6, 6), (6, 7)\}$ be a relation define on the space X. Observe that $\mathcal{R}^* = \{(0, 2), (5, 2)\}$. Then only element which satisfies the Suzuki condition is $(5, 2)$. If we set $\theta(t) = e^t$ and $\zeta(t, s) = \frac{s}{t\phi(s)}$ where the map $\phi : [1, \infty) \to [1, \infty)$ is defined by

$$\phi(s) = \begin{cases} e & \text{if } s \leq e^2, \\ s^{\frac{1}{3}} & \text{if } s > e^2, \end{cases}$$

then, T satisfy the Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contraction. But it is not $\mathcal{L}_{\mathcal{R}}$-contraction because if we choose $(0, 2) \in \mathcal{R}^*$ then $\zeta(\theta(d(T0, T2)), \theta(d(0, 2))) < 1$.

Proposition 3.14. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ and $T : X \to X$. For a given $\zeta \in \mathcal{L}$, $\theta \in \Theta^*$, the following are equivalent:*

(i) $\forall\, x, y \in X$ *with* $(x, y) \in \mathcal{R}^* \implies \zeta(\theta(d(Tx, Ty)), \theta(\mathcal{M}(x, y))) \geq 1$;
(ii) $\forall\, x, y \in X$ *with* $[x, y] \in \mathcal{R}^* \implies \zeta(\theta(d(Tx, Ty)), \theta(\mathcal{M}(x, y))) \geq 1$.

4. Main results

Given a binary relation $\mathcal{R}$ and a self-mapping T on a nonempty set X, we employ the following notations:

(i) $X(T;\mathcal{R}) := \{x \in X : (x, Tx) \in \mathcal{R}\}$,
(ii) $\Upsilon(x, y, \mathcal{R})$: the class of all paths in $\mathcal{R}$ from x to y,
(iii) $F(T)$: set of all fixed points.

Now, we state and prove our main result.

Theorem 4.1. *Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X and $T : X \to X$. Suppose that the following conditions hold:*

(i) *(X, d) is $\mathcal{R}$-complete,*
(ii) *$\mathcal{R}$ is T-closed and T-transitive,*
(iii) *$X(T;\mathcal{R})$ is non-empty,*
(iv) *either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed,*
(v) *T is Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contraction w.r.t. some $\zeta \in \mathcal{L}$.*

Then T has a fixed point. Moreover, for each $x_0 \in X(T;\mathcal{R})$, the Picard sequence $T^n(x_0)$ for all $n \in \mathbb{N}$, converges to a fixed point of T.

Proof. Since $X(T;\mathcal{R}) \neq \emptyset$, let x_0 be an arbitrary point such that $x_0 \in X(T;\mathcal{R})$. Now define a sequence (x_n) by $x_n = T^n x_0$, for all $n \in \mathbb{N}_0$. Since $(x_0, Tx_0) \in \mathcal{R}$, then due to the T-closedness of $\mathcal{R}$, we have

$$(x_n, Tx_n) \in \mathcal{R} \text{ for all } n \in \mathbb{N}_0. \tag{6}$$

Now, if there exists some $n_0 \in \mathbb{N}_0$ such that $d(x_{n_0}, Tx_{n_0}) = 0$, then the result follows immediately. Otherwise, for all $n \in \mathbb{N}_0$, $x_n \neq x_{n+1}$ *i.e.*, $d(Tx_{n-1}, Tx_n) > 0$ which enable us to conclude that $(x_{n-1}, x_n) \in \mathcal{R}^*$. Now we see that $\frac{1}{2}d(x_{n-1}, Tx_{n-1}) = \frac{1}{2}d(x_{n-1}, x_n) < d(x_{n-1}, x_n)$ and $(x_{n-1}, x_n) \in \mathcal{R}^*$, then by condition (v), we have

$$\begin{aligned} 1 &\leq \zeta\Big(\theta(d(Tx_{n-1}, Tx_n)), \theta(\mathcal{M}(x_{n-1}, x_n))\Big) \\ &< \frac{\theta(\mathcal{M}(x_{n-1}, x_n))}{\theta(d(Tx_{n-1}, Tx_n))} \end{aligned}$$

or,

$$\begin{aligned} \theta(d(Tx_{n-1}, Tx_n)) &< \theta(\mathcal{M}(x_{n-1}, x_n)) \\ &< \theta(max(d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{1}{2}\{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})\})) \\ &< \theta(max(d(x_{n-1}, x_n), d(x_n, x_{n+1}))) \end{aligned}$$

then due to (θ_1), we deduce $d(x_n, x_{n+1}) < max(d(x_{n-1}, x_n), d(x_n, x_{n+1}))$ for all $n \in \mathbb{N}$. So if $max(d(x_{n-1}, x_n), d(x_n, x_{n+1})) = d(x_n, x_{n+1})$. Then we have

$$d(x_n, x_{n+1}) < d(x_n, x_{n+1})$$

which is a contradiction. Hence, we have

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{7}$$

Therefore, $\{d(x_n, x_{n+1})\}_{n=0}^{\infty}$ is a monotonically decreasing sequence of positive real numbers, and hence there exists $l \geq 0$, such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = l$.
Now, we show that $l = 0$. On contrary, suppose that $l > 0$ then by using (θ_4), we obtain

$$\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = \lim_{n\to\infty} \theta(d(x_{n+1}, x_{n+2})) = \theta(l).$$

