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On dynamical systems and fixed point theory in Eilenberg-Jachymski spaces

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Abstract. We introduce the sets of periodic, recurrent, ω-limit and nonwandering points for a selfmap defined on a nonempty set endowed with an Eilenberg-Jachymski collection. Then, under some appropriate conditions, we show that these sets all coincide. Moreover, we establish fixed and periodic point theorems for a new class of φ-contractive mappings in ψ-dislocated metric spaces, and generalize some results obtained by Edelstein, Matkowski and Bessenyei-Páles.

1. Introduction and preliminaries

The last two authors have developed in [1] new fixed point results for mappings defined on an EJ-space, which is a nonempty set endowed with a structure not necessarily a base of uniformity. They studied the existence of fixed points for a self map $f\colon X \to X$, where X is a nonempty set endowed with an EJ-collection $\mathcal{R} = (P, R, B)$, where P is a partially ordered set, R is a family of binary relations on X indexed by P and B is an auxiliary binary relation on X. Observe that the EJ-spaces generalize the uniform spaces, and that the assumptions on $\mathcal{R}$ play an important role in developing new fixed point results. For some recent developments on the minimal structure required by the fixed point theorems, see for instance [4, 8].

In this paper, we establish new results of dynamical systems and fixed point theory in appropriate EJ-spaces. More precisely, we consider a nonempty set X endowed with an EJ-collection $\mathcal{R} = (P, R, B)$, where P is a real interval, R is weakly nested and $B = X \times X$. We first introduce the sets $\mathcal{R}$-periodic, $\mathcal{R}$-recurrent, $\mathcal{R}$-limit and $\mathcal{R}$-nonwandering, and study the relationship between them. In particular, when f is an $\mathcal{R}$-contractive mapping, we show that $\Omega(f, \mathcal{R}) = \Lambda(f, \mathcal{R}) = \mathrm{R}(f, \mathcal{R}) = \mathrm{P}(f, \mathcal{R})$ (see Definitions 1.10 and 1.11). Then, we exhibit a bijection between a subclass of EJ-collections and the set of ψ-dislocated metrics on a same set X, in order to understand their interconnection. Finally, we present new fixed and periodic

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point theorems in ψ-dislocated metric spaces, which generalize some results of Edelstein [3], Matkowski [6] and Bessenyei-Páles [2].

The paper is organized as follows. Section 1 presents some basic definitions and notations. Section 2 studies the connection between EJ-spaces and ψ-dislocated metric spaces. Section 3 contains proofs of the main results. Some consequences in ψ-dislocated metric spaces are presented in Section 4.

Let X be a non-empty set and f be a self-mapping of X. Let $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}_+$ be the set of all positive integers, the set of all non-negative integers and the set of non-negative real numbers, respectively. For $n \in \mathbb{N}_0$, denote by f^n the n-th iterate of f, where f^0 is the identity mapping. A point $x \in X$ is called *periodic* of period $n \in \mathbb{N}$ if $f^n x = x$ and $f^i x \neq x$ for $1 \leq i < n$; if $n = 1$, x is called a *fixed point* of f, that is, $fx = x$. Denote by $\mathrm{P}(f)$ the set of periodic points of f and by $\mathrm{Fix}(f)$ the set of all fixed points of f. For any $x \in X$, the orbit of x under f is the set $O(x, f) = \left\{f^n x : n \in \mathbb{N}_0\right\}$. A binary relation on X is a subset of $X \times X$. In particular, the diagonal relation on X is denoted by $\Delta := \left\{(x, x) : x \in X\right\}$. Let S be a binary relation on X and denote its symmetric by $S^{-1} := \left\{(y, x) \in X \times X : (x, y) \in S\right\}$. A composition of two binary relations S and Q is given by

$$S \circ Q := \left\{(x, y) \in X \times X : \text{ there exists } z \in X \text{ such that } (x, z) \in S \text{ and } (z, y) \in Q\right\}.$$

Definition 1.1 ([1]). *A poset P is said to be* extended *if it has a greatest element. We define the extension of a poset P, and we denote it by $\overline{P}$, the poset given by:*

(i) *$\overline{P} = P$, if P is extended.*

(ii) *If P is not extended, then $\overline{P} = P \cup \{\top\}$, where $\top$ is an extra element added to P with $x \leq \top$ for all $x \in P$, that is, $\top$ is the greatest element of $\overline{P}$.*

Denote by $\top$ the greatest element of $\overline{P}$ whether P is extended or not. The following definition generalizes that of EJ-collection given in [1].

Definition 1.2. *Let X be a nonempty set. An* Eilenberg-Jachymski collection *on X (shortly EJ-collection) is a triple (P, R, B) satisfying the following assumptions:*

(i) *$(P, \leq)$ is a poset.*

(ii) *$R = \left\{R_\lambda\right\}_{\lambda \in \overline{P}}$ is a family of binary relations over X with*

$$R_\top \subseteq \Delta := \left\{(x, x) \in X \times X\right\}.$$

(iii) *B is a nonempty binary relation over X.*

A nonempty set endowed with an EJ-collection is called an EJ-space.

In the sequel, we assume that X is a nonempty set, $f : X \to X$ is a mapping and I is a closed subset of $\mathbb{R}_+$ endowed with the dual order of $\mathbb{R}$ such that $0 \in I$. In this case, we have $I = \overline{I}$, $\top = 0$ and any EJ-collection $\mathcal{R} = (I, R, X \times X)$ can be identified with its family of binary relations

$$R = \left\{R_\lambda\right\}_{\lambda \in I}.$$

Denote by $\mathcal{J}(I)$ the set of all families of binary relations indexed by I and satisfying (ii) of Definition 1.2.

Definition 1.3. *Let $\widehat{I} = I \cup \{\infty\}$ and $\psi : \widehat{I} \times \widehat{I} \to \widehat{I}$ be a function. A family $\mathcal{R} \in \mathcal{J}(I)$ is said to be:*

- *Symmetric if $R_\lambda^{-1} = R_\lambda$ for all $\lambda \in I$.*
- *ψ-transitive if $R_\lambda \circ R_\mu \subseteq R_{\psi(\lambda,\mu)}$ for all $\lambda, \mu \in \widehat{I}$, where $R_\infty = \bigcup_{\lambda \in I} R_\lambda$.*

Definition 1.4. *A family $\mathcal{R} \in \mathcal{J}(I)$ is said to be weakly-nested (resp. nested) if for every non-increasing sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq I$ converges to λ , we have*

$$\bigcap_{n\in\mathbb{N}} R_{\lambda_n} \subseteq R_\lambda \quad (\textit{resp.} \bigcap_{n\in\mathbb{N}} R_{\lambda_n} = R_\lambda).$$

Remark 1.5. *Note that if $\mathcal{R} \in \mathcal{J}(I)$ is nested, then*

$$\lambda, \mu \in I : \lambda \leq \mu \implies R_\lambda \subseteq R_\mu.$$

Definition 1.6. *Let $\psi\colon \widehat{I} \times \widehat{I} \to \widehat{I}$ be a function. We define the following sets:*

- $\mathcal{J}^t(I) := \{\mathcal{R} \in \mathcal{J}(I) : R_\infty = X \times X\}$.
- $\mathcal{J}^s(I, \psi) := \{\mathcal{R} \in \mathcal{J}(I) : \mathcal{R}$ *is symmetric and ψ-transitive* $\}$.
- $\mathcal{J}^n(I, \psi) := \{\mathcal{R} \in \mathcal{J}^s(I, \psi) : \mathcal{R}$ *is nested* $\}$.
- $\mathcal{J}^w(I, \psi) := \{\mathcal{R} \in \mathcal{J}^s(I, \psi) : \mathcal{R}$ *is weakly-nested* $\}$.

Clearly, we have $\mathcal{J}^n(I, \psi) \subseteq \mathcal{J}^w(I, \psi) \subseteq \mathcal{J}^s(I, \psi)$.

Remark 1.7. *Recall that a family $\mathcal{R}$ of binary relations indexed by I is said to be a base of uniformity for X if:*

(i) $\Delta \subseteq R_\lambda$ *for all* $\lambda \in I$.

(ii) *For all $\lambda \in I$, there exists $\mu \in I$ such that $R_\mu \subseteq R_\lambda^{-1}$.*

(iii) *For all $\lambda \in I$, there exists $\mu \in I$ such that $R_\mu \circ R_\mu \subseteq R_\lambda$.*

(iv) *For all $\lambda, \lambda' \in I$ there exists $\mu \in I$ such that $R_\mu \subseteq R_\lambda \cap R_{\lambda'}$.*

For further details on uniform spaces, we refer the reader to [5, Chapter 6]. If $\mathcal{R} \in \mathcal{J}^n(I, \psi)$ and $R_0 = \Delta$, then $\mathcal{R}$ contains a base of uniformity of X. However, if $R_0 \neq \Delta$ or if $R_0 = \Delta$ and $\mathcal{R} \in \mathcal{J}^w(I, \psi)$, then R does not necessarily contains a base of uniformity of X, since (i) *may fail.*

Definition 1.8. *For $\mathcal{R} \in \mathcal{J}(I)$ and $\varepsilon \in \overline{\mathbb{R}}_+ \setminus \{0\}$, let $I_\varepsilon := I \cap [0, \varepsilon)$ and $\mathcal{R}_\varepsilon = \{R_\lambda\}_{\lambda \in I_\varepsilon}$.*

- *We say that $\mathcal{R}_\varepsilon$ is f-invariant if for all $\lambda \in I_\varepsilon$, we have*

$$(x, y) \in R_\lambda \setminus \Delta \implies (fx, fy) \in R_\lambda. \tag{1}$$

- *We say that f is $\mathcal{R}_\varepsilon$-contractive if $\mathcal{R}_\varepsilon$ is f-invariant and there exists a mapping $m\colon X \times X \times I \to \mathbb{N}$ such that for all $\lambda \in I_\varepsilon$, we have*

$$(x, y) \in R_\lambda \setminus \Delta \implies \exists\, \mu < \lambda : (f^{m(x,y,\lambda)}x, f^{m(x,y,\lambda)}y) \in R_\mu. \tag{2}$$

- *We say that f is $\mathcal{R}$-contractive, if* (1) *and* (2) *are satisfied for all $\lambda \in I$.*

Remark 1.9. *Any $\mathcal{R}$-contractive mapping is also $\mathcal{R}_\varepsilon$-contractive for all $\varepsilon \in \overline{\mathbb{R}}_+ \setminus \{0\}$.*

