Filomat 39:1 (2025), 313–324
https://doi.org/10.2298/FIL2501313T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Abilov's inequalities in the Laguerre hypergroup

Othman Tyr[a,*], Larbi Rakhimi[b]

[a]*Laboratory of Sciences and Research Technology (LSRT), The Higher School of Education and Training, University Ibnou Zohr, Agadir, Morocco*
[b]*Department of Mathematics, Faculty of Sciences Aïn Chock, University Hassan II, Casablanca, Morocco*

Abstract. Let $(\mathbb{K}, *_\alpha)$ be the Laguerre hypergroup where $\mathbb{K} = [0, +\infty) \times \mathbb{R}$ and $*_\alpha$ a convolution product on $\mathbb{K}$ coming from the product formula satisfied by the Laguerre functions. In this work, we give new estimates for the Laguerre kernel. We obtain new inequalities for the Fourier-Laguerre transform in the space $L^2_\alpha(\mathbb{K})$, by using a generalized translation operator to prove these estimates in certain classes of functions characterized by a generalized continuity modulus.

1. Introduction and Preliminaries

Let $f : \mathbb{R} \to \mathbb{C}$ be a square-integrable function in Lebesgue's sense over $\mathbb{R}$ ($f \in L^2(\mathbb{R})$). Let us introduce the finite differences of the higher orders $k \in \mathbb{N}$ by

$$\Delta_h^k(f;x) = (F_h - E)^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} F_h^i f(x), \quad x \in \mathbb{R},$$

where F_h is the operator defined by

$$F_h f(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt, \quad h > 0, \tag{1}$$

and E is the unit operator in $L^2(\mathbb{R})$.

For a given positive real number δ, the k^{th}-order generalized continuity modulus is defined for f by

$$\Omega_k(f,\delta) = \sup_{0<h\leq\delta} \|\Delta_h^k(f;x)\|_{L^2(\mathbb{R})}.$$

Let $W^{r,k}_{2,\Phi}(D)$, $(r = 0, 1, \ldots, k = 1, 2, \ldots)$ denote the class of functions $f \in L^2(\mathbb{R})$ having the generalized partial derivatives

$$\frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots, \frac{\partial^r f}{\partial x^r}$$

2020 *Mathematics Subject Classification*. Primary 43A62 ; Secondary 42B35, 26A16.
Keywords. Laguerre Hypergroup, Fourier-Laguerre transform, generalized translation operator, Abilov's estimates.
Received: 11 November 2023; Revised: 19 July 2024; Accepted: 04 Decenber 2024
Communicated by Miodrag Spalević
* Corresponding author: Othman Tyr
Email addresses: otywac@hotmail.fr (Othman Tyr), rakhimilarbi@gmail.com (Larbi Rakhimi)
ORCID iDs: https://orcid.org/0000-0002-7261-8809 (Othman Tyr), https://orcid.org/0000-0002-1253-3010 (Larbi Rakhimi)

in the sense of Levi (see [15, p. 172]) that belong to $L^2(\mathbb{R})$. They are estimated by

$$\Omega_k(D^r f,\delta) = O(\Phi(\delta^k)) \quad \text{as} \quad \delta \to 0,$$

where $D = \dfrac{\partial}{\partial x}$,

$$D^0 f = f, \quad D^i f = D(D^{i-1}f), \quad i = 1,2,\ldots,r,$$

and Φ is a steadily increasing continuous function on $[0,+\infty)$ with $\Phi(0) = 0$.

The following theorem is an analogue of Jackson's direct theorem in the classical theory of approximation of function (see [15, Ch. 5]).

Theorem 1.1. *[1] It holds that*

$$\sup_{f\in W^{r,k}_{2,\Phi}(D)} \sqrt{\int_{|\lambda|\geq N} |\widehat{f}(\lambda)|^2 d\lambda} = O\left(N^{-r}\Phi\left[\left(\frac{2}{N}\right)^k\right]\right),$$

as $N \to +\infty$, where $r = 0,1,\ldots$, $k = 1,2,\ldots$, and $\widehat{f}$ stands for the Fourier transform of f.

In the case where $\Phi(t) = t^\nu, \nu > 0$, Abilov and al. characterized the functions $f \in L^2(\mathbb{R})$ by the following equivalence:

Theorem 1.2. *[1] Let $\Phi(t) = t^\nu (\nu > 0)$. Then,*

$$\sqrt{\int_{|\lambda|\geq N} |\widehat{f}(\lambda)|^2 d\lambda} = O(N^{-r-k\nu}) \ \text{ as } \ N \to +\infty \Leftrightarrow f \in W^{r,k}_{2,t^\nu}(D),$$

where $r = 0,1,\ldots$, $k = 1,2,\ldots$, and $0 < \nu < 2$.

Considerable attention has been devoted to discovering generalizations to new contexts for Theorem 1.1 and Theorem 1.2. In [2], the authors proved these estimates for the classical Fourier transform in the space of multivariate square integrable functions on certain classes of functions characterized by the generalized continuity modulus. We emphasize that these estimates have been generalized in [5] to the multidimensional case for the Fourier-transform in the space $L^2(\mathbb{R}^n)$, using the spherical mean operator instead of the operator defined by (1). Recently, it has also been extended in the case of noncompact rank 1 Riemannian symmetric spaces for the Helgason Fourier transform [6]. An extension of these estimates using different differential operators has been given, where considering generalized Fourier transforms: Fourier-Bessel transform [3], Cherednik-Opdam transform [8], q-Dunkl transform [7, 20] and Clifford-Fourier transform [17], Jacobi–Dunkl Expansions [18, 19], etc.