Now, if we set $x_n = \theta(d(x_n, x_{n+1}))$, $y_n = \theta(d(x_{n+1}, x_{n+2}))$ then $y_n < x_n$, for all $n \in \mathbb{N}$ (by (7)) and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n > 1$. Then by (ζ_3), we obtain

$$1 \le \limsup_{n\to\infty} \zeta\Big(\theta(d(Tx_n, Tx_{n+1})), \theta(d(x_n, x_{n+1}))\Big) < 1,$$

which is a contradiction and hence $l = 0$, *i.e.*,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \tag{8}$$

by (θ_2), we also have

$$\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = 1.$$

Next, we show that (x_n) is Cauchy sequence. To do this, on contrary let (x_n) is not Cauchy, then there exists $\epsilon > 0$ and $l_0 \in \mathbb{N}_0$ with $m(l) > n(l) > l \ge l_0$, such that

$$d(x_{m(l)}, x_{n(l)}) \ge \epsilon \ \text{ and } \ d(x_{m(l)-1}, x_{n(l)}) < \epsilon.$$

Thus, we obtain

$$\epsilon \le d(x_{m(l)}, x_{n(l)}) \le d(x_{m(l)}, x_{m(l)-1}) + d(x_{m(l)-1}, x_{n(l)}) < d(x_{m(l)}, x_{m(l)-1}) + \epsilon$$

taking $l \to \infty$ and using (8), we get

$$\lim_{l\to\infty} d(x_{m(l)}, x_{n(l)}) = \epsilon \ \text{ or } \ \lim_{l\to\infty} \theta(d(x_{m(l)}, x_{n(l)})) = \theta(\epsilon), \tag{9}$$

and hence

$$\lim_{l\to\infty} d(x_{m(l)+1}, x_{n(l)+1}) = \epsilon \text{ or } \lim_{l\to\infty} \theta(d(x_{m(l)+1}, x_{n(l)+1})) = \theta(\epsilon). \tag{10}$$

As the sequence (x_n) is $\mathcal{R}$-preserving and $\mathcal{R}$ is T-transitive, therefore $(x_{m(l)}, x_{n(l)}) \in \mathcal{R}^*$ and we have from (8) that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_{m(l)}, x_{m(l)+1}) < \epsilon \ \text{ for all } \ l > N.$$

Thus we can say that

$$\frac{1}{2} d(x_{m(l)}, Tx_{m(l)}) \le \frac{1}{2} d(x_{m(l)}, x_{m(l)+1}) < \epsilon \le d(x_{m(l)}, x_{n(l)}).$$

Then from the definition of Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contraction, we have

$$\zeta\Big(\theta(d(Tx_{m(l)}, Tx_{n(l)})), \theta(\mathcal{M}(x_{m(l)}, x_{n(l)}))\Big) \ge 1.$$

Now taking $l \to \infty$ and on using (9), (10) and (ζ_3), we get

$$1 \le \limsup_{l\to\infty} \zeta\Big(\theta(d(Tx_{m(l)}, Tx_{n(l)})), \theta(\mathcal{M}(x_{m(l)}, x_{n(l)}))\Big) < 1,$$

which is a contradiction. Thus, the sequence (x_n) is an $\mathcal{R}$-preserving Cauchy sequence in X. Owing to the $\mathcal{R}$-completeness of X, there exists a $x^* \in X$ such that $x_n \xrightarrow{d} x^*$.
If T is $\mathcal{R}$-continuous, then we have

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tx^*,$$

and hence x^* is a fixed point of T.

Otherwise, suppose that $\mathcal{R}$ is d-self-closed. Then, there exists a subsequence $(x_{n(l)})$ of (x_n) with $[x_{n(l)}, x^*] \in \mathcal{R}$, for all $l \in \mathbb{N}_0$. Now, without loss of generality, we may assume that $x_{n(l)} \neq x^*$, for all $l \in \mathbb{N}$. Now, we claim that (for all $l \in \mathbb{N}_0$),

$$\frac{1}{2}d(x_{n(l)}, x_{n(l)+1}) < d(x_{n(l)}, x^*) \text{ or } \frac{1}{2}d(x_{n(l)+1}, x_{n(l)+2}) < d(x_{n(l)+1}, x^*). \tag{11}$$

Arguing by contradiction, we assume that (for some $l_1 \in \mathbb{N}_0$)

$$\frac{1}{2}d(x_{n(l_1)}, x_{n(l_1)+1} > d(x_{n(l_1)}, x^*) \text{ and } \frac{1}{2}d(x_{n(l_1)+1}, x_{n(l_1)+2}) > d(x_{n(l_1)+1}, x^*)$$