Definition 1.10. *Let $\mathcal{R} = \{R_\lambda\} \in \mathcal{J}(I)$. The $\omega_{\mathcal{R}}$-limit set of $x \in X$ under f will be denoted by:*

$$\omega_{\mathcal{R}}(x, f) := \left\{ y \in X : \exists \{n_k\} \subset \mathbb{N}, \{\lambda_k\} \in I \;\middle|\; \begin{array}{l} \{n_k\} \text{ is increasing,} \\ \{\lambda_k\} \text{ is convergent to } 0, \\ (f^{n_k}x, y) \in R_{\lambda_k},\ \forall k \in \mathbb{N} \end{array} \right\}.$$

The set of $\mathcal{R}$-limit points of f is

$$\Lambda(f, \mathcal{R}) := \bigcup_{x\in X} \omega_{\mathcal{R}}(x, f).$$

Definition 1.11. *Let $\mathcal{R} = \{R_\lambda\} \in \mathcal{J}(I)$.*

- *If $x \in \mathrm{Fix}(f)$ and $(x, x) \in R_0$, we say that x is an $\mathcal{R}$-fixed point of f.*
- *If $x \in \mathrm{P}(f)$ and $(x, x) \in R_0$, we say that x is an $\mathcal{R}$-periodic point of f.*
- *If $x \in \omega_{\mathcal{R}}(x, f)$, we say that x is an $\mathcal{R}$-recurrent point of f.*
- *If there exist a sequence $\{x_k\}$ in X, a non-decreasing sequence of positive integers $\{n_k\}$ and two convergent sequences $\{\lambda_k\}$ and $\{\mu_k\}$ in I to zero such that for all k, we have*

  $$(x_k, x) \in R_{\mu_k} \text{ and } (f^{n_k} x_k, x) \in R_{\lambda_k},$$

  we say that x is an $\mathcal{R}$-nonwandering point of f.

The sets of $\mathcal{R}$-fixed, $\mathcal{R}$-periodic, $\mathcal{R}$-recurrent and $\mathcal{R}$-nonwandering points of f will be denoted respectively by $\mathrm{Fix}(f, \mathcal{R})$, $\mathrm{P}(f, \mathcal{R})$, $\mathrm{R}(f, \mathcal{R})$ and $\Omega(f, \mathcal{R})$.

The following Lemma is a direct consequence of Definitions 1.10 and 1.11.

Lemma 1.12. *The following inclusions hold:*

$$\mathrm{Fix}(f, \mathcal{R}) \subseteq \mathrm{P}(f, \mathcal{R}) \subseteq \mathrm{R}(f, \mathcal{R}) \subseteq \Lambda(f, \mathcal{R}) \subseteq \Omega(f, \mathcal{R}).$$

Proof. The inclusions are obvious except the last one. Let $x \in X$ and $y \in \Lambda(f, \mathcal{R})$ be such that $y \in \omega_{\mathcal{R}}(x, f)$. Consider the sequences $\{n_k\}$ and $\{\lambda_k\}$ given by Definition 1.10. As $\{n_k\}$ is an increasing sequence (up to extraction of a subsequence, if necessary), we may assume that the sequence $\{m_k\}$, defined by $m_k = n_k - n_{k-1}$ for all $k > 0$, is non-decreasing sequence of positive integers. Then for $x_k = f^{n_{k-1}} x$, we have

$$(x_k, y) \in R_{\lambda_{k-1}} \quad \text{and} \quad (f^{m_k} x_k, y) \in R_{\lambda_k}.$$

Hence, $y \in \Omega(f, \mathcal{R})$. □

Definition 1.13. *Let $\psi \colon \widehat{I} \times \widehat{I} \to \widehat{I}$ be a mapping. We say that $\delta \colon X \times X \to I$ is a ψ-dislocated metric if for all $x, y, z \in X$, we have*

(i) $\delta(x, y) = 0$ *implies* $x = y$.

(ii) $\delta(x, y) = \delta(y, x)$.

(iii) $\delta(x, y) \leq \psi(\delta(x, z), \delta(z, y))$.

We say that (X, δ) is a ψ-dislocated metric space. We denote by $\mathcal{D}(I, \psi)$ the set of all ψ-dislocated metrics on X. If $I = \mathbb{R}_+$ and $\psi(\lambda, \mu) = \lambda + \mu$, the pair (X, δ) is called dislocated metric space.

Question 1.1. *Let $\delta \colon X \times X \to \mathbb{R}_+$ be a mapping satisfying (i) and (ii) of Definition 1.13. Is there a ψ function for which δ is a ψ-dislocated metric?*

Remark 1.14. *In case where δ is a semi-metric on X, the response of Question 1.1 is positive, and ψ is the basic triangular function introduced by Bessenyei and Páles [2].*

We later show how to derive a ψ-dislocated metric from a family of binary relations in $\mathcal{J}^t(I) \cap \mathcal{J}^w(I, \psi)$.

Definition 1.15. *A function $\psi \colon \widehat{I} \times \widehat{I} \to \widehat{I}$ is said to be monotone if it is increasing in both of its arguments.*

Definition 1.16. *For $\delta \in \mathcal{D}(I, \psi)$ and $\varepsilon \in \overline{\mathbb{R}}_+ \setminus \{0\}$.*

- *We say that f is δ_ε-nonexpansive, if for all $x \neq y$, we have*

$$\delta(x,y) \in I_\varepsilon \implies \delta(fx, fy) \leq \delta(x,y). \tag{3}$$

- *We say that f is δ_ε-contractive, if f is δ_ε-nonexpansive and there exists $m\colon X \times X \times I \to \mathbb{N}$ such that for all $x \neq y$, we have*

$$\delta(x,y) \in I_\varepsilon \implies \delta(f^{m(x,y,\delta(x,y))}x, f^{m(x,y,\delta(x,y))}y) < \delta(x,y). \tag{4}$$

- *We say that f is δ-contractive, if* (3) *and* (4) *are satisfied for all $x, y \in X$ such that $x \neq y$.*

Remark 1.17. *Any δ-contractive mapping is also δ_ε-contractive for all $\varepsilon \in \overline{\mathbb{R}}_+ \setminus \{0\}$.*

Definition 1.18. *Let $\delta \in \mathcal{D}(I, \psi)$. The ω_δ-limit set of a point $x \in X$ under f will be denoted by:*

$$\omega_\delta(x,f) := \left\{ y \in X : \exists \left\{n_k\right\} \subset \mathbb{N} \;\middle|\; \begin{array}{l} \left\{n_k\right\} \text{ is increasing} \\ \lim\limits_{k\to+\infty} \delta(f^{n_k}x, y) = 0 \end{array} \right\}.$$

The set of δ-limit points of f is

$$\Lambda(f,\delta) := \bigcup_{x \in X} \omega_\delta(x,f).$$

Definition 1.19. *Let $\delta \in \mathcal{D}(I, \psi)$.*

- *If $x \in \mathrm{Fix}(f)$ and $\delta(x,x) = 0$, we say that x is a δ-fixed point of f.*
- *If $x \in \mathrm{P}(f)$ and $\delta(x,x) = 0$, we say that x is a δ-periodic point of f.*
- *If $x \in \omega_\delta(x,f)$, we say that x is a δ-recurrent point of f.*
- *If there exist a sequence $\{x_k\}$ in X, a non-decreasing sequence of positive integers $\left\{n_k\right\}$ such that*

$$\lim_{k\to\infty} \delta(x_k, x) = \lim_{k\to\infty} \delta(f^{n_k}x_k, x) = 0,$$

we say that x is a δ-nonwandering point of f.

The sets of δ-fixed, δ-periodic, δ-recurrent and δ-nonwandering points of f will be denoted respectively by $\mathrm{Fix}(f,\delta)$, $\mathrm{P}(f,\delta)$, $\mathrm{R}(f,\delta)$ and $\Omega(f,\delta)$.

The following lemma is a direct consequence of Definitions 1.18 and 1.19.

Lemma 1.20. *The following inclusions hold:*

$$\mathrm{Fix}(f,\delta) \subseteq \mathrm{P}(f,\delta) \subseteq \mathrm{R}(f,\delta) \subseteq \Lambda(f,\delta) \subseteq \Omega(f,\delta).$$

Definition 1.21. *Let $\delta \in \mathcal{D}(I, \psi)$.*

- *A sequence $\left\{x_n\right\}$ is said to be convergent to $x \in X$ if $\lim\limits_{n\to\infty} \delta(x_n, x) = 0$.*
- *A sequence $\left\{x_n\right\}$ is said to be Cauchy if for all $\varepsilon > 0$, there exists an integer N such that for all $n, m \geq N$, we have $\delta(x_n, x_m) < \varepsilon$.*
- *(X, δ) is complete if every Cauchy sequence is convergent.*

2. Connections between the sets $\mathcal{J}^t(I)$, $\mathcal{J}^n(I,\psi)$, $\mathcal{J}^w(I,\psi)$ and $\mathcal{D}(I,\psi)$.

In this section, we point out some connections between the sets given in Definitions 1.6 and 1.13. More precisely, we show that any element of $\mathcal{J}^t(I)\cap\mathcal{J}^w(I,\psi)$ induces a ψ-dislocated metric on X. Moreover, we compare the sets given in Definitions 1.10 and 1.11 to those of Definitions 1.18 and 1.19.

Proposition 2.1. *Let $\mathcal{R}=\left\{R_\lambda\right\}\in\mathcal{J}^t(I)$. Then the mapping $\delta_\mathcal{R}: X\times X\to I$ given by:*

$$\delta_\mathcal{R}(x,y)=\inf\left\{\lambda\in I:(x,y)\in R_\lambda\right\},$$

satisfies the following properties:

(i) *If $\mathcal{R}$ is weakly-nested, then for all $x,y\in X$, we have $(x,y)\in R_{\delta_\mathcal{R}(x,y)}$. In particular, if $\delta_\mathcal{R}(x,y)=0$, then $x=y$. Moreover, if $R_0=\Delta$, then $\delta_\mathcal{R}(x,x)=0$ for all $x\in X$.*

(ii) *If $\mathcal{R}$ is symmetric, then $\delta_\mathcal{R}(x,y)=\delta_\mathcal{R}(y,x)$ for all $x,y\in X$.*

(iii) *If $\mathcal{R}$ is weakly-nested and ψ-transitive, then for all $x,y,z\in X$, we have*

$$\delta_\mathcal{R}(x,y)\le\psi\Big(\delta_\mathcal{R}(x,z),\delta_\mathcal{R}(z,y)\Big).$$

In particular, if $\mathcal{R}\in\mathcal{J}^t(I)\cap\mathcal{J}^w(I,\psi)$, then $\delta_\mathcal{R}\in\mathcal{D}(I,\psi)$.

