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An efficient scheme to study the approximation of BS ideals in semigroups

Nosheen Malik[a], Muhammad Shabir[a], Rizwan Gul[a,*], Sanaa Ahmed Bajri[b], Hamiden Abd El-Wahed Khalifa[c]

[a]*Department of Mathematics, Quaid-i-Azam University, Islamabad, 45320, Pakistan*
[b]*Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia*
[c]*Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia*

Abstract. Rough set (RS) theory is a valuable mathematical tool to address incomplete knowledge. In contrast, BS sets (BSSs) can handle the vagueness and the bipolarity of the data in a variety of scenarios. The main goal of the current article is to analyze the concept of rough BS (BS) ideals in semigroup ($\mathcal{S}_{\mathfrak{g}}$) by combining RS theory and BSSs. To this end, the ideas of BS left ideals (BS-LIs), BS right ideals (BS-RIs), BS two-sided ideals (BS-2SIs), BS interior ideals (BS-IIs) and BS bi-ideals (BS-BIs) over the $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta}$ are developed. Some significant properties associated with these notions are highlighted with several concrete illustrations. Furthermore, it has been noted that the congruence relation ($\mathbb{C}_{\mathfrak{r}}$) and complete congruence relation ($\mathbb{CC}_{\mathfrak{r}}$) are found to be essential for creating rough approximations of BS ideals. As a result, their affiliated features are evaluated through $\mathbb{C}_{\mathfrak{r}}$ and $\mathbb{CC}_{\mathfrak{r}}$.

1. Introduction

By expanding on conventional set theory, Zadeh's concept of FSs [50] transforms science, technology, logic, and mathematics. The fuzzy membership function (M_f) is a key component of FS theory, which enables us to figure out the membership degree (M_d) of an item of a set. FS theory not only copes with uncertainty but also can interpret human linguistic terms statistically. The introduction of FSs provided mathematicians with a fresh outlook on handling ambiguity in uncertain dilemmas. In recent decades, research on FSs has been extremely active and making tremendous progress. Zhang et al. [4] constructed a discrete switched scheme and fuzzy robust control of a dynamic supply chain network. Al-shami [7] manifested the paradigm of (2,1)-FSs and their applications to decision analysis. Zhang et al. [26] projected a fuzzy control technique and simulation for a nonlinear supply chain system with lead times.

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Communicated by Miodrag Spalević
* Corresponding author: Rizwan Gul
Email addresses: nosheenmalik@math.qau.edu.pk (Nosheen Malik), mshabirbhatti@yahoo.co.uk (Muhammad Shabir), rgul@math.qau.edu.pk (Rizwan Gul), sabajri@pnu.edu.sa (Sanaa Ahmed Bajri), Ha.Ahmed@qu.edu.sa (Hamiden Abd El-Wahed Khalifa)
ORCID iDs: https://orcid.org/0000-0002-3161-8482 (Muhammad Shabir), https://orcid.org/0000-0002-6139-1872 (Rizwan Gul), https://orcid.org/0009-0008-4187-5205 (Sanaa Ahmed Bajri), https://orcid.org/0000-0002-8269-8822 (Hamiden Abd El-Wahed Khalifa)

Pawlak put forward the foundational concept of RS philosophy [36] as an effective framework for addressing uncertainty in data processing. The lower and upper approximations that built the root of this theory permit the extraction of prospective information from the analyzed data. The RS theory relies on an equivalence relation ($\mathcal{E}_r$) that reflects the indiscernibility among items. The computational approach in RS philosophy has attracted significant interest in recent years, leading to its application in critical fields. Despite the fruitful use of RS theory in numerous fields, some restrictions may limit its application spectrum. The association between the items cannot be exactly reflected by an $\mathcal{E}_r$ in many real situations across different realms. As a result, weaker versions of binary relations have a weakened $\mathcal{E}_r$ requirement, which has yields in several other generic RS variants. Zhu [52] predicted the prospect of generalized RSs based on relations. The fuzzy RS (FRS) variant was manifested by Dubois and Prade [53] by swapping the crisp relation with the fuzzy relation. Yao [48] described relational interpretations of neighborhood and RS approximations. Wang et al. [45] devised an attribute reduction for covering-based RSs. Qurashi et al. [25] manifested rough substructures via overlaps of successors in quantales using serial fuzzy relations. Bashir et al. [27] framed a conflict resolution scheme via game-theoretic RSs.

Soft set ($\mathcal{SS}$) theory [34] was created by Molodtsov in 1999 as an innovative mathematical approach to deal with ambiguity. When assessing data, the attributes information is crucial. The $\mathcal{SS}$ hypothesis shows promise as an effective parameterization mechanism. Thus, this technique effectively surmounts several challenges associated with conventional theories. The number of study $\mathcal{SS}$s has significantly increased over the past ten years. Maji et al. [33] delineated various operations of $\mathcal{SS}$s. The authors of [5] introduced a variety of innovative operations pertaining to $\mathcal{SS}$s. Furthermore, Ali et al. [6] invented algebraic frameworks of $\mathcal{SS}$s associated with innovative operators. In [9], the authors recommended a platform for the hybridization of $\mathcal{SS}$s, RSs, and FSs. Ayub et al. [54] studied modules of fractions in the framework of FSs and $\mathcal{SS}$s.

Rosenfeld [38] was the first to develop fuzzy groups. The paradigm of fuzzy $\mathcal{S}_\mathfrak{g}$ was examined by Kuroki [22]. The soft group was initially established by Aktaş and Çağman [3]. The authors of [39] identified soft and normalistic soft groups. The framework of a fuzzy soft group was manifested by the authors in [8]. The development of soft-ordered $\mathcal{S}_\mathfrak{g}$ was accomplished by Jun et al. [16].

The significance of bipolarity in data is evident across numerous real-life scenarios. Bipolarity refers to the dual nature of certain situations, encompassing both positive and negative aspects. Zhang [51] proposed the concept of bipolar FSs (BFSs) to simultaneously address bipolarity and fuzziness in these circumstances. In BFSs, each item is allocated a positive M_f within the interval $[0, 1]$ and a negative M_f within the interval $[-1, 0]$. Numerous scholars have demonstrated significant interest in BFSs since the development of BFS theory, and they have examined various potential applications across multiple realms. The authors of [31] framed a decision analytic mechanism via BFSs. Han et al. [14] postulated a YinYang bipolar fuzzy cognitive TOPSIS scheme for bipolar disorder diagnosis. Lee [24] worked on the basic operations of BFSs. Riaz and Tehrim [37] came up with the VIKOR approach for BFSs via connection numbers of SPA theory-based metric spaces. Gul [58] framed an extension of the VIKOR approach for multi-criteria decision making using bipolar fuzzy preference δ-covering based bipolar fuzzy RS model.

A multitude of research on FSs and BFSs have been undertaken. Malik et al. [57] described rough bipolar fuzzy ideals in $\mathcal{S}_\mathfrak{g}$s. Additionally, Shabir et al. [43] described interval-valued fuzzy ideals within hemirings. The authors in [32] presented bipolar fuzzy subgroups. In [49], Yiarayong investigated various algebraic structures on $\mathcal{S}_\mathfrak{g}$s employing BFSs. The authors in [11] analyzed $\mathcal{S}_\mathfrak{g}$s via cubic bipolar fuzzy ideals. Yaqoob [47] established bipolar fuzzy ideals in LA-$\mathcal{S}_\mathfrak{g}$s.

An $\mathcal{SS}$ may or may not include an element of the universe in the image set corresponding to the attribute. In some cases, a universe item is not part of an image set, and the image set complement is associated with an attribute. To capture such instances, Shabir and Naz [44] launched the idea of a BSS. Karaaslan and Karataş [18] reformulated a form of BSSs. Mahmood [55] defined an innovative version of BSSs, which is entitled T-BSSs and discusses their applications in decision analysis. In [35], the scholars developed algebraic features of fuzzy BSSs. Multiple endeavors have been undertaken to integrate RS paradigm and BFSs [46]. The authors of [28] invented a consensus scheme using rough bipolar fuzzy approximations. Malik et al. [30] revealed a medical decision making technique based on BS information. Malik and Shabir [29] put forward rough fuzzy BSSs. Karaaslan and Çağman [20] initiated the paradigm of BS rough sets

(BSRSs) and their applications. Shabir and Gul [41] postulated the notion of modified rough BSSs. Gul et al. [12] projected multigranulation modified rough BSSs with decision making applications. Besides, Gul et al. [13] planned a novel technique roughness of BSSs and their applications. Malik et al. [57] recommended a novel decision analytic technique based on T-rough BFSs.

The study of ideals is fundamental in $\mathcal{S}_{\mathfrak{g}}$s, which make up a substantial portion of algebra. Numerous researchers have integrated ideals within $\mathcal{S}_{\mathfrak{g}}$s into the FSs in diverse ways. The authors of [2] discovered the fuzzy quasi-ideals in $\mathcal{S}_{\mathfrak{g}}$s. Fuzzy interior ideals in $\mathcal{S}_{\mathfrak{g}}$s were analyzed by Hong et al. [15]. In [40], the authors explored soft and generalized fuzzy ideals in $\mathcal{S}_{\mathfrak{g}}$s. Karaaslan et al. [19] introduced BS groups. The authors in [42] explored $\mathcal{S}_{\mathfrak{g}}$s defined by $(\hbar, \hbar \vee qk)$-fuzzy ideals. Rough ideals in $\mathcal{S}_{\mathfrak{g}}$s were devised by Kuroki [23]. Kim et al. [21] examined bipolar fuzzy ideals in $\mathcal{S}_{\mathfrak{g}}$s. Yiarayong [49] examined a method to address the algebraic features of $\mathcal{S}_{\mathfrak{g}}$s via BFSs. Feng et al. [10] examined the utilization of soft relations ($\mathcal{SR}$s) in $\mathcal{S}_{\mathfrak{g}}$s. In [17], the authors focused on the approximation in $\mathcal{S}_{\mathfrak{g}}$s through $\mathcal{SR}$s. Authors of [1] introduced bipolar fuzzy soft Γ-ideals within a Γ-$\mathcal{S}_{\mathfrak{g}}$.

According to the literature review, many researchers have looked at the approximation of different algebraic structures within the frameworks of FSs, RSs, and $\mathcal{SS}$s. To our understanding, no previous literature has examined ideals in the context of rough BSSs. As a result, in this article, we try to bridge the knowledge gap by introducing the notion of rough BS ideals in $\mathcal{S}_{\mathfrak{g}}$s. In this regard, we have proposed the notions of BSSs and BS ideals (BS ideals) in a $\mathcal{S}_{\mathfrak{g}}$. Additionally, we assessed the roughness in the BSSs via a $\mathbb{C}_r$ characterized on the $\mathcal{S}_{\mathfrak{g}}$ and analyzed many of their associated features. Moreover, this script elucidates and explores the concepts of rough BS ideals (RBS ideals), rough BS interior ideals ($\mathcal{RBS}-\mathcal{II}$s), and rough BS bi-ideals (RBS-BIs) within the context of $\mathcal{S}_{\mathfrak{g}}$s.

This article is organized as follows:

1. briefly reviews the core literature utilized in this work.
2. Section 3 presents the conception of BSSs in $\mathcal{S}_{\mathfrak{g}}$s.
3. Section 4 explains the idea of BS Ideals in $\mathcal{S}_{\mathfrak{g}}$s.
4. In Section 5, we analyzed the framework of rough BSSs in $\mathcal{S}_{\mathfrak{g}}$s.
5. Section 6 delineates paradigm of the rough BS left ideal ($\mathcal{RBS}-\mathcal{LI}$), rough BS right ideal ($\mathcal{RBS}-\mathcal{RI}$), rough BS two-sided ideal ($\mathcal{RBS}-2\mathcal{SI}$), rough BS interior ideal ($\mathcal{RBS}-\mathcal{II}$) and rough BS bi-ideal (RBS-BI) in $\mathcal{S}_{\mathfrak{g}}$s.
6. Section 7 presents a comparative analysis of the proposed work with certain prevalent schemes.
7. The article is finally concluded in Section 8.

2. Preliminaries

This segment recaps some key terminology related to the current work.

Definition 2.1. *[36] An approximation space is expressed as the pair $(\tilde{\Omega}, \varsigma)$, where $\tilde{\Omega}$ be a non-finite set and ς is an $\hbar_r$ over $\tilde{\Omega}$. Utilizing ς, the lower and upper approximations of $\mathfrak{S} \subseteq \tilde{\Omega}$ are articulated as:*

$$\underline{\mathfrak{S}}_{\varsigma} = \left\{ \mathfrak{b} \in \tilde{\Omega} : [\mathfrak{b}]_{\varsigma} \subseteq \mathfrak{S} \right\}, \tag{2.1}$$

$$\overline{\mathfrak{S}}^{\varsigma} = \left\{ \mathfrak{b} \in \tilde{\Omega} : [\mathfrak{b}]_{\varsigma} \cap \mathfrak{S} \neq \varnothing \right\}. \tag{2.2}$$

The pair $\left(\underline{\mathfrak{S}}_{\varsigma}, \overline{\mathfrak{S}}^{\varsigma}\right)$ is titled an RS of $\mathfrak{S}$ in $\tilde{\Omega}$.

Let $\widehat{\wp}$ be a non-empty assembling of parameters and $2^{\tilde{\Omega}}$ signify the family of all subsets of $\tilde{\Omega}$. Then, we characterize an SS using a set-valued mapping, as shown beneath.

Definition 2.2. *An structure $(\pounds, \overset{\star}{\Pi})$ is entitled an $\mathcal{SS}$ on $\tilde{\Omega}$, where $\overset{\star}{\Pi} \subseteq \widehat{\wp}$ and $\pounds : \overset{\star}{\Pi} \longrightarrow 2^{\tilde{\Omega}}$.*

Definition 2.3. *[44] The NOT set of parameters* $\overset{\star}{\Pi}$ *is postulated as:*

$$\neg\overset{\star}{\Pi} = \{\neg\hbar : \hbar \in \overset{\star}{\Pi}\}, \ \textit{where } \neg\hbar = \textit{not } \hbar \textit{ for } \hbar \in \overset{\star}{\Pi}.$$

Definition 2.4. *[44] A BSS over* $\tilde{\Omega}$ *is an object having the form* $\eth = (£, \Gamma; \overset{\star}{\Pi})$*, where* $£ : \overset{\star}{\Pi} \longrightarrow 2^{\tilde{\Omega}}$ *and* $\Gamma : \neg\overset{\star}{\Pi} \longrightarrow 2^{\tilde{\Omega}}$ *such that* $\forall\ \hbar \in \overset{\star}{\Pi}$, $£(\hbar) \cap \Gamma(\neg\hbar) = \emptyset$.

We specify the set of all BSSs over $\tilde{\Omega}$ by $\mathfrak{BSS}(\tilde{\Omega})$.

Definition 2.5. *[44] A BSS* $\Phi_n = (\Phi_n, \tilde{\Omega}_n; \overset{\star}{\Pi}) \in \mathfrak{BSS}(\ddot{\Theta})$ *is called the relative null BSS over* $\ddot{\Theta}$*, if* $\Phi_n(\hbar) = \emptyset$ *and* $\tilde{\Omega}_n(\neg\hbar) = \tilde{\Omega}\ \forall\ \hbar \in \overset{\star}{\Pi}$.

Definition 2.6. *[44] A BSS* $\tilde{\Omega}_w = (\tilde{\Omega}_w, \Phi_w; \overset{\star}{\Pi}) \in \mathfrak{BSS}(\ddot{\Theta})$ *is the relative whole BSS over* $\ddot{\Theta}$*, if* $\tilde{\Omega}_w(\hbar) = \tilde{\Omega}$ *and* $\Phi_w(\neg\hbar) = \emptyset\ \forall\ \hbar \in \overset{\star}{\Pi}$.

Definition 2.7. *[44] Let* $\eth_1 = (£_1, \Gamma_1; \overset{\star}{\Pi}_1), \eth_1 = (£_2, \Gamma_2; \overset{\star}{\Pi}_2) \in \mathfrak{BSS}(\tilde{\Omega})$*, then* $\eth_1 \subseteq \eth_2$ *if*

1. $\overset{\star}{\Pi}_1 \subseteq \overset{\star}{\Pi}_2$,
2. $£_1(\hbar) \subseteq £_2(\hbar)$ *and* $\Gamma_1(\neg\hbar) \supseteq \Gamma_2(\neg\hbar)\ \forall\ \hbar \in \overset{\star}{\Pi}$.