Applying the triangle inequality of metric, we obtain

$$\begin{aligned} d(x_{n(l_1)}, x_{n(l_1)+1}) &\leq d(x_{n(l_1)}, x^*) + d(x^*, x_{n(l_1)+1}) \\ &< \frac{1}{2}d(x_{n(l_1)}, x_{n(l_1)+1}) + \frac{1}{2}d(x_{n(l_1)+1}, x_{n(l_1)+2}) \\ &< \frac{1}{2}d(x_{n(l_1)}, x_{n(l_1)+1}) + \frac{1}{2}d(x_{n(l_1)}, x_{n(l_1)+1}) \\ &< \frac{1}{2}\{d(x_{n(l_1)}, x_{n(l_1)+1}) + d(x_{n(l_1)}, x_{n(l_1)+1})\} = d(x_{n(l_1)}, x_{n(l_1)+1}), \end{aligned}$$

which is a contradiction. Therefore, (11) holds true for all $l \in \mathbb{N}_0$ immediately.
Since T is Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contraction then from Equation (5) and Proposition 3.14, we have

$$\zeta\Big(\theta(d(Tx_{n(l)}, Tx^*)), \theta(\mathcal{M}(x_{n(l)}, x^*))\Big) \geq 1, \quad \forall\, l \in \mathbb{N}_0. \tag{12}$$

We show that x^* is a fixed point of T. On contrary, suppose that it is not the case then $d(Tx^*, x^*) > 0$. By using $(\zeta_1), (\zeta_2)$ and (12), we obtain

$$\begin{aligned} 1 &\leq \zeta(\theta(d(Tx_{n(l)}, Tx^*)), \theta(\mathcal{M}(x_{n(l)}, x^*))) \\ &< \frac{\theta(\mathcal{M}(x_{n(l)}, x^*))}{\theta(d(Tx_{n(l)}, Tx^*))} \\ \theta(d(Tx_{n(l)}, Tx^*)) &< \theta(\mathcal{M}(x_{n(l)}, x^*)) \\ d(Tx_{n(l)}, Tx^*) &< \mathcal{M}(x_{n(l)}, x^*) \\ \limsup_{l\to\infty} d(x_{n(l)+1}, Tx^*) &< \limsup_{l\to\infty} \mathcal{M}(x_{n(l)}, x^*) \end{aligned}$$

Case I: If $\mathcal{M}(x_{n(l)}, x^*) = d(x^*, Tx^*)$ then we obtain $d(x^*, Tx^*) < d(x^*, Tx^*)$ which is a contradiction.
Case II: If $\mathcal{M}(x_{n(l)}, x^*) = d(x_{n(l)}, x_{n(l)+1})$ then we obtain $d(x^*, Tx^*) < 0$ which is a contradiction.
Case III: If $\mathcal{M}(x_{n(l)}, x^*) = \frac{1}{2}\{d(x_{n(l)}, Tx^*) + d(x^*, Tx_{n(l)})\}$ then we obtain $d(x^*, Tx^*) < d(x^*, Tx^*)$ which is a contradiction.
Case IV: So if $\mathcal{M}(x_{n(l)}, x^*) = d(x_{n(l)}, x^*)$ then we obtain $d(x^*, Tx^*) < 0$ which is a contradiction.
Then we can say that x^* is a fixed point of T. □

Theorem 4.2. *In addition to the assumptions of Theorem 4.1, if $\Upsilon(x, y; \mathcal{R}|_{T(X)})$ is non-empty for all $x, y \in T(X)$ then T admits a unique fixed point.*

Proof. On the lines of the proof of Theorem 4.1, one can show that $F(T)$ is non-empty. Now, if $F(T)$ is singleton then the proof is obvious. Otherwise, there exists two distinct elements $x^*, y^* \in F(T)$. As $\Upsilon(x, y; \mathcal{R}|_{T(X)})$ is non-empty for all $x, y \in T(X)$, there exists a path of some finite length n from x^* to y^* in $\mathcal{R}|_{T(X)}$ say $\{Tx_0, Tx_1, Tx_2, \ldots, Tx_n\}$ such that $x^* = Tx_0, y^* = Tx_n$ with $(Tx_i, Tx_{i+1}) \in \mathcal{R}|_{T(X)}$ for each $i \in \{0, 1, 2, \ldots, n-1\}$. As $\mathcal{R}$ is T-transitive, we obtain

$$(x^*, Tx_1) \in \mathcal{R}, (Tx_1, Tx_2) \in \mathcal{R}, \ldots, (Tx_{n-1}, y^*) \in \mathcal{R} \text{ implies } (x^*, y^*) \in \mathcal{R}.$$