In the following, we denote $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$, the sets of non-negative integers, real numbers and complex numbers respectively and $\mathbb{K} = [0,+\infty[\times\mathbb{R}$.

In our current research, we are interested in the Laguerre hypergroup $\mathbb{K} = [0,+\infty) \times \mathbb{R}$ which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group [9]. The aim is to generalize these estimates in the framework of the Laguerre hypergroup, and establish some new results by means of the Fourier-Laguerre analysis for some classes of functions characterized by a generalized modulus of continuity, using the basic properties of Fourier-Laguerre transform.

In this paper, we consider the following partial differential operators

$$\begin{cases} \mathcal{D} = \dfrac{\partial}{\partial t}, \\ \mathcal{L} = \dfrac{\partial^2}{\partial x^2} + \dfrac{2\alpha+1}{x}\dfrac{\partial}{\partial x} + x^2\dfrac{\partial^2}{\partial t^2}, \end{cases}$$

with $(x,t) \in \mathbb{K}$ and $\alpha \geq 0$.

For $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, the initial value problem

$$\begin{cases} \mathcal{D}u = i\lambda u, \\ \mathcal{L}u = -4|\lambda|\left(m + \dfrac{\alpha+1}{2}\right)u, \\ u(0,0) = 1, \ \dfrac{\partial u}{\partial x}(0,t) = 0 \ \text{ for all } \ t \in \mathbb{R}, \end{cases}$$

has a unique solution $\varphi_{\lambda,m}$ given by

$$\forall (x,t) \in \mathbb{K}, \ \varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathfrak{L}_m^{(\alpha)}(|\lambda| x^2), \tag{2}$$

where $\mathfrak{L}_m^{(\alpha)}$ is the Laguerre function defined on $\mathbb{R}_+$ by

$$\mathfrak{L}_m^{(\alpha)}(x) = e^{-\frac{x}{2}} \frac{L_m^{\alpha}(x)}{L_m^{\alpha}(0)}, \tag{3}$$

and L_m^{α} is the Laguerre polynomial of degree m and order α, given by

$$L_m^{\alpha}(x) = \sum_{k=0}^{m} (-1)^k \frac{\Gamma(m+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{1}{k!(m-k)!} x^k. \tag{4}$$

Let $\alpha \geq 0$ be a fixed number and m_α the weighted Lebesgue measure on $\mathbb{K}$, given by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}}{\pi\Gamma(\alpha+1)} dxdt. \tag{5}$$

For $(x,t) \in \mathbb{K}$, the generalized translation operator $T_{(x,t)}^{(\alpha)}$ is defined for $\alpha = 0$ by

$$T_{(x,t)}^{(\alpha)}(f)(y,s) = \frac{1}{2\pi} \int_0^{2\pi} f(\sqrt{x^2 + y^2 + 2xy\cos(\theta)}, t + s + xy\sin(\theta))d\theta,$$

and for $\alpha > 0$ by

$$\begin{aligned} &T_{(x,t)}^{(\alpha)}(f)(y,s) \\ &= \frac{\alpha}{\pi} \int_0^{2\pi} \int_0^1 f(\sqrt{x^2 + y^2 + 2xyr\cos(\theta)}, t + s + xyr\sin(\theta)) r(1-r^2)^{\alpha-1} drd\theta. \end{aligned}$$

Let $M_b(\mathbb{K})$ denote the space of bounded Radon measures on $\mathbb{K}$. The convolution on $M_b(\mathbb{K})$ is defined by (see [14, Definition I.2])

$$(\mu_1 *_\alpha \mu_2)(f) = \int_{\mathbb{K}\times\mathbb{K}} T_{(x,t)}^{(\alpha)}(f)(y,s) d\mu_1(x,t) d\mu_2(y,s).$$

This convolution is commutative. If $f, g \in L_\alpha^1(\mathbb{K})$ and $\mu_1 = fm_\alpha$, $\mu_2 = gm_\alpha$, then $\mu_1 *_\alpha \mu_2 = (f *_\alpha g)m_\alpha$, where $f *_\alpha g$ is the convolution of functions f and g, defined by (see [14, Proposition I.2])

$$(f *_\alpha g)(x,t) = \int_{\mathbb{K}} T_{(x,t)}^{(\alpha)}(f)(y,s) g(y,-s) dm_\alpha(y,s).$$

For all $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, the kernel $\varphi_{\lambda,m}$ verifies the following product formula (see [14, Proposition II.2])

$$\varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,s) = T_{x,t}^{(\alpha)}(\varphi_{\lambda,m})(y,s), \ \ (x,t),(y,s) \in \mathbb{K},$$

and has the property

$$\forall (\lambda, m) \in \mathbb{R} \times \mathbb{N}, \ \sup_{(x,t)\in\mathbb{K}} |\varphi_{\lambda,m}(x,t)| = 1. \tag{6}$$