Definition 2.8. *[44] Let* $\eth_1 = (£_1, \Gamma_1; \overset{\star}{\Pi}_1), \eth_2 = (£_2, \Gamma_2; \overset{\star}{\Pi}_2) \in \mathfrak{BSS}(\tilde{\Omega})$*. Then,*

1. *The extended union of* $\eth_1$ *and* $\eth_2$ *is a BSS*

$$\eth_1 \widetilde{\sqcup}_\varepsilon \eth_2 = (£_1 \widetilde{\cup}_\varepsilon £_2, \Gamma_1 \widetilde{\cap}_\varepsilon \Gamma_2; \overset{\star}{\Pi}_1 \cup \overset{\star}{\Pi}_2)$$

over $\tilde{\Omega}$*, where* $£_1 \widetilde{\cup}_\varepsilon £_2 : \overset{\star}{\Pi}_1 \cup \overset{\star}{\Pi}_2 \longrightarrow 2^{\tilde{\Omega}}$ *is described as:*

$$(£_1 \widetilde{\cup}_\varepsilon £_2)(\hbar) = \begin{cases} £_1(\hbar) & \textit{if } \hbar \in \overset{\star}{\Pi}_1 - \overset{\star}{\Pi}_2 \\ £_2(\hbar) & \textit{if } \hbar \in \overset{\star}{\Pi}_2 - \overset{\star}{\Pi}_1 \\ £_1(\hbar) \cup £_2(\hbar) & \textit{if } \hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2 \end{cases}$$

and $\Gamma_1 \widetilde{\cap}_\varepsilon \Gamma_2 : \neg(\overset{\star}{\Pi}_1 \cup \overset{\star}{\Pi}_2) \longrightarrow 2^{\tilde{\Omega}}$ *is expressed as:*

$$(\Gamma_1 \widetilde{\cap}_\varepsilon \Gamma_2)(\neg\hbar) = \begin{cases} \Gamma_1(\neg\hbar) & \textit{if } \neg\hbar \in (\neg\overset{\star}{\Pi}_1) - (\neg\overset{\star}{\Pi}_2) \\ \Gamma_2(\neg\hbar) & \textit{if } \neg\hbar \in (\neg\overset{\star}{\Pi}_2) - (\neg\overset{\star}{\Pi}_1) \\ \Gamma_1(\neg\hbar) \cap \Gamma_2(\neg\hbar) & \textit{if } \neg\hbar \in \neg(\overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2) \end{cases}$$

2. *The extended intersection of* $\eth_1$ *and* $\eth_2$ *is a BSS*

$$\eth_1 \widetilde{\sqcap}_\varepsilon \eth_2 = (£_1 \widetilde{\cap}_\varepsilon £_2, \Gamma_1 \widetilde{\cup}_\varepsilon \Gamma_2; \overset{\star}{\Pi}_1 \cup \overset{\star}{\Pi}_2)$$

over $\tilde{\Omega}$*, where* $£_1 \widetilde{\cap}_\varepsilon £_2 : \overset{\star}{\Pi}_1 \cup \overset{\star}{\Pi}_2 \longrightarrow 2^{\tilde{\Omega}}$ *is given as:*

$$(£_1 \widetilde{\cap}_\varepsilon £_2)(\hbar) = \begin{cases} £_1(\hbar) & \textit{if } \hbar \in \overset{\star}{\Pi}_1 - \overset{\star}{\Pi}_2 \\ £_2(\hbar) & \textit{if } \hbar \in \overset{\star}{\Pi}_2 - \overset{\star}{\Pi}_1 \\ £_1(\hbar) \cap £_2(\hbar) & \textit{if } \hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2 \end{cases}$$

and $\Gamma_1 \widetilde{\cup}_\varepsilon \Gamma_2 : \neg(\breve{A}_1 \cup \breve{A}_2) \longrightarrow 2^{\tilde{\Omega}}$ *is defined as:*

$$(\Gamma_1 \widetilde{\cup}_\varepsilon \Gamma_2)(\neg\hbar) = \begin{cases} \Gamma_1(\neg\hbar) & \text{if } \neg\hbar \in (\neg\overset{\star}{\Pi}_1) - (\neg\overset{\star}{\Pi}_2) \\ \Gamma_2(\neg\hbar) & \text{if } \neg\hbar \in (\neg\overset{\star}{\Pi}_2) - (\neg\overset{\star}{\Pi}_1) \\ \Gamma_1(\neg\hbar) \cup \Gamma_2(\neg\hbar) & \text{if } \neg\hbar \in \neg(\overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2) \end{cases}$$

3. *The restricted union of* $\eth_1$ *and* $\eth_2$ *is a BSS*

$$\eth_1 \widetilde{\sqcup}_r \eth_2 = (£_1 \widetilde{\cup}_r £_2, \Gamma_1 \widetilde{\cap}_r \Gamma_2; \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2)$$

over $\tilde{\Omega}$, *where* $£_1 \widetilde{\cup}_r £_2 : \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2 \longrightarrow 2^{\tilde{\Omega}}$ *is expressed as* $(£_1 \widetilde{\cup}_r £_2)(\hbar) = £_1(\hbar) \cup £_2(\hbar)\ \forall\ \hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$, *and* $\Gamma_1 \widetilde{\cap}_r \Gamma_2 : \neg(\overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2) \longrightarrow 2^{\tilde{\Omega}}$ *is given as* $(\Gamma_1 \widetilde{\cap}_r \Gamma_2)(\neg\hbar) = \Gamma_1(\neg\hbar) \cap \Gamma_2(\neg\hbar)\ \forall\ \neg\hbar \in \neg(\overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2)$, *provided* $\overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2 \neq \emptyset$.

4. *The restricted intersection of* $\eth_1$ *and* $\eth_2$ *is a BSS*

$$\eth_1 \widetilde{\sqcap}_r \eth_2 = (£_1 \widetilde{\cap}_r £_2, \Gamma_1 \widetilde{\cup}_r \Gamma_2; \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2)$$

over $\tilde{\Omega}$, *where* $£_1 \widetilde{\cap}_r £_2 : \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2 \longrightarrow 2^{\tilde{\Omega}}$ *is described as* $(£_1 \widetilde{\cap}_r £_2)(\hbar) = £_1(\hbar) \cap £_2(\hbar)\ \forall\ \hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$, *and* $\Gamma_1 \widetilde{\cup}_r \Gamma_2 : \neg(\overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2) \longrightarrow 2^{\tilde{\Omega}}$ *is expressed as* $(\Gamma_1 \widetilde{\cup}_r \Gamma_2)(\neg\hbar) = \Gamma_1(\neg\hbar) \cup \Gamma_2(\neg\hbar)\ \forall\ \neg\hbar \in \neg(\overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2)$, *provided* $\overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2 \neq \emptyset$.

Malik et al. [30] originated the concept of RBSSs and investigated their basic properties, which are given as follows:

Theorem 2.9. *[30] Take an approximation space* $(\tilde{\Omega}, \mathfrak{R})$ *and let* $\eth = (£, \Gamma; \overset{\star}{\Pi}) \in \mathfrak{BSS}(\tilde{\Omega})$. *Then,*

1. $\underline{\eth}_{\mathfrak{R}} \widetilde{\subseteq} \eth \widetilde{\subseteq} \overline{\eth}^{\mathfrak{R}}$;
2. $\underline{\Phi_n}_{\mathfrak{R}} = \Phi_n = \overline{\Phi_n}^{\mathfrak{R}}$;
3. $\underline{\tilde{\Omega}_w}_{\mathfrak{R}} = \tilde{\Omega}_w = \overline{\tilde{\Omega}_w}^{\mathfrak{R}}$;
4. $\underline{\underline{\eth}_{\mathfrak{R}}}_{\mathfrak{R}} = \underline{\eth}_{\mathfrak{R}} = \overline{\left(\underline{\eth}_{\mathfrak{R}}\right)}^{\mathfrak{R}}$;
5. $\underline{(\overline{\eth}^{\mathfrak{R}})}_{\mathfrak{R}} = \overline{\eth}^{\mathfrak{R}} = \overline{(\overline{\eth}^{\mathfrak{R}})}^{\mathfrak{R}}$;
6. $\overline{\eth^c}^{\mathfrak{R}} = \left(\underline{\eth}_{\mathfrak{R}}\right)^c$;
7. $\underline{\eth^c}_{\mathfrak{R}} = \left(\overline{\eth}^{\mathfrak{R}}\right)^c$.

Theorem 2.10. *[30] Let* $(\tilde{\Omega}, \mathfrak{R})$. *Then, the subsequent assertions are true* $\forall\ \eth_1 = (£_1, \Gamma_1; \overset{\star}{\Pi}_1), \eth_2 = (£_2, \Gamma_2; \overset{\star}{\Pi}_2) \in \mathfrak{BSS}(\tilde{\Omega})$.

1. $\eth_1 \widetilde{\subseteq} \eth_2$ *implies that* $\underline{\eth_1}_{\mathfrak{R}} \widetilde{\subseteq} \underline{\eth_2}_{\mathfrak{R}}$ *and* $\overline{\eth_1}^{\mathfrak{R}} \widetilde{\subseteq} \overline{\eth_2}^{\mathfrak{R}}$,
2. $\underline{\eth_1 \widetilde{\sqcap}_r \eth_2}_{\mathfrak{R}} = \underline{\eth_1}_{\mathfrak{R}} \widetilde{\sqcap}_r \underline{\eth_2}_{\mathfrak{R}}$,
3. $\underline{\eth_1 \widetilde{\sqcup}_r \eth_2}_{\mathfrak{R}} \widetilde{\supseteq} \underline{\eth_1}_{\mathfrak{R}} \widetilde{\sqcup}_r \underline{\eth_2}_{\mathfrak{R}}$,
4. $\overline{\eth_1 \widetilde{\sqcap}_r \eth_2}^{\mathfrak{R}} \widetilde{\subseteq} \overline{\eth_1}^{\mathfrak{R}} \widetilde{\sqcap}_r \overline{\eth_2}^{\mathfrak{R}}$,
5. $\overline{\eth_1 \widetilde{\sqcup}_r \eth_2}^{\mathfrak{R}} = \overline{\eth_1}^{\mathfrak{R}} \widetilde{\sqcup}_r \overline{\eth_2}^{\mathfrak{R}}$.

2.1. Ideals in $\mathcal{S}_{\mathfrak{g}}$

An $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta}$ is a non-empty set having an associative binary operation ($\mathcal{BO}$). For $\emptyset \neq \mathcal{W} \subseteq \ddot{\Theta}$, we have

1. $\mathcal{W}$ is titled a subsemigroup ($\mathcal{SS}_{\mathfrak{g}}$) of $\ddot{\Theta}$, if $\forall$ $\mathfrak{p}, \mathfrak{q} \in \mathcal{W}$, $\mathfrak{p}\mathfrak{q} \in \mathcal{W}$ (i.e., $\mathcal{W}\mathcal{W} \subseteq \mathcal{W}$).
2. $\mathcal{W}$ is a left ideal of $\ddot{\Theta}$, if $\mathfrak{r}\mathfrak{p} \in \mathcal{W}$ (i.e., $\ddot{\Theta}\mathcal{W} \subseteq \mathcal{W}$) $\forall$ $\mathfrak{p} \in \mathcal{W}$ and $\mathfrak{r} \in \ddot{\Theta}$.
3. $\mathcal{W}$ is the right ideal of $\ddot{\Theta}$, if $\mathfrak{p}\mathfrak{r} \in \mathcal{W}$ (i.e., $\mathcal{W}\ddot{\Theta} \subseteq \mathcal{W}$) $\forall$ $\mathfrak{p} \in \mathcal{W}$ and $\mathfrak{r} \in \ddot{\Theta}$.
4. $\mathcal{W}$ is named an ideal of $\ddot{\Theta}$ when it left and right ideal of $\ddot{\Theta}$.
5. $\mathcal{W}$ is an interior ideal of $\ddot{\Theta}$, if $\mathfrak{r}\mathfrak{p}\mathfrak{s} \in \mathcal{W}$ $\forall$ $\mathfrak{p} \in \mathcal{W}$ and $\mathfrak{r}, \mathfrak{s} \in \ddot{\Theta}$ (i.e., $\ddot{\Theta}\mathcal{W}\ddot{\Theta} \subseteq \mathcal{W}$).
6. $\mathcal{W}$ is a bi-ideal of Θ, if $\mathcal{W}$ is a $\mathcal{SS}_{\mathfrak{g}}$ of $\ddot{\Theta}$ and $\mathfrak{p}\mathfrak{r}\mathfrak{q} \in \mathcal{W}$ $\forall$ $\mathfrak{p}, \mathfrak{q} \in \mathcal{W}$ and $\mathfrak{r} \in \ddot{\Theta}$ (i.e., $\ddot{\Theta}\mathcal{W} \subseteq \mathcal{W}$).

Definition 2.11. *A $\mathbb{C}_{\mathfrak{r}}$ $\beth$ on a $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta}$ is an $\mathcal{E}_r$ on $\ddot{\Theta}$ which is right and left compatible. That is, if $(\mathfrak{r}, \mathfrak{s}) \in \beth$ then $(p\mathfrak{r}, p\mathfrak{s}), (\mathfrak{r}p, \mathfrak{s}p) \in \beth$ $\forall$ $p, \mathfrak{r}, \mathfrak{s} \in \ddot{\Theta}$. Presume that $[\mathfrak{r}]_{\beth}$ be the $\beth$-congruence class ($\beth$-cng-cl) of $\mathfrak{r} \in \ddot{\Theta}$. For a $\mathbb{C}_{\mathfrak{r}}$ $\beth$ on $\ddot{\Theta}$, it follows $[\mathfrak{r}]_{\beth}[\mathfrak{s}]_{\beth} \subseteq [\mathfrak{r}\mathfrak{s}]_{\mathfrak{R}}$ $\forall$ $\mathfrak{r}, \mathfrak{s} \in \ddot{\Theta}$. A $\mathbb{C}_{\mathfrak{r}}$ $\beth$ on $\ddot{\Gamma}$ isnamed, $\mathbb{CC}_{\mathfrak{r}}$, if $[\mathfrak{r}]_{\beth}[\mathfrak{s}]_{\beth} = [\mathfrak{r}\mathfrak{s}]_{\beth}$ $\forall$ $\mathfrak{r}, \mathfrak{s} \in \ddot{\Theta}$.*

Example 2.12. *Presume that $\ddot{\Theta} = \{\mathfrak{s}, \tau, \theta, \mathfrak{v}\}$ be a $\mathcal{S}_{\mathfrak{g}}$ with $\mathcal{BO}$ displayed in Table 1.*

.	$\mathfrak{s}$	τ	θ	$\mathfrak{v}$
$\mathfrak{s}$	$\mathfrak{s}$	τ	θ	$\mathfrak{v}$
τ	τ	τ	θ	$\mathfrak{v}$
θ	θ	θ	θ	$\mathfrak{v}$
$\mathfrak{v}$	$\mathfrak{v}$	$\mathfrak{v}$	$\mathfrak{v}$	$\mathfrak{u}$

Table 1: $\mathcal{BO}$ of $\mathcal{S}_{\mathfrak{g}}$

Consider $\mathbb{C}_{\mathfrak{r}}$s $\beth_1$ and $\beth_2$ on $\ddot{\Theta}$ as:

$$\beth_1 = \Big\{(\mathfrak{s},\mathfrak{s}),(\tau,\tau),(\theta,\theta),(\mathfrak{v},\mathfrak{v}),(\theta,\mathfrak{v}),(\mathfrak{v},\theta)\Big\},$$
$$\beth_2 = \Big\{(\mathfrak{s},\mathfrak{s}),(\tau,\tau),(\theta,\theta),(\mathfrak{v},\mathfrak{v}),(\theta,\tau),(\tau,\theta)\Big\}.$$

Then, $\beth_1$ describes the cng-cls $\{\mathfrak{s}\}, \{\tau\}$ and $\{\theta, \mathfrak{v}\}$, while $\beth_2$ characterizing the cng-cls $\{\mathfrak{s}\}, \{\tau, \theta\}$ and $\{\mathfrak{v}\}$. It can be confirmed that, $[\mathfrak{r}]_{\beth_1}[\mathfrak{s}]_{\beth_1} = [\mathfrak{x}\mathfrak{v}]_{\beth_1}$ $\forall$ $\mathfrak{r}, \mathfrak{s} \in \ddot{\Theta}$, i.e, $\beth_1$ is a $\mathbb{CC}_{\mathfrak{r}}$ on $\ddot{\Theta}$. Further, $[\mathfrak{v}]_{\beth_2}[\mathfrak{v}]_{\beth_2} \subsetneq [\mathfrak{v}\mathfrak{v}]_{\beth_2}$ $\forall$ $\mathfrak{v} \in \ddot{\Theta}$, since $[\mathfrak{v}]_{\beth_2} = \{\mathfrak{v}\}$, thus, $[\mathfrak{v}]_{\beth_2}[\mathfrak{v}]_{\beth_2} = \{\mathfrak{v}\mathfrak{v}\} = \{\theta\}$ and $[\mathfrak{v}\mathfrak{v}]_{\beth_2} = [\theta]_{\beth_2} = \{\tau, \theta\}$. This reveals that, $\beth_2$ is not a $\mathbb{CC}_{\mathfrak{r}}$.

Definition 2.13. *[23] Presume that $\ddot{\Theta}$ be a $\mathcal{S}_{\mathfrak{g}}$ and $\beth$ be a $\mathbb{C}_{\mathfrak{r}}$ on $\ddot{\Theta}$. The lower and upper approximations of $\mathcal{W} \subseteq \ddot{\Theta}$ are postulated as:*

$$\underline{\mathcal{W}}_{\beth} = \Big\{\mathfrak{r} \in \ddot{\Theta} : [\mathfrak{r}]_{\beth} \subseteq \mathcal{W}\Big\}, \tag{2.3}$$
$$\overline{\mathcal{W}}^{\beth} = \Big\{\mathfrak{r} \in \ddot{\Theta} : [\mathfrak{r}]_{\beth} \cap \mathcal{W} \neq \emptyset\Big\}. \tag{2.4}$$

$\mathcal{W}$ is titled a RS of the $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta}$, if $\underline{\mathcal{W}}_{\beth} \neq \overline{\mathcal{W}}^{\beth}$; otherwise, $\mathcal{W}$ is exact in $\ddot{\Theta}$. Additionally,

(1) $\mathcal{W}$ is named a lower (an upper) rough $\mathcal{SS}_{\mathfrak{g}}$ of $\ddot{\Theta}$, if $\underline{\mathcal{W}}_{\beth}$ ($\overline{\mathcal{W}}^{\beth}$) is a $\mathcal{SS}_{\mathfrak{g}}$ of $\ddot{\Theta}$.

(2) $\mathcal{W}$ is a lower rough left (right, two-sided, interior, bi-) ideal of $\ddot{\Theta}$ when $\underline{\mathcal{W}}_{\beth}$ is a left (right, two-sided, interior, bi-) ideal of $\ddot{\Theta}$ and $\mathcal{W}$ is an upper rough left (right, two-sided, interior, bi-) ideal of $\ddot{\Theta}$ if $\overline{\mathcal{W}}^{\beth}$ is a left (right, two-sided, interior, bi-) ideal of $\ddot{\Theta}$.

(3) $\mathcal{W}$ is called a rough left (right, two-sided, interior, bi-) ideal of $\ddot{\Theta}$ when it is both lower and upper left (right, two-sided, interior, bi-) ideal of $\ddot{\Theta}$.