Proof. Firstly, observe that the mapping $\delta_\mathcal{R}$ is well defined, since $R_\infty=X\times X$. Let $x,y,z\in X$.

(i) By definition of $\delta_\mathcal{R}(x,y)$, there exists a non-increasing sequence $\left\{\alpha_n\right\}_{n\in\mathbb{N}}\subset\left\{\lambda\in I:(x,y)\in R_\lambda\right\}$, which converges to $\delta_\mathcal{R}(x,y)$. Since $\mathcal{R}$ is weakly-nested, then

$$(x,y)\in\bigcap_{n\in\mathbb{N}}R_{\alpha_n}\subseteq R_{\delta_\mathcal{R}(x,y)}.$$

In particular, if $\delta_\mathcal{R}(x,y)=0$, then $(x,y)\in R_0\subseteq\Delta$. Moreover, if $R_0=\Delta$, then by definition of $\delta_\mathcal{R}$, we have $\delta_\mathcal{R}(x,x)=0$.

(ii) If $\mathcal{R}$ is symmetric, then $(x,y)\in R_\lambda$ if and only if $(y,x)\in R_\lambda$. Thus, we have

$$\left\{\lambda\in I:(x,y)\in R_\lambda\right\}=\left\{\lambda\in I:(y,x)\in R_\lambda\right\},$$

which implies $\delta_\mathcal{R}(x,y)=\delta_\mathcal{R}(y,x)$.

(iii) Again since $\mathcal{R}$ is weakly-nested, it follows that $(x,z)\in R_{\delta_\mathcal{R}(x,z)}$ and $(z,y)\in R_{\delta_\mathcal{R}(z,y)}$. Then using the ψ-transitivity of $\mathcal{R}$, we obtain $(x,y)\in R_{\psi(\delta_\mathcal{R}(x,z),\delta_\mathcal{R}(z,y))}$, which implies $\delta_\mathcal{R}(x,y)\le\psi(\delta_\mathcal{R}(x,z),\delta_\mathcal{R}(z,y))$.
□

Theorem 2.2. *The mapping $F\colon\mathcal{J}^t(I)\cap\mathcal{J}^w(I,\psi)\to\mathcal{D}(I,\psi)$, $\mathcal{R}\longmapsto\delta_\mathcal{R}$ is onto. Moreover, the restriction of F to $\mathcal{J}^t(I)\cap\mathcal{J}^n(I,\psi)$ is a bijection.*

Proof. By Proposition 2.1, F is well defined. Let H be the restriction of F to $\mathcal{J}^t(I)\cap\mathcal{J}^n(I,\psi)$. Thus, to conclude, it suffices to show that H is a bijection. Let $\mathcal{R}=\left\{R_\lambda\right\}$, $\mathcal{R}'=\left\{R'_\lambda\right\}\in\mathcal{J}^t(I)\cap\mathcal{J}^n(I,\psi)$ such that $H(\mathcal{R})=H(\mathcal{R}')$, that is, $\delta_\mathcal{R}=\delta_{\mathcal{R}'}$. By nestedness, for all $\lambda\in\widehat{I}$, we have

$$(x,y)\in R_\lambda\iff\delta_\mathcal{R}(x,y)\le\lambda\iff\delta_{\mathcal{R}'}(x,y)\le\lambda\iff(x,y)\in R'_\lambda,$$

so $R_\lambda=R'_\lambda$. Hence $\mathcal{R}=\mathcal{R}'$ and therefore H is one-to-one. Let $\delta\in\mathcal{D}(I,\psi)$. Define the set of binary relations $\mathcal{R}=\left\{R_\lambda\right\}_{\lambda\in\widehat{I}}$ on X as follow:

$$R_\lambda=\left\{(x,y):\delta(x,y)\le\lambda\right\}.$$

It is not difficult to see that $\mathcal{R} \in \mathcal{J}^t(I) \cap \mathcal{J}^n(I,\psi)$. Now we shall prove that $H(\mathcal{R}) = \delta$. Indeed, it's enough to prove that $\delta_{\mathcal{R}} = \delta$. Let $(x, y) \in X \times X$,

$$\begin{aligned}\delta_{\mathcal{R}}(x,y) &= \inf\left\{\lambda \in I : (x,y) \in R_\lambda\right\}\\ &= \inf\left\{\lambda \in I : \delta(x,y) \le \lambda\right\} \ge \delta(x,y)\end{aligned}$$

Conversely, as $(x, y) \in R_{\delta(x,y)}$, so by definition of $\delta_{\mathcal{R}}$, we get $\delta_{\mathcal{R}}(x, y) \le \delta(x, y)$. □

Proposition 2.3. *For each $\mathcal{R} \in \mathcal{J}^t(I) \cap \mathcal{J}^w(I,\psi)$, we have*

(i) $\omega_{\mathcal{R}}(x, f) = \omega_{\delta_{\mathcal{R}}}(x, f)$, *for all $x \in X$.*

(ii) $\mathrm{Fix}(f, \mathcal{R}) = \mathrm{Fix}(f, \delta_{\mathcal{R}})$.

(iii) $\mathrm{P}(f, \mathcal{R}) = \mathrm{P}(f, \delta_{\mathcal{R}})$.

(iv) $\mathrm{R}(f, \mathcal{R}) = \mathrm{R}(f, \delta_{\mathcal{R}})$.

(v) $\Lambda(f, \mathcal{R}) = \Lambda(f, \delta_{\mathcal{R}})$.

(vi) $\Omega(f, \mathcal{R}) = \Omega(f, \delta_{\mathcal{R}})$.

Proof. Let $\mathcal{R} = \left\{R_\lambda\right\} \in \mathcal{J}^t(I) \cap \mathcal{J}^w(I,\psi)$. Let $x \in X$ and assume that $y \in \omega_{\mathcal{R}}(x, f)$. Consider $\left\{n_k\right\}_{k\in\mathbb{N}}$ be an increasing sequence and $\left\{\lambda_k\right\}_{k\in\mathbb{N}}$ be a sequence convergent to 0 such that $(f^{n_k}x, y) \in R_{\lambda_k}$ for all $k \in \mathbb{N}$. By definition of $\delta_{\mathcal{R}}$, we have $\delta_{\mathcal{R}}(f^{n_k}x, y) \le \lambda_k$, for all $k \in \mathbb{N}$. Then $\lim_{k\to\infty} \delta_{\mathcal{R}}(f^{n_k}x, y) = 0$, which implies that $y \in \omega_{\delta_{\mathcal{R}}}(x, f)$. Conversely, let $y \in \omega_{\delta_{\mathcal{R}}}(x, f)$. Then there exists an increasing sequence $\left\{n_k\right\}$ such that $\lim_{k\to\infty} \delta_{\mathcal{R}}(f^{n_k}x, y) = 0$. Consider the sequence $\left\{\lambda_k\right\}_{k\in\mathbb{N}} \subset I$ defined by $\lambda_k = \delta_{\mathcal{R}}(f^{n_k}x, y)$. Hence $\left\{\lambda_k\right\}_{k\in\mathbb{N}} \subset I$, converges to 0 and by Proposition 2.1-(i), we have $(f^{n_k}x, y) \in R_{\lambda_k}$ for all $k \in \mathbb{N}$, that is, $y \in \omega_{\mathcal{R}}(x, f)$, which proves (i). A point $x \in \mathrm{P}(f, \delta_{\mathcal{R}})$ (resp. $\mathrm{Fix}(f, \delta_{\mathcal{R}})$) if and only if $x \in \mathrm{P}(f)$ (resp. $x \in \mathrm{Fix}(f)$) and $\delta_{\mathcal{R}}(x, x) = 0$, which is equivalent to $x \in \mathrm{P}(f)$ (resp. $x \in \mathrm{Fix}(f)$) and $(x, x) \in R_0$, that is, $x \in \mathrm{P}(f, \mathcal{R})$ (resp. $x \in \mathrm{Fix}(f, \mathcal{R})$), then (ii) (resp. (iii)) holds. The assertions (iv) and (v) follow from (i). Finally, $x \in \Omega(f, \delta_{\mathcal{R}})$ if and only if there exist a sequence $\{x_k\}$ in X and non-decreasing sequence of integers $\left\{n_k\right\}$ such that $\lim\limits_{k\to\infty} \delta_{\mathcal{R}}(x_k, x) = \lim\limits_{k\to\infty} \delta_{\mathcal{R}}(f^{n_k}x_k, x) = 0$, which is equivalent to $(x_k, x) \in R_{\lambda_k}$ and $(f^{n_k}x_k, x) \in R_{\mu_k}$, where $\lambda_k = \delta_{\mathcal{R}}(x_k, x)$ and $\mu_k = \delta_{\mathcal{R}}(f^{n_k}x_k, x)$, that is, $x \in \Omega(f, \mathcal{R})$, which proves (vi). □

Proposition 2.4. *Let $\mathcal{R} \in \mathcal{J}^t(I) \cap \mathcal{J}^w(I,\psi)$ and $\varepsilon \in \overline{\mathbb{R}}_+ \setminus \{0\}$. If f is $\mathcal{R}_\varepsilon$-contractive, then f is $(\delta_{\mathcal{R}})_\varepsilon$-contractive. In addition, if $\mathcal{R}$ is nested, then the converse is true.*

Proof. Suppose that f is $\mathcal{R}_\varepsilon$-contractive. Let $x, y \in X \setminus \Delta$ satisfying $\delta_{\mathcal{R}}(x, y) \in I_\varepsilon$. From the weak-nestedness of $\mathcal{R}$, we have $(x, y) \in R_{\delta_{\mathcal{R}}(x,y)}\setminus\Delta$. Then

$$(fx, fy) \in R_{\delta_{\mathcal{R}}(x,y)} \quad \text{and} \quad (f^{m(x,y,\delta_{\mathcal{R}}(x,y))}x, f^{m(x,y,\delta_{\mathcal{R}}(x,y))}y) \in R_\mu,$$

for some $\mu < \delta_{\mathcal{R}}(x, y)$. Hence by definition of $\delta_{\mathcal{R}}$, we obtain

$$\delta_{\mathcal{R}}(fx, fy) \le \delta_{\mathcal{R}}(x, y) \quad \text{and} \quad \delta_{\mathcal{R}}(f^{m(x,y,\delta_{\mathcal{R}}(x,y))}x, f^{m(x,y,\delta_{\mathcal{R}}(x,y))}y) \le \mu < \delta_{\mathcal{R}}(x, y).$$

Thus f is $(\delta_{\mathcal{R}})_\varepsilon$-contractive.