We use the following notations:

- $|x,t| = |(x,t)|_{\mathbb{K}} = (x^4 + 4t^2)^{\frac{1}{4}}$ is the homogeneous norm on $\mathbb{K}$ with respect to the family of dilations $(\delta_r)_{r>0}$, $\delta_r(x,t) = (rx, r^2t)$ (cf. [16, Formula 1.17]).
- $|\lambda, m| = |(\lambda, m)|_{\mathbb{R}\times\mathbb{N}} = 4|\lambda|(m + \frac{\alpha+1}{2})$ is the quasinorm on $\mathbb{R}\times\mathbb{N}$ (cf. [16]).
- $\mathbb{B}_r$ is the ball centered on 0 and of radius r, defined by

$$\mathbb{B}_r = \{(\lambda, m) \in \mathbb{R}\times\mathbb{N};\ |\lambda, m| < r\} \text{ and } \mathbb{B}_r^c = (\mathbb{R}\times\mathbb{N})\backslash\mathbb{B}_r.$$

- $L^p_\alpha(\mathbb{K}) = L^p(\mathbb{K}, dm_\alpha), p \in [1,+\infty]$, the space of measurable functions $f : \mathbb{K} \to \mathbb{C}$, such that $||f||_{p,\alpha} < +\infty$, where

$$||f||_{p,\alpha} = \left(\int_{\mathbb{K}} |f(x,t)|^p dm_\alpha(x,t)\right)^{1/p} \text{ if } p \in [1,+\infty[,$$

$$||f||_{\infty,\alpha} = \underset{(x,t)\in\mathbb{K}}{\text{ess sup}}|f(x,t)|.$$

- $L^p_\alpha(\mathbb{R}\times\mathbb{N}), p \in [1,+\infty]$, the space of measurable functions $g : \mathbb{R}\times\mathbb{N} \to \mathbb{C}$, such that $||g||_{L^p_\alpha} < +\infty$, where

$$||g||_{L^p_\alpha} = \left(\int_{\mathbb{R}\times\mathbb{N}} |g(\lambda,m)|^p d\gamma_\alpha(\lambda,m)\right)^{1/p} \text{ if } p \in [1,+\infty[,$$

$$||g||_{L^\infty_\alpha} = \underset{(\lambda,m)\in\mathbb{R}\times\mathbb{N}}{\text{ess sup}} |g(\lambda,m)|,$$

  where $d\gamma_\alpha$ is the positive measure defined on $\mathbb{R}\times\mathbb{N}$ by (see [14])

$$\int_{\mathbb{R}\times\mathbb{N}} g(\lambda,m)d\gamma_\alpha(\lambda,m) = \sum_{m=0}^{+\infty} L^\alpha_m(0)\int_{\mathbb{R}} g(\lambda,m)|\lambda|^{\alpha+1}d\lambda.$$

The Fourier-Laguerre transform of a function in $L^1_\alpha(\mathbb{K})$ is given by

$$\mathcal{F}_L(f)(\lambda,m) = \int_{\mathbb{K}} f(x,t)\varphi_{-\lambda,m}(x,t)dm_\alpha(x,t),\ \ (\lambda,m) \in \mathbb{R}\times\mathbb{N}.$$

It is well known that the Fourier-Laguerre transform $\mathcal{F}_L$ satisfies the following properties (see [14]).

- We have the following Plancherel formula:

$$||f||_{2,\alpha} = ||\mathcal{F}_L(f)||_{L^2_\alpha} \text{ for } f \in L^1_\alpha(\mathbb{K}) \cap L^2_\alpha(\mathbb{K}).$$

- We also have the inverse formula of the Fourier-Laguerre transform:

$$f(x,t) = \int_{\mathbb{R}\times\mathbb{N}} \mathcal{F}_L(f)(\lambda,m)\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m),\ \ (x,t) \in \mathbb{K},$$

  provided $\mathcal{F}_L(f) \in L^1_\alpha(\mathbb{R}\times\mathbb{N})$.

- For all $f \in L^1_\alpha(\mathbb{K})$ and $(x,t) \in \mathbb{K}$, we have

$$\mathcal{F}_L(T^{(\alpha)}_{(x,t)}f)(\lambda,m) = \varphi_{\lambda,m}(x,t)\mathcal{F}_L(f)(\lambda,m),\ \ (\lambda,m) \in \mathbb{R}\times\mathbb{N}. \tag{7}$$

- For $f \in L^p_\alpha(\mathbb{K}), p \in [1,+\infty]$, we have $T^{(\alpha)}_{(x,t)}(f) \in L^p_\alpha(\mathbb{K})$ and

$$||T^{(\alpha)}_{(x,t)}(f)||_{p,\alpha} \le ||f||_{p,\alpha}.$$

Now, we define the finite differences of order $k \in \mathbb{N}$ and step $(x,t) \in \mathbb{K}$ by

$$\Delta^k_{(x,t)} f(y,s) = (T^{(\alpha)}_{(x,t)} - I)^k f(y,s), \tag{8}$$

where I denotes the unit operator on $\mathbb{K}$ and $(x,t) \neq (0,0)$.