Theorem 2.14. *[23] Suppose that* $\beth$ *be a* $\mathbb{C}_r$ *on a* $\mathcal{S}_g$ $\ddot{\Theta}$*. Then,*

1. *Every* $\mathcal{SS}_g$ *of* $\ddot{\Theta}$ *is an upper rough* $\mathcal{SS}_g$ *of* $\ddot{\Theta}$*.*
2. *Every left (right, bi-) ideal of* $\ddot{\Theta}$ *is an upper rough left (right, bi-) ideal of* $\ddot{\Theta}$*.*

Theorem 2.15. *[23] Assume that* $\beth$ *be a* $\mathbb{CC}_r$ *on a* $\mathcal{S}_g$ $\ddot{\Theta}$*. Then,*

1. *Every* $\mathcal{SS}_g$ $\mathcal{W}$ *of* $\ddot{\Theta}$ *is a lower rough* $\mathcal{SS}_g$ *of* $\ddot{\Theta}$*, if* $\underline{\mathcal{W}_\beth} \neq \emptyset$*.*
2. *Every left (right, bi-) ideal* $\mathcal{W}$ *of* $\ddot{\Theta}$ *is a lower rough left (right, bi-) ideal of* $\ddot{\Theta}$*, if* $\underline{\mathcal{W}_\beth} \neq \emptyset$*.*

3. Bipolar Soft Sets over Semigroups

The BSSs in $\mathcal{S}_g$s are formulated by integrating the RBS approximations of the BSSs with $\mathcal{S}_g$s. Throughout this section, $\ddot{\Theta}$ is a $\mathcal{S}_g$ and $\widehat{\wp}$ is the collection of attributes for $\ddot{\Theta}$.

Definition 3.1. *[35] A triplet* $\eth = (\pounds, \Gamma; \overset{\star}{\Pi})$ *is termed a BSS on a* $\mathcal{S}_g$ Θ*, where* $\breve{A} \subseteq \widehat{\wp}$*,* $\pounds : \overset{\star}{\Pi} \longrightarrow 2^{\ddot{\Theta}}$ *and* $\Gamma : \neg\overset{\star}{\Pi} \longrightarrow 2^{\ddot{\Theta}}$ *such that* $\pounds(\hbar) \cap \Gamma(\neg\hbar) = \emptyset \ \forall \ \hbar \in \overset{\star}{\Pi}$*.*

Definition 3.2. *Let* $\eth_1 = (\pounds_1, \Gamma_1; \overset{\star}{\Pi}_1), \eth_2 = (\pounds_2, \Gamma_2; \overset{\star}{\Pi}_2) \in \mathfrak{BSS}(\ddot{\Theta})$ *for a* $\mathcal{S}_g$ $\ddot{\Theta}$*. The product of* $\eth_1$ *and* $\eth_2$ *is a BSS* $\eth_1 \widehat{*} \eth_2 = (\pounds_1 * \pounds_2, \Gamma_1 * \Gamma_2; \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2)$ *over* $\ddot{\Theta}$*, where*

$$\begin{aligned} (\pounds_1 * \pounds_2)(\hbar) &= \pounds_1(\hbar)\pounds_2(\hbar), \\ (\Gamma_1 * \Gamma_2)(\neg\hbar) &= \left(\Gamma_1'(\neg\hbar)\Gamma_2'(\neg\hbar)\right)' \end{aligned}$$

$\forall \ \hbar \in \overset{\star}{\Pi}$.

Here, $\Gamma_1'(\neg\hbar)$ denotes the crisp compliment $\ddot{\Theta} - \Gamma_1(\neg\hbar)$ of $\Gamma_1(\neg\hbar)$.

Definition 3.3. *A BSS* $\eth$ *over a* $\mathcal{S}_g$ $\ddot{\Theta}$ *is a BS subsemigroup (BS-*$\mathcal{SS}_g$*) over* $\ddot{\Theta}$ *if and only if* $\eth \widehat{*} \eth \widetilde{\subseteq} \eth$*.*

Theorem 3.4. *A BSS* $\eth = (\pounds, \Gamma; \overset{\star}{\Pi}) \in \mathfrak{BSS}(\ddot{\Theta})$ *over a* $\mathcal{S}_g$ $\ddot{\Theta}$ *is a BS-*$\mathcal{SS}_g$ *over* $\ddot{\Theta}$ *if and only if* $\pounds(\hbar)$ *and* $\Gamma'(\neg\hbar)$ *are SSGs of* $\ddot{\Theta} \ \forall \ \hbar \in \overset{\star}{\Pi}$*.*

Proof. Let $\eth = (\pounds, \Gamma; \overset{\star}{\Pi})$ be a BS-$\mathcal{SS}_g$ over $\ddot{\Theta}$. Then, $\eth \widehat{*} \eth \widetilde{\subseteq} \eth$. That is,

$$(\pounds * \pounds, \Gamma * \Gamma; \overset{\star}{\Pi}) \widetilde{\subseteq} (\pounds, \Gamma; \overset{\star}{\Pi}).$$

This gives $(\pounds * \pounds)(\hbar) \subseteq \pounds(\hbar)$ and $(\Gamma * \Gamma)(\neg\hbar) \supseteq \Gamma(\neg\hbar) \ \forall \ \hbar \in \overset{\star}{\Pi}$. Which yields $\pounds(\hbar)\pounds(\hbar) \subseteq \pounds(\hbar)$ and $\left(\Gamma'(\neg\hbar)\Gamma'(\neg\hbar)\right)' \supseteq \Gamma(\neg\hbar)$, that is, $\Gamma'(\neg\hbar)\Gamma'(\neg\hbar) \subseteq \Gamma'(\neg\hbar) \ \forall \ \hbar \in \overset{\star}{\Pi}$. Hence, $\pounds(\hbar)$ and $\Gamma'(\neg\hbar)$ are $\mathcal{SS}_g$s of $\ddot{\Theta} \ \forall \ \hbar \in \overset{\star}{\Pi}$.

Converse follows by reversing the above steps. □

Theorem 3.5. *Let* $\eth_1$ *and* $\eth_2$ *be any two BS-*$\mathcal{SS}_g$*s over a* $\mathcal{S}_g$ $\ddot{\Theta}$*. Then,* $\eth_1 \widetilde{\sqcap}_\varepsilon \eth_2$ *and* $\eth_1 \widetilde{\sqcap}_r \eth_2$ *are also BS-*$\mathcal{SS}_g$*s on* $\ddot{\Theta}$*.*

Proof. Let $\eth_1 = (\pounds_1, \Gamma_1; \overset{\star}{\Pi}_1)$ and $\eth_2 = (\pounds_2, \Gamma_2; \overset{\star}{\Pi}_2)$ be BS-$\mathcal{SS}_g$s on $\ddot{\Theta}$. From Theorem 3.4, $\pounds_1(\hbar_1)$, $\pounds_2(\hbar_2)$, $\Gamma_1'(\neg\hbar_1)$ and $\Gamma_2'(\neg\hbar_2)$ are $\mathcal{SS}_g$s of $\ddot{\Theta} \ \forall \ \hbar_1 \in \breve{A}_1$ and $\hbar_2 \in \breve{A}_2$. The extended intersection of $\eth_1$ and $\eth_2$ is

$$\eth_1 \widetilde{\sqcap}_\varepsilon \eth_2 = (\pounds_1 \widetilde{\cap}_\varepsilon \pounds_2, \Gamma_1 \widetilde{\cup}_\varepsilon \Gamma_2; \overset{\star}{\Pi}_1 \cup \overset{\star}{\Pi}_2).$$

The case is obvious when $\hbar \in \overset{\star}{\Pi}_1 - \overset{\star}{\Pi}_2$ or $\hbar \in \overset{\star}{\Pi}_2 - \overset{\star}{\Pi}_1$. Take $\hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$. Then, $£_1(\hbar) \cap £_2(\hbar)$ and $\Gamma_1'(\neg\hbar) \cap \Gamma_2'(\neg\hbar)$ are $\mathcal{SS}_{\mathfrak{g}}$s of $\ddot{\Theta}$. But,

$$\Gamma_1'(\neg\hbar) \cap \Gamma_2'(\neg\hbar) = \left(\Gamma_1(\neg\hbar) \cup \Gamma_2(\neg\hbar)\right)'.$$

Thus, $£_1(\hbar) \cap £_2(\hbar)$ and $\left(\Gamma_1(\neg\hbar) \cup \Gamma_2(\neg\hbar)\right)'$ are $\mathcal{SS}_{\mathfrak{g}}$s of $\ddot{\Theta}$ $\forall$ $\hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$. Hence, $ð_1 \widetilde{\sqcap}_\varepsilon ð_2$ is a BS-$\mathcal{SS}_{\mathfrak{g}}$ over $\ddot{\Theta}$.

Similar is the proof of the restricted intersection $ð_1 \widetilde{\sqcap}_r ð_2$. □

Remark 3.6. *Note that, if* $ð_1 = (£_1, \Gamma_1; \overset{\star}{\Pi}_1)$ *and* $ð_2 = (£_2, \Gamma_2; \overset{\star}{\Pi}_2)$ *are BS-*$\mathcal{SS}_{\mathfrak{g}}$*s over* $\ddot{\Theta}$ *and* $\overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2 = \varnothing$, *then* $ð_1 \widetilde{\sqcup}_\varepsilon ð_2$ *is surely a BS-*$\mathcal{SS}_{\mathfrak{g}}$ *over* $\ddot{\Theta}$. *But generally, the restricted (or extended) union of two BS-*$\mathcal{SS}_{\mathfrak{g}}$*s over* $\ddot{\Theta}$ *may not be a BS-*$\mathcal{SS}_{\mathfrak{g}}$ *over* $\ddot{\Theta}$. *This is established in the subsequent example.*

Example 3.7. *Let* $\ddot{\Theta} = \{1, a, b, c\}$ *be a* $\mathcal{S}_{\mathfrak{g}}$ *with* $\mathcal{BO}$ *is listed in Table 2.*

.	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	b	a	1

Table 2: Multiplication table of $\mathcal{S}_{\mathfrak{g}}$

Let $\widehat{\wp} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}$. *We consider two BS-*$\mathcal{SS}_{\mathfrak{g}}$*s* $ð_1 = (£_1, \Gamma_1; \overset{\star}{\Pi}_1)$ *and* $ð_2 = (£_2, \Gamma_2; \overset{\star}{\Pi}_2)$ *over* $\ddot{\Theta}$ *with* $\overset{\star}{\Pi}_1 = \{\hbar_1, \hbar_2\}$ *and* $\overset{\star}{\Pi}_2 = \{\hbar_1, \hbar_2, \hbar_3\}$, *defined as:*

$$£_1(\hbar) = \begin{cases} \{1, c\} & \text{if } \hbar = \hbar_1 \\ \{a\} & \text{if } \hbar = \hbar_2 \end{cases} \quad \Gamma_1(\neg\hbar) = \begin{cases} \varnothing & \text{if } \neg\hbar = \neg\hbar_1 \\ \{c\} & \text{if } \neg\hbar = \neg\hbar_2 \end{cases}$$

$$£_2(\hbar) = \begin{cases} \{b\} & \text{if } \hbar = \hbar_1 \\ \{b\} & \text{if } \hbar = \hbar_2 \\ \{1, a, b\} & \text{if } \hbar = \hbar_3 \end{cases} \quad \Gamma_2(\neg\hbar) = \begin{cases} \{a, c\} & \text{if } \neg\hbar = \neg\hbar_1 \\ \{c\} & \text{if } \neg\hbar = \neg\hbar_2 \\ \{c\} & \text{if } \neg\hbar = \neg\hbar_3 \end{cases}$$

The restricted union $ð_1 \widetilde{\sqcup}_r ð_2 = (£_1 \widetilde{\cup}_r £_2, \Gamma_1 \widetilde{\cap}_r \Gamma_2; \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2)$ *is calculated for* $\overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2 = \{\hbar_1, \hbar_2\}$, *as:*

$$(£_1 \widetilde{\cup}_r £_2)(\hbar) = \begin{cases} \{1, b, c\} & \text{if } \hbar = \hbar_1 \\ \{a, b\} & \text{if } \hbar = \hbar_2 \end{cases} \quad (\Gamma_1 \widetilde{\cap}_r \Gamma_2)(\neg e) = \begin{cases} \varnothing & \text{if } \neg\hbar = \neg\hbar_1 \\ \{c\} & \text{if } \neg\hbar = \neg\hbar_2 \end{cases}$$

We find that $b, c \in (£_1 \widetilde{\cup}_r £_2)(\hbar_1)$. *But,* $cb = a \notin (£_1 \widetilde{\cup}_r £_2)(\hbar_1)$. *So,* $ð_1 \widetilde{\sqcup}_r ð_2$ *(and similarly,* $ð_1 \widetilde{\sqcup}_\varepsilon ð_2$*) is not a BS-*$\mathcal{SS}_{\mathfrak{g}}$ *over* $\ddot{\Theta}$.

4. Bipolar Soft Ideals over Semigroups

In the present section, we define the BS left ideals (BS-LIs), BS right ideals (BS-RIs), BS two-sided ideals (BS-2SIs), BS interior ideals (BS-IIs) and BS bi-ideals (BS-BIs) over the $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta}$. Some characterizations of these ideals with concrete illustrations are also discussed in this section.

Definition 4.1. *A BSS* $ð = (£, \Gamma; \overset{\star}{\Pi})$ *over a SG* $\ddot{\Theta}$ *is a BS-LI (resp. BS-RI) over* $\ddot{\Theta}$ *if and only if* $\tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*} ð \mathrel{\widetilde{\sqsubseteq}} ð$ *(resp.* $ð \widehat{*} \tilde{\Omega}_{\overset{\star}{\Pi}} \mathrel{\widetilde{\sqsubseteq}} ð$*).*

A BSS $\eth \in \mathfrak{B}\mathfrak{S}\mathfrak{S}(\ddot{\Theta})$ is a BS-2SI over $\ddot{\Theta}$ when it is both, BS-LI and BS-RI over $\ddot{\Theta}$.

Definition 4.2. *A BSS $\tilde{\Omega}_{\overset{\star}{\Pi}} = (\tilde{\Omega}, \Phi; \overset{\star}{\Pi}) \in \mathfrak{B}\mathfrak{S}\mathfrak{S}(\ddot{\Theta})$ is the relative whole BSS over $\ddot{\Theta}$, if $\tilde{\Omega}(\hbar) = \ddot{\Theta}$ and $\Phi(\neg\hbar) = \varnothing$ $\forall\, \hbar \in \overset{\star}{\Pi}$.*

Theorem 4.3. *A BSS $\eth = (\pounds, \Gamma; \overset{\star}{\Pi})$ on a $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta}$ is a BS-LI (resp. BS-RI, BS-2SI) on $\ddot{\Theta}$ if and only if $\pounds(\hbar)$ and $\Gamma'(\neg\hbar)$ are left (resp. right, two-sided) ideals of $\ddot{\Theta}$ $\forall\, \hbar \in \overset{\star}{\Pi}$.*

Proof. Presume that $\eth = (\pounds, \Gamma; \overset{\star}{\Pi})$ be a BS-LI over $\ddot{\Theta}$. Then, $\tilde{\Omega}_{\overset{\star}{\Pi}} \,\widehat{*}\, \eth \mathrel{\widetilde{\subseteq}} \eth$. That is, $(\tilde{\Omega} * \pounds, \Phi * \Gamma; \overset{\star}{\Pi}) \mathrel{\widetilde{\subseteq}} (\pounds, \Gamma; \overset{\star}{\Pi})$. This gives

$$(\tilde{\Omega} * \pounds)(\hbar) = \tilde{\Omega}(\hbar)\pounds(\hbar) = \ddot{\Theta}.\pounds(\hbar) \subseteq \pounds(\hbar)$$

and

$$\begin{aligned}(\Phi * \Gamma)(\neg\hbar) &= \left(\Phi'(\neg\hbar)\Gamma'(\neg\hbar)\right)' = (\varnothing'\Gamma'(\neg\hbar))' \\ &= \left(\ddot{\Theta}\,\Gamma'(\neg\hbar)\right)' \supseteq \Gamma(\neg\hbar)\end{aligned}$$

$\forall\, \hbar \in \overset{\star}{\Pi}$. Which yields that, $\ddot{\Theta}\pounds(\hbar) \subseteq \pounds(\hbar)$ and $\ddot{\Theta}\Gamma'(\neg\hbar) \subseteq \Gamma'(\neg\hbar)$ $\forall\, \hbar \in \overset{\star}{\Pi}$.

Hence, $\pounds(\hbar)$ and $\Gamma'(\neg\hbar)$ are left ideals of $\ddot{\Theta}$ $\forall\, \hbar \in \overset{\star}{\Pi}$.

Converse follows by reversing the above steps.

Analogously, the cases of the BS-RI and the BS-2SI can be verified. □

Theorem 4.4. *Let $\eth_1$ and $\eth_2$ be any two BS-LI (BS-RI, BS-2SI) over a $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta}$. Then, $\eth_1 \widetilde{\sqcap}_\varepsilon \eth_2$ and $\eth_1 \widetilde{\sqcap}_r \eth_2$ are also BS-LI (BS-RI, BS-2SI) over $\ddot{\Theta}$.*

Proof. Same as the proof of the above theorem. □

Theorem 4.5. *Let $\eth_1$ and $\eth_2$ be any two BS-LIs (BS-RIs, BS-2SIs) over a $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta}$. Then, $\eth_1 \widetilde{\sqcup}_\varepsilon \eth_2$ and $\eth_1 \widetilde{\sqcup}_r \eth_2$ are also BS-LI (BS-RI, BS-2SI) over $\ddot{\Theta}$.*

Proof. Let $\eth_1 = (\pounds_1, \Gamma_1; \overset{\star}{\Pi}_1)$ and $\eth_2 = (\pounds_2, \Gamma_2; \overset{\star}{\Pi}_2)$ be BS-LIs over $\ddot{\Theta}$. From Theorem 4.3, $\pounds_1(\hbar_1)$, $\pounds_2(\hbar_2)$, $\Gamma_1'(\neg\hbar_1)$ and $\Gamma_2'(\neg\hbar_2)$ are left ideals of $\ddot{\Theta}$ $\forall\, \hbar_1 \in \overset{\star}{\Pi}_1$ and $\hbar_2 \in \overset{\star}{\Pi}_2$. The extended union of $\eth_1$ and $\eth_2$ is

$$\eth_1 \widetilde{\sqcup}_\varepsilon \eth_2 = (\pounds_1 \widetilde{\cup}_\varepsilon \pounds_2, \Gamma_1 \widetilde{\cap}_\varepsilon \Gamma_2; \overset{\star}{\Pi}_1 \cup \overset{\star}{\Pi}_2).$$

The case is obvious when $\hbar \in \overset{\star}{\Pi}_1 - \overset{\star}{\Pi}_2$ or $\hbar \in \overset{\star}{\Pi}_2 - \overset{\star}{\Pi}_1$. Take $\hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$. Then, $\pounds_1(\hbar) \cup \pounds_2(\hbar)$ and $\Gamma_1'(\neg\hbar) \cup \Gamma_2'(\neg\hbar)$ are left ideals of $\ddot{\Theta}$. But,

$$\Gamma_1'(\neg\hbar) \cup \Gamma_2'(\neg\hbar) = \left(\Gamma_1(\neg\hbar) \cap \Gamma_2(\neg\hbar)\right)'.$$

Thus, $(\pounds_1 \widetilde{\cup}_\varepsilon \pounds_2)(\hbar)$ and $(\Gamma_1 \widetilde{\cap}_\varepsilon \Gamma_2)'(\neg\hbar)$ are left ideals of $\ddot{\Theta}$ $\forall\, \hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$. Hence, $\eth_1 \widetilde{\sqcup}_\varepsilon \eth_2$ is a BS-LI over $\ddot{\Theta}$.