Now, assume that $\mathcal{R} \in \mathcal{J}^t(I) \cap \mathcal{J}^n(I,\psi)$ and f is $(\delta_{\mathcal{R}})_\varepsilon$-contractive. Consider the mapping $m' : X \times X \times I \to I$ defined by

$$m'(x,y,\lambda) = \begin{cases} m(x,y,\delta_{\mathcal{R}}(x,y)) & \text{for all } \lambda \ge \delta_{\mathcal{R}}(x,y),\\ 0 & \text{otherwise.}\end{cases}$$

Let $\lambda \in I_\varepsilon$ and $(x, y) \in R_\lambda \backslash \Delta$. By definition of $\delta_\mathcal{R}$, it follows that $0 < \delta_\mathcal{R}(x, y) \leq \lambda < \varepsilon$. Therefore, by the nestedness of $\mathcal{R}$, we have

$$\delta_\mathcal{R}(fx, fy) \leq \delta_\mathcal{R}(x, y) \leq \lambda \implies (fx, fy) \in R_\lambda,$$

and

$$\delta_\mathcal{R}(f^{m(x,y,\delta_\mathcal{R}(x,y))}x, f^{m(x,y,\delta_\mathcal{R}(x,y))}y) < \delta_\mathcal{R}(x, y) \leq \lambda,$$

if and only if,

$$\delta_\mathcal{R}(f^{m'(x,y,\lambda)}x, f^{m'(x,y,\lambda)}y) < \delta_\mathcal{R}(x, y) \leq \lambda,$$

which implies that

$$(f^{m'(x,y,\lambda)}x, f^{m'(x,y,\lambda)}y) \in R_\mu,$$

where $\mu = \delta_\mathcal{R}(f^{m'(x,y,\lambda)}x, f^{m'(x,y,\lambda)}y)$. Thus, f is $\mathcal{R}_\varepsilon$-contractive. □

3. Main results

The first result of this section deals with the special case where 0 is an isolated point of I.

Theorem 3.1. *Let $\mathcal{R} \in \mathcal{J}(I)$. If 0 is an isolated point of I, then*

$$\mathrm{P}(f, \mathcal{R}) = \mathrm{R}(f, \mathcal{R}) = \Lambda(f, \mathcal{R}) = \Omega(f, \mathcal{R}).$$

Proof. Let $\mathcal{R} = \left\{R_\lambda\right\} \in \mathcal{J}(I)$. From Lemma 1.12, we have $\mathrm{P}(f, \mathcal{R}) \subseteq \mathrm{R}(f, \mathcal{R}) \subseteq \Lambda(f, \mathcal{R}) \subseteq \Omega(f, \mathcal{R})$. To conclude, we have to show that $\mathrm{P}(f, \mathcal{R}) \supseteq \Omega(f, \mathcal{R})$. Indeed, for $x \in \Omega(f, \mathcal{R})$ there exist a sequence $\left\{x_k\right\}$ in X, a non-decreasing sequence of positive integers $\left\{n_k\right\}$ and two sequences $\left\{\lambda_k\right\}$ and $\left\{\mu_k\right\}$ in I convergent to 0, such that $(x_k, x) \in R_{\lambda_k}$ and $(f^{n_k}x_k, x) \in R_{\mu_k}$ for all k. As, 0 is an isolated point of I, we deduce that $\mu_k = \lambda_k = 0$, for k sufficiently large, thus $(x_k, x) \in R_0$ and $(f^{n_k}x_k, x) \in R_0$. Hence, $x_k = x$, $(x, x) \in R_0$ and $f^{n_k}x = x$, which means that $x \in \mathrm{P}(f, \mathcal{R})$. □

Until the end of this section, we will assume that I contains a decreasing sequence convergent to zero and ψ is continuous at $(0, 0)$ with $\psi(0, 0) = 0$.

Theorem 3.2. *Let $\mathcal{R} \in \mathcal{J}^w(I, \psi)$ and $\varepsilon \in \overline{\mathbb{R}}_+ \setminus \{0\}$. If $\mathcal{R}_\varepsilon$ is f-invariant, then*

$$\mathrm{R}(f, \mathcal{R}) = \Lambda(f, \mathcal{R}) = \Omega(f, \mathcal{R}).$$

Proof. It suffices to prove that $\mathrm{R}(f, \mathcal{R}) \supseteq \Omega(f, \mathcal{R})$. Let $\mathcal{R} = \left\{R_\lambda\right\}$, $\{x_k\}$ be a sequence in X, $\left\{n_k\right\}$ be a non-decreasing sequence of positive integers and $\left\{\lambda_k\right\}$, $\left\{\mu_k\right\}$ be two sequences in I convergent to 0 such that $(x_k, x) \in R_{\lambda_k}$ and $(f^{n_k}x_k, x) \in R_{\mu_k}$. By taking a subsequence if necessary, we may assume that λ_k, $\mu_k < \varepsilon$ for all k. From the invariance assumption, we get $(f^{n_k}x_k, f^{n_k}x) \in R_{\lambda_k}$. Using the transitivity hypothesis, we obtain $(f^{n_k}x, x) \in R_{\alpha_k}$, where $\alpha_k = \psi(\lambda_k, \mu_k)$. By continuity of ψ at $(0, 0)$, we see that the sequence $\left\{\alpha_k\right\}$ is convergent to 0. If $\left\{n_k\right\}$ contains an increasing subsequence, then we deduce that $x \in \omega_\mathcal{R}(x, f)$, that is, $x \in \mathrm{R}(f, \mathcal{R})$. Otherwise we may suppose that $\left\{n_k\right\}$ is constant with $n_k = n_0$ for all k. Then $(f^{n_0}x, x) \in R_{\alpha_k}$. Using the weak-nestedness property, we get

$$(f^{n_0}x, x) \in \bigcap_{k \in \mathbb{N}} R_{\alpha_k} \subseteq R_0.$$

Hence $f^{n_0}x = x$ and $(x, x) \in R_0$, that is, $x \in \mathrm{P}(f, \mathcal{R}) \subseteq \mathrm{R}(f, \mathcal{R})$. □

The following example shows, under the hypotheses of Theorem 3.2, that in general $\mathrm{P}(f,\mathcal{R})$ and $\mathrm{R}(f,\mathcal{R})$ are different.

Example 3.3. *Let $X = \mathbb{D}^1$ be the unit disc in $\mathbb{R}^2$, endowed by the euclidean metric d, and let $I = \mathbb{R}_+$. The family $\mathcal{R} = \left\{R_\lambda\right\}_{\lambda\in\widehat{I}}$ of binary relations defined by*

$$R_\lambda := \left\{(x,y) \in \mathbb{D}^1 \times \mathbb{D}^1 : d(x,y) \le \lambda\right\}, \text{ for all } \lambda \in I$$

is an element of $\mathcal{J}^t(I) \cap \mathcal{J}^n(I,\psi) \subseteq \mathcal{J}^w(I,\psi)$, where and $\psi(\lambda,\mu) = \lambda + \mu$. Let f be an irrational rotation. For all $\varepsilon \in \overline{\mathbb{R}}_+ \setminus \{0\}$, $\mathcal{R}_\varepsilon$ is f-invariant and $\mathrm{P}(f,\mathcal{R}) = P(f,d) = \left\{(0,0)\right\}$. However, $\mathrm{R}(f,\mathcal{R}) = \mathrm{R}(f,d) = \mathbb{D}^1$.

Theorem 3.4. *Let $\mathcal{R} \in \mathcal{J}^w(I,\psi)$ and $\varepsilon \in \overline{\mathbb{R}}_+ \setminus \{0\}$ such that f is an $\mathcal{R}_\varepsilon$-contractive mapping. If 0 is the unique cluster point of I, then*

$$\mathrm{P}(f,\mathcal{R}) = \mathrm{R}(f,\mathcal{R}) = \Lambda(f,\mathcal{R}) = \Omega(f,\mathcal{R}).$$

Theorem 3.5. *Let $\mathcal{R} \in \mathcal{J}^w(I,\psi)$ and $\varepsilon \in \overline{\mathbb{R}}_+ \setminus \{0\}$. Assume that,*

(i) *ψ is continuous at $(0,x)$ for all $x \in I$.*

(ii) *$\psi(0,x) \le x$ for all $x \in I$.*

(iii) *f is $\mathcal{R}_\varepsilon$-contractive.*

Then, $\mathrm{P}(f,\mathcal{R}) = \mathrm{R}(f,\mathcal{R}) = \Lambda(f,\mathcal{R}) = \Omega(f,\mathcal{R})$.

To prove our results we need a few lemmas.

Lemma 3.6. *Let q be a non-negative integer, $\mathcal{R} = \left\{R_\lambda\right\} \in \mathcal{J}(I)$ and $\varepsilon \in \overline{\mathbb{R}}_+ \setminus \{0\}$ such that $\mathcal{R}_\varepsilon$ is f-invariant. Assume that $O(x,f)$ is infinite and $y \in \omega_{\mathcal{R}}(x,f)$. Then the sequences $\left\{n_k\right\}_{k\in\mathbb{N}}$ and $\left\{\lambda_k\right\}_{k\in\mathbb{N}}$ given by Definition 1.10 can be chosen so that*

$$(f^{n_k+i}x, f^i y) \in R_{\lambda_k} \setminus \Delta, \quad k \in \mathbb{N} \text{ and } 0 \le i \le q. \tag{5}$$

Proof. Observe that for all $i \ge 0$ there exists at most an integer m_i such that $f^{m_i}x = f^i y$, otherwise the orbit $O(x,f)$ is finite. For $M = \max\left\{m_i : 0 \le i \le q\right\}$, we have $f^n x \ne f^i y$ for all $n > M$ and $0 \le i \le q$. As $\left\{n_k\right\}$ is increasing, there exists k_0 such that $n_k + i > M$ for all $k > k_0$ and $0 \le i \le q$. Consequently, $(f^{n_k+i}x, f^i y) \notin \Delta$, for all $k > k_0$ and $0 \le i \le q$. Now, as $\left\{\lambda_k\right\}$ converges to 0, there exists k_1 such that $\lambda_k < \varepsilon$ for all $k > k_1$. Then by the f-invariance of $\mathcal{R}_\varepsilon$, we obtain $(f^{n_k+i}x, f^i y) \in R_{\lambda_k} \setminus \Delta$ for all $k > k_2 = \max\left\{k_0, k_1\right\}$ and $0 \le i \le q$. Consequently, the subsequences $\left\{n_k\right\}$ and $\left\{\lambda_k\right\}$, for $k > k_2$ give the desired result. □