Remark 1.3. *For all $k \in \mathbb{N}$, we have*

$$\Delta^k_{(x,t)} f(y,s) = \sum_{0\le i\le k} (-1)^{k-i} \binom{k}{i} (T^{(\alpha)}_{(x,t)})^i f(y,s). \tag{9}$$

Lemma 1.4. *For a fixed $(x,t) \in \mathbb{K}$ with $(x,t) \neq (0,0)$, we have*

$$\mathcal{F}_L(\Delta^k_{(x,t)} f)(\lambda, m) = (\varphi_{\lambda,m}(x,t) - 1)^k \mathcal{F}_L(f)(\lambda, m), \tag{10}$$

for all $k \in \mathbb{N}$.

Proof. The proof follows immediately from (7) and an iteration for k □

The k^{th} order generalized modulus of continuity of the function $f \in L^2_\alpha(\mathbb{K})$ is defined as

$$\Omega_k(f,\delta) = \sup_{0<|x,t|\le\delta} \|\Delta^k_{(x,t)} f\|_{2,\alpha}. \tag{11}$$

Let $W^{r,k}_{2,\phi}(\mathcal{L})$ denote the class of functions $f \in L^2_\alpha(\mathbb{K})$ that have generalized derivatives satisfying the estimate

$$\Omega_k(\mathcal{L}^r f, \delta) = O\left(\phi\left(\delta^k\right)\right), \quad \delta \to 0,$$

i.e:

$$W^{r,k}_{2,\phi}(\mathcal{L}) = \{f \in L^2_\alpha(\mathbb{K}) / \ \mathcal{L}^r f \in L^2_\alpha(\mathbb{K}) \ \text{ and } \ \Omega_k(\mathcal{L}^r f, \delta) = O\left(\phi\left(\delta^k\right)\right),\ \delta \to 0\}, \tag{12}$$

where ϕ is any continuous nonnegative function given on $[0,\infty)$. For the Laguerre operator $\mathcal{L}$, we have $\mathcal{L}^0 f = f$, $\mathcal{L}^r f = \mathcal{L}\left(\mathcal{L}^{r-1} f\right)$, $r = 1, 2, \ldots$
From ([**?** , Remark 1]), we obtain

$$\|\Delta^k_{(x,t)}(\mathcal{L}^r f)\|^2_{2,\alpha} = \int_{\mathbb{R}\times\mathbb{N}} |\varphi_{\lambda,m}(x,t) - 1|^{2k} |\lambda, m|^{2r} |\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda, m), \tag{13}$$

where $r = 0, 1, \ldots, k$.

2. Main results

In this Section, taking into account what has been presented in the previous Section, for some classes of functions characterized by the generalized modulus of continuity, we can prove two estimates for the integral:

$$\int_{\mathbb{B}^c_N} |\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m),$$

which are useful in applications.

In the remainder of this paper, we refer to $c_1, c_2, c_3, \ldots$, as positive constants which are generally different in different places and which may depend on k, r, α and other inessential parameters.

To prove the main results, we need to rely on some preliminary results.

Lemma 2.1. *For all $x > 0$ and $t \in \mathbb{R}$, we have*

$$\lim_{|\lambda,m|\to+\infty} \varphi_{\lambda,m}(x,t) = 0. \tag{14}$$

Proof. See [12, Lemma 4.3]. □

Lemma 2.2. *The following assertions are verified:*

(1) *There exist $c_1 > 0$ such that for all $(\lambda, m) \in \mathbb{R}\times\mathbb{N}$ and $(x,t) \in \mathbb{K}$,*

$$|\varphi_{\lambda,m}(x,t) - 1| \le c_1|\lambda,m||x,t|. \tag{15}$$

(2) *There exist $c_2 > 0$ such that for all $(\lambda, m) \in \mathbb{B}_N^c$ and $(x,t) \in \mathbb{K}$,*

$$|\varphi_{\lambda,m}(x,t)| \le c_2(|\lambda,m|x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}. \tag{16}$$

Proof. **(1)** From [9, Proposition 7], we deduce that for every $(\lambda, m) \in \mathbb{R}\times\mathbb{N}$ and $(x,t) \in \mathbb{K}$, we have

$$\varphi_{\lambda,m}(x,t) = 1 + i\lambda t - \frac{|\lambda,m|}{4(\alpha+1)}x^2 + \frac{|\lambda,m|^2}{16}\mathcal{R}^{\alpha}_{\lambda,m}(x,t), \tag{17}$$

with

$$|\mathcal{R}^{\alpha}_{\lambda,m}(x,t)| \le (4 + |\lambda,m|)\left(1 + (x^2+|t|)^2 + t^2(x^2+|t|)\right). \tag{18}$$

Therefore, we obtain

$$\begin{aligned}&|\varphi_{\lambda,m}(x,t) - 1|^2\\ &= |\lambda t|^2 + \frac{|\lambda,m|^2x^4}{16(\alpha+1)^2} + \frac{|\lambda,m|^4}{256}(\mathcal{R}^{\alpha}_{\lambda,m}(x,t))^2 - \frac{|\lambda,m|^3x^2}{32(\alpha+1)}\mathcal{R}^{\alpha}_{\lambda,m}(x,t).\end{aligned}$$