Similarly, the cases of BS-RIs (BS-2SIs) and the restricted union $\eth_1 \widetilde{\sqcup}_r \eth_2$ can be verified. □

Theorem 4.6. *Let $\ddot{\Theta}$ be a $\mathcal{S}_{\mathfrak{g}}$. Then, the following assertion holds for every BS-RI $\eth_1$ and BS-LI $\eth_2$ over $\ddot{\Theta}$.*

$$\eth_1 \widehat{*}\, \eth_2 \mathrel{\widetilde{\subseteq}} \eth_1 \widetilde{\sqcap}_r \eth_2.$$

Proof. Let $\eth_1 = (£_1, \Gamma_1; \overset{\star}{\Pi}_1)$ be a BS-RI and $\eth_2 = (£_2, \Gamma_2; \overset{\star}{\Pi}_2)$ be a BS-LI over $\ddot{\Theta}$. We have

$$\begin{aligned} \eth_1 \widehat{*} \eth_2 &= (£_1 * £_2, \Gamma_1 * \Gamma_2; \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2), \\ \eth_1 \widetilde{\sqcap}_r \eth_2 &= (£_1 \widetilde{\cap}_r £_2, \Gamma_1 \widetilde{\cup}_r \Gamma_2; \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2). \end{aligned}$$

From Theorem 4.3, $£_1(\hbar)$ and $\Gamma_1'(\neg\hbar)$ are right ideals of $\ddot{\Theta}$ $\forall\ \hbar \in \overset{\star}{\Pi}_1$, while $£_2(\hbar)$ and $\Gamma_2'(\neg\hbar)$ are left ideals of $\ddot{\Theta}$ $\forall\ \hbar \in \overset{\star}{\Pi}_2$. Then, $\forall\ \hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$, we have

$$\begin{aligned} (£_1 * £_2)(\hbar) &= £_1(\hbar)£_2(\hbar) \\ &\subseteq £_1(\hbar) \cap £_2(\hbar) = (£_1 \widetilde{\cap}_r £_2)(\hbar) \end{aligned} \tag{1}$$

and

$$\begin{aligned} \Gamma_1'(\neg\hbar)\Gamma_2'(\neg\hbar) &\subseteq \Gamma_1'(\neg\hbar) \cap \Gamma_2'(\neg\hbar) \\ &= \big(\Gamma_1(\neg\hbar) \cup \Gamma_2(\neg\hbar)\big)'. \end{aligned} \tag{2}$$

Equation (2) gives

$$\begin{aligned} (\Gamma_1 * \Gamma_2)(\neg\hbar) &= \big(\Gamma_1'(\neg\hbar)\Gamma_2'(\neg\hbar)\big)' \\ &\supseteq \Gamma_1(\neg\hbar) \cup \Gamma_2(\neg\hbar) \\ &= (\Gamma_1 \widetilde{\cup}_r \Gamma_2)(\neg\hbar). \end{aligned} \tag{3}$$

The expressions (1) and (3) yield that,

$$\eth_1 \widehat{*} \eth_2 \,\widetilde{\subseteq}\, \eth_1 \widetilde{\sqcap}_r \eth_2$$

$\forall$ BS-RI $\eth_1$ and BS-LI $\eth_2$ over $\ddot{\Theta}$. □

Corollary 4.7. *Let $\ddot{\Theta}$ be an $\mathcal{S}_g$. Then, $\forall$ BS-RI $\eth_1$ and BS-LI $\eth_2$ over $\ddot{\Theta}$, we have*

$$\eth_1 \widehat{*} \eth_2 \,\widetilde{\subseteq}\, \eth_1 \widetilde{\sqcap}_\varepsilon \eth_2.$$

Proof. This is follows directly from Theorem 4.6, as $\eth_1 \widetilde{\sqcap}_r \eth_2 \,\widetilde{\subseteq}\, \eth_1 \widetilde{\sqcap}_\varepsilon \eth_2$. □

Definition 4.8. *A BSS $\eth = (£, \Gamma; \overset{\star}{\Pi})$ over a $\mathcal{S}_g$ $\ddot{\Theta}$ is a BS-II over $\ddot{\Theta}$ if and only if $\tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*} \eth \widehat{*} \tilde{\Omega}_{\overset{\star}{\Pi}} \,\widetilde{\subseteq}\, \eth$.*

Theorem 4.9. *A BSS $\eth = (£, \Gamma; \overset{\star}{\Pi})$ over a $\mathcal{S}_g$ $\ddot{\Theta}$ is a BS-II over $\ddot{\Theta}$ if and only if $£(\hbar)$ and $\Gamma'(\neg\hbar)$ are interior ideals of $\ddot{\Theta}$ $\forall\ \hbar \in \overset{\star}{\Pi}$.*

Proof. Presume that $\eth = (£, \Gamma; \overset{\star}{\Pi})$ be a BS-II over $\ddot{\Theta}$. Then, $\tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*} \eth \widehat{*} \tilde{\Omega}_{\overset{\star}{\Pi}} \,\widetilde{\subseteq}\, \eth$. Which gives

$$(\tilde{\Omega} * £ * \tilde{\Omega}, \Phi * \Gamma * \Phi; \overset{\star}{\Pi}) \,\widetilde{\subseteq}\, (£, \Gamma; \overset{\star}{\Pi}).$$

That is, $(\tilde{\Omega} * £ * \tilde{\Omega})(\hbar) \subseteq £(\hbar)$ and $(\Phi * \Gamma * \Phi)(\neg\hbar) \supseteq \Gamma(\neg\hbar)$ $\forall\ \hbar \in \overset{\star}{\Pi}$. This yields $\tilde{\Omega}(\hbar)£(\hbar)\tilde{\Omega}(\hbar) \subseteq £(\hbar)$ and $\big(\Phi'(\neg\hbar)\Gamma'(\neg\hbar)\Phi'(\neg\hbar)\big)' \supseteq \Gamma(\neg\hbar)$ $\forall\ \hbar \in \overset{\star}{\Pi}$. Thus, it follows that $\ddot{\Theta}£(\hbar)\ddot{\Theta} \subseteq £(\hbar)$ and $\ddot{\Theta}\Gamma'(\neg\hbar)\ddot{\Theta} \subseteq \Gamma'(\neg\hbar)$ $\forall\ \hbar \in \overset{\star}{\Pi}$. Hence, proved that $£(\hbar)$ and $\Gamma'(\neg\hbar)$ are interior ideals of $\ddot{\Theta}$ $\forall\ \hbar \in \overset{\star}{\Pi}$.

Converse follows by reversing the above steps. □

Theorem 4.10. *Let $\eth_1$ and $\eth_2$ be any two BS-IIs over a $\mathcal{S}_g$ $\ddot{\Theta}$. Then, $\eth_1 \widetilde{\sqcap}_\varepsilon \eth_2$ and $\eth_1 \widetilde{\sqcap}_r \eth_2$ are the BS-IIs on $\ddot{\Theta}$.*

Proof. Similar to the proof of the previous result. □

Theorem 4.11. *Suppose that* $ð_1$ *and* $ð_2$ *be any two BS-IIs over a* $\mathcal{S}_g$ $\ddot{\Theta}$. *Then,* $ð_1\widetilde{\sqcup}_\varepsilon ð_2$ *and* $ð_1\widetilde{\sqcup}_r ð_2$ *are also the BS-IIs on* $\ddot{\Theta}$.

Proof. Same as the proof of Theorem 4.5. □

Definition 4.12. *A BS-*$\mathcal{SS}_g$ $ð = (£, \Gamma; \overset{\star}{\Pi})$ *on a* $\mathcal{S}_g$ $\ddot{\Theta}$ *is a BS-BI over* $\ddot{\Theta}$ *if and only if* $ð \,\widehat{*}\, \tilde{\Omega}_{\overset{\star}{\Pi}} \,\widehat{*}\, ð \mathrel{\widetilde{\subseteq}} ð$.

Theorem 4.13. *A BSS* $ð = (£, \Gamma; \overset{\star}{\Pi})$ *on a* $\mathcal{S}_g$ $\ddot{\Theta}$ *is a BS-BI on* $\ddot{\Theta}$ *if and only if* $£(\hbar)$ *and* $\Gamma'(\neg\hbar)$ *are bi-ideals of* $\ddot{\Theta}$ $\forall$ $\hbar \in \overset{\star}{\Pi}$.

Proof. Let $ð = (£, \Gamma; \overset{\star}{\Pi})$ be a BS-BI on $\ddot{\Theta}$. Then, ð is a BS-$\mathcal{SS}_g$ on $\ddot{\Theta}$. From Theorem 3.4, $£(\hbar)$ and $\Gamma'(\neg\hbar)$ are $\mathcal{SS}_g$s of $\ddot{\Theta}$ $\forall$ $\hbar \in \overset{\star}{\Pi}$. Since ð is a BS-BI over $\ddot{\Theta}$, so, $ð \,\widehat{*}\, \tilde{\Omega}_{\overset{\star}{\Pi}} \,\widehat{*}\, ð \mathrel{\widetilde{\subseteq}} ð$. Which gives

$$(£ * \tilde{\Omega} * £, \Gamma * \Phi * \Gamma; \overset{\star}{\Pi}) \mathrel{\widetilde{\subseteq}} (£, \Gamma; \overset{\star}{\Pi}).$$

That is, $(£ * \tilde{\Omega} * £)(\hbar) \subseteq £(\hbar)$ and $(\Gamma * \Phi * \Gamma)(\neg\hbar) \supseteq \Gamma(\neg\hbar)$ $\forall$ $\hbar \in \overset{\star}{\Pi}$. This yields $£(\hbar)\tilde{\Omega}(\hbar)£(\hbar) \subseteq £(\hbar)$ and $\left(\Gamma'(\neg\hbar)\Gamma'(\neg\hbar)\Gamma'(\neg\hbar)\right)' \supseteq \Gamma(\neg\hbar)$ $\forall$ $\hbar \in \overset{\star}{\Pi}$. Thus, we have $£(\hbar)\ddot{\Theta}£(\hbar) \subseteq £(\hbar)$ and $\Gamma'(\neg\hbar)\ddot{\Theta}\Gamma'(\neg\hbar) \subseteq \Gamma'(\neg\hbar)$ $\forall$ $\hbar \in \overset{\star}{\Pi}$. Hence, $£(\hbar)$ and $\Gamma'(\neg\hbar)$ are bi-ideals of $\ddot{\Theta}$ $\forall$ $\hbar \in \overset{\star}{\Pi}$.

Converse follows by reversing the above steps. □

Theorem 4.14. *Let* $ð_1$ *and* $ð_2$ *be any two BS-BIs over a* $\mathcal{S}_g$ $\ddot{\Theta}$. *Then,* $ð_1\widetilde{\sqcap}_\varepsilon ð_2$ *and* $ð_1\widetilde{\sqcap}_r ð_2$ *are still BS-BIs on* $\ddot{\Theta}$.

Proof. Identical to the proof of Theorem 3.5. □

The extended or restricted union of $ð_1$ and $ð_2$ might not be BS-BIs on $\ddot{\Theta}$, since they are not BS-$\mathcal{SS}_g$s on $\ddot{\Theta}$, as exhibited in Example 3.7.

5. Rough Bipolar Soft Sets over Semigroups

The rough BSSs (RBSSs) are described through lower and upper RBS approximations of a BSS on an $\mathcal{S}_g$ $\ddot{\Theta}$, on which a $\mathbb{C}_r$ $\mathfrak{R}$ is specified. These approximations are characterized in this segment. The RBS subsemigroups (RBS-$\mathcal{SS}_g$s) over $\ddot{\Theta}$ are also given in this segment.

Definition 5.1. *Let* $\mathfrak{R}$ *be a* $\mathbb{C}_r$ *on a* $\mathcal{S}_g$ $\ddot{\Theta}$ *and* $ð = (£, \Gamma; \overset{\star}{\Pi}) \in \mathfrak{BSS}(\ddot{\Theta})$. *The lower and upper RBS approximations of* ð *regarding* $(\ddot{\Theta}, \mathfrak{R})$ *are the BSSs symbolized by* $\underline{ð}_{\mathfrak{R}} = (\underline{£}_{\mathfrak{R}}, \underline{\Gamma}_{\mathfrak{R}}; \overset{\star}{\Pi})$ *and* $\overline{ð}^{\mathfrak{R}} = (\overline{£}^{\mathfrak{R}}, \overline{\Gamma}^{\mathfrak{R}}; \overset{\star}{\Pi})$, *respectively, where* $\underline{£}_{\mathfrak{R}}$, $\overline{£}^{\mathfrak{R}}$ *are defined as:*

$$\underline{£}_{\mathfrak{R}}(\hbar) = \left\{u \in \ddot{\Theta} : [u]_{\mathfrak{R}} \subseteq £(\hbar)\right\}, \tag{5.1}$$

$$\overline{£}^{\mathfrak{R}}(\hbar) = \left\{u \in \ddot{\Theta} : [u]_{\mathfrak{R}} \cap £(\hbar) \neq \varnothing\right\} \tag{5.2}$$

$\forall$ $\hbar \in \overset{\star}{\Pi}$, *and* $\underline{\Gamma}_{\mathfrak{R}}$, $\overline{\Gamma}^{\mathfrak{R}}$ *are defined as:*

$$\underline{\Gamma}_{\mathfrak{R}}(\neg\hbar) = \left\{u \in \ddot{\Theta} : [u]_{\mathfrak{R}} \cap \Gamma(\neg\hbar) \neq \varnothing\right\}, \tag{5.3}$$

$$\overline{\Gamma}^{\mathfrak{R}}(\neg\hbar) = \left\{u \in \ddot{\Theta} : [u]_{\mathfrak{R}} \subseteq \Gamma(\neg\hbar)\right\} \tag{5.4}$$

$\forall$ $\neg\hbar \in \neg\overset{\star}{\Pi}$.

If $\underline{ð}_{\mathfrak{R}} = \overline{ð}^{\mathfrak{R}}$, *then,* ð *is said to be* $\mathfrak{R}$*-definable; otherwise,* ð *is an RBSS over* $\ddot{\Theta}$.

In [30], some characterizations of the RBSSs over $\varnothing \neq \tilde{\Omega}$ having an $\mathcal{E}_r$ $\mathfrak{R}$ are given. These characterizations also hold when $\tilde{\Omega}$ is swapped with the $\mathcal{S}_g$ $\ddot{\Theta}$ and the $\hbar_r$ on $\tilde{\Omega}$ is replaced by a $\mathbb{C}_r$ on $\ddot{\Theta}$. Thus, the outcomes in [30] are also valid for RBS approximations of a BSS on the $\mathcal{S}_g$ $\ddot{\Theta}$, given in the Definition 3.1 of [30].

Theorem 5.2. *Presume that $\mathfrak{R}$ be a $\mathbb{C}_r$ on a $\mathcal{S}_g$ $\ddot{\Theta}$ and $\eth_1, \eth_2 \in \mathfrak{BSS}(\ddot{\Theta})$. Then,*

$$\overline{\eth_1}^{\mathfrak{R}} \widehat{*} \overline{\eth_2}^{\mathfrak{R}} \mathrel{\widetilde{\subseteq}} \overline{\eth_1 \widehat{*} \eth_2}^{\mathfrak{R}}.$$

Proof. Let $\eth_1 = (\pounds_1, \Gamma_1; \overset{\star}{\Pi}_1), \eth_2 = (\pounds_2, \Gamma_2; \overset{\star}{\Pi}_2) \in \mathfrak{BSS}(\ddot{\Theta})$. We have

$$\begin{aligned} \overline{\eth_1}^{\mathfrak{R}} \widehat{*} \overline{\eth_2}^{\mathfrak{R}} &= (\overline{\pounds_1}^{\mathfrak{R}} * \overline{\pounds_2}^{\mathfrak{R}}, \overline{\Gamma_1}^{\mathfrak{R}} * \overline{\Gamma_2}^{\mathfrak{R}}; \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2), \\ \overline{\eth_1 \widehat{*} \eth_2}^{\mathfrak{R}} &= (\overline{\pounds_1 * \pounds_2}^{\mathfrak{R}}, \overline{\Gamma_1 * \Gamma_2}^{\mathfrak{R}}; \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2). \end{aligned}$$

Take $\hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$ and $s \in (\overline{\pounds_1}^{\mathfrak{R}} * \overline{\pounds_2}^{\mathfrak{R}})(\hbar) = \overline{\pounds_1}^{\mathfrak{R}}(\hbar)\overline{\pounds_2}^{\mathfrak{R}}(\hbar)$. Then, $s = ab$ for some $a \in \overline{\pounds_1}^{\mathfrak{R}}(\hbar)$ and $b \in \overline{\pounds_2}^{\mathfrak{R}}(\hbar)$. Which gives $[a]_{\mathfrak{R}} \cap \pounds_1(\hbar) \neq \varnothing$ and $[b]_{\mathfrak{R}} \cap \pounds_2(\hbar) \neq \varnothing$. Let $c \in [a]_{\mathfrak{R}} \cap \pounds_1(\hbar)$ and $d \in [b]_{\mathfrak{R}} \cap \pounds_2(\hbar)$. Then, $cd \in [a]_{\mathfrak{R}}[b]_{\mathfrak{R}} \subseteq [ab]_{\mathfrak{R}}$ and also $cd \in \pounds_1(\hbar)\pounds_2(\hbar)$, as, $c \in \pounds_1(\hbar)$ and $d \in \pounds_2(\hbar)$. This yields $[a]_{\mathfrak{R}}[b]_{\mathfrak{R}} \cap \pounds_1(\hbar)\pounds_2(\hbar) \neq \varnothing$. That is, $[ab]_{\mathfrak{R}} \cap (\pounds_1 * \pounds_2)(\hbar) \neq \varnothing$. So, $ab = s \in (\overline{\pounds_1 * \pounds_2}^{\mathfrak{R}})(\hbar)$. Hence, we get

$$(\overline{\pounds_1}^{\mathfrak{R}} * \overline{\pounds_2}^{\mathfrak{R}})(\hbar) \subseteq (\overline{\pounds_1 * \pounds_2}^{\mathfrak{R}})(\hbar) \tag{5.5}$$

$\forall\, \hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$.

