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Numerical solution of a class of quadratic matrix equations

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Abstract. Quadratic matrix equations arise in many fields of scientific computing and engineering applications. In this paper, we study a class of quadratic matrix equations. Firstly, we prove the existence of minimal nonnegative solution for this quadratic matrix equation under a certain condition. Then, we propose some numerical methods for solving it. Finally, convergence analysis and numerical examples are given to verify the theories and the numerical methods of this paper.

1. Introduction

In this paper, we study quadratic matrix equations of the form

$$X^2 - BX + C = 0, \tag{1}$$

where $B \in \mathbb{R}^{n\times n}$ is a nonsingular M-matrix and $C \in \mathbb{R}^{n\times n}$ is a nonnegative matrix. Quadratic matrix equations usually arise in many fields of scientific computing and engineering applications, such as the quadratic eigenvalue problem in the analysis of damped structural systems and vibration problems [19], the Quasi-Birth-Death problem in telecommunication computer performance and inventory control [4, 6, 15], and the noisy Wiener-Hopf problem in Markov chains [10] and so on. In particular, equation (1) is motivated from the study of quadratic eigenvalue problem and noisy Wiener-Hopf problem.

The study on the theories and numerical methods for quadratic matrix equations is very earlier and extensive. Davis [8] first considered Newton method for solving quadratic matrix equations. Higham and Kim [12, 13] improved the convergence of Newton method by incorporating with exact line searches. Some modifications of Newton method are proposed in [9, 17]. In general, Newton method is not competitive in terms of CPU time, since a Sylvester equation is needed to solve at each iteration which is very expensive, while Bernoulli method is usually linearly convergent and sometimes can be very slow. Bai et al [1, 2] constructed a modified Bernoulli method for quadratic matrix equations. For a class of quadratic matrix equation with an M-matrix, Yu et al [20, 21] proved the existence and uniqueness of the maximal nonpositive solution, and discussed the convergence analysis of Newton method and Bernoulli method in details. Kim et al [14] developed a diagonal update method for solving a quadratic matrix equation, which is a modification

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of the Bernoulli method and is a little faster than Bernoulli method. Chen [7] proposed a structure-preserving doubling algorithm for a quadratic matrix equation with an M-matrix, which is quadratically convergent and less expensive than Newton method. For more discussions on quadratic matrix equations one can refer to [5, 16, 18].

In the following, we give some notations and definitions that are needed in the sequel. For more details, we refer to [3].

Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$. If $a_{ij} \geq 0$ for all i, j, then A is called a nonnegative matrix, denoted by $A \geq 0$. If $a_{ij} > 0$ for all i, j, then A is called a positive matrix, denoted by $A > 0$. For $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{m\times n}$, we write $A \geq B$ if $A - B \geq 0$, and $A > B$ if $A - B > 0$. For $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, if $a_{ij} \leq 0$ for all $i \neq j$, then A is called a Z-matrix. A Z-matrix A is called an M-matrix if there exists a nonnegative matrix B such that $A = sI - B$ and $s \geq \rho(B)$ where $\rho(B)$ is the spectral radius of B. In particular, A is called a nonsingular M-matrix if $s > \rho(B)$ and singular M-matrix if $s = \rho(B)$.

Definition 1.1. *([11]) Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix. Then A is said to be regular if $Av \geq 0$ for some $v > 0$.*

Lemma 1.2. *([3]) Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix. Then the following statements are equivalent:*
(1) A is a nonsingular M-matrix;
(2) $A^{-1} \geq 0$;
(3) $Av > 0$ for some vector $v > 0$.

Lemma 1.3. *([3]) Let A and B be Z-matrices. If A is a nonsingular M-matrix and $A \leq B$, then B is also a nonsingular M-matrix.*

It is easy to verify that nonsingular M-matrices are always regular M-matrices. Any Z-matrix A such that $Av \geq 0$ for some $v > 0$ is a regular M-matrix.