Consequently, we deduce the behavior in 0 of the characters $\varphi_{\lambda,m}(x,t)$ by the following relation

$$|\varphi_{\lambda,m}(x,t) - 1|^2 = |\lambda t|^2 + \frac{|\lambda,m|^2x^4}{16(\alpha+1)^2} + o(|\lambda|^2|x,t|^4).$$

In consequence, there exist $C > 0$ and $\eta > 0$ such that for all $(x,t) \in \mathbb{K}$,

$$|\lambda,m||x,t|^2 < \eta \Rightarrow |\varphi_{\lambda,m}(x,t) - 1|^2 \le C|\lambda,m|^2|x,t|^2. \tag{19}$$

Then, we have

$$|\lambda,m||x,t|^2 < \eta \Rightarrow |\varphi_{\lambda,m}(x,t) - 1| \le \sqrt{C}|\lambda,m||x,t|.$$

On the other hand, it follows from Lemma 2.1 that

$$\frac{|\varphi_{\lambda,m}(x,t) - 1|^2}{|\lambda,m|^2|x,t|^2} \to 0 \quad \text{as} \quad |\lambda,m| \to +\infty.$$

Hence, there exist $c > 0$ and $A > 0$, such that

$$|\lambda,m| > A \Rightarrow |\varphi_{\lambda,m}(x,t) - 1|^2 \le c|\lambda,m|^2|x,t|^2. \tag{20}$$

If $\frac{\eta}{|x,t|^2} < A$. Take

$$M = \max_{\frac{\eta}{|x,t|^2}\le|\lambda,m|\le A} \frac{|\varphi_{\lambda,m}(x,t) - 1|^2}{|\lambda,m|^2|x,t|^2}.$$

Therefore for all $(\lambda, m) \in \mathbb{B}^c_{\frac{\eta}{|x,t|^2}}$, we have

$$|\varphi_{\lambda,m}(x,t) - 1| \leq k|\lambda, m||x,t|,$$

where $k = \min(\sqrt{c}, \sqrt{M})$. Hence we have the result where $c_1 = \max(\sqrt{C}, k)$.

(2) From [10, Page 87], we have the asymptotic formula

$$L_m^\alpha(x) \approx \frac{\Gamma(m+\alpha+1)}{m!} e^{x/2} \left(\left(m + \frac{\alpha+1}{2}\right)x\right)^{-\frac{\alpha}{2}} J_\alpha\left(2\sqrt{\left(m+\frac{\alpha+1}{2}\right)x}\right), \tag{21}$$

as $m \to +\infty$. On the other hand, it was shown in [3] and also in [21, p. 355], the following estimate

$$\sqrt{x}J_p(x) = O(1), \quad x \geq 0; \tag{22}$$

where $J_p(x)$ is the Bessel function of the first kind (see [4]). Therefore, it follows from (3), (21) and (22) that

$$\mathfrak{L}_m^\alpha(|\lambda|x^2) = O\left((|\lambda, m|x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}\right).$$

Thus the proof is completed. □

Theorem 2.3. *Given ϕ, r, k and $f \in \mathrm{W}^{r,k}_{2,\phi}(\mathcal{L})$. Then there exists a constant $c_3 > 0$ such that the following inequality holds, for all $N > 0$*

$$\int_{\mathbb{B}^c_N} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m) = O\left(N^{-2r}\left(\phi\left(c_3 N^{-k}\right)\right)^2\right), \tag{23}$$

as $N \to +\infty$, where the constant in the O-symbol depends only on r, k, α.

Proof. For a given $f \in \mathrm{W}^{r,k}_{2,\phi}(\mathcal{L})$ and $N > 0$, we have

$$\int_{\mathbb{B}^c_N} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m) \leq I_1 + I_2, \tag{24}$$

where

$$I_1 = \int_{\mathbb{B}^c_N} |\varphi_{\lambda,m}(x,t)||\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m),$$

and

$$I_2 = \int_{\mathbb{B}^c_N} |\varphi_{\lambda,m}(x,t) - 1||\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m).$$

From (16) of Lemma 2.2, we have

$$I_1 \leq c_2(Nx^2)^{-\frac{\alpha}{2}-\frac{1}{4}} \int_{\mathbb{B}^c_N} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m). \tag{25}$$

By combining the relations (24) and (25) and by choosing a constant c_4 such that the number $c_5 = 1 - c_2 c_4^{-\frac{\alpha}{2}-\frac{1}{4}}$ is positive. Setting $|x,t| = \frac{c_4}{N}$ in the inequality (24), we have

$$c_5 \int_{\mathbb{B}^c_N} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m) \leq I_2. \tag{26}$$

By Hölder inequality, the second term in (26) satisfies

$$\begin{aligned}
I_2 &\leq \left(\int_{\mathbb{B}_N^c} |\varphi_{\lambda,m}(x,t)-1|^{2k}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{2k}} \\
&\quad \times \left(\int_{\mathbb{B}_N^c} |\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\right)^{1-\frac{1}{2k}} \\
&= \left(\int_{\mathbb{B}_N^c} |\lambda,m|^{-2r}|\varphi_{\lambda,m}(x,t)-1|^{2k}|\lambda,m|^{2r}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{2k}} \\
&\quad \times \left(\int_{\mathbb{B}_N^c} |\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\right)^{1-\frac{1}{2k}} \\
&\leq N^{-\frac{r}{k}}\left(\int_{\mathbb{B}_N^c} |\varphi_{\lambda,m}(x,t)-1|^{2k}|\lambda,m|^{2r}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{2k}} \\
&\quad \times \left(\int_{\mathbb{B}_N^c} |\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\right)^{1-\frac{1}{2k}}.
\end{aligned}$$