Now, let $x \in (\overline{\Gamma_1 * \Gamma_2}^{\mathfrak{R}})(\neg\hbar)$. Then,

$$[x]_{\mathfrak{R}} \subseteq (\Gamma_1 * \Gamma_2)(\neg\hbar) = \left(\Gamma_1'(\neg\hbar)\Gamma_2'(\neg\hbar)\right)'.$$

This means that,

$$[x]_{\mathfrak{R}} \cap \Gamma_1'(\neg\hbar)\Gamma_2'(\neg\hbar) = \varnothing. \tag{5.6}$$

We claim that, $x \notin \left(\overline{\Gamma_1}^{\mathfrak{R}}(\neg\hbar))'(\overline{\Gamma_2}^{\mathfrak{R}}(\neg\hbar)\right)'$. As if, $x \in \left(\overline{\Gamma_1}^{\mathfrak{R}}(\neg\hbar)\right)'\left(\overline{\Gamma_2}^{\mathfrak{R}}(\neg\hbar)\right)'$, then we can write $x = ab$ for some $a \in \left(\overline{\Gamma_1}^{\mathfrak{R}}(\neg\hbar)\right)'$ and $b \in \left(\overline{\Gamma_2}^{\mathfrak{R}}(\neg\hbar)\right)'$. That is, $a \notin \overline{\Gamma_1}^{\mathfrak{R}}(\neg\hbar)$ and $b \notin \overline{\Gamma_2}^{\mathfrak{R}}(\neg\hbar)$. Which means that, $[a]_{\mathfrak{R}} \not\subseteq \Gamma_1(\neg\hbar)$ and $[b]_{\mathfrak{R}} \not\subseteq \Gamma_2(\neg\hbar)$. Let $c \in [a]_{\mathfrak{R}}$, $c \notin \Gamma_1(\neg\hbar)$, $d \in [b]_{\mathfrak{R}}$ and $d \notin \Gamma_2(\neg\hbar)$. Then, $cd \in [a]_{\mathfrak{R}}[b]_{\mathfrak{R}} \subseteq [ab]_{\mathfrak{R}} \subseteq [x]_{\mathfrak{R}}$ and $cd \in \left(\Gamma_1(\neg\hbar)\right)'\left(\Gamma_2(\neg\hbar)\right)'$. Therefore, $[x]_{\mathfrak{R}} \cap \left(\Gamma_1(\neg\hbar)\right)'\left(\Gamma_2(\neg\hbar)\right)' \neq \varnothing$. Which contradicts Equation 5.6. Hence our claim is true, that is,

$$x \in \left(\left(\overline{\Gamma_1}^{\mathfrak{R}}(\neg\hbar)\right)'\left(\overline{\Gamma_2}^{\mathfrak{R}}(\neg\hbar)\right)'\right)' = (\overline{\Gamma_1}^{\mathfrak{R}} * \overline{\Gamma_2}^{\mathfrak{R}})(\neg\hbar).$$

So, we have

$$(\overline{\Gamma_1 * \Gamma_2}^{\mathfrak{R}})(\neg\hbar) \subseteq (\overline{\Gamma_1}^{\mathfrak{R}} * \overline{\Gamma_2}^{\mathfrak{R}})(\neg\hbar) \tag{5.7}$$

$\forall\, \hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$. The assertions (5.5) and (5.7) prove that $\overline{\eth_1}^{\mathfrak{R}} \widehat{*} \overline{\eth_2}^{\mathfrak{R}} \mathrel{\widetilde{\subseteq}} \overline{\eth_1 \widehat{*} \eth_2}^{\mathfrak{R}}$. □

Corollary 5.3. *Let $\ddot{\Theta}$ be a $\mathcal{S}_g$ and $\{\eth_i : 1 \leq i \leq n\} \subset BSS(\ddot{\Theta})$. Then, for a $\mathbb{C}_r$ $\mathfrak{R}$ on $\ddot{\Theta}$, we have*

$$\widehat{*}_{i=1}^{n} \overline{\eth_i}^{\mathfrak{R}} \mathrel{\widetilde{\subseteq}} \overline{\widehat{*}_{i=1}^{n} \eth_i}^{\mathfrak{R}}.$$

Here, $\widehat{*}_{i=1}^{n} \eth_i$ denotes the finite product $\eth_1 \widehat{*} \eth_2 \widehat{*} \cdots \widehat{*} \eth_n$.

Theorem 5.4. *For a* $\mathcal{S}_g$ $\ddot{\Theta}$*, a* $\mathbb{CC}_r$ $\mathfrak{R}$ *on* $\ddot{\Theta}$ *and* $\eth_1, \eth_2 \in \mathfrak{BSS}(\ddot{\Theta})$*, we have*

$$\underline{\eth_1}_{\mathfrak{R}} \widehat{*} \underline{\eth_2}_{\mathfrak{R}} \widetilde{\subseteq} \underline{\eth_1 \widehat{*} \eth_2}_{\mathfrak{R}}.$$

Proof. Let $\eth_1 = (£_1, \Gamma_1; \overset{\star}{\Pi}_1), \eth_2 = (£_2, \Gamma_2; \overset{\star}{\Pi}_2) \in \mathfrak{BSS}(\ddot{\Theta})$. We have

$$\begin{aligned} \underline{\eth_1}_{\mathfrak{R}} \widehat{*} \underline{\eth_2}_{\mathfrak{R}} &= (\underline{£_1}_{\mathfrak{R}} * \underline{£_2}_{\mathfrak{R}}, \underline{\Gamma_1}_{\mathfrak{R}} * \underline{\Gamma_2}_{\mathfrak{R}}; \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2), \\ \underline{\eth_1 \widehat{*} \eth_2}_{\mathfrak{R}} &= (\underline{£_1 * £_2}_{\mathfrak{R}}, \underline{\Gamma_1 * \Gamma_2}_{\mathfrak{R}}, \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2). \end{aligned}$$

Take $\hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$ and let $s \in (\underline{£_1}_{\mathfrak{R}} * \underline{£_2}_{\mathfrak{R}})(\hbar) = \underline{£_1}_{\mathfrak{R}}(\hbar)\underline{£_2}_{\mathfrak{R}}(\hbar)$. Then, we can write $s = ab$ for some $a \in \underline{£_1}_{\mathfrak{R}}(\hbar)$ and $b \in \underline{£_2}_{\mathfrak{R}}(\hbar)$. Which gives $[a]_{\mathfrak{R}} \subseteq £_1(\hbar)$ and $[b]_{\mathfrak{R}} \subseteq £_2(\hbar)$. Then, $[a]_{\mathfrak{R}}[b]_{\mathfrak{R}} \subseteq £_1(\hbar)£_2(\hbar)$. Since $\mathfrak{R}$ is a $\mathbb{CC}_r$, so, $[a]_{\mathfrak{R}}[b]_{\mathfrak{R}} = [ab]_{\mathfrak{R}}$. Which gives $[ab]_{\mathfrak{R}} \subseteq £_1(\hbar)£_2(\hbar)$. That is, $[s]_{\mathfrak{R}} = [ab]_{\mathfrak{R}} \subseteq (£_1 * £_2)(\hbar)$. So, $s \in (\underline{£_1 * £_2}_{\mathfrak{R}})(\hbar)$. Hence,

$$(\underline{£_1}_{\mathfrak{R}} * \underline{£_2}_{\mathfrak{R}})(\hbar) \subseteq (\underline{£_1 * £_2}_{\mathfrak{R}})(\hbar) \tag{5.8}$$

$\forall\ \hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$.

Now, let $x \in (\underline{\Gamma_1 * \Gamma_2}_{\mathfrak{R}})(\neg\hbar)$. Then, $[x]_{\mathfrak{R}} \cap (\Gamma_1 * \Gamma_2)(\neg\hbar) \neq \emptyset$. That is,

$$[x]_{\mathfrak{R}} \cap \left(\Gamma_1'(\neg\hbar)\Gamma_2'(\neg\hbar)\right)' \neq \emptyset. \tag{5.9}$$

We claim that,

$$x \in (\underline{\Gamma_1}_{\mathfrak{R}} * \underline{\Gamma_2}_{\mathfrak{R}})(\neg\hbar) = \left(\left(\underline{\Gamma_1}_{\mathfrak{R}}(\neg\hbar)\right)'\left(\underline{\Gamma_2}_{\mathfrak{R}}(\neg\hbar)\right)'\right)'.$$

As if, $x \notin \left(\left(\underline{\Gamma_1}_{\mathfrak{R}}(\neg\hbar)\right)'\left(\underline{\Gamma_2}_{\mathfrak{R}}(\neg\hbar)\right)'\right)'$, that is, $x \in \left(\underline{\Gamma_1}_{\mathfrak{R}}(\neg\hbar)\right)'\left(\underline{\Gamma_2}_{\mathfrak{R}}(\neg\hbar)\right)'$. Then, we can write $x = yz$, for some $y \in \left(\underline{\Gamma_1}_{\mathfrak{R}}(\neg\hbar)\right)'$ and $z \in \left(\underline{\Gamma_2}_{\mathfrak{R}}(\neg\hbar)\right)'$. So, $[y]_{\mathfrak{R}} \cap \Gamma_1(\neg\hbar) = \emptyset$ and $[z]_{\mathfrak{R}} \cap \Gamma_2(\neg\hbar) = \emptyset$. That is, $[y]_{\mathfrak{R}} \subseteq \Gamma_1'(\neg\hbar)$ and $[z]_{\mathfrak{R}} \subseteq \Gamma_2'(\neg\hbar)$. Thus, $[y]_{\mathfrak{R}}[z]_{\mathfrak{R}} \subseteq \Gamma_1'(\neg\hbar)\Gamma_2'(\neg\hbar)$. As $\mathfrak{R}$ is a $\mathbb{CC}_r$, so, $[y]_{\mathfrak{R}}[z]_{\mathfrak{R}} = [yz]_{\mathfrak{R}} = [x]_{\mathfrak{R}}$. Which gives $[x]_{\mathfrak{R}} \subseteq \Gamma_1'(\neg\hbar)\Gamma_2'(\neg\hbar)$. This contradicts Equation (5.9). So, our claim is true. That is, $x \in (\underline{\Gamma_1}_{\mathfrak{R}} * \underline{\Gamma_2}_{\mathfrak{R}})(\neg\hbar)$. So

$$(\underline{\Gamma_1 * \Gamma_2}_{\mathfrak{R}})(\neg\hbar) \subseteq (\underline{\Gamma_1}_{\mathfrak{R}} * \underline{\Gamma_2}_{\mathfrak{R}})(\neg\hbar) \tag{5.10}$$

$\forall\ \hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$. The assertions (5.8) and (5.10) prove that $\underline{\eth_1}_{\mathfrak{R}} \widehat{*} \underline{\eth_2}_{\mathfrak{R}} \widetilde{\subseteq} \underline{\eth_1 \widehat{*} \eth_2}_{\mathfrak{R}}$. □

Corollary 5.5. *For a* $\mathcal{S}_g$ $\ddot{\Theta}$*, a* $\mathbb{CC}_r$ $\mathfrak{R}$ *on* $\ddot{\Theta}$ *and* $\{\eth_i : 1 \leq i \leq n\} \subset BSS(\ddot{\Theta})$*, we have*

$$\widehat{*}_{i=1}^{n} \underline{\eth_i}_{\mathfrak{R}} \widetilde{\subseteq} \underline{\widehat{*}_{i=1}^{n} \eth_i}_{\mathfrak{R}}.$$

Theorem 5.6. *Presume that* $\mathfrak{R}$ *be a* $\mathbb{C}_r$ *on a* $\mathcal{S}_g$ $\ddot{\Theta}$*. Then, the subsequent assertion is valid* $\forall$ *BS-RI* $\eth_1$ *and BS-LI* $\eth_2$ *over* $\ddot{\Theta}$*.*

$$\overline{\eth_1 \widehat{*} \eth_2}^{\mathfrak{R}} \widetilde{\subseteq} \overline{\eth_1}^{\mathfrak{R}} \widetilde{\sqcap}_r \overline{\eth_2}^{\mathfrak{R}}.$$

Proof. Let $\eth_1 = (£_1, \Gamma_1; \overset{\star}{\Pi}_1)$ be a BS-RI and $\eth_2 = (£_2, \Gamma_2; \overset{\star}{\Pi}_2)$ be a BS-LI over $\ddot{\Theta}$. Then $£_1(\hbar)$ and $\Gamma_1'(\neg\hbar)$ are right ideals $\forall\ \hbar \in \overset{\star}{\Pi}_1$, while $£_2(\hbar)$ and $\Gamma_2'(\neg\hbar)$ are left ideals of $\ddot{\Theta}$, $\forall\ e \in \overset{\star}{\Pi}_2$. We have

$$\begin{aligned} \overline{\eth_1 \widehat{*} \eth_2}^{\mathfrak{R}} &= (\overline{£_1 * £_2}^{\mathfrak{R}}, \overline{\Gamma_1 * \Gamma_2}^{\mathfrak{R}}; \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2), \\ \overline{\eth_1}^{\mathfrak{R}} \widetilde{\sqcap}_r \overline{\eth_2}^{\mathfrak{R}} &= (\overline{£_1}^{\mathfrak{R}} \widetilde{\cap}_r \overline{£_2}^{\mathfrak{R}}, \overline{\Gamma_1}^{\mathfrak{R}} \widetilde{\cup}_r \overline{\Gamma_2}^{\mathfrak{R}}; \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2). \end{aligned}$$

Take $\hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$ and $s \in (\overline{\pounds_1 * \pounds_2}^{\mathfrak{R}})(\hbar)$. Then $[s]_{\mathfrak{R}} \cap (\pounds_1 * \pounds_2)(\hbar) \neq \emptyset$, that is, $[s]_{\mathfrak{R}} \cap \pounds_1(\hbar)\pounds_2(\hbar) \neq \emptyset$. Let $t \in [s]_{\mathfrak{R}} \cap \pounds_1(\hbar)\pounds_2(\hbar)$. Then $t \in [s]_{\mathfrak{R}}$ and $t \in \pounds_1(\hbar)\pounds_2(\hbar)$. We can write $t = ab$, where $a \in \pounds_1(\hbar)$ and $b \in \pounds_2(\hbar)$. Now $ab = t \in \pounds_1(\hbar)$, as $\pounds_1(\hbar)$ is right ideal of $\ddot{\Theta}$ and $ab = t \in \pounds_2(\hbar)$, as $\pounds_2(\hbar)$ is left ideal of $\hat{S}$. So, we have $t \in [s]_{\mathfrak{R}} \cap \pounds_1(\hbar)$ and $t \in [s]_{\mathfrak{R}} \cap \pounds_2(\hbar)$. Which means that,

$$[s]_{\mathfrak{R}} \cap (\pounds_1)(\hbar) \neq \emptyset \neq [s]_{\mathfrak{R}} \cap (\pounds_2)(\hbar).$$

That is, $s \in \overline{\pounds_1}^{\mathfrak{R}}(\hbar) \cap \overline{\pounds_2}^{\mathfrak{R}}(\hbar) = (\overline{\pounds_1}^{\mathfrak{R}} \widetilde{\cap}_r \overline{\pounds_2}^{\mathfrak{R}})(\hbar)$. So, we have

$$(\overline{\pounds_1 * \pounds_2}^{\mathfrak{R}})(\hbar) \subseteq (\overline{\pounds_1}^{\mathfrak{R}} \widetilde{\cap}_r \overline{\pounds_2}^{\mathfrak{R}})(\hbar) \tag{5.11}$$

$\forall\, \hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$.