Lemma 3.7. *Let $x \in X$ and $\mathcal{R} \in \mathcal{J}(I)$. If $\mathcal{R}$ is weakly-nested and $O(x,f)$ is finite, then*

$$\omega_{\mathcal{R}}(x,f) \subseteq \mathrm{P}(f,\mathcal{R}).$$

Proof. Assume that $\mathcal{R} = \left\{R_\lambda\right\}$. If $\omega_{\mathcal{R}}(x,f) = \emptyset$, we are done. Otherwise, let $y \in \omega_{\mathcal{R}}(x,f)$, so there exist $\left\{n_k\right\} \subset \mathbb{N}$ and a sequence $\left\{\lambda_k\right\} \subset I$ which converges to 0 such that

$$(f^{n_k}x, y) \in R_{\lambda_k}, \quad k \in \mathbb{N}.$$

As $O(x,f)$ is finite, without loss of generality, modulo a choice of a subsequence, we may assume that $\left\{f^{n_k}x\right\}$ is constant, that is, $f^{n_k}x = f^{n_0}x$ for all k and $f^{n_0}x \in \mathrm{P}(f)$. Now, we have

$$(f^{n_0}x, y) \in R_{\lambda_k}, \quad k \in \mathbb{N}.$$

From the weak-nestedness of $\mathcal{R}$, we deduce that $(f^{n_0}x, y) \in R_0$, so $y \in \mathrm{P}(f,\mathcal{R})$. □

Lemma 3.8. *Let* $\mathcal{R} = \left\{R_\lambda\right\} \in \mathcal{J}^s(I, \psi)$, $\varepsilon \in \overline{\mathbb{R}}_+ \setminus \{0\}$ *and* $x \in X$. *Assume that* $O(x, f)$ *is infinite and* $\omega_{\mathcal{R}}(x, f)$ *is nonempty. Then there exists* $p \in \mathbb{N}$ *such that the set*

$$\mathcal{L}_p(x, \varepsilon) := \left\{\beta \in I_\varepsilon \;:\; \exists\, m \in \mathbb{N},\, (f^{m+p}x, f^m x) \in R_\beta\right\},$$

is nonempty and $0 \notin \mathcal{L}_p(x, \varepsilon)$. *If in addition,*

(i) f *is* $\mathcal{R}_\varepsilon$*-invariant and there exists* $y \in \omega_{\mathcal{R}}(x, f)$ *such that* $y = f^p y$, *then*

$$\inf \mathcal{L}_p(x, \varepsilon) = 0.$$

(ii) f *is* $\mathcal{R}_\varepsilon$*-contractive, then* $\inf \mathcal{L}_p(x, \varepsilon) \notin \mathcal{L}_p(x, \varepsilon)$.

Proof. Let $y \in \omega_{\mathcal{R}}(x, f)$, $\left\{n_k\right\}$ be an increasing sequence in $\mathbb{N}$ and $\left\{\lambda_k\right\} \subseteq I$ be a convergent sequence to 0 such that

$$(f^{n_k}x, y) \in R_{\lambda_k}, \quad \text{for all } k \in \mathbb{N}.$$

Since ψ is continuous at $(0, 0)$ and $\psi(0, 0) = 0$, then $\psi(\lambda_k, \lambda_{k+1})$ tends to zero. Thus, there exists $q \in \mathbb{N}$ such that $\psi(\lambda_q, \lambda_{q+1}) < \varepsilon$. Let $\ell = n_q$ and $p = n_{q+1} - \ell$. Then,

$$(f^\ell x, y) \in R_{\lambda_q} \text{ and } (f^{\ell+p}x, y) \in R_{\lambda_{q+1}},$$

and by the symmetry and the ψ-transitivity of $\mathcal{R}$, we deduce that $(f^{\ell+p}x, f^\ell x) \in R_\alpha$, where $\alpha = \psi(\lambda_q, \lambda_{q+1})$. Hence, $\mathcal{L}_p(x, \varepsilon) \neq \emptyset$. If $0 \in \mathcal{L}_p(x, \varepsilon)$, then there exists m such that $f^m x = f^{m+p}x$, and therefore $O(x, f)$ is finite, which is a contradiction. Now, assume that $y = f^p y$ and f is $\mathcal{R}_\varepsilon$-invariant, then

$$(f^{n_k}x, y) \in R_{\lambda_k},\; (f^{n_k+p}x, y) \in R_{\lambda_k}.$$

Using the symmetry and the ψ-transitivity of $\mathcal{R}$, we obtain

$$(f^{n_k}x, f^{n_k+p}x) \in R_{\psi(\lambda_k, \lambda_k)}.$$

Hence, $\psi(\lambda_k, \lambda_k) \in \mathcal{L}_p(x, \varepsilon)$ for all $k \in \mathbb{N}$. Since, the sequence $\{\lambda_k\}$ converges to zero, ψ is continuous at $(0, 0)$ and $\psi(0, 0) = 0$, we deduce that $\inf \mathcal{L}_p(x, \varepsilon) = 0$. Finally, assume that f is $\mathcal{R}_\varepsilon$-contractive and let $\tau = \inf \mathcal{L}_p(x, \varepsilon)$. If $\tau \in \mathcal{L}_p(x, \varepsilon)$, then there exists $m \in \mathbb{N}$ such that $(f^{m+p}x, f^m x) \in R_\tau$. Using the $\mathcal{R}_\varepsilon$-contractive condition, we obtain a contradiction with the minimality of τ, so $\tau \notin \mathcal{L}_p(x, \varepsilon)$. □

Lemma 3.9. *Under the hypotheses of Lemma 3.8, assume that* f *is* $\mathcal{R}_\varepsilon$*-contractive. Then, for* $y \in \omega_{\mathcal{R}}(x, f)$, *there exist a sequence* $\left\{\lambda_k\right\}_{k\in\mathbb{N}} \subseteq I$ *convergent to* 0, *and a sequence* $\left\{\alpha_k\right\}_{k\in\mathbb{N}} \subset \mathcal{L}_p(x, \varepsilon)$ *convergent to* $\tau := \inf \mathcal{L}_p(x, \varepsilon)$, *for some integer* p *such that*

$$(f^p y, y) \in R_{\psi(\lambda_k, \psi(\lambda_k, \alpha_k))}, \quad \textit{for all } k \in \mathbb{N}. \tag{6}$$

Proof. Let $y \in \omega_{\mathcal{R}}(x, f)$, $\left\{n_k\right\}$ be an increasing sequence in $\mathbb{N}$ and $\left\{\lambda_k\right\}$ be a sequence in I convergent to 0 such that

$$(f^{n_k}x, y) \in R_{\lambda_k}, \quad \text{for all } k \in \mathbb{N}.$$

By Lemma 3.8, there exists $p \in \mathbb{N}$ such that $\mathcal{L}_p(x, \varepsilon)$ is nonempty and $\tau \notin \mathcal{L}_p(x, \varepsilon)$. Therefore, there exist a decreasing sequence $\left\{\alpha_k\right\}_{k\in\mathbb{N}} \subset \mathcal{L}_p(x, \varepsilon)$ convergent to τ and $\left\{m_k\right\}_{k\in\mathbb{N}}$ a sequence in $\mathbb{N}$ such that $(f^{m_k+p}x, f^{m_k}x) \in R_{\alpha_k}$, for all $k \in \mathbb{N}$. Up to extraction of a subsequence of $\left\{n_k\right\}_{k\in\mathbb{N}}$, we may assume that $n_k \geq m_k$ for all k. Since $\mathcal{R}_\varepsilon$ is f-invariant, then by applying $f^{n_k - m_k}$, we obtain

$$(f^{n_k+p}x, f^{n_k}x) \in R_{\alpha_k}. \tag{7}$$

Using the symmetry and the ψ-transitivity, by applying Lemma 3.6 for $i = 0$ and $i = p$ combined with (7), we conclude that (6) holds. □

Proof. [**Proof of Theorem 3.4**] From Theorem 3.2, we have $\mathrm{R}(f,\mathcal{R}) = \Lambda(f,\mathcal{R}) = \Omega(f,\mathcal{R})$. Then, it suffices to prove that $\Lambda(f,\mathcal{R}) \subseteq \mathrm{P}(f,\mathcal{R})$, since the reverse inclusion is obvious. Let $y \in \Lambda(f,\mathcal{R})$, so there exists $x \in X$ such that $y \in \omega_{\mathcal{R}}(x,f)$. According to Lemma 3.7, the result holds when $O(x,f)$ is finite. Assume now that $O(x,f)$ is infinite. By Lemma 3.9, there exist an integer p, a sequence $\left\{\lambda_k\right\}_{k\in\mathbb{N}} \subseteq I$, which converges to 0, and a sequence $\left\{\alpha_k\right\}_{k\in\mathbb{N}} \subset \mathcal{L}_p(x,\varepsilon)$ convergent to $\tau := \inf \mathcal{L}_p(x,\varepsilon)$ such that

$$(f^p y, y) \in R_{\psi(\lambda_k,\psi(\lambda_k,\alpha_k))}, \quad k \in \mathbb{N}.$$

By Lemma 3.8, $\tau \notin \mathcal{L}_p(x,\varepsilon)$. According to the fact that $\left\{\alpha_k\right\} \subseteq I$ is convergent to τ, then τ is a cluster point of I. Since 0 is the unique cluster point of I, then the sequence $\left\{\alpha_k\right\}$ converges to 0. Using the continuity of ψ at $(0,0)$ and the weak-nestedness property, it follows that $(f^p y, y) \in R_0 \subseteq \Delta$, therefore $y \in \mathrm{P}(f,\mathcal{R})$. □