We have seen that

$$\int_{\mathbb{B}_N^c} |\varphi_{\lambda,m}(x,t)-1|^{2k}|\lambda,m|^{2r}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \leq \|\Delta_{(x,t)}^k(\mathcal{L}^r f)\|_{2,\alpha}^2.$$

Therefore

$$I_2 \leq N^{-\frac{r}{k}}\left(\|\Delta_{(x,t)}^k(\mathcal{L}^r f)\|_{2,\alpha}\right)^{\frac{1}{k}}\left(\int_{\mathbb{B}_N^c} |\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\right)^{1-\frac{1}{2k}}.$$

For $f \in W_{2,\phi}^{r,k}(\mathcal{L})$, there exist a constant $c_6 > 0$ such that

$$\|\Delta_{(x,t)}^k(\mathcal{L}^r f)\|_{2,\alpha}^2 \leq c_6(\phi(\delta^k))^2 \quad \text{as} \quad \delta \to 0,$$

by virtue of (11) and (12). For $\delta = \frac{c_4}{N}$, we obtain

$$\begin{aligned}
&c_5 \int_{\mathbb{B}_N^c} |\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \\
&\quad \leq N^{-\frac{r}{k}}\left(c_6\phi\left[\left(\frac{c_4}{N}\right)^k\right]\right)^{\frac{1}{k}}\left(\int_{\mathbb{B}_N^c} |\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\right)^{1-\frac{1}{2k}}.
\end{aligned}$$

Then,

$$c_5\left(\int_{\mathbb{B}_N^c} |\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{2k}} \leq N^{-\frac{r}{k}}\left(c_6\phi\left[\left(\frac{c_4}{N}\right)^k\right]\right)^{\frac{1}{k}}.$$

Therefore,

$$\int_{\mathbb{B}_N^c} |\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) = O\left(N^{-2r}\left(\phi\left[\left(\frac{c_4}{N}\right)^k\right]\right)^2\right),$$

for all $N > 0$. Thus this theorem is proved with $c_3 = c_4^k$. □

Theorem 2.4. *Let* $\phi(t) = t^v$. *Then*

$$\sqrt{\int_{\mathbb{B}_N^c} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)} = O(N^{-r-kv}) \ \ as \ \ N \to +\infty \Leftrightarrow f \in \mathrm{W}^{r,k}_{2,\phi}(\mathcal{L}),$$

where $r = 0, 1, \ldots, k = 1, 2, \ldots,$ *and* $0 < v < 1$.

Proof. If $f \in \mathrm{W}^{r,k}_{2,\phi}(\mathcal{L})$, then by using Theorem 2.3, we get

$$\sqrt{\int_{\mathbb{B}_N^c} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m)} = O(N^{-r-kv}). \tag{27}$$

Now we prove the opposite implication. From relation (13), we obtain

$$\begin{aligned} \|\Delta^k_{(x,t)}(\mathcal{L}^r f)\|^2_{2,\alpha} &= \int_{\mathbb{R}\times\mathbb{N}} |\varphi_{\lambda,m}(x,t) - 1|^{2k} |\lambda, m|^{2r} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m) \\ &= J_1 + J_2, \end{aligned}$$

where

$$\begin{aligned} J_1 &= \int_{\mathbb{B}_N} |\varphi_{\lambda,m}(x,t) - 1|^{2k} |\lambda, m|^{2r} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m), \\ J_2 &= \int_{\mathbb{B}_N^c} |\varphi_{\lambda,m}(x,t) - 1|^{2k} |\lambda, m|^{2r} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m), \end{aligned}$$

and $N = E\left(\frac{1}{|x,t|}\right)$ is the integer part of the number $\frac{1}{|x,t|}$.

From (6), we have the estimate

$$\begin{aligned} J_2 &\le c_7 \int_{\mathbb{B}_N^c} |\lambda, m|^{2r} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m) \\ &= c_7 \sum_{l=0}^{+\infty} \int_{\mathbb{B}_{N+l+1}\setminus\mathbb{B}_{N+l}} |\lambda, m|^{2r} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m) \\ &\le c_7 \sum_{l=0}^{+\infty} (N + l + 1)^{2r} \int_{\mathbb{B}_{N+l+1}\setminus\mathbb{B}_{N+l}} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m) \\ &= c_7 \sum_{l=0}^{+\infty} a_l \left(\mathcal{J}_l - \mathcal{J}_{l+1}\right). \end{aligned}$$

with $a_l = (N + l + 1)^{2r}$ and

$$\mathcal{J}_l = \int_{\mathbb{B}_{N+l}^c} |\mathcal{F}_L(f)(\lambda, m)|^2 d\gamma_\alpha(\lambda, m).$$