The rest of the paper is organized as follows. In Section 2, under a weak condition, we prove the existence of minimal nonnegative solution for equation (1). In Section 3, we propose some numerical methods for solving equation (1), and discuss convergence analysis of them. In Section 4, we use some numerical examples to validate the theories and numerical behaviours of the methods. Conclusions are given in Section 5.

2. Theoretical analysis

In applications, the solution of practical interest for equation (1) is the minimal nonnegative solution. In this section, we show the existence of minimal nonnegative solution for equation (1) under certain conditions.

To achieve this goal, we first write equation (1) as a fixed-point form

$$X = B^{-1}(X^2 + C),$$

and then consider the iteration

$$X_{k+1} = B^{-1}(X_k^2 + C), \quad X_0 = 0. \tag{2}$$

Since B is a nonsingular M-matrix, it is evident that the sequence $\{X_k\}$ is well defined. In addition, we have the following results.

Theorem 2.1. *For equation (1), if $B - I - C$ is a regular M-matrix, then the sequence $\{X_k\}$ generated by (2) is monotonically increasing and is bounded from above.*

Proof. We prove the assertion by induction.

(1) First we show that the sequence $\{X_k\}$ is monotonically increasing. When $k = 0$, we have $X_1 = B^{-1}C \geq 0 = X_0$. If $X_k \geq X_{k-1}$ holds true, then from

$$X_{k+1} = B^{-1}(X_k^2 + C), \quad X_k = B^{-1}(X_{k-1}^2 + C),$$

we have

$$X_{k+1} - X_k = B^{-1}(X_k^2 - X_{k-1}^2) \geq 0.$$

By induction, the sequence $\{X_k\}$ is monotonically increasing.

(2) Next we show that the sequence $\{X_k\}$ is bounded from above. Since $B - I - C$ is a regular M-matrix, there is a positive vector $u > 0$ such that

$$(B - I - C)u = Bu - u - Cu \geq 0.$$

Hence we have $Cu \leq Bu - u < Bu$. Now we will show that $X_k u < u$ holds for all $k \geq 0$. When $k = 0$, it is evident. If $X_k u < u$ holds for k, then

$$\begin{aligned} X_{k+1}u &= B^{-1}(X_k^2 + C)u \\ &\leq B^{-1}(X_k^2 u + Bu - u) \\ &= B^{-1}X_k^2 u + u - B^{-1}u \\ &< B^{-1}u + u - B^{-1}u \\ &= u. \end{aligned}$$

By induction, $X_k u < u$ hold for all $k \geq 0$. □

Theorem 2.2. *If $B - I - C$ is a regular M-matrix, then equation (1) has a minimal nonnegative solution X, and $B - X$ is a regular M-matrix. In particular, if $B - I - C$ is nonsingular, $B - X$ is also nonsingular.*

Proof. (1) We have shown in Theorem 2.1 that the sequence $\{X_k\}$ generated by (2) is monotonically increasing and is bounded from above. Thus it has a limit $\lim_{k\to\infty} X_k = X$. Taking limit on both sides of (2), we know that X is a solution of (1). Since $X \geq 0$, it is a nonnegative solution. If $Y \geq 0$ is another nonnegative solution of (1), we can prove by induction as in Theorem 2.1 that $X_k \leq Y$. Taking limit we have $X \leq Y$. Hence X is the minimal nonnegative solution.

(2) In Theorem 2.1, we have shown that $Cu \leq Bu - u$ and that $X_k u < u$ holds for all $k \geq 0$. Taking limit we have $Xu \leq u$. Since $B - X$ is a Z-matrix, and

$$\begin{aligned} (B - X)u &= Bu - Xu \geq Bu - u \\ &\geq Cu \geq 0, \end{aligned}$$

we can conclude that $B - X$ is a regular M-matrix. If $B - I - C$ is nonsingular, we can obtain similarly that $Cu < Bu - u$, and

$$\begin{aligned} (B - X)u &= Bu - Xu \geq Bu - u \\ &> Cu \geq 0. \end{aligned}$$

Hence by Lemma 1.2, $B - X$ is a nonsingular M-matrix. □

In the following, a few examples are given to verify Theorem 2.2.