Also, we have

$$\frac{1}{2}d(x^*, Tx^*) = 0 < d(x^*, y^*).$$

Now, as T is Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contraction, we have

$$1 \leq \zeta\Big(\theta(d(Tx^*, Ty^*)), \theta(\mathcal{M}(x^*, y^*))\Big) < \frac{\theta(\mathcal{M}(x^*, y^*))}{\theta(d(Tx^*, Ty^*))} = 1,$$

a contradiction. Therefore, the fixed point of T is unique. □

Example 4.3. *Let $(X = (0, \infty), d)$ be a metric space endowed with a binary relation*

$$\mathcal{R} := \{(1,2), (2,2), (2,3), (2,6), (3,2), (3,3), (3,5), (3,6), (2,5)\},$$

where $d(x, y) = |x - y|$, for all $x, y \in X$. Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} 3 & \text{if } x \in (0,5), \\ 2 & \text{if } x = 5, \\ \frac{x}{2} - 1 & \text{if } x > 5, \end{cases}$$

then $X(T; \mathcal{R}) \neq \emptyset$ as $(2, T2) = (2, 3) \in \mathcal{R}$, T is $\mathcal{R}$-continuous which is not continuous in usual sense. Also, X is $\mathcal{R}$-complete, $\mathcal{R}$ is T-closed and $\mathcal{R}$ is T-transitive but not transitive.

Now, choose $\theta(\beta) = e^{\sqrt{\beta}}$ for all $\beta > 0$ and if we take $\zeta^*(x, y) = \frac{y^k}{x}$ for all $x, y \in [1, \infty)$ and $k \in \left(\frac{2}{3}, 1\right)$. Since, we have $\mathcal{R}^* = \{(2,6), (3,5), (3,6), (2,5)\}$, then the following four cases are arise:

Case (I): If we take $x = 2, y = 6$, then we have $\frac{1}{2}d(2, T2) = \frac{1}{2}d(2, 3) = \frac{1}{2} < 4 = d(2, 6)$ and

$$\begin{aligned} \mathcal{M}(2,6) &= max\{d(2,6), d(2,T2), d(6,T6), \frac{1}{2}(d(2,T6) + d(6,T2))\} \\ &= max\{d(2,6), d(2,3), d(6,2), \frac{1}{2}(d(2,2) + d(6,3))\} \\ &= max\{4, 1, 4, \frac{1}{2}(0+3)\} \\ &= 4 \end{aligned}$$

Then by condition (v) *of Theorem 4.1, we have*

$$\zeta^*(\theta(d(T2, T6)), \theta(\mathcal{M}(2,6))) = \zeta^*(\theta(1), \theta(4)) = \zeta^*(e, e^2) = \frac{e^{2k}}{e} = e^{2k-1} > 1.$$

Case (II): If we take $x = 3, y = 5$, then we have $\frac{1}{2}d(3, T3) = \frac{1}{2}d(3, 3) = 0 < 2 = d(3, 5)$ and

$$\begin{aligned} \mathcal{M}(3,5) &= max\{d(3,5), d(3,T3), d(5,T5), \frac{1}{2}(d(3,T5) + d(5,T3))\} \\ &= max\{d(3,5), d(3,3), d(5,2), \frac{1}{2}(d(3,2) + d(5,3))\} \\ &= max\{2, 0, 3, \frac{1}{2}(1+2)\} \\ &= 3 \end{aligned}$$

Then by condition (v) *of Theorem 4.1, we have*

$$\zeta^*(\theta(d(T3, T5)), \theta(\mathcal{M}(3,5))) = \zeta^*(\theta(1), \theta(3)) = \zeta^*(e, e^{\sqrt{3}}) = \frac{e^{\sqrt{3}k}}{e} = e^{\sqrt{3}k-1} > 1.$$

Case (III): If we take $x = 3, y = 6$, *then we have* $\frac{1}{2}d(3, T3) = \frac{1}{2}d(3, 3) = 0 < 3 = d(3, 6)$ *and*

$$\begin{aligned}\mathcal{M}(3,6) &= max\{d(3,6), d(3,T3), d(6,T6), \frac{1}{2}(d(3,T6) + d(6,T3))\}\\ &= max\{d(3,6), d(3,3), d(6,2), \frac{1}{2}(d(3,2) + d(6,3))\}\\ &= max\{3, 0, 4, \frac{1}{2}(1+3)\}\\ &= 4\end{aligned}$$

Then by condition (v) *of Theorem 4.1, we have*

$$\zeta^*(\theta(d(T3,T6)), \theta(\mathcal{M}(3,6))) = \zeta^*(\theta(1), \theta(4)) = \zeta^*(e, e^2) = \frac{e^{2k}}{e} = e^{2k-1} > 1.$$

Case (IV): If we take $x = 2, y = 5$, *then we have* $\frac{1}{2}d(2, T2) = \frac{1}{2}d(2, 3) = \frac{1}{2} < 3 = d(2, 5)$ *and*

$$\begin{aligned}\mathcal{M}(2,5) &= max\{d(2,5), d(2,T2), d(5,T5), \frac{1}{2}(d(2,T5) + d(5,T2))\}\\ &= max\{d(2,5), d(2,3), d(5,2), \frac{1}{2}(d(2,2) + d(5,3))\}\\ &= max\{3, 1, 3, \frac{1}{2}(0+2)\}\\ &= 3\end{aligned}$$