For all integers $M \ge 1$, the Abel transformation shows

$$\begin{aligned} \sum_{l=0}^{M} a_l \left(\mathcal{J}_l - \mathcal{J}_{l+1}\right) &= a_0\mathcal{J}_0 - a_M\mathcal{J}_{M+1} + \sum_{l=1}^{M} (a_l - a_{l-1})\mathcal{J}_l \\ &\le a_0\mathcal{J}_0 + \sum_{l=1}^{M} (a_l - a_{l-1})\mathcal{J}_l, \end{aligned}$$

because $a_M\mathcal{J}_{M+1} \geq 0$. Moreover by the finite increments theorem, we have

$$a_l - a_{l-1} \leq 2r(N+l+1)^{2r-1}.$$

On the other hand, by (27), there exists $c_8 > 0$ such that, for all $N > 0$

$$\mathcal{J}_l \leq c_8(N+l)^{-2r-2k\nu}.$$

For $N \geq 1$, we have

$$\begin{aligned}\sum_{l=0}^{M} a_l(\mathcal{J}_l - \mathcal{J}_{l+1}) &\leq a_0\mathcal{J}_0 + \sum_{l=1}^{M}(a_l - a_{l-1})\mathcal{J}_l \\ &\leq c_8\left(1+\frac{1}{N}\right)^{2r} N^{-2k\nu} + 2rc_8\sum_{l=1}^{M}\left(1+\frac{1}{N+l}\right)^{2r-1}(N+l)^{-1-2k\nu} \\ &\leq c_8 2^{2r}N^{-2k\nu} + 2^{2r}rc_8\sum_{l=1}^{M}(N+l)^{-1-2k\nu}.\end{aligned}$$

Finally, by the integral comparison test, we have

$$\sum_{l=1}^{M}(N+l)^{-1-2k\nu} \leq \sum_{\mu=N+1}^{+\infty}\mu^{-1-2k\nu} \leq \int_N^{+\infty} t^{-1-2k\nu}dt = \frac{1}{2k\nu}N^{-2k\nu}.$$

Letting $M \to +\infty$, we see that, for $r \geq 0$ and $k, \nu > 0$, there exists a constant c_9 such that, for all $N \geq 1$,

$$J_2 \leq c_9 N^{-2k\nu}.$$

Consequently, for all $|x,t| > 0$, we get

$$J_2 \leq c_9|x,t|^{2k\nu} \text{ as } |x,t| \to 0. \tag{28}$$

Now, we estimate J_1. From Lemma 2.2 we have

$$\begin{aligned}J_1 &= \int_{\mathbb{B}_N}|\varphi_{\lambda,m}(x,t)-1|^{2k}|\lambda,m|^{2r}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \\ &\leq c_1|x,t|^{2k}\int_{\mathbb{B}_N}|\lambda,m|^{2k+2r}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \\ &= c_1|x,t|^{2k}\sum_{l=0}^{N-1}\int_{\mathbb{B}_{l+1}\setminus\mathbb{B}_l}|\lambda,m|^{2k+2r}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \\ &\leq c_1|x,t|^{2k}\sum_{l=0}^{N-1}(l+1)^{2k+2r}\int_{\mathbb{B}_{l+1}\setminus\mathbb{B}_l}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \\ &= c_1|x,t|^{2k}\sum_{l=0}^{N-1} b_l\left(\mathcal{I}_l - \mathcal{I}_{l+1}\right),\end{aligned}$$

with $b_l = (l+1)^{2k+2r}$ and

$$\mathcal{I}_l = \int_{\mathbb{B}_l^c}|\mathcal{F}_L(f)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m).$$

Using a summation by parts transforms and proceeding as with J_2 and the fact that $\mathcal{I}_l \leq c_8 l^{-2r-2k\nu}$ by hypothesis, we obtain

$$\begin{aligned} J_1 &\leq c_1|x,t|^{2k} \sum_{l=0}^{N-1} b_l \left(\mathcal{I}_l - \mathcal{I}_{l+1}\right) \\ &\leq c_1|x,t|^{2k} \left(b_0 \mathcal{I}_0 + \sum_{l=1}^{N-1} \mathcal{I}_l (b_l - b_{l-1}) \right) \\ &\leq c_1|x,t|^{2k} \left(\mathcal{I}_0 + c_8(2r+2k) \sum_{l=1}^{N-1} (l+1)^{2r+2k-1} l^{-2r-2k\nu} \right). \end{aligned}$$

From the inequality $l+1 \leq 2l$, we conclude that

$$J_1 \leq c_1|x,t|^{2k} \left(\mathcal{I}_0 + c_{10} \sum_{l=1}^{N-1} l^{2k-2k\nu-1} \right).$$

As a consequence of a series comparison, we have the inequality,

$$\mu \sum_{l=1}^{N-1} l^{\mu-1} \leq N^{\mu} \quad \text{for} \quad \mu > 0 \text{ and } N \geq 2.$$