Now, let $x \in (\overline{\Gamma_1}^{\mathfrak{R}} \widetilde{\cup}_r \overline{\Gamma_2}^{\mathfrak{R}})(\neg\hbar) = (\overline{\Gamma_1}^{\mathfrak{R}})(\neg\hbar) \cup (\overline{\Gamma_2}^{\mathfrak{R}})(\neg\hbar)$. Then, $x \in (\overline{\Gamma_1}^{\mathfrak{R}})(\neg\hbar)$ or $x \in (\overline{\Gamma_2}^{\mathfrak{R}})(\neg\hbar)$. Which means that,

$$[x]_{\mathfrak{R}} \subseteq \Gamma_1(\neg\hbar) \text{ or } [x]_{\mathfrak{R}} \subseteq \Gamma_2(\neg\hbar). \tag{5.12}$$

We claim that $[x]_{\mathfrak{R}} \cap \Gamma_1'(\neg\hbar)\Gamma_2'(\neg\hbar) = \emptyset$. As if there is some $y \in [x]_{\mathfrak{R}} \cap \Gamma_1'(\neg\hbar)\Gamma_2'(\neg\hbar)$. Then, $y = cd$ for some $c \in \Gamma_1'(\neg\hbar)$ and $d \in \Gamma_2'(\neg\hbar)$. But $\Gamma_1'(\neg\hbar)$ is a right ideal and $\Gamma_2'(\neg\hbar)$ is a left ideal of $\ddot{\Theta}$. So, $y = cd \in \Gamma_1'(\neg\hbar)$ and $y = cd \in \Gamma_2'(\neg\hbar)$. Which yields $y \in [x]_{\mathfrak{R}} \cap \Gamma_1'(\neg\hbar)$ and $y \in [x]_{\mathfrak{R}} \cap \Gamma_2'(\neg\hbar)$. This contradicts the assertion (5.12). Thus, $[x]_{\mathfrak{R}} \cap \Gamma_1'(\neg\hbar)\Gamma_2'(\neg\hbar) = \emptyset$. This gives $[x]_{\mathfrak{R}} \subseteq \left(\Gamma_1'(\neg\hbar)\Gamma_2'(\neg\hbar)\right)'$. Which means that, $x \in (\overline{\Gamma_1 * \Gamma_2}^{\mathfrak{R}})(\neg\hbar)$. Thus, we have

$$(\overline{\Gamma_1}^{\mathfrak{R}} \widetilde{\cup}_r \overline{\Gamma_2}^{\mathfrak{R}})(\neg\hbar) \subseteq (\overline{\Gamma_1 * \Gamma_2}^{\mathfrak{R}})(\neg\hbar) \tag{5.13}$$

$\forall\, \hbar \in \overset{\star}{\Pi}_1 \cap \overset{\star}{\Pi}_2$. The statements (5.11) and (5.13) prove that $\overline{\eth_1 \widehat{*} \eth_2}^{\mathfrak{R}} \widetilde{\subseteq} \overline{\eth_1}^{\mathfrak{R}} \widetilde{\sqcap}_r \overline{\eth_2}^{\mathfrak{R}}$. □

Theorem 5.7. *Let $\mathfrak{R}$ be a $\mathbb{CC}_r$ on a $\mathcal{S}_g$ $\ddot{\Theta}$. Then, the subsequent assertion is true $\forall$ BS-RI $\eth_1$ and BS-LI $\eth_2$ over $\ddot{\Theta}$.*

$$\underline{\eth_1 \widehat{*} \eth_2}_{\mathfrak{R}} \widetilde{\subseteq} \underline{\eth_1}_{\mathfrak{R}} \widetilde{\sqcap}_r \underline{\eth_2}_{\mathfrak{R}}.$$

Proof. Identical to the proof of Theorem 5.6. □

Definition 5.8. *Let $\mathfrak{R}$ be a $\mathbb{C}_r$ on a $\mathcal{S}_g$ $\ddot{\Theta}$ and $\eth \in \mathfrak{BSS}(\ddot{\Theta})$. Then, $\eth$ is a lower (or upper) RBS-$\mathcal{SS}_g$ over $\ddot{\Theta}$, if $\underline{\eth}_{\mathfrak{R}}$ (or $\overline{\eth}^{\mathfrak{R}}$) is a BS-$\mathcal{SS}_g$ over $\ddot{\Theta}$.*

A BSS $\eth = (\pounds, \Gamma; \overset{\star}{\Pi})$ on $\ddot{\Theta}$ which is lower and upper RBS-$\mathcal{SS}_g$ on $\ddot{\Theta}$, is named an RBS-$\mathcal{SS}_g$ on $\ddot{\Theta}$.

Theorem 5.9. *Suppose that $\mathfrak{R}$ be a $\mathbb{C}_r$ on a $\mathcal{S}_g$ $\ddot{\Theta}$. Then, every BS-$\mathcal{SS}_g$ on $\ddot{\Theta}$ is an upper RBS-$\mathcal{SS}_g$ on $\ddot{\Theta}$.*

Proof. Let $\eth \in \mathfrak{BSS}(\ddot{\Theta})$ be a BS-$\mathcal{SS}_g$ over $\ddot{\Theta}$. Then, $\eth \widehat{*} \eth \widetilde{\subseteq} \eth$. From Theorems 2.10 and 5.2, we have

$$\overline{\eth}^{\mathfrak{R}} \widehat{*} \overline{\eth}^{\mathfrak{R}} \widetilde{\subseteq} \overline{\eth \widehat{*} \eth}^{\mathfrak{R}} \widetilde{\subseteq} \overline{\eth}^{\mathfrak{R}}.$$

This shows that $\overline{\eth}^{\mathfrak{R}}$ is a BS-$\mathcal{SS}_g$ over $\ddot{\Theta}$. Thus, $\eth$ is an upper RBS-$\mathcal{SS}_g$ over $\ddot{\Theta}$. □

The next illustration highlights that the converse assertion of the previous result is not universally valid.

.	a	b	c	d
a	a	b	b	d
b	b	b	b	d
c	b	b	b	d
d	d	d	d	d

Table 3: $\mathcal{BO}$ for $\mathcal{S}_{\mathfrak{g}}$

Example 5.10. *Presume that $\ddot{\Theta} = \{a, b, c, d\}$ be a $\mathcal{S}_{\mathfrak{g}}$ with $\mathcal{BO}$ described in Table 3.*

Let $\widehat{\wp} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}$ be the set of attributes for $\ddot{\Theta}$. Take a $\mathbb{C}_{\mathrm{r}}$ $\mathfrak{R}$ on $\ddot{\Theta}$, describing the cng-cls $\{a\}, \{b, d\}$ and $\{c\}$. We take a BSS $\eth = (\pounds, \Gamma; \overset{\star}{\Pi})$ over $\ddot{\Theta}$ with $\overset{\star}{\Pi} = \{\hbar_1, \hbar_2\}$, given as follows:

$$\pounds(\hbar) = \begin{cases} \{c, d\} & \text{if } \hbar = \hbar_1 \\ \{b, c\} & \text{if } \hbar = \hbar_2 \end{cases} \quad \Gamma(\neg\hbar) = \begin{cases} \{a\} & \text{if } \neg\hbar = \neg\hbar_1 \\ \{a, d\} & \text{if } \neg\hbar = \neg\hbar_2 \end{cases}$$

Note that $\eth$ is not a BS-$\mathcal{SS}_{\mathfrak{g}}$ over $\ddot{\Theta}$ because $c \in \pounds(\hbar_1)$, but $cc = b \notin \pounds(\hbar_1)$. The upper RBS approximation $\overline{\mathfrak{R}}(\eth) = (\overline{\pounds}^{\mathfrak{R}}, \overline{\Gamma}^{\mathfrak{R}}; \overset{\star}{\Pi})$ of $\eth$ regarding $\mathfrak{R}$ is computed as:

$$\overline{\pounds}^{\mathfrak{R}}(\hbar) = \begin{cases} \{b, c, d\} & \text{if } \hbar = \hbar_1 \\ \{b, c, d\} & \text{if } \hbar = \hbar_2 \end{cases} \quad \overline{\Gamma}^{\mathfrak{R}}(\neg\hbar) = \begin{cases} \{a\} & \text{if } \neg\hbar = \neg\hbar_1 \\ \{a\} & \text{if } \neg\hbar = \neg\hbar_2 \end{cases}$$

Routine calculations reveal that $\overline{\mathfrak{R}}(\eth)$ is a BS-$\mathcal{SS}_{\mathfrak{g}}$ over $\ddot{\Theta}$. So, $\eth$ is not a BS-$\mathcal{SS}_{\mathfrak{g}}$ over $\ddot{\Theta}$, despite it is an upper RBS-$\mathcal{SS}_{\mathfrak{g}}$ on $\ddot{\Theta}$.

Theorem 5.11. *Let $\mathfrak{R}$ be a $\mathbb{CC}_{\mathrm{r}}$ on a $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta}$. Then, every BS-$\mathcal{SS}_{\mathfrak{g}}$ on $\ddot{\Theta}$ is a lower RBS-$\mathcal{SS}_{\mathfrak{g}}$ on $\ddot{\Theta}$.*

Proof. Suppose that $\mathfrak{R}$ be a $\mathbb{CC}_{\mathrm{r}}$ on $\ddot{\Theta}$ and $\eth$ be a BS-$\mathcal{SS}_{\mathfrak{g}}$ over $\ddot{\Theta}$. Then, $\eth \widehat{*} \eth \mathrel{\widetilde{\subseteq}} \eth$. From Theorems 2.10 and 5.4, we have

$$\underline{\eth}_{\mathfrak{R}} \widehat{*} \underline{\eth}_{\mathfrak{R}} \mathrel{\widetilde{\subseteq}} \underline{\eth \widehat{*} \eth}_{\mathfrak{R}} \mathrel{\widetilde{\subseteq}} \underline{\eth}_{\mathfrak{R}}.$$

This verifies that $\underline{\eth}_{\mathfrak{R}}$ is a BS-$\mathcal{SS}_{\mathfrak{g}}$ over $\ddot{\Theta}$. Consequently, $\eth$ is a lower RBS-$\mathcal{SS}_{\mathfrak{g}}$ on $\ddot{\Theta}$. □

The converse of Theorem 5.11 is not necessarily valid, as demonstrated in the subsequent illustration.

Example 5.12. *Let $\ddot{\Theta} = \{s, \tau, \theta, v\}$ be a $\mathcal{S}_{\mathfrak{g}}$ whose $\mathcal{BO}$ is portrayed in Table 4.*

.	s	τ	θ	v
s	s	τ	θ	v
τ	τ	τ	θ	v
θ	θ	θ	θ	v
v	v	v	v	θ

Table 4: $\mathcal{BO}$ of $\mathcal{S}_{\mathfrak{g}}$

Consider $\widehat{\wp} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5\}$ for $\ddot{\Theta}$ and $\mathfrak{R}$ be a $\mathbb{CC}_{\mathrm{r}}$ over $\ddot{\Theta}$, describing cng-cls $\{s\}, \{\tau\}$ and $\{\theta, v\}$. Consider a BSS $\eth = (\pounds, \Gamma; \overset{\star}{\Pi})$ over $\ddot{\Theta}$ with $\overset{\star}{\Pi} = \{\hbar_1\}$, described as:

$$\pounds(\hbar_1) = \{\tau, v\}, \ \Gamma(\neg\hbar_1) = \{s\}.$$

Note that ð *is not a BS-*$\mathcal{SS}_{\mathrm{g}}$ *over* $\ddot{\Theta}$ *because* $v \in \pounds(\hbar_1)$*, but* $vv = \theta \notin \pounds(\hbar_1)$*. The lower RBS approximation* $\underline{\mathfrak{R}}(ð) = (\underline{\pounds}_{\mathfrak{R}}, \underline{\Gamma}_{\mathfrak{R}}; \overset{\star}{\Pi})$ *of* ð *regarding* $\mathfrak{R}$ *is determined as:*

$$\underline{\pounds}_{\mathfrak{R}}(\hbar_1) = \{\tau\},\ \underline{\Gamma}_{\mathfrak{R}}(\neg\hbar_1) = \{s\}.$$

Simple calculations indicate $\underline{\mathfrak{R}}(ð)$ *is a BS-*$\mathcal{SS}_{\mathrm{g}}$ *over* $\ddot{\Theta}$*. So,* ð *is not a BS-*$\mathcal{SS}_{\mathrm{g}}$ *over* $\ddot{\Theta}$*, even though, it is a lower RBS-*$\mathcal{SS}_{\mathrm{g}}$ *on* $\ddot{\Theta}$*.*

The next illustration confirms that the Theorem 5.11 is not true when $\mathfrak{R}$ is not a $\mathbb{CC}_{\mathrm{r}}$.

Example 5.13. *Take the* $\mathcal{S}_{\mathrm{g}}$ $\ddot{\Theta} = \{a, b, c, d\}$ *with* $\widehat{\wp}$ *and the* $\mathbb{C}_{\mathrm{r}}$ $\mathfrak{R}$ *as provided in Example 5.10. Then,* $\mathfrak{R}$ *is not* $\mathbb{CC}_{\mathrm{r}}$*. We consider a BS-*$\mathcal{SS}_{\mathrm{g}}$ $ð = (\pounds, \Gamma; \overset{\star}{\Pi})$ *over* $\ddot{\Theta}$ *with* $\overset{\star}{\Pi} = \{\hbar_2\}$*, postulated as:*

$$\pounds(\hbar_2) = \{b, c\},\ \Gamma(\neg\hbar_2) = \{a\}.$$

The lower RBS approximation $\underline{\mathfrak{R}}(ð) = (\underline{\pounds}_{\mathfrak{R}}, \underline{\Gamma}_{\mathfrak{R}}; \overset{\star}{\Pi})$ *of* ð *regarding* $\mathfrak{R}$ *is computed as:*

$$\underline{\pounds}_{\mathfrak{R}}(\hbar_2) = \{c\},\ \underline{\Gamma}_{\mathfrak{R}}(\neg\hbar_2) = \{a\}.$$

We find that $\underline{\mathfrak{R}}(ð)$ *is not BS-*$\mathcal{SS}_{\mathrm{g}}$ *over* $\ddot{\Theta}$*, because* $c \in \underline{\pounds}_{\mathfrak{R}}(\hbar_2)$*, but we have* $cc = b \notin \underline{\pounds}_{\mathfrak{R}}(\hbar_2)$*. So,* ð *is not a lower RBS-*$\mathcal{SS}_{\mathrm{g}}$ *on* $\ddot{\Theta}$*.*

6. Rough Bipolar Soft Ideals over Semigroups

This section delineates the ideas of the $\mathcal{RBS}-\mathcal{LI}$, $\mathcal{RBS}-\mathcal{RI}$, $\mathcal{RBS}-2\mathcal{SI}$, $\mathcal{RBS}-\mathcal{II}$ and RBS-BI $\ddot{\Theta}$ and examines some of their properties with concrete illustrations.

Definition 6.1. *Let* $\mathfrak{R}$ *be a* $\mathbb{C}_{\mathrm{r}}$ *on a* $\mathcal{S}_{\mathrm{g}}$ $\ddot{\Theta}$ *and* $ð \in \mathfrak{BSS}(\ddot{\Theta})$*. Then,* ð *is a lower (resp. upper)* $\mathcal{RBS}-\mathcal{LI}$ *(*$\mathcal{RBS}-\mathcal{RI}$*,* $\mathcal{RBS}-2\mathcal{SI}$*) over* $\ddot{\Theta}$*, if* $\underline{ð}_{\mathfrak{R}}$ *(resp.* $\overline{ð}^{\mathfrak{R}}$*) is a BS-LI (BS-RI, BS-2SI) on* $\ddot{\Theta}$*.*

A BSS ð in $\ddot{\Theta}$ which is lower and upper $\mathcal{RBS}-\mathcal{LI}$ ($\mathcal{RBS}-\mathcal{RI}$, $\mathcal{RBS}-2\mathcal{SI}$) on $\ddot{\Theta}$ is titled an $\mathcal{RBS}-\mathcal{LI}$ ($\mathcal{RBS}-\mathcal{RI}$, $\mathcal{RBS}-2\mathcal{SI}$) on $\ddot{\Theta}$.

Theorem 6.2. *Let* $\mathfrak{R}$ *be a* $\mathbb{C}_{\mathrm{r}}$ *on a* $\mathcal{S}_{\mathrm{g}}$ $\ddot{\Theta}$*. Then every BS-LI (BS-RI, BS-2SI) on* $\ddot{\Theta}$ *is an upper* $\mathcal{RBS}-\mathcal{LI}$ *(*$\mathcal{RBS}-\mathcal{RI}$*,* $\mathcal{RBS}-2\mathcal{SI}$*) on* $\ddot{\Theta}$*.*

Proof. Let $ð \in \mathfrak{BSS}(\ddot{\Theta})$ be a BS-LI over $\ddot{\Theta}$. Then, $\tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*}\, ð \mathrel{\tilde{\subseteq}} ð$. From Theorem 2.10, we have $\overline{\tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*}\, ð}^{\mathfrak{R}} \mathrel{\tilde{\subseteq}} \overline{ð}^{\mathfrak{R}}$. Now, from Theorems 2.9 and 5.2, we have

$$\begin{aligned} \tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*}\, \overline{ð}^{\mathfrak{R}} &= \overline{\tilde{\Omega}_{\overset{\star}{\Pi}}}^{\mathfrak{R}} \widehat{*}\, \overline{ð}^{\mathfrak{R}} \\ &\mathrel{\tilde{\subseteq}} \overline{\tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*}\, ð}^{\mathfrak{R}} \mathrel{\tilde{\subseteq}} \overline{ð}^{\mathfrak{R}}. \end{aligned}$$

This verifies that $\overline{ð}^{\mathfrak{R}}$ is a BS-LI over $\ddot{\Theta}$. Thus, ð is an upper $\mathcal{RBS}-\mathcal{LI}$ over $\ddot{\Theta}$. Analogously, the scenarios of BS-RI and the BS-2SI over $\ddot{\Theta}$ can be confirmed. □

The converse of the Theorem 6.2 is usually invalid, as revealed by the below illustration.