Then by condition (v) *of Theorem 4.1, we have*

$$\zeta^*(\theta(d(T2,T5)), \theta(\mathcal{M}(2,5))) = \zeta^*(\theta(1), \theta(3)) = \zeta^*(e, e^{\sqrt{3}}) = \frac{e^{\sqrt{3}k}}{e} = e^{\sqrt{3}k-1} > 1.$$

Hence, T is Suzuki generalized $\mathcal{L}_{\mathcal{R}}$*-contraction w.r.t.* $\zeta^* \in \mathcal{L}$*. Therefore, all the required conditions of Theorems 4.1 and 4.2 are fulfilled and consequently T has a unique fixed point, i.e.,* $T(3) = 3$*. It is worth mentioning here that T is not Suzuki generalized* $\mathcal{L}$*-contraction w.r.t. any* $\theta \in \Theta^*$ *and* $\zeta \in \mathcal{L}$ *(by Remark 3.7, as T is an isometry for* $x, y > 5$*), so we cannot apply Theorem 4 of [7]. The present example demonstrates the utility of our results over the known relevant results especially in the context of contraction condition.*

Below is the graphical representation of the fixed point for the given mapping provided in Example 4.3 for the space $X = (0, 10)$.

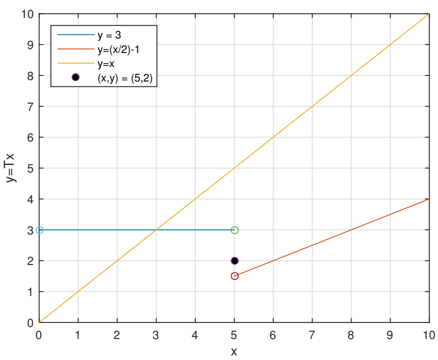


Figure 1: Fixed point of T

In view of the Remark 3.11, we have the analog version of Theorems 4.1 and 4.2 as corollary which is stated as follows:

Corollary 4.4. *Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X and $T : X \to X$. Suppose that the following conditions hold:*

(i) *$X(T; \mathcal{R})$ is non-empty,*
(ii) *$\mathcal{R}$ is T-closed and T-transitive,*
(iii) *T is Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contraction w.r.t. some $\zeta \in \mathcal{L}$,*
(iv) *(X, d) is $\mathcal{R}$-complete,*
(v) *either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed.*

Then T has a fixed point. Moreover, for each $x_0 \in X(T; \mathcal{R})$, the Picard sequence $T^n(x_0)$ for all $n \in \mathbb{N}$, converges to a fixed point of T. In addition to the assumptions of Theorem 4.1, if $\Upsilon(x, y; \mathcal{R}|_{T(X)})$ is non-empty for all $x, y \in T(X)$ then T admits a unique fixed point.

If we take $\zeta(x, y) = \frac{y^k}{x}$ for all $x, y \in [1, \infty)$, where $k \in (0, 1)$ in Theorem 4.1 and 4.2 we have the following corollary.

Corollary 4.5. *Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$ and $T : X \to X$. Suppose that the following conditions hold:*

(i) *$X(T; \mathcal{R})$ is non-empty,*
(ii) *$\mathcal{R}$ is T-closed and T-transitive,*
(iii) *there exists $\theta \in \Theta^*$ such that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \implies \theta(d(Tx, Ty)) \leq \theta(\mathcal{M}(x, y))^k$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}^*$ and $k \in (0, 1)$,
(iv) *(X, d) is $\mathcal{R}$-complete,*
(v) *either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed.*

Then T has a fixed point. Moreover, for each $x_0 \in X(T; \mathcal{R})$, the Picard sequence $T^n(r_0)$ for all $n \in \mathbb{N}$, converges to a fixed point of T. In addition, if $\Upsilon(x, y; \mathcal{R}|_{T(X)})$ is non-empty for all $x, y \in T(X)$ then T admits a unique fixed point.

Remark 4.6. *If $\mathcal{M}(x, y) = d(x, y)$, then Corollary 4.5 is sharpen version of Theorem 2.2 by Ahmad et al. [1] in context of relational notion and Suzuki condition.*

If we choose $\zeta(x, y) = \zeta_2(x, y) = \frac{y}{x\phi(y)}$ in Theorem 4.1 and 4.2, we have the following corollary.

Corollary 4.7. *Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$ and $T : X \to X$. Suppose that the following conditions hold:*

(i) *$X(T; \mathcal{R})$ is non-empty,*
(ii) *$\mathcal{R}$ is T-closed and T-transitive,*
(iii) *there exists $\theta \in \Theta^*$ such that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \implies \theta(d(Tx, Ty)) \leq \frac{\theta(\mathcal{M}(x, y))}{\varphi(\theta(\mathcal{M}(x, y)))}$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}^*$ and $\varphi : [1, \infty) \to [1, \infty)$ is a lower semi continuous and non decreasing function such that $\varphi^{-1}(\{1\}) = \{1\}$,
(iv) *(X, d) is $\mathcal{R}$-complete,*
(v) *either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed.*

Then T has a fixed point. Moreover, if $\Upsilon(x, y; \mathcal{R}|_{T(X)})$ is non-empty for all $x, y \in T(X)$ then T admits a unique fixed point.