Proof. [**Proof of Theorem 3.5**] By Theorem 3.2, it's enough to show that $\Lambda(f,\mathcal{R}) \subseteq \mathrm{P}(f,\mathcal{R})$. Indeed, let $y \in \Lambda(f,\mathcal{R})$, so there exists $x \in X$ such that $y \in \omega_{\mathcal{R}}(x,f)$. If $O(x,f)$ is finite, then by Lemma 3.7, we get $y \in \mathrm{P}(f,\mathcal{R})$. Assume that $O(x,f)$ is infinite, by Lemma 3.9, we obtain (6). Let $\tau = \inf \mathcal{L}_p(x,\varepsilon)$. If $\tau = 0$, up to extraction of a subsequence, for $\nu_k = \psi(\lambda_k, \psi(\lambda_k,\alpha_k))$ for all $k \in \mathbb{N}$, we can suppose that the sequence $\left\{\nu_k\right\}$ is decreasing to zero. Then using the week-nestedness of $\mathcal{R}$, we deduce that $(f^p y, y) \in R_0$, then $y \in P(f,\mathcal{R})$. If $\tau \neq 0$, by lemma 3.8-(i), we have $(y, f^p y) \notin \Delta$. By condition (i), the sequence ν_k converges to $\mu = \psi(0,\psi(0,\tau))$ and by (ii), $\mu \le \tau$. Now, we distinguish two cases each of them leads to a contradiction.
Case 1. $\tau \neq 0$ and $\nu_k \ge \mu$, for infinitely many $k \in \mathbb{N}$. By taking a subsequence if necessary, we may assume that the sequence ν_k is non-increasing. Hence, from the weak-nestedness of $\mathcal{R}$ it follows that

$$(f^p y, y) \in R_\mu \setminus \Delta,$$

and from (iii), we deduce that there exists $0 \le \gamma < \mu$ such that

$$(f^{m(f^p y,y,\mu)+p} y, f^{m(f^p y,y,\mu)} y) \in R_\gamma. \tag{8}$$

Now, by Lemma 3.6, we can choose the sequences $\left\{n_k\right\}$ and $\left\{\lambda_k\right\}$, so that

$$(f^{n_k+i} x, f^i y) \in R_{\lambda_k} \setminus \Delta, \text{ for } 0 \le i \le m(f^p y, y, \mu) + p \text{ and } k \in \mathbb{N}. \tag{9}$$

And from the condition (i), $\left\{\psi(\lambda_k, \psi(\lambda_k,\gamma))\right\}_{k\in\mathbb{N}}$ converges to $\psi(0,\psi(0,\gamma))$ such that $\psi(0,\psi(0,\gamma)) \le \gamma < \mu$. Then, there exists $k_0 \in \mathbb{N}$ such that $\psi(\lambda_k,\psi(\lambda_k,\gamma)) < \mu \le \tau$, for all $k \ge k_0$. By the symmetry and the ψ-transitivity of $\mathcal{R}$, it follows from (8) and (9) for $i = m(f^p y, y, \mu) + p$ that

$$(f^{n_{k_0}+m(f^p y,y,\mu)+p} x, f^{m(f^p y,y,\mu)} y) \in R_{\psi(\lambda_{k_0},\gamma)}. \tag{10}$$

Combining (9) for $i = m(f^p y, y, \mu)$ and (10), we get

$$(f^{n_{k_0}+m(f^p y,y,\mu)+p} x, f^{n_{k_0}+m(f^p y,y,\mu)} x) \in R_{\psi(\lambda_{k_0},\psi(\lambda_{k_0},\gamma))}.$$

So $\psi(\lambda_{k_0}, \psi(\lambda_{k_0},\gamma)) \in \mathcal{L}_p(x,\varepsilon)$, and we have $\psi(\lambda_{k_0},\psi(\lambda_{k_0},\gamma)) < \tau$, then we obtain a contradiction.
Case 2. $\tau \neq 0$ and there exists k_0 such that $\nu_k < \mu$ for all $k \ge k_0$. As $(f^p y, y) \in R_{\nu_k}$ for all $k \ge k_0$. Then by (iii), there exists $\beta < \nu_{k_0}$ such that

$$(f^{m(f^p y,y,\nu_{k_0})+p} y, f^{m(f^p y,y,\nu_{k_0})} y) \in R_\beta.$$

By the symmetry and the ψ-transitivity of $\mathcal{R}$, and using (5) of Lemma 3.6 for $i = m(f^p y, y, \nu_{k_0})$ and $i = m(f^p y, y, \nu_{k_0}) + p$, we obtain

$$(f^{n_k+m(f^p y,y,\nu_{k_0})+p} x, f^{n_k+m(f^p y,y,\nu_{k_0})} x) \in R_{\psi(\lambda_k,\psi(\lambda_k,\beta))}.$$

Then $\psi(\lambda_k, \psi(\lambda_k,\beta)) \in \mathcal{L}_p(x,\varepsilon)$ for all $k \ge k_0$. Finally, from (i)-(ii), there exists $k_1 \ge k_0$ such that $\psi(\lambda_{k_1}, \psi(\lambda_{k_1},\beta)) < \tau$, which is a contradiction. □

Corollary 3.10. *Under hypotheses of Theorem 3.4 or Theorem 3.5, assume that:*

(i) $\mathrm{Gr}(f) \subseteq \bigcup_{\lambda\in I_\varepsilon} R_\lambda$, *where* $\mathrm{Gr}(f)$ *is the graph of* f.

Then $\Omega(f,\mathcal{R}) = \mathrm{Fix}(f,\mathcal{R})$. *If in addition, the following conditions hold:*

(ii) f *is* $\mathcal{R}$*-contractive.*

(iii) $\mathcal{R} \in \mathcal{J}^t(I)$ *and* $\omega_{\mathcal{R}}(x,f)$ *is nonempty for some* $x \in X$.

Then, f *has a unique* $\mathcal{R}$*-fixed point.*

Proof. By Theorem 3.4 or Theorem 3.5, we have $\Omega(f,\mathcal{R}) = \mathrm{P}(f,\mathcal{R})$. Assume that there exists $x \in \mathrm{P}(f,\mathcal{R})$ such that the period of x is $p > 1$. By (i), there exists $\lambda \in I_\varepsilon$ such that $(x, fx) \in R_\lambda$. Then, the set

$$\mathcal{S} := \left\{\lambda \in I_\varepsilon : (f^k x, f^r x) \in R_\lambda \text{ with } 0 \le k < r < p\right\}.$$

is nonempty. Let $\alpha := \inf \mathcal{S}$ and consider a non-increasing sequence $\left\{\lambda_n\right\} \subseteq \mathcal{S}$, which converges to α. By definition of $\mathcal{S}$, there exist two sequences $\left\{k_n\right\}$ and $\left\{r_n\right\}$ satisfying $0 \le k_n < r_n < p$ such that $(f^{k_n}x, f^{r_n}x) \in R_{\lambda_n}$, for all n. By taking a subsequence, we can suppose that $k_n = k$ and $r_n = r$ are constants. Using the weak-nestedness of $\mathcal{R}$, we deduce that $(f^k x, f^r x) \in R_\alpha$. Then by the $\mathcal{R}_\varepsilon$-contractive condition, it follows that $\alpha = 0$ and hence $(f^k x, f^r x) \in R_0$. Thus, we have

$$f^k x = f^r x \implies f^p x = f^{p+(r-k)} x \implies x = f^{(r-k)} x,$$

which implies that $p \le r - k$, a contradiction. Then $p = 1$ and x is a fixed point, that is, $x \in \mathrm{Fix}(f,\mathcal{R})$. Moreover, if $\omega_{\mathcal{R}}(x,f) \ne \emptyset$ for some $x \in X$ then $\mathrm{Fix}(f,\mathcal{R}) \ne \emptyset$. If, in addition, $\mathcal{R} \in \mathcal{J}^t(I)$ and $x, y \in X$ are two $\mathcal{R}$-fixed points, then the set

$$\mathcal{T} := \left\{\lambda \in I : (x,y) \in R_\lambda\right\},$$

is nonempty. Let $\beta := \inf \mathcal{T}$, so by the weak-nestedness, we obtain $(x,y) \in R_\beta$. Consequently, it follows from the $\mathcal{R}$-contractive condition that $\beta = 0$ and hence $x = y$. □

4. Some consequences in ψ-dislocated metric space.

4.1. Matkowski-Edelstein type results.

The following result is a consequence of Theorem 3.5 and extends [3, Theorem 2] of Edelstein to a class of ψ-dislocated metric spaces.

Theorem 4.1. *Let* $\delta \in \mathcal{D}(I,\psi)$ *and* $\varepsilon \in \overline{\mathbb{R}}_+ \setminus \{0\}$ *such that* f *is* δ_ε*-contractive mapping. Assume that:*

(i) ψ *is continuous at* $(0,x)$ *for all* $x \in I$.

(ii) $\psi(0,x) \le x$ *for all* $x \in I$.

Then, $\Omega(f,\delta) = \mathrm{P}(f,\delta)$. *In addition, if* 0 *is a cluster point of* I, *then the following assertions hold:*

(iii) *If for all* $x \in X$, $\delta(x,fx) < \varepsilon$, *then* $\Omega(f,\delta) = \mathrm{Fix}(f,\delta)$.

(iv) *If* f *is* δ*-contractive and there exists* $x \in X$ *such that* $\omega(x,f,\delta) \ne \emptyset$, *then* f *has a unique* δ*-fixed point.*

Proof. Since $\delta \in \mathcal{D}(I,\psi)$, by Theorem 2.2, there exists $\mathcal{R} \in \mathcal{J}^t(I) \cap \mathcal{J}^n(I,\psi)$ such that $\delta = \delta_{\mathcal{R}}$. If 0 is an isolated point of I, then by Theorem 3.1, $\Omega(f,\mathcal{R}) = \mathrm{P}(f,\mathcal{R})$. Otherwise, By Proposition 2.4, f is $\mathcal{R}_\varepsilon$-contractive, and according to Theorem 3.5, we obtain also $\Omega(f,\mathcal{R}) = \mathrm{P}(f,\mathcal{R})$. So by applying Proposition 2.3, we get the result. Moreover, if the hypothesis of (iii) is satisfied, then $\mathrm{Gr}(f) \subseteq \bigcup_{\lambda\in I_\varepsilon} R_\lambda$. We conclude by Corollary 3.10-(i) and Proposition 2.3 that $\Omega(f,\delta) = \mathrm{Fix}(f,\delta)$. Finally, if the hypothesis of (iv) is verified, we deduce the result by Corollary 3.10-(iii). □

Denote by $\overline{\Phi}(I)$ the set of all monotone functions $\varphi\colon I \to I$ satisfying

$$\lim_n \varphi^n(t) = 0, \text{ for all } t \in I.$$

Note that $\varphi(t) < t$ for all $t \in I \setminus \{0\}$. Let $t_0 \in I$, $\delta \in \mathcal{D}(I, \psi)$, $\varphi \in \overline{\Phi}(I)$ and define the following sets:

$$I_{\varphi,t_0} := \Big\{0\Big\} \cup \Big\{\lambda_n := \varphi^n(t_0) : n \in \mathbb{N}_0\Big\} \quad \textit{and} \quad \widehat{I}_{\varphi,t_0} := I_{\varphi,t_0} \cup \{\infty\}.$$

Denote $\psi_1 \colon \widehat{I}_{\varphi,t_0} \times \widehat{I}_{\varphi,t_0} \to \widehat{I}_{\varphi,t_0}$ the mapping given by

$$\psi_1(\lambda,\mu) = \begin{cases} \lambda_k & \text{if there exists } k \text{ such that } \lambda_{k+1} < \psi(\lambda,\mu) \le \lambda_k \\ 0 & \text{if } \psi(\lambda,\mu) = 0 \\ t_0 & \text{otherwise.} \end{cases}$$