If $\mu = 2k - 2k\nu > 0$ for $0 < \nu < 1$, then we obtain

$$J_1 \leq c_1|x,t|^{2k} \left(\mathcal{I}_0 + c_{11} N^{2k-2k\nu} \right) \leq c_1|x,t|^{2k} \left(\mathcal{I}_0 + c_{11}|x,t|^{2k\nu-2k} \right),$$

since $N \leq 1/|x,t|$. If $|x,t|$ is sufficiently small then $\mathcal{I}_0 \leq c_{11}|x,t|^{2k\nu-2k}$. Then we have

$$J_1 \leq c_{12}|x,t|^{2k\nu}. \tag{29}$$

Combining the estimates (28) and (29) for J_1 and J_2 gives

$$\|\Delta^k_{(x,t)}(\mathcal{L}^r f)\|_{2,\alpha} = O(|x,t|^{k\nu}) \text{ as } |x,t| \to 0.$$

Consequently

$$\Omega_k\left(\mathcal{L}^r f, \delta\right) = O(\delta^{k\nu}) = O(\phi(\delta^k)) \text{ as } \delta \to 0.$$

Therefore the necessity is proved and the proof of this theorem is completed. □

Acknowledgements

The author would like to thank the anonymous referees for their useful comments and suggestions for improving the presentation of this paper.

Availability of data and materials

No data were used to support this study.

References

[1] V. A. Abilov, F. V. Abilova, M. K. Kerimov, *Some Remarks concerning the Fourier transform in the space* $L_2(\mathbb{R})$, Comp. Math. Math. Phys. **48**(6) (2008), 885–891.

[2] V. A. Abilov, F. V. Abilova, M. K. Kerimov, *Some Remarks concerning the Fourier transform in the space* $L_2(\mathbb{R}^n)$, Comp. Math. Math. Phys. **48**(12) (2008), 2146–2153.

[3] V. A. Abilov, F. V. Abilova, M. K. Kerimov, *On Estimates for the Fourier-Bessel integral in the space* $L^2(\mathbb{R}^+)$, Comp. Math. Math. Phys. **49**(7) (2009), 1103–1110.

[4] H. Bateman, A. Erdélyi, *Higher Transcendental Functions, Vol II*, Mc Graw-Hill, New York, 1953, Nauka, Moscow, 1974.

[5] R. Daher, M. El Hamma, *On estimates for the Fourier transform in the space* $L^2(\mathbb{R}^n)$, C. R. Math. **352** (2014), 235–240.

[6] R. Daher, S. El Ouadih, *Some new estimates for the Helgason Fourier transform on rank 1 symmetric spaces*, Proc. Math. Sci. **128**(37) (2018), 1–11.

[7] R. Daher, O. Tyr, *Modulus of Smoothness and Theorems Concerning Approximation in the Space* $L^2_{q,\alpha}(\mathbb{R}_q)$ *with Power Weight*. Mediterr. J. Math. **18**(69) (2021), 1–18.

[8] S. El Ouadih, R. Daher, *Some new estimates for the Cherednik-Opdam transform in the space* $L^2_{\alpha,\beta}(\mathbb{R})$. J. Pseudo-Differ. Oper. Appl. **7**, (2016), 47–58.

[9] J. Faraut, K. Harzallah, *Deux cours d'Analyse Harmonique. In: Ecole d'été d'Analyse Harmonique de Tunis*, Birkhäuser, Boston, 1984.

[10] N. N. Lebedev, *Special functions and their applications*, Dover Publications, 1972.

[11] B. Levi, *Sul Principio di Dirichlet*, Rend. Circolo Mat. di Palermo **22** (1906), 293–359.

[12] S. Negzaoui, *Lipschitz Conditions in Laguerre Hypergroup*. Mediterr. J. Math. **14**(191) (2017), 1-12.

[13] M. M. Nessibi, *A local central limit theorem on the Laguerre hypergroup*, J. Math. Anal. Appl. **354**(2) (2009), 630–640.

[14] M. M. Nessibi, K. Trimèche, *Inversion of the Radon Transform on the Laguerre hypergroup by using generalized wavelets*. J. Math. Anal. Appl. **208** (1997), 337-363.

[15] S. M. Nikol'skii, *Approximation of functions of several variables and embedding theorems*, (2nd ed., Nauka, Moscow 1977; English transl. of 1st ed., Grundlehren Math. Wiss. Vol. 205), Springer-Verlag, New York-Heidelberg, 1975.

[16] K. Stempak, *Mean summability methods for Laguerre series*, Trans. Am. Math. Soc. **322**(2) (1990), 129–147.

[17] O. Tyr, R. Daher, *Abilov's estimates for the Clifford–Fourier transform in real Clifford algebras analysis*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **69** (2023), 227–243.

[18] O. Tyr, R. Daher, *On the Approximation of Functions by Jacobi–Dunkl Expansions in the Weighted Space* $\mathbb{L}_2^{(\alpha,\beta)}$, Ukr. Math. J. **74** (2023), 1629–1644.

[19] O. Tyr, R. Daher, *Some direct and inverse theorems of approximation of functions in Jacobi-Dunkl discrete harmonic analysis*, J. Math. Sci. **266** (2022), 534–553.

[20] O. Tyr, R. Daher, S. El Ouadih, O. El Fourchi, *On the Jackson-type inequalities in approximation theory connected to the q-Dunkl operators in the weighted space* $L^2_{q,\alpha}(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$, Bol. Soc. Mat. Mex. **27**(51) (2021), 1–21.

[21] V. S. Vladimirov, *Equations of Mathematical Physics*, Marcel Dekker, New York, 1971.