Remark 4.8. *Corollary 4.7 is sharpen version of Corollary 3.5 by Cho [8] in the context of metric space and relational notion.*

Remark 4.9. *Corollary 4.7 is sharpen version of Corollary 3.6 by Cho [8] in the context of metric space and relational notion if $\mathcal{M}(x, y) = d(x, y)$.*

Remark 4.10. *Theorems 4.1 and 4.2 are generalized version of main result of Cho [8] in the setting of relation theoretic metric notions.*

Remark 4.11. *Theorems 4.1 and 4.2 are sharpen and generalized version of main result of Hasanuzzaman et al. [9] in the setting of Suzuki metric contractions.*

If we take $\zeta = \zeta_2(x, y) = \frac{\eta(y)}{x}$ in Theorems 4.1 and 4.2, we have the following corollary.

Corollary 4.12. *Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$ and $T : X \to X$. Suppose that the following conditions hold:*

(i) *$X(T; \mathcal{R})$ is non-empty,*
(ii) *$\mathcal{R}$ is T-closed and T-transitive,*
(iii) *there exists $\theta \in \Theta^*$ such that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \implies \theta(d(Tx, Ty)) \leq \eta(\theta(\mathcal{M}(x, y)))$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}^*$ and where $\eta : [1, \infty) \to [1, \infty)$ is upper semi-continuous with $\eta(x) < x$, $\forall x \geq 1$ and $\eta(x) = x$ if and only if $x = 1$,
(iv) *(X, d) is $\mathcal{R}$-complete,*
(v) *either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed.*

Then T has a fixed point. Moreover, if $\Upsilon(x, y; \mathcal{R}|_{T(X)})$ is non-empty for all $x, y \in T(X)$ then T admits a unique fixed point.

Remark 4.13. *Corollary 4.12 is sharpen version of Corollary 3.9 of Cho [8] in the context of relation theoretic metrical notions.*

Remark 4.14. *Corollary 4.12 is sharpen version of Corollary 3.10 of Cho [8] in the context of relation theoretic metrical notions if $\mathcal{M}(x, y) = d(x, y)$.*

5. Application to fractional thermostat model

The thermostat model is a problem that depicts the stationary state of a heated bar of finite length which is insulated at an initial stage of time(i.e., $t = 0$) and has a controller at final stage of time(i.e., $t = T$) that can add or remove heat depending on the temperature detected by a point sensor attach at the initial stage of time. Using our relation-theoretic metrical fixed point Theorem 4.1, we are interested in finding a positive solution to a fractional thermostat model in this section under specific circumstances.
The fractional thermostat model given in [13] is as follows

$$^{C}D^{\alpha}w(t) = -g(t, w(t)) \qquad (0 \leq t \leq 1, 1 < \alpha \leq 2) \tag{13}$$

with boundary conditions

$$w'(0) = 0, \beta\, {}^{C}D^{\alpha}w(t) + w(\mu) = w'(0) \tag{14}$$

where $\beta > 0, 0 \leq \mu \leq 1$ are given constants. Any function $w(t) \in C[0,1]$ is a solution of Equation (13), as demonstrated by the authors of [13] if and only if

$$w(t) = \int_0^1 K(t,r) g(r, w(r)) dr, \tag{15}$$

where $K(t,r)$ is the Green's function (depending on α) is given by

$$K(t,r) = \beta + G_\mu(r) - G_t(r) \tag{16}$$

and for $t \in [0,1]$, $G_t : [0,1] \to \mathbb{R}$ is defined by

$$G_t(r) = \begin{cases} \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)} & \text{for } r \leq t, \\ 0 & \text{for } r > t. \end{cases}$$

Let Φ be the collection of all mappings $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(φ_1) φ is non-decreasing;
(φ_2) $\varphi(t) \leq t$, for all $t \in [0,\infty)$.

Using our relation-theoretic metrical fixed point result (Theorem 4.1), we can use it to determine the positive solution of Equation (13) with boundary condition (14) with the help of following theorem:

Theorem 5.1. *Let $w(t) \in C[0,1]$, c be a positive constant and $g : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ a continuous function satisfying Equations* (13) *and* (14)*, such that*

$$\beta\Gamma(\alpha) \geq (1-\mu)^{\alpha-1},$$

and

$$|g(t, w(t)) - g(t, z(t))| \leq M\varphi\left(\frac{|w(t) - z(t)|}{(1 + c\sqrt{\|w - z\|})^2}\right), \tag{17}$$

for each $w, z \in C[0,1]$ such that $w(t)z(t) \geq 0$ for all $t \in [0,1]$ and M is a constant with $M\left(\beta + \frac{2}{\Gamma(\alpha+1)}\right) \leq 1$.