We define a family of binary relations $\mathcal{R}_{(\delta,\varphi,t_0)} = \Big\{R_\lambda\Big\}_{\lambda \in \widehat{I}_{\varphi,t_0}}$ by

$$R_\infty = R_{t_0} = X \times X \text{ and } R_\lambda = \Big\{(x,y) \in X \times X : \delta(x,y) \le \lambda\Big\}, \text{ for all } \lambda \in I_{\varphi,t_0} \setminus \{t_0\}.$$

Remark 4.2. *Observe that if $\varphi(t) = 0$ for some $t > 0$, then I_{φ,t_0} is finite. Otherwise, 0 is the unique cluster point of I_{φ,t_0}.*

Lemma 4.3. *Let $t_0 \in I \setminus \{0\}$, $\varphi \in \overline{\Phi}(I)$ and $\delta \in \mathcal{D}(I,\psi)$ such that ψ is monotone. Then,*

(i) *$\mathcal{R}_{(\delta,\varphi,t_0)} \in \mathcal{R}^t(I_{\varphi,t_0}) \cap \mathcal{R}^n(I_{\varphi,t_0}, \psi_1)$.*

(ii) *for $\delta_1 = F(\mathcal{R}_{(\delta,\varphi,t_0)})$ and $x, y \in X$, we have*

$$\delta_1(x,y) = 0 \textit{ if and only if } \delta(x,y) = 0.$$

In particular, $\mathrm{P}(f,\delta_1) = \mathrm{P}(f,\delta)$ *and* $\mathrm{Fix}(f,\delta_1) = \mathrm{Fix}(f,\delta)$.

(iii) *If for some $x, y \in X$ there exists a subsequence $\Big\{f^{n_k}x\Big\}$ in $O(x,f)$ such that $\lim_{k\to+\infty} \delta(f^{n_k}x, y) = 0$, then $\Lambda(f, \mathcal{R}_{(\delta,\varphi,t_0)})$ is nonempty.*

Proof. It is clear from its definition that $\mathcal{R}_{(\delta,\varphi,t_0)}$ is nested, symmetric and belongs to $\mathcal{R}^t(I_{\varphi,t_0})$. To prove (i), we have to show that $\mathcal{R}_{(\delta,\varphi,t_0)}$ is ψ_1-transitive. Let $(x,y) \in R_\lambda$ and $(y,z) \in R_\mu$, by monotony of ψ, we have

$$\delta(x,z) \le \psi(\delta(x,y), \delta(y,z)) \le \psi(\lambda,\mu).$$

If there exists k such that $\lambda_{k+1} < \psi(\lambda,\mu) \le \lambda_k$ or $\psi(\lambda,\mu) = 0$, then $\psi(\lambda,\mu) \le \psi_1(\lambda,\mu)$ and we deduce that $(x,z) \in R_{\psi_1(\lambda,\mu)}$. Otherwise, $\psi(\lambda,\mu) > t_0$, in this case $\psi_1(\lambda,\mu) = t_0$ and we have $(x,z) \in R_{t_0} = X \times X$. To show (ii), let $x, y \in X$ such that $\delta_1(x,y) = 0$. Then,

$$0 = \inf\Big\{\lambda \in I_{\varphi,t_0} : (x,y) \in R_\lambda\Big\} = \inf\Big\{\lambda \in I_{\varphi,t_0} : \delta(x,y) \le \lambda\Big\} \ge \delta(x,y),$$

which implies that $\delta(x,y) = 0$. Conversely, if $\delta(x,y) = 0$, then $(x,y) \in R_0$ and hence $\delta_1(x,y) = 0$, by definition of δ_1. To prove (iii), since $\lim_{n\to+\infty} \delta(f^{n_k}x, y) = 0$, then by taking a subsequence if necessary, we may assume that $\delta(f^{n_k}x, y) \le \lambda_k$, for all $\lambda_k \in I_{\varphi,t_0}$, that is, $(f^{n_k}x, y) \in R_{\lambda_k}$, thus we conclude that $y \in \omega_{\mathcal{R}_{(\delta,\varphi,t_0)}}(x,f)$. □

Lemma 4.4. *Let $t_0 \in I \setminus \{0\}$, $\varepsilon \in (0, t_0)$, $\varphi \in \overline{\Phi}(I)$ and $\delta \in \mathcal{D}(I,\psi)$ such that ψ is monotone.*

(i) *If f is a δ-nonexpansive mapping such that for all $x \in X$ there exists an integer $p = p(x)$ satisfying*

$$\delta(f^p x, f^p y) \le \varphi\Big(\delta(x,y)\Big), \text{ for all } y \in X, \tag{11}$$

then f is $\mathcal{R}_{(\delta,\varphi,t_0)}$-contractive.

(ii) *If f is δ_ε-nonexpansive mapping such that for all $x \in X$ there exists an integer $p = p(x)$ satisfying*

$$\delta(x,y) < \varepsilon \implies \delta(f^p x, f^p y) \leq \varphi\Big(\delta(x,y)\Big), \text{ for all } y \in X, \tag{12}$$

then f is $(\mathcal{R}_{(\delta,\varphi,t_0)})_\varepsilon$-contractive.

Proof. We observe that if f is δ_ε-nonexpansive (resp. δ-nonexpansive), then $(\mathcal{R}_{(\delta,\varphi,t_0)})_\varepsilon$ (resp. $\mathcal{R}_{(\delta,\varphi,t_0)}$) is f-invariant .

(i): For $(x,y) \in X \times X$, consider

$$(x_0, y_0) = (x,y) \text{ and } (x_{n+1}, y_{n+1}) = (f^{p(x_n)} x_n, f^{p(x_n)} y_n).$$

Then obviously from (11), we get

$$\delta(x_{n+1}, y_{n+1}) \leq \varphi(\delta(x_n, y_n)) \leq \cdots \leq \varphi^{n+1}(\delta(x,y)).$$

Define then $m(x,y,\lambda) = \sum_{i=0}^{n(x,y)} p(x_i)$, for all $\lambda \in I$, where

$$n(x,y) = \min\Big\{k \in \mathbb{N} : \varphi^k(\delta(x,y)) \leq t_0\Big\}.$$

Note that, as $\lim\limits_{n\to+\infty} \varphi^n(t) = 0$ for all $t > 0$, the integer $n(x,y)$ is well defined. Now, assume that $(x,y) \in R_\lambda \setminus \Delta$. If $\lambda = \varphi^n(t_0)$ for some positive integer n, then $n(x,y) = 0$ and from (11), the monotony of φ and the definition of R_λ, we obtain

$$\delta(f^{m(x,y,\lambda)} x, f^{m(x,y,\lambda)} y) = \delta(f^{p(x)} x, f^{p(x)} y) \leq \varphi(\delta(x,y)) \leq \varphi^{n+1}(t_0).$$

Thus, for $\mu = \varphi^{n+1}(t_0)$, we have $\mu < \lambda$ and $(f^{m(x,y,\lambda)} x, f^{m(x,y,\lambda)} y) \in R_\mu$. If $\lambda = t_0$, then by definition of $n(x,y)$, we have

$$\delta(f^{m(x,y,\lambda)} x, f^{m(x,y,\lambda)} y) \leq \varphi^{n(x,y)+1}(\delta(x,y)) \leq \varphi(t_0).$$

Then for $\mu = \varphi(t_0)$, we have $\mu < \lambda$ and $(f^{m(x,y,\lambda)} x, f^{m(x,y,\lambda)} y) \in R_\mu$.

(ii): Let $(x,y) \in X \times X$. If $\lambda \geq \varepsilon$, then take $m(x,y,\lambda) = 1$ (or any other values in $\mathbb{N}$). Otherwise, we define $m(x,y,\lambda)$ as in (i). Now, if $(x,y) \in R_\lambda \setminus \Delta$ and $\lambda < \varepsilon$. Then there exists n such that $\lambda = \varphi^n(t_0)$ and we have $n(x,y) = 0$. From (12), the monotony of φ and the definition of R_λ, using the same argument as in (i), we obtain $(f^{m(x,y,\lambda)} x, f^{m(x,y,\lambda)} y) \in R_\mu$ for $\mu = \varphi^{n+1}(t_0)$. □
□

Theorem 4.5. *Let $\varepsilon > 0$, $\varphi \in \overline{\Phi}(I)$, $\delta \in \mathcal{D}(I,\psi)$ such that (X,δ) is complete and f is δ_ε-nonexpansive. Assume that:*

(i) *ψ is monotone and continuous at $(0,0)$ with $\psi(0,0) = 0$.*

(ii) *There exists $x_0 \in X$ such that* $\mathrm{Diam}(O(x_0,f)) < \varepsilon$, *where*

$$\mathrm{Diam}(O(x_0,f)) := \sup\Big\{\delta(y,z) : y,z \in O(x_0,f)\Big\}.$$

(iii) *For all $x \in X$ there exists an integer $p = p(x)$ such that for all $y \in X$ satisfying $\delta(x,y) < \varepsilon$, we have*

$$\delta(f^p x, f^p y) \leq \varphi\Big(\delta(x,y)\Big). \tag{13}$$

Then f has a δ-periodic point. Moreover,

(a) *If* $\sup\Big\{\delta(x,fx) : x \in X\Big\} \leq \varphi(\varepsilon)$, *then f has a δ-fixed point.*

(b) *If f is δ-nonexpansive and* (13) *is satisfied for all $x, y \in X$, then f has a unique δ-fixed point.*