.	$\mathfrak{p}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{s}$
$\mathfrak{p}$	$\mathfrak{p}$	$\mathfrak{p}$	$\mathfrak{p}$	$\mathfrak{s}$
$\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{q}$	$\mathfrak{p}$	$\mathfrak{s}$
$\mathfrak{r}$	$\mathfrak{p}$	$\mathfrak{p}$	$\mathfrak{r}$	$\mathfrak{s}$
$\mathfrak{s}$	$\mathfrak{s}$	$\mathfrak{s}$	$\mathfrak{s}$	$\mathfrak{s}$

Table 5: $\mathcal{BO}$ of $\mathcal{S}_{\mathfrak{g}}$

Example 6.3. *Let $\ddot{\Theta} = \{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}\}$ be an $\mathcal{S}_{\mathfrak{g}}$ with the following $\mathcal{BO}$ displayed in Table 5.*

Think about $\widehat{\wp} = \{\hbar_1, \hbar_2, \hbar_3\}$ and $\mathfrak{R}$ be a $\mathbb{C}_{\mathfrak{r}}$ on $\ddot{\Theta}$, describing cng-cls $\{\mathfrak{p}, \mathfrak{q}, \mathfrak{s}\}$ and $\{\mathfrak{r}\}$. Think about a BSS $\eth = (\pounds, \Gamma; \overset{\star}{\Pi})$ on $\ddot{\Theta}$ with $\overset{\star}{\Pi} = \{\hbar_1, \hbar_3\}$, given as:

$$\pounds(\hbar) = \begin{cases} \{\mathfrak{p}, \mathfrak{q}\} & \text{if } \hbar = \hbar_1 \\ \{\mathfrak{q}, \mathfrak{s}\} & \text{if } \hbar = \hbar_3 \end{cases} \quad \Gamma(\neg\hbar) = \begin{cases} \{\mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_1 \\ \{\mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_3 \end{cases}$$

Note that $\eth$ is not a BS-LI over $\ddot{\Theta}$ because $\mathfrak{p} \in \pounds(\hbar_1)$, but $\mathfrak{s}\mathfrak{p} = \mathfrak{s} \notin \pounds(\hbar_1)$. The upper RBS approximation $\overline{\mathfrak{R}}(\eth) = (\overline{\pounds}^{\mathfrak{R}}, \overline{\Gamma}^{\mathfrak{R}}; \overset{\star}{\Pi})$ of $\eth$ regarding $\mathfrak{R}$ is evaluated as:

$$\overline{\pounds}^{\mathfrak{R}}(\hbar) = \begin{cases} \{\mathfrak{p}, \mathfrak{q}, \mathfrak{s}\} & \text{if } \hbar = \hbar_1 \\ \{\mathfrak{p}, \mathfrak{q}, \mathfrak{s}\} & \text{if } \hbar = \hbar_3 \end{cases} \quad \overline{\Gamma}^{\mathfrak{R}}(\neg\hbar) = \begin{cases} \{\mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_1 \\ \{\mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_3 \end{cases}$$

Simple calculations verify that $\overline{\mathfrak{R}}(\eth)$ is a BS-LI over $\ddot{\Theta}$. So, $\eth$ is not a BS-LI over $\ddot{\Theta}$, nevertheless, it is an upper $\mathcal{RBS} - \mathcal{LI}$ on $\ddot{\Theta}$.

Theorem 6.4. *Presume that $\mathfrak{R}$ be a $\mathbb{CC}_{\mathfrak{r}}$ on an $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta}$. Then, every BS-LI (BS-RI, BS-2SI) over $\ddot{\Theta}$ is a lower $\mathcal{RBS} - \mathcal{LI}$ ($\mathcal{RBS} - \mathcal{RI}$, $\mathcal{RBS} - 2\mathcal{SI}$) on $\ddot{\Theta}$.*

Proof. Presume that $\mathfrak{R}$ be a $\mathbb{CC}_{\mathfrak{r}}$ on $\ddot{\Theta}$ and let $\eth$ be a BS-LI over $\ddot{\Theta}$. Then, $\tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*}\, \eth \mathrel{\widetilde{\sqsubseteq}} \eth$. From Theorem 2.10, we have $\underline{\tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*}\, \eth}_{\mathfrak{R}} \mathrel{\widetilde{\sqsubseteq}} \underline{\eth}_{\mathfrak{R}}$. Now, from Theorems 2.9 and 5.4, we have

$$\begin{aligned} \tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*}\, \underline{\eth}_{\mathfrak{R}} &= \underline{\tilde{\Omega}_{\overset{\star}{\Pi}}}_{\mathfrak{R}} \widehat{*}\, \underline{\eth}_{\mathfrak{R}} \\ &\mathrel{\widetilde{\sqsubseteq}} \underline{\tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*}\, \eth}_{\mathfrak{R}} \mathrel{\widetilde{\sqsubseteq}} \underline{\eth}_{\mathfrak{R}}. \end{aligned}$$

This verifies that $\underline{\eth}_{\mathfrak{R}}$ is a BS-LI over $\ddot{\Theta}$. Hence, $\eth$ is a lower $\mathcal{RBS} - \mathcal{LI}$ over $\ddot{\Theta}$. Similarly, the instances of BS-RI and the BS-2SI on $\ddot{\Theta}$ can be confirmed. □

The converse statement of Theorem 6.4 is not true in general, as demonstrated in the next illustration.

Example 6.5. *Consider the $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta} = \{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}\}$ as indicated in Example 6.3. Consider a $\mathbb{CC}_{\mathfrak{r}}$ $\mathfrak{R}$ on $\ddot{\Theta}$, defining cng-cls $\{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}\}$ and $\{\mathfrak{s}\}$. Let $\eth = (\pounds, \Gamma; \overset{\star}{\Pi})$ be a BSS over $\ddot{\Theta}$ such that $\overset{\star}{\Pi} = \{\hbar_2, \hbar_3\}$, described as:*

$$\pounds(\hbar) = \begin{cases} \{\mathfrak{q}, \mathfrak{r}, \mathfrak{s}\} & \text{if } \hbar = \hbar_2 \\ \{\mathfrak{q}, \mathfrak{s}\} & \text{if } \hbar = \hbar_3 \end{cases} \quad \Gamma(\neg\hbar) = \begin{cases} \{\mathfrak{p}\} & \text{if } \neg\hbar = \neg\hbar_2 \\ \{\mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_3 \end{cases}$$

Note that $\eth$ is not a BS-LI over $\ddot{\Theta}$ because $m \in \pounds(\hbar_2)$, but $\ell m = k \notin \pounds(\hbar_2)$. The lower RBS approximation $\underline{\mathfrak{R}}(\eth) = (\underline{\pounds}_{\mathfrak{R}}, \underline{\Gamma}_{\mathfrak{R}}; \overset{\star}{\Pi})$ of $\eth$ regarding $\mathfrak{R}$ is assessed as:

$$\underline{\pounds}_{\mathfrak{R}}(\hbar) = \begin{cases} \{\mathfrak{s}\} & \text{if } \hbar = \hbar_2 \\ \{\mathfrak{s}\} & \text{if } \hbar = \hbar_3 \end{cases} \quad \underline{\Gamma}_{\mathfrak{R}}(\neg\hbar) = \begin{cases} \{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_2 \\ \{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_3 \end{cases}$$

Simple calculations show that $\underline{\mathfrak{R}}(\eth)$ is a BS-LI over $\ddot{\Theta}$. So, $\eth$ is not a BS-LI over $\ddot{\Theta}$, even though it is a lower $\mathcal{RBS}-\mathcal{LI}$ over $\ddot{\Theta}$.

The next illustration shows that the above result is invalid when $\mathfrak{R}$ is not a $\mathbb{CC}_{\mathrm{r}}$.

Example 6.6. *Revisit the $\mathcal{S}_{\mathrm{g}}$ $\ddot{\Theta}=\{s,\tau,\theta,v\}$ as revealed in Example 5.12. Take a $\mathbb{C}_{\mathrm{r}}$ $\mathfrak{R}$ on $\ddot{\Theta}$, defining cng-cls $\{s\},\{\tau,\theta\}$, and $\{v\}$. Then, $\mathfrak{R}$ is not $\mathbb{CC}_{\mathrm{r}}$. We consider a BS-LI $\eth=(\pounds,\Gamma;\overset{\star}{\Pi})$ over $\ddot{\Theta}$ with $\overset{\star}{\Pi}=\{\hbar_4\}$, given as:*

$$\pounds(\hbar_4)=\{\theta,v\},\ \Gamma(\neg\hbar_4)=\{s\}.$$

The lower RBS approximation $\underline{\mathfrak{R}}(\eth)=(\underline{\pounds}_{\mathfrak{R}},\underline{\Gamma}_{\mathfrak{R}};\overset{\star}{\Pi})$ of $\eth$ regarding $\mathfrak{R}$ is computed as:

$$\underline{\pounds}_{\mathfrak{R}}(\hbar_4)=\{v\},\ \underline{\Gamma}_{\mathfrak{R}}(\neg\hbar_4)=\{s\}.$$

We find that $\underline{\mathfrak{R}}(\eth)$ is not BS-LI over $\ddot{\Theta}$, because $v\in\underline{\pounds}_{\mathfrak{R}}(\hbar_4)$, but we have $vv=\theta\notin\underline{\pounds}_{\mathfrak{R}}(\hbar_4)$. So, $\eth$ is not a lower $\mathcal{RBS}-\mathcal{LI}$ on $\ddot{\Theta}$.

Definition 6.7. *Let $\mathfrak{R}$ be a $\mathbb{C}_{\mathrm{r}}$ on a $\mathcal{S}_{\mathrm{g}}$ $\ddot{\Theta}$ and $\eth\in\mathfrak{BSS}(\ddot{\Theta})$. Then, $\eth$ is a lower (or upper) $\mathcal{RBS}-\mathcal{II}$ over $\ddot{\Theta}$, if $\underline{\eth}_{\mathfrak{R}}$ (or $\overline{\eth}^{\mathfrak{R}}$) is a BS-II over $\ddot{\Theta}$.*

A BSS $\eth$ over $\ddot{\Theta}$ which is lower and upper $\mathcal{RBS}-\mathcal{II}$ on $\ddot{\Theta}$, is named an $\mathcal{RBS}-\mathcal{II}$ on $\ddot{\Theta}$.

Theorem 6.8. *Let $\mathfrak{R}$ be a $\mathbb{C}_{\mathrm{r}}$ on a $\mathcal{S}_{\mathrm{g}}$ $\ddot{\Theta}$. Then, every BS-II on $\ddot{\Theta}$ is an upper $\mathcal{RBS}-\mathcal{II}$ on $\ddot{\Theta}$.*

Proof. Let $\eth$ be a BS-II over $\ddot{\Theta}$. Then, $\tilde{\Omega}_{\overset{\star}{\Pi}}\widehat{*}\eth\widehat{*}\tilde{\Omega}_{\overset{\star}{\Pi}}\,\widetilde{\subseteq}\,\eth$. From Theorem 2.10, we have $\overline{\tilde{\Omega}_{\overset{\star}{\Pi}}\widehat{*}\eth\widehat{*}\tilde{\Omega}_{\overset{\star}{\Pi}}}^{\mathfrak{R}}\,\widetilde{\subseteq}\,\overline{\eth}^{\mathfrak{R}}$. Now, from Theorem 2.9 and Corollary 5.3, we have

$$\begin{aligned}\tilde{\Omega}_{\overset{\star}{\Pi}}\widehat{*}\overline{\eth}^{\mathfrak{R}}\widehat{*}\tilde{\Omega}_{\overset{\star}{\Pi}} &= \overline{\tilde{\Omega}_{\overset{\star}{\Pi}}}^{\mathfrak{R}}\widehat{*}\overline{\eth}^{\mathfrak{R}}\widehat{*}\overline{\tilde{\Omega}_{\overset{\star}{\Pi}}}^{\mathfrak{R}}\\ &\widetilde{\subseteq}\,\overline{\tilde{\Omega}_{\overset{\star}{\Pi}}\widehat{*}\eth\widehat{*}\tilde{\Omega}_{\overset{\star}{\Pi}}}^{\mathfrak{R}}\,\widetilde{\subseteq}\,\overline{\eth}^{\mathfrak{R}}.\end{aligned}$$

This verifies that $\overline{\eth}^{\mathfrak{R}}$ is a BS-II over $\ddot{\Theta}$. Therefore, $\eth$ is an upper $\mathcal{RBS}-\mathcal{II}$ on $\ddot{\Theta}$. □

The converse of the above result 6.8 is not true universally, as demonstrated in the subsequent illustration.

Example 6.9. *Consider the $\mathcal{S}_{\mathrm{g}}$ $\ddot{\Theta}=\{\mathfrak{p},\mathfrak{q},\mathfrak{r},\mathfrak{s}\}$, as provided in Example 6.3. Consider a BSS $\eth=(\pounds,\Gamma;\overset{\star}{\Pi})$ over $\ddot{\Theta}$ such that $\overset{\star}{\Pi}=\{\hbar_2,\hbar_3\}$, described as:*

$$\pounds(\hbar)=\begin{cases}\{\mathfrak{q},\mathfrak{s}\} & \text{if } \hbar=\hbar_2\\ \{\mathfrak{p},\mathfrak{q}\} & \text{if } \hbar=\hbar_3\end{cases}\qquad \Gamma(\neg\hbar)=\begin{cases}\{\mathfrak{r}\} & \text{if } \neg\hbar=\neg\hbar_2\\ \{\mathfrak{r}\} & \text{if } \neg\hbar=\neg\hbar_3\end{cases}$$

Note that $\eth$ is not a BS-II over $\ddot{\Theta}$ as $l\in\pounds(\hbar_2)$, but $m\ell m=k\notin\pounds(\hbar_2)$. The upper RBS approximation $\overline{\mathfrak{R}}(\eth)=(\overline{\pounds}^{\mathfrak{R}},\overline{\Gamma}^{\mathfrak{R}};\overset{\star}{\Pi})$ of $\eth$ regarding $\mathfrak{R}$ is calculated as:

$$\overline{\pounds}^{\mathfrak{R}}(\hbar)=\begin{cases}\{\mathfrak{p},\mathfrak{q},\mathfrak{s}\} & \text{if } \hbar=\hbar_2\\ \{\mathfrak{p},\mathfrak{q},\mathfrak{s}\} & \text{if } \hbar=\hbar_3\end{cases}\qquad \overline{\Gamma}^{\mathfrak{R}}(\neg\hbar)=\begin{cases}\{\mathfrak{r}\} & \text{if } \neg\hbar=\neg\hbar_2\\ \{\mathfrak{r}\} & \text{if } \neg\hbar=\neg\hbar_3\end{cases}$$

Routine calculations demonstrates that $\overline{\mathfrak{R}}(\eth)$ is a BS-II over $\ddot{\Theta}$. So, $\eth$ is not a BS-II over $\ddot{\Theta}$, nonetheless, it is an upper $\mathcal{RBS}-\mathcal{II}$ on $\ddot{\Theta}$.

Theorem 6.10. *Let $\mathfrak{R}$ be a $\mathbb{CC}_{\mathfrak{r}}$ on a $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta}$. Then, every BS-II on $\ddot{\Theta}$ is a lower $\mathcal{RBS}-\mathcal{II}$ on $\ddot{\Theta}$.*

Proof. Assume that $\mathfrak{R}$ be a $\mathbb{CC}_{\mathfrak{r}}$ on $\ddot{\Theta}$ and ð be a BS-II over $\ddot{\Theta}$. Then, $\tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*}\, ð \,\widehat{*}\, \tilde{\Omega}_{\overset{\star}{\Pi}} \,\widetilde{\subseteq}\, ð$. From Theorem 2.10, we have $\underline{\tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*}\, ð \,\widehat{*}\, \tilde{\Omega}_{\overset{\star}{\Pi}}}_{\mathfrak{R}} \,\widetilde{\subseteq}\, \underline{ð}_{\mathfrak{R}}$. Now, according to Theorem 2.9 and Corollary 5.5, it follows that

$$\begin{aligned} \tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*}\, \underline{ð}_{\mathfrak{R}} \,\widehat{*}\, \tilde{\Omega}_{\overset{\star}{\Pi}} &= \underline{\tilde{\Omega}_{\overset{\star}{\Pi}}}_{\mathfrak{R}} \,\widehat{*}\, \underline{ð}_{\mathfrak{R}} \,\widehat{*}\, \underline{\tilde{\Omega}_{\overset{\star}{\Pi}}}_{\mathfrak{R}} \\ &\widetilde{\subseteq} \underline{\tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*}\, ð \,\widehat{*}\, \tilde{\Omega}_{\overset{\star}{\Pi}}}_{\mathfrak{R}} \,\widetilde{\subseteq}\, \underline{ð}_{\mathfrak{R}}. \end{aligned}$$

This verifies that $\underline{ð}_{\mathfrak{R}}$ is a BS-II over $\ddot{\Theta}$. Thus, ð is a lower $\mathcal{RBS}-\mathcal{II}$ on $\ddot{\Theta}$. □

The converse statement of the above result is not true universally, as confirmed in the accompanying illustration.

Example 6.11. *Revisit the $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta} = \{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}\}$ and the $\mathbb{CC}_{\mathfrak{r}}$ $\mathfrak{R}$ on $\ddot{\Theta}$, as considered in Example 6.5. Think about a BSS $ð = (£, \Gamma; \overset{\star}{\Pi})$ over $\ddot{\Theta}$ with $\overset{\star}{\Pi} = \{\hbar_1, \hbar_2\}$ expressed as:*

$$£(\hbar) = \begin{cases} \{\mathfrak{r}, \mathfrak{s}\} & \text{if } \hbar = \hbar_1 \\ \{\mathfrak{q}, \mathfrak{s}\} & \text{if } \hbar = \hbar_2 \end{cases} \qquad \Gamma(\neg\hbar) = \begin{cases} \{\mathfrak{q}\} & \text{if } \neg\hbar = \neg\hbar_1 \\ \{\mathfrak{p}\} & \text{if } \neg\hbar = \neg\hbar_2 \end{cases}$$

Clearly, ð is not a BS-II over $\ddot{\Theta}$ because $m \in £(\hbar_1)$, but $km\ell = k \notin £(\hbar_1)$. The lower RBS approximation $\underline{\mathfrak{R}}(ð) = (\underline{£}_{\mathfrak{R}}, \underline{\Gamma}_{\mathfrak{R}}; \overset{\star}{\Pi})$ of ð regarding $\mathfrak{R}$ is obtained as:

$$\underline{£}_{\mathfrak{R}}(\hbar) = \begin{cases} \{\mathfrak{s}\} & \text{if } \hbar = \hbar_1 \\ \{\mathfrak{s}\} & \text{if } \hbar = \hbar_2 \end{cases} \qquad \underline{\Gamma}_{\mathfrak{R}}(\neg\hbar) = \begin{cases} \{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_1 \\ \{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_2 \end{cases}$$

Simple calculations shows that $\underline{\mathfrak{R}}(ð)$ is a BS-II over $\ddot{\Theta}$. So, ð is not a BS-II over $\ddot{\Theta}$, even though it is a lower $\mathcal{RBS}-\mathcal{II}$ on $\ddot{\Theta}$.