Proof. Let $X = C[0,1]$ together with the metric $d(w,z) = \sup_{t\in[0,1]} \|w(t) - z(t)\|$ is a complete metric space. Define a self mapping T on X by

$$Tw(t) = \int_0^1 K(t,r) g(r, w(r)) dr, \tag{18}$$

where $K(t,r)$ is a Green's function defined on (16). Clearly the solutions of (13) with boundary condition (14) are nothing but the fixed point of T defined on (18).
Now we define the binary relation $\mathcal{R}$ on X as

$$\mathcal{R} := \{(w,z) \in \mathcal{R} \Leftrightarrow w(t)z(t) \geq 0 \text{ for all } w, z \in X \text{ and } t \in [0,1]\}.$$

Now, it is clear that X is $\mathcal{R}$-complete metric space. Since $\beta\Gamma(\alpha) \geq (1-\mu)^{\alpha-1}$ implies that $K(t,r) \geq 0$ (See, Lemma 2.6 of [13]), one can easily verify that $Tw(t) \geq 0$. So, for any $w(t) \geq 0$ we have $w(t)Tw(t) \geq 0$, which implies that $(w(t), Tw(t)) \in \mathcal{R}$. Hence, $X(T,\mathcal{R})$ is non-empty. Next, for any $w(t) \geq 0$ we have $w(t)Tw(t) \geq 0$, then if $(w(t), z(t)) \in \mathcal{R}$ then $Tw(t)Tz(t) \geq 0$ implies that $(Tw(t), Tz(t)) \in \mathcal{R}$. Hence, $\mathcal{R}$ is T-closed. By definition of our involved binary relation, we have $\mathcal{R}$ is transitive and consequently it is T-transitive. It is easy to verify that the mapping T is $\mathcal{R}$-continuous. Now, we have to verify the contractive condition, let for all

$t \in [0,1]$, $(w(t), z(t)) \in \mathcal{R}^* = \{(w,z) \in \mathcal{R} \text{ with } Tw \neq Tz\}$. Then, we have

$$\begin{aligned}
|(Tw)(t) - (Tz)(t)| &= \left| \int_0^1 K(t,r)g(r,w(r))dr - \int_0^1 K(t,r)g(r,z(r))dr \right| \\
&\leq \left| \beta \int_0^1 f(r,w(r))dr + \int_0^\mu \frac{(\mu-r)^{\alpha-1}}{\Gamma(\alpha)} f(r,w(r))dr - \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)} f(r,w(r))dr \right. \\
&\quad \left. -\beta \int_0^1 g(r,z(r))dr - \int_0^\mu \frac{(\mu-r)^{\alpha-1}}{\Gamma(\alpha)} g(r,z(r))dr + \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)} g(r,z(r))dr \right| \\
&\leq \beta \left| \int_0^1 g(r,w(r))dr - \int_0^1 g(r,z(r))dr \right| \\
&\quad + \left| \int_0^\mu \frac{(\mu-r)^{\alpha-1}}{\Gamma(\alpha)} g(r,w(r))dr - \int_0^\mu \frac{(\mu-r)^{\alpha-1}}{\Gamma(\alpha)} g(r,z(r))dr \right| \\
&\quad + \left| \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)} g(r,w(r))dr - \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)} g(r,z(r))dr \right| \\
&\leq \beta \int_0^1 \left| g(r,w(r)) - g(r,z(r)) \right| dr + \int_0^\mu \frac{(\mu-r)^{\alpha-1}}{\Gamma(\alpha)} \left| g(r,w(r)) - g(r,z(r)) \right| dr \\
&\quad + \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)} \left| g(r,w(r)) - g(r,z(r)) \right| dr \\
&\leq \beta \int_0^1 M\varphi\left(\frac{|w(t)-z(t)|}{(1+c\sqrt{\|w-z\|})^2} \right) dr + \int_0^\mu \frac{(\mu-r)^{\alpha-1}}{\Gamma(\alpha)} M\varphi\left(\frac{|w(t)-z(t)|}{(1+c\sqrt{\|w-z\|})^2} \right) dr \\
&\quad + \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)} M\varphi\left(\frac{|w(t)-z(t)|}{(1+c\sqrt{\|w-z\|})^2} \right) dr \\
&\leq \beta \int_0^1 M\varphi\left(\frac{\|w-z\|}{(1+c\sqrt{\|w-z\|})^2} \right) dr + \int_0^\mu \frac{(\mu-r)^{\alpha-1}}{\Gamma(\alpha)} M\varphi\left(\frac{\|w-z\|}{(1+c\sqrt{\|w-z\|})^2} \right) dr \\
&\quad + \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)} M\varphi\left(\frac{\|w-z\|}{(1+c\sqrt{\|w-z\|})^2} \right) dr \\
&\leq \beta M \int_0^1 \left(\frac{\|w-z\|}{(1+c\sqrt{\|w-z\|})^2} \right) dr + M \int_0^\mu \frac{(\mu-r)^{\alpha-1}}{\Gamma(\alpha)} \left(\frac{\|w-z\|}{(1+c\sqrt{\|w-z\|})^2} \right) dr \\
&\quad + M \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)} \left(\frac{\|w-z\|}{(1+c\sqrt{\|w-z\|})^2} \right) dr \\
&\leq \beta M \left(\frac{\|w-z\|}{(1+c\sqrt{\|w-z\|})^2} \right) \int_0^1 dr + M \left(\frac{\|w-z\|}{(1+c\sqrt{\|w-z\|})^2} \right) \int_0^\mu \frac{(\mu-r)^{\alpha-1}}{\Gamma(\alpha)} dr \\
&\quad + M \left(\frac{\|w-z\|}{(1+c\sqrt{\|w-z\|})^2} \right) \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)} dr \\
&\leq \beta M \left(\frac{\|w-z\|}{(1+c\sqrt{\|w-z\|})^2} \right) + \frac{2M}{\Gamma(\alpha+1)} \left(\frac{\|w-z\|}{(1+c\sqrt{\|w-z\|})^2} \right) \\
&\leq \left(\frac{\|w-z\|}{(1+c\sqrt{\|w-z\|})^2} \right) M \left(\beta + \frac{2}{\Gamma(\alpha+1)} \right) \\
&\leq \left(\frac{\|w-z\|}{(1+c\sqrt{\|w-z\|})^2} \right).
\end{aligned}$$