Proof. Fix a positive real t_0 such that $t_0 > \varepsilon$. By Lemma 4.3-(i), $\mathcal{R}_{(\delta,\varphi,t_0)} \in \mathcal{R}^n(I_{\varphi,t_0}, \psi_1)$, and from Lemma 4.4-(ii), we see that f is $(\mathcal{R}_{(\delta,\varphi,t_0)})_\varepsilon$-contractive. If 0 is the unique cluster point of I_{φ,t_0}, by Proposition 2.3-(iii), Lemma 4.3-(ii) and Theorem 3.4, we have $\Lambda(f, \mathcal{R}_{(\delta,\varphi,t_0)}) = \mathrm{P}(f,\delta)$. If 0 is an isolated point of I_{φ,t_0}, then by Proposition 2.3-(iii), Lemma 4.3-(ii) and Theorem 3.1, we have also $\Lambda(f, \mathcal{R}_{(\delta,\varphi,t_0)}) = \mathrm{P}(f,\delta)$. To conclude that f has a δ-periodic point, it is enough to show that the set $\Lambda(f, \mathcal{R}_{(\delta,\varphi,t_0)})$ is nonempty. Let $\{x_n\}$ be the sequence defined by $x_{n+1} = f^{p(x_n)}x_n$. Now, we shall prove that $\{x_n\}$ is convergent. Denote $p_n = p(x_n)$ and for $k, n \in \mathbb{N}$ let $s_{k,n} = \sum_{i=n}^{n+k-1} p_i$. By condition (i), we obtain

$$\begin{aligned}
\delta(x_n, x_{n+k}) = \delta(x_n, f^{s_{n,k}}x_n) &= \delta(f^{p_{n-1}}x_{n-1}, f^{s_{n,k}}f^{p_{n-1}}x_{n-1}) \\
&\le \varphi\Big(\delta(x_{n-1}, f^{s_{n,k}}x_{n-1})\Big) \\
&\le \varphi\Big(\delta(f^{p_{n-2}}x_{n-2}, f^{p_{n-2}}f^{s_{n,k}}x_{n-2})\Big) \\
&\le \varphi^2\Big(\delta(x_{n-2}, f^{s_{n,k}}x_{n-2})\Big) \\
&\;\;\vdots \\
&\le \varphi^n\Big(\delta(x_0, f^{s_{n,k}}x_0)\Big) \\
&\le \varphi^n(\varepsilon)
\end{aligned}$$

Then $\{x_n\}$ is a Cauchy sequence. By completeness, the sequence $\{x_n\}$ is convergent. It follows from Lemma 4.3-(iii) that $\Lambda(f, \mathcal{R}_{(\delta,\varphi,t_0)}) \neq \emptyset$. Assume now that the assumption of (a) is satisfied. As $\varepsilon < t_0$, there exists a non-negative integer n such that

$$\varphi^{n+1}(t_0) \le \varepsilon < \varphi^n(t_0).$$

Then, by monotony of φ, we obtain

$$\delta(x, fx) \le \varphi(\varepsilon) \le \varphi^{n+1}(t_0), \quad x \in X.$$

It comes out that, for $I_\varepsilon := I_{\varphi,t_0} \cap (0, \varepsilon)$, we have

$$Gr(f) \subseteq R_{\varphi^{n+1}(t_0)} \subseteq \bigcup_{\lambda \in I_\varepsilon} R_\lambda.$$

Thus we conclude by Proposition 2.3-(ii) and Corollary 3.10-(i) that $\Lambda(f, \mathcal{R}_{(\delta,\varphi,t_0)})$ is equal to $\mathrm{Fix}(f,\delta)$ and hence f has a δ-fixed point. Finally, if the hypothesis of (b) holds, then from Lemma 4.4-(i), we deduce that f is $\mathcal{R}_{(\delta,\varphi,t_0)}$-contractive. Since $\mathcal{R}_{(\delta,\varphi,t_0)} \in \mathcal{R}^t(I_{\varphi,t_0})$, we conclude that f has a unique δ-fixed point, by Corollary 3.10. □

4.2. *Some fixed point results*

If I is bounded, then the condition (ii) of Theorem 4.5 is obviously satisfied. In the context where I is not bounded, we state the following upshot.

Corollary 4.6. *Let $\delta \in \mathcal{D}(I, \psi)$ such that (X, δ) is complete, $\varphi \in \overline{\Phi}(I)$ and f be a δ-nonexpansive mapping. If the following conditions hold:*

(i) *ψ is monotone and continuous at $(0,0)$ with $\psi(0,0) = 0$.*

(ii) *For all $h \in I$ there exists $c \in I$, $c > h$ such that*

$$t > c \implies t > \psi(h, \varphi(t)).$$

(iii) *For all $x \in X$ there exists an integer $p = p(x)$ such that for all $y \in X$, we have*

$$\delta(f^p x, f^p y) \leq \varphi\big(\delta(x, y)\big).$$

Then f has a unique δ-fixed point.

Proof. Note that the conditions (i) and (b) of Theorem 4.5 are satisfied. Let $x \in X$, to conclude, we have to show that $\mathrm{Diam}\big(O(x, f)\big) < M$, for some $M > 0$. If I is bounded, then obviously $\mathrm{Diam}\big(O(x, f)\big)$ is finite. Otherwise, define

$$u_k = \delta(x, f^k x) \text{ and } h = \max\big\{u_0, \ldots, u_p\},$$

where $p = p(x)$. By (i), it follows that there exists $c > h$ such that

$$t > c \implies t > \psi(h, \varphi(t)).$$

Assume that there exists $k > p$ such that $u_k > c$. Let $j > p$ be the smallest integer such that $u_j > c$. Consider $q \geq 0$ and $0 \leq r < p$ such that $j = pq + r$. Hence, by (iii), the monotony of ψ and φ,

$$\begin{aligned} u_j = \delta(x, f^j x) &\leq \psi\Big(\delta(x, f^p x), \delta(f^p x, f^p f^{(q-1)p+r} x)\Big) \\ &\leq \psi\Big(\delta(x, f^p x), \varphi(\delta(x, f^{(q-1)p+r} x))\Big) \\ &= \psi\Big(u_p, \varphi(u_{(q-1)p+r})\Big) \\ &\leq \psi\Big(h, \varphi(u_j)\Big), \end{aligned}$$

that is, $u_j \leq \psi\big(h, \varphi(u_j)\big)$ which a contraction. Then $u_j \leq c$ for all $j \geq 0$. Therefore, $\delta(x, f^j x) < c$ for all $j \geq 0$. Now, by the monotony of ψ, for all i, j, we have

$$\delta(f^i x, f^j x) \leq \psi\big(u_i, u_j\big) \leq \psi(c, c).$$

Thus, the orbit of x is bounded. □

Remark 4.7. *In Corollary 4.6 , if δ is a dislocated metric, then the condition* (ii) *is equivalent to $\lim_{t \to +\infty} t - \varphi(t) = +\infty$. Consequently, the next results extend Matkowski fixed point [7, Theorem 2], to the context of dislocated metric spaces, under some additional conditions.*

Corollary 4.8. *Let (X, δ) be a complete dislocated metric space, f be a δ-nonexpansive mapping and $\varphi \in \overline{\Phi}(\mathbb{R}_+)$. Assume that:*

(i) $\lim_{t \to +\infty} t - \varphi(t) = +\infty$.

(ii) *for all $x \in X$ there exists an integer $p = p(x)$ such that for all $y \in X$, we have*

$$\delta(f^p x, f^p y) \leq \varphi\big(\delta(x, y)\big).$$

Then f has a unique δ-fixed point.

It has been proved in [2] that any semimetric $d \colon X \times X \to \mathbb{R}_+$ has a monotone triangular function $\psi \colon \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$. The following result generalizes [2, Theorem 1] to the context of ψ-dislocated metric space.

Corollary 4.9. *Let $\varphi \in \overline{\Phi}(I)$, $\delta \in \mathcal{D}(I, \psi)$ such that (X, δ) is complete. Assume that:*

(i) *ψ is monotone and continuous at $(0, 0)$ with $\psi(0, 0) = 0$.*

(ii) $\delta(fx, fy) \leq \varphi\Big(\delta(x, y)\Big)$ *for all* $x, y \in X$.

Then f *has a unique* δ*-fixed point.*

Proof. From Theorem 4.5, it suffices to shows that the orbit of some $x \in X$ is bounded. Let $x \in X$, $x_0 = x$ and $x_{n+1} = fx_n$, then from the monotony of ψ, we deduce

$$\delta(x_{n+k}, x_n) \leq \varphi^n\Big(\delta(f^k x_0, x_0)\Big), \quad n, k \in \mathbb{N}. \tag{14}$$

Let $\varepsilon > 0$, by continuity of ψ there exist an integer n_ε and $\eta(\varepsilon) > 0$ such that

$$\varphi^{n_\varepsilon}(\varepsilon) < \eta(\varepsilon) \text{ and } \psi(u, v) < \varepsilon, \text{ for all } u, v \leq \eta(\varepsilon).$$

Similarly, there exists $n_0 > 0$ such that

$$\varphi^{n_0}(\delta(x_0, f^k x_0)) < \min\Big\{\eta(\varepsilon), \varepsilon\Big\}, \text{ for all } k = 0, \dots, n_\varepsilon. \tag{15}$$

Let $B(x_{n_0}, \varepsilon)$ the ball of center x_{n_0} and radius ε. Then from (14) and (15), we have $x_{n_0+k} \in B(x_{n_0}, \varepsilon)$ for all $k = 0, \dots, n_\varepsilon - 1$. For $z \in B(x_{n_0}, \varepsilon)$, we have

$$\begin{aligned} \delta(f^{n_\varepsilon} z, x_{n_0}) &\leq \psi\Big(\delta(f^{n_\varepsilon} z, f^{n_\varepsilon} x_{n_0}), \delta(f^{n_\varepsilon} x_{n_0}, x_{n_0})\Big) \\ &\leq \psi\Big(\varphi^{n_\varepsilon}(\delta(z, x_{n_0})), \varphi^{n_0}(\delta(f^{n_\varepsilon} x_0, x_0))\Big) \\ &\leq \psi\Big(\eta(\varepsilon), \eta(\varepsilon)\Big) < \varepsilon \end{aligned}$$

In particular, $f^{n_\varepsilon}\Big(B(x_{n_0}, \varepsilon)\Big) \subseteq B(x_{n_0}, \varepsilon)$. Now, for any positive integer $m = qn_\varepsilon + k$, $0 \leq k < n_\varepsilon$, we have

$$x_{n_0+m} = f^m x_{n_0} = f^{qn_\varepsilon} f^k x_{n_0} = f^{qn_\varepsilon} x_{n_0+k} \in f^{qn_\varepsilon}\Big(B(x_{n_0}, \varepsilon)\Big) \subseteq B(x_{n_0}, \varepsilon).$$

Consequently, $\delta(x_{n_0+m}, x_{n_0}) < \varepsilon$, for all $m > 0$, so x_n is a Cauchy sequence. Then $O(x, f)$ is bounded. □

Finally, if (X, δ) is a complete dislocated metric space, then the condition (i) of Corollary 4.9 is obviously satisfied, and we obtain a generalized version of the Matkowski result [6, Theorem 1.2] to the context of dislocated metric space.

Corollary 4.10. *Let* $\varphi \in \overline{\Phi}(\mathbb{R}_+)$ *and* (X, δ) *be a complete dislocated metric space. Assume that*

$$\delta(fx, fy) \leq \varphi\Big(\delta(x, y)\Big), \text{ for all } x, y \in X.$$

Then f *has a unique fixed point.*

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