The next illustration indicates that the Theorem 6.10 is invalid when $\mathfrak{R}$ is not a $\mathbb{CC}_{\mathfrak{r}}$.

Example 6.12. *Take the $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta} = \{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}\}$ and the attribute set $\widehat{\wp}$ as provided in Example 6.3. Take the $\mathbb{C}_{\mathfrak{r}}$ $\mathfrak{R}$ on $\ddot{\Theta}$ defining the cng-cls $\{\mathfrak{p}, \mathfrak{r}, \mathfrak{s}\}$ and $\{\mathfrak{q}\}$. Then, $\mathfrak{R}$ is not a $\mathbb{CC}_{\mathfrak{r}}$. We take a BS-II $ð = (£, \Gamma; \overset{\star}{\Pi})$ over $\ddot{\Theta}$ with $\overset{\star}{\Pi} = \{\hbar_2, \hbar_3\}$, articulated as:*

$$£(\hbar) = \begin{cases} \{\mathfrak{p}, \mathfrak{q}, \mathfrak{s}\} & \text{if } \hbar = \hbar_2 \\ \{\mathfrak{s}\} & \text{if } \hbar = \hbar_3 \end{cases} \qquad \Gamma(\neg\hbar) = \begin{cases} \{\mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_2 \\ \{\mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_3 \end{cases}$$

The lower RBS approximation $\underline{\mathfrak{R}}(ð) = (\underline{£}_{\mathfrak{R}}, \underline{\Gamma}_{\mathfrak{R}}; \overset{\star}{\Pi})$ of ð under $\mathfrak{R}$ is acquired as:

$$\underline{£}_{\mathfrak{R}}(\hbar) = \begin{cases} \{\mathfrak{q}\} & \text{if } \hbar = \hbar_2 \\ \varnothing & \text{if } \hbar = \hbar_3 \end{cases} \qquad \underline{\Gamma}_{\mathfrak{R}}(\neg\hbar) = \begin{cases} \{\mathfrak{p}, \mathfrak{r}, \mathfrak{s}\} & \text{if } \neg\hbar = \neg\hbar_2 \\ \{\mathfrak{p}, \mathfrak{r}, \mathfrak{s}\} & \text{if } \neg\hbar = \neg\hbar_3 \end{cases}$$

We find that $\underline{\mathfrak{R}}(ð)$ is not BS-II over $\ddot{\Theta}$, because $\mathfrak{q} \in \underline{£}_{\mathfrak{R}}(\hbar_2)$, but we have $\mathfrak{pqr} = \mathfrak{p} \notin \underline{£}_{\mathfrak{R}}(\hbar_2)$. So, ð is not a lower $\mathcal{RBS}-\mathcal{II}$ on $\ddot{\Theta}$.

Definition 6.13. *Let $\mathfrak{R}$ be a $\mathbb{C}_{\mathfrak{r}}$ on a $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta}$ and $ð \in BSS(\ddot{\Theta})$. Then, ð is a lower (or upper) RBS bi-ideal on $\ddot{\Theta}$, when $\underline{ð}_{\mathfrak{R}}$ (or $\overline{ð}^{\mathfrak{R}}$) is a BS-BI over $\ddot{\Theta}$.*

A BSS ð over $\ddot{\Theta}$ which is lower and upper RBS-BI on $\ddot{\Theta}$, is termed an RBS-BI on $\ddot{\Theta}$.

Theorem 6.14. *Presume that $\mathfrak{R}$ be a $\mathbb{C}_{\mathfrak{r}}$ on a $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta}$. Then, every BS-BI on $\ddot{\Theta}$ is an upper RBS-BI on $\ddot{\Theta}$.*

Proof. Assume that $\eth \in \mathfrak{BSS}(\ddot{\Theta})$ be a BS-BI over $\ddot{\Theta}$. Then, $\eth$ is BS-$\mathcal{SS}_{\mathfrak{g}}$ over $\ddot{\Theta}$ and $\eth \widehat{*} \tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*} \eth \mathrel{\widetilde{\subseteq}} \eth$. So, $\overline{\eth}^{\mathfrak{R}}$ is a BS-$\mathcal{SS}_{\mathfrak{g}}$ over $\ddot{\Theta}$ from Theorem 5.9 and $\overline{\eth \widehat{*} \tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*} \eth}^{\mathfrak{R}} \mathrel{\widetilde{\subseteq}} \overline{\eth}^{\mathfrak{R}}$ from Theorem 2.10. Now, in the light of Theorem 2.9 and Corollary 5.3, we have

$$\begin{aligned} \overline{\eth}^{\mathfrak{R}} \widehat{*} \tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*} \overline{\eth}^{\mathfrak{R}} &= \overline{\eth}^{\mathfrak{R}} \widehat{*} \overline{\tilde{\Omega}_{\overset{\star}{\Pi}}}^{\mathfrak{R}} \widehat{*} \overline{\eth}^{\mathfrak{R}} \\ &\mathrel{\widetilde{\subseteq}} \overline{\eth \widehat{*} \tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*} \eth}^{\mathfrak{R}} \mathrel{\widetilde{\subseteq}} \overline{\eth}^{\mathfrak{R}}. \end{aligned}$$

This verifies that $\overline{\eth}^{\mathfrak{R}}$ is a BS-BI over $\ddot{\Theta}$. Hence, $\eth$ is an upper RBS-BI on $\ddot{\Theta}$. □

The succeeding illustration shows that the contrary of Theorem 6.14 is incorrect universally.

Example 6.15. *Consider the $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta} = \{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}\}$ and the $\mathbb{C}_{\mathfrak{r}}$ $\mathfrak{R}$ over $\ddot{\Theta}$ as stated in Example 6.3. Take a BSS $\eth = (\pounds, \Gamma; \overset{\star}{\Pi})$ on $\ddot{\Theta}$ with $\ddot{\Theta} = \{\hbar_1, \hbar_2\}$, characterized as:*

$$\pounds(\hbar) = \begin{cases} \{\mathfrak{q}, \mathfrak{p}\} & \text{if } \hbar = \hbar_1 \\ \{\mathfrak{q}, \mathfrak{s}\} & \text{if } \hbar = \hbar_2 \end{cases} \quad \Gamma(\neg\hbar) = \begin{cases} \{\mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_1 \\ \{\mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_2 \end{cases}$$

Note that $\eth$ is not a BS-BI over $\ddot{\Theta}$ because $\ell \in \pounds(\hbar_1)$, but $\ell n \ell = n \notin \pounds(\hbar_1)$. The upper RBS approximation $\overline{\mathfrak{R}}(\eth) = (\overline{\pounds}^{\mathfrak{R}}, \overline{\Gamma}^{\mathfrak{R}}; \overset{\star}{\Pi})$ of $\eth$ regarding $\mathfrak{R}$ is determined as:

$$\overline{\pounds}^{\mathfrak{R}}(\hbar) = \begin{cases} \{\mathfrak{q}, \mathfrak{p}, \mathfrak{s}\} & \text{if } \hbar = \hbar_1 \\ \{\mathfrak{q}, \mathfrak{p}, \mathfrak{s}\} & \text{if } \hbar = \hbar_2 \end{cases} \quad \overline{\Gamma}^{\mathfrak{R}}(\neg\hbar) = \begin{cases} \{\mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_1 \\ \{\mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_2 \end{cases}$$

Routine calculations demonstrates that $\overline{\mathfrak{R}}(\eth)$ is a BS-BI over $\ddot{\Theta}$. Thus, $\eth$ is not a BS-BI over $\ddot{\Theta}$, even though, it is an upper RBS-BI on $\ddot{\Theta}$.

Theorem 6.16. *Let $\mathfrak{R}$ be a $\mathbb{CC}_{\mathfrak{r}}$ on a $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta}$. Then, every BS-BI on $\ddot{\Theta}$ is a lower RBS-BI on $\ddot{\Theta}$.*

Proof. Let $\mathfrak{R}$ be a $\mathbb{CC}_{\mathfrak{r}}$ on $\ddot{\Theta}$ and let $\eth$ be a BS-BI over $\ddot{\Theta}$. Then, $\eth$ is BS-SSG over $\ddot{\Theta}$ and $\eth \widehat{*} \tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*} \eth \mathrel{\widetilde{\subseteq}} \eth$. So, $\underline{\eth}_{\mathfrak{R}}$ is a BS-$\mathcal{SS}_{\mathfrak{g}}$ over $\ddot{\Theta}$ from Theorem 5.11 and $\underline{\eth \widehat{*} \tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*} \eth}_{\mathfrak{R}} \mathrel{\widetilde{\subseteq}} \underline{\eth}_{\mathfrak{R}}$ from Theorem 2.10. Now, by Theorem 2.9 and Corollary 5.5, we have

$$\begin{aligned} \underline{\eth}_{\mathfrak{R}} \widehat{*} \tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*} \underline{\eth}_{\mathfrak{R}} &= \underline{\eth}_{\mathfrak{R}} \widehat{*} \underline{\tilde{\Omega}_{\overset{\star}{\Pi}}}_{\mathfrak{R}} \widehat{*} \underline{\eth}_{\mathfrak{R}} \\ &\mathrel{\widetilde{\subseteq}} \underline{\eth \widehat{*} \tilde{\Omega}_{\overset{\star}{\Pi}} \widehat{*} \eth}_{\mathfrak{R}} \mathrel{\widetilde{\subseteq}} \underline{\eth}_{\mathfrak{R}}. \end{aligned}$$

This verifies that $\underline{\eth}_{\mathfrak{R}}$ is a BS-BI over $\ddot{\Theta}$. As a result, $\eth$ is a lower RBS-BI on $\ddot{\Theta}$. □

The converse assertion of the above result is generally invalid generally, as demonstrated in the subsequent illustration.

Example 6.17. *Revisit the $\mathcal{S}_{\mathfrak{g}}$ $\ddot{\Theta} = \{s, \tau, \theta, v\}$ and the $\mathbb{CC}_{\mathfrak{r}}$ $\mathfrak{R}$ on $\ddot{\Theta}$, as portrayed in Example 5.12. Let us take a BSS $\eth = (\pounds, \Gamma; \overset{\star}{\Pi})$ over $\ddot{\Theta}$ such that $\overset{\star}{\Pi} = \{\hbar_5\}$, described as:*

$$\pounds(\hbar_5) = \{s, \theta\}, \ \Gamma(\neg\hbar_5) = \{\tau, v\}.$$

Clearly, $\eth$ is not a BS-BI over $\ddot{\Theta}$ because $s, \theta \in \pounds(\hbar_5)$, but $sv\theta = v \notin \pounds(\hbar_5)$. The lower RBS approximation $\underline{\mathfrak{R}}(\eth) = (\underline{\pounds}_{\mathfrak{R}}, \underline{\Gamma}_{\mathfrak{R}}; \overset{\star}{\Pi})$ of $\eth$ regarding $\mathfrak{R}$ is calculated as:

$$\underline{£}_{\mathfrak{R}}(\hbar_5) = \{s\},\ \underline{\Gamma}_{\mathfrak{R}}(\neg\hbar_5) = \{t, \theta, v\}.$$

Simple calculations reveal that $\underline{\mathfrak{R}}(ð)$ is a BS-BI over $\ddot{\Theta}$. Thus, ð is not a BS-BI on $\ddot{\Theta}$, however, it is a lower RBS-BI on $\ddot{\Theta}$.

The next illustration shows that Theorem 6.16 is invalid when $\mathfrak{R}$ is not a $\mathbb{CC}_\mathfrak{r}$.

Example 6.18. *Recall the $\mathcal{S}_\mathfrak{g}$ $\ddot{\Theta} = \{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}\}$, the parameter set $\widehat{\wp}$ as taken in Example 6.3. Take the $\mathbb{C}_\mathfrak{r}$ $\mathfrak{R}$ on $\ddot{\Theta}$ defining the classes $\{\mathfrak{p}, \mathfrak{r}, \mathfrak{s}\}$ and $\{\mathfrak{q}\}$. Then, $\mathfrak{R}$ is not a $\mathbb{CC}_\mathfrak{r}$. We take a BS-BI $ð = (£, \Gamma; \overset{\star}{\Pi})$ over $\ddot{\Theta}$ with $\overset{\star}{\Pi} = \{\hbar_1, \hbar_3\}$, expressed as:*

$$£(\hbar) = \begin{cases} \{\mathfrak{p}, \mathfrak{q}, \mathfrak{s}\} & \text{if } \hbar = \hbar_1 \\ \{\mathfrak{r}\} & \text{if } \hbar = \hbar_3 \end{cases} \quad \Gamma(\neg\hbar) = \begin{cases} \{\mathfrak{r}\} & \text{if } \neg\hbar = \neg\hbar_1 \\ \{\mathfrak{p}, \mathfrak{q}, \mathfrak{s}\} & \text{if } \neg\hbar = \neg\hbar_3 \end{cases}$$

The lower RBS approximation $\underline{\mathfrak{R}}(ð) = (\underline{£}_{\mathfrak{R}}, \underline{\Gamma}_{\mathfrak{R}}; \overset{\star}{\Pi})$ of ð under $\mathfrak{R}$ is calculated as:

$$\underline{£}_{\mathfrak{R}}(\hbar) = \begin{cases} \{\mathfrak{q}\} & \text{if } \hbar = \hbar_1 \\ \varnothing & \text{if } \hbar = \hbar_3 \end{cases} \quad \underline{\Gamma}_{\mathfrak{R}}(\neg\hbar) = \begin{cases} \{\mathfrak{p}, \mathfrak{r}, \mathfrak{s}\} & \text{if } \neg\hbar = \neg\hbar_1 \\ \ddot{\Theta} & \text{if } \neg\hbar = \neg\hbar_3 \end{cases}$$

$\underline{\mathfrak{R}}(ð)$ is not BS-BI over $\ddot{\Theta}$, because $\mathfrak{q} \in \underline{£}_{\mathfrak{R}}(\hbar_1)$, but $\mathfrak{q}\mathfrak{r}\mathfrak{q} = \mathfrak{p} \notin \underline{£}_{\mathfrak{R}}(\hbar_2)$. Therefore, ð is not a lower RBS-BI over $\ddot{\Theta}$.

7. Comparative Study

Rough BS ideals in $\mathcal{S}_\mathfrak{g}$s is a unique idea that combines RSs and BSSs to examine ideals in $\mathcal{S}_\mathfrak{g}$s. In this script, we present an innovative framework for approximating the BS ideals in $\mathcal{S}_\mathfrak{g}$s. To the best of our knowledge, $\mathcal{S}_\mathfrak{g}$s have not commonly implemented BSS approximations to date. Consequently, consideration of a novel scheme of approximations of BS ideals in $\mathcal{S}_\mathfrak{g}$s is reasonable and indispensable. This method provides a more flexible and enhanced approach to addressing uncertainty in $\mathcal{S}_\mathfrak{g}$ philosophy by examining rough approximations in the formulation of BS ideals. Rough BS ideals in $\mathcal{S}_\mathfrak{g}$s possess various unique advantages compared to other theories, including:

1. One of the primary benefits of rough BS ideals is their capacity to encapsulate uncertainty and vagueness. Integrating RSs with BSSs enables the management of a broader spectrum of data and enhances the precision in capturing the imprecision and bipolarity commonly encountered in actual dilemmas.
2. FSs and $\mathcal{SS}$s are merged with ideals in $\mathcal{S}_\mathfrak{g}$s in many ways (see [2, 15, 40, 43]). Numerous researchers have examined ideals in $\mathcal{S}_\mathfrak{g}$s [17, 22]. Nonetheless, their outcomes fail to consider the bipolarity, which is a vital element of human perception. In contrast, rough BS ideals can more naturally and comprehensibly represent both the good and negative attributes of the data, together with the associated trade-offs.
3. A multitude of studies on the bipolarity in fuzzy ideals and groups are stated in [19, 21, 32, 47], but the roughness of the proposed ideals is not studied in these papers. Our proposed study is a unification of RSs, BSSs, and the bipolarity of data employed in the ideal theory of $\mathcal{S}_\mathfrak{g}$s. In this study, we implement the notions of roughness to the BS ideals in $\mathcal{S}_\mathfrak{g}$s, which is the uniqueness and originality of our recommended approach.

8. Concluding Remarks

In the present paper, we brought out the notion of the ideals of $\mathcal{S}_\mathfrak{g}$s in the context of BSSs. The idea is further generalized to RBSSs in $\mathcal{S}_\mathfrak{g}$s by establishing the lower and upper approximations of BSSs in $\mathcal{S}_\mathfrak{g}$s. Also, the idea of rough BS ideals over $\mathcal{S}_\mathfrak{g}$s is established. The $\mathbb{C}_\mathfrak{r}$ and $\mathbb{CC}_\mathfrak{r}$ are employed to evaluate rough approximations of BS ideals in $\mathcal{S}_g$s. These relations provide us with a useful method to contemplate roughness in BS ideals of $\mathcal{S}_g$s. However, it is probably evident that in the case of lower approximation, the $\mathbb{C}_r$ fails to acquire the desired outcomes. To fix this complication, a $\mathbb{C}\mathbb{C}_r$ is being considered. Furthermore, approximations are implemented to the BS-LIs, BS-RIs, BS-2SIs, BS-IIs, and BS-BIs of $\mathcal{S}_g$s using $\mathbb{C}_r$ and $\mathbb{C}\mathbb{C}_r$. Various significant aspects of these concepts are studied comprehensively. We have incorporated numerous illustrations to aid readers in understanding the issues under discussion.

In future research, we will explore the following related topics:

1. We would like to concentrate on the roughness of BS ideals in ternary $\mathcal{S}_g$s and ternary semirings.
2. We think that researchers will look into the features of rough BS ideals in a number of different algebraic contexts, such as normal groups, semirings, and hemirings.
3. We will give attention to the roughness of BS hyperideals of semihyper groups.
4. The devised scheme of roughness can be carried out for the other two notions of BSSs, proposed by Mahmood [55] and Karaaslan and Karataş [18].

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