Thus, we obtain $\quad |(Tw)(t) - (Tz)(t)| \leq \left(\frac{\|w-z\|}{(1+c\sqrt{\|w-z\|})^2} \right)$ for all $t \in [0,1]$.

Taking supremum both side, we get

$$\|Tw - Tz\| \leq \left(\frac{\|w - z\|}{(1 + c\sqrt{\|w - z\|})^2} \right).$$

Then we can rewrite the above inequality as

$$\begin{aligned}
\frac{(1 + c\sqrt{\|w - z\|})^2}{\|w - z\|} &\leq \frac{1}{\|Tw - Tz\|} \\
\left(c + \frac{1}{\sqrt{\|w - z\|}} \right)^2 &\leq \frac{1}{\|Tw - Tz\|} \\
c + \frac{1}{\sqrt{\|w - z\|}} &\leq \frac{1}{\sqrt{\|Tw - Tz\|}} \\
\frac{-1}{\sqrt{d(Tw, Tz)}} &\leq -\ln k - \frac{1}{\sqrt{d(w, z)}} \quad (\text{for } c = \ln k) \\
e^{\frac{-1}{\sqrt{d(Tw,Tz)}}} &\leq k e^{-\frac{1}{\sqrt{d(w,z)}}} \quad (\text{apply exponent both side}) \\
e^{e^{\frac{-1}{\sqrt{d(Tw,Tz)}}}} &\leq \left(e^{e^{-\frac{1}{\sqrt{d(w,z)}}}} \right)^k \quad (\text{apply exponent again both side}) \\
\implies \qquad \theta(d(Tw, Tz)) &\leq (\theta(d(w, z))^k \quad (\text{for } \ \theta(\beta) = e^{e^{\frac{-1}{\sqrt{\beta}}}}) \\
\implies \qquad \frac{(\theta(d(w, z))^k}{\theta(d(Tw, Tz))} &\geq 1
\end{aligned}$$

Therefore, we get

$$\zeta(\theta(d(Tw, Tz)), \theta(d(w, z))) \geq 1 \quad \text{for} \ \ \zeta(w, z) = \frac{z^k}{w}.$$

So we see that for $\mathcal{M}(w, z) = d(w, z)$, $\theta(\beta) = e^{e^{\frac{-1}{\sqrt{\beta}}}}$ and $\zeta(w, z) = \frac{z^k}{w}$ for all $w, z \in [1, \infty)$ such that $k \in (0, 1)$, the self-mapping T is Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contraction. By Theorem 4.1, the mapping T have a fixed point and consequently the fractional thermostat model given by Equation (13) along with boundary condition (14) has a positive solution. □

Remark 5.2. *If we set* $\phi(t) = \left(\frac{1}{\sqrt{t}} - c \right)^{-2}$ *and* $\lambda = \beta + \frac{2}{\Gamma(\alpha+1)}$ *such that* $M\lambda < 1$ *in* (17)*, then we deduce Theorem 3.3 due to Senapati and Dey [16].*

6. Conclusion

We have presented the existence and uniqueness of the fixed points for the Suzuki generalized $\mathcal{L}_{\mathcal{R}}$-contraction in $\mathcal{R}$-complete metric spaces in the form of Theorems 4.1 and 4.2. We can deduce some established outcomes as corollaries and remarks. In order to establish the genuineness of our newly proved results, a demonstrative example is furnished. Additionally, we have established the existence of a positive solution for a fractional thermostat model under a suitable setting using our main result. In addition, using a weaker type of transitivity (of the involved binary relation), one can extend these results in various spaces such as b-metric, quasi-metric space, etc. besides employing various control functions, especially due to Matkowski and, Boyd and Wong, etc.

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