Example 2.1. Consider equation (1) with

$$B = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

In this example, $B - I - C$ is an irreducible singular M-matrix. By direct computation, we can find four solutions

$$X_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix},$$

$$X_3 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad X_4 = \frac{1}{2}\begin{pmatrix} 7 & -3 \\ -3 & 7 \end{pmatrix}.$$

Here X_3 is the minimal nonnegative solution and $B - X_3$ is a nonsingular M-matrix.

Example 2.2. Consider equation (1) with

$$B = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this example, $B - I - C$ is an irreducible singular M-matrix. By direct computation, we can find two solutions

$$X_1 = \frac{\sqrt{3}-1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}, \quad X_2 = \frac{\sqrt{3}+1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ -1 & \sqrt{3} \end{pmatrix}.$$

Here X_1 is the minimal nonnegative solution and it is easy to verify that $B - X_1$ is a nonsingular M-matrix.

3. Newton method and Bernoulli method

In this section, we propose some numerical methods for solving equation (1), and then discuss convergence analysis of them.

It is evident that the fixed-point iteration (2) in Section 2 can be used to compute the minimal nonnegative solution of equation (1). However, the fixed-point iteration method is usually very slow to converge and requires a lot of iterations. So we need consider some quick iteration methods.

Newton method has been used to solve general quadratic matrix equations. Apply Newton method to equation (1), we have the following iteration:

$$(B - X_k)X_{k+1} - X_{k+1}X_k = C - X_k^2, \quad X_0 = 0. \tag{3}$$

However, a Sylvester equation is must to be solved in each iteration, which will cost $60n^3$ if we use the Bartels-Stewart algorithm. Hence Newton method is a little expensive.

In the following, we consider Bernoulli iteration method. Write equation (1) as a fixed-point form

$$(B - X)X = C,$$

then we have Bernoulli's iteration as follows

$$(B - X_k)X_{k+1} = C, \quad X_0 = 0. \tag{4}$$

At each iteration, a linear matrix equation is to be solved, which will cost $8n^3/3$ and is cheaper than Newton method.

In the following, we give convergence analysis of the Bernoulli method.

Lemma 3.1. *For the equation (1), if $B - I - C$ is a regular M-matrix and $B - X$ is a nonsingular M-matrix, then the sequence $\{X_k\}$ generated by (4) is well-defined and satisfies*

$$0 \le X_k \le X, \quad X_k \le X_{k+1}, \quad k \ge 0, \tag{5}$$

where X is the minimal nonnegative solution.

Proof. We prove (5) by induction.

When $k = 0$, we have $0 = X_0 \le X$. Since $BX_1 = C$, X_1 is well-defined and satisfies $X_1 = B^{-1}C \ge 0 = X_0$. Thus (5) is true for $k = 0$.

Suppose now that (5) is true for $k-1$. Since $B-X$ is a nonsingular M-matrix and $0 \le X_{k-1} \le X$, we know from Lemma 1.3 that $B - X_{k-1}$ is also a nonsingular M-matrix, and thus X_k is well-defined. In addition, we have

$$\begin{aligned} X - X_k &= (B-X)^{-1}C - (B-X_{k-1})^{-1}C \\ &= (B-X)^{-1}[(B-X_{k-1}) - (B-X)](B-X_{k-1})^{-1}C \\ &= (B-X)^{-1}(X - X_{k-1})(B-X_{k-1})^{-1}C \\ &\ge 0. \end{aligned}$$

Since $0 \le X_k \le X$, $B - X_k$ is a nonsingular M-matrix. Thus X_{k+1} is well-defined and

$$\begin{aligned} X_{k+1} - X_k &= (B-X_k)^{-1}C - (B-X_{k-1})^{-1}C \\ &= (B-X_k)^{-1}[(B-X_{k-1}) - (B-X_k)](B-X_{k-1})^{-1}C \\ &= (B-X_k)^{-1}(X_k - X_{k-1})(B-X_{k-1})^{-1}C \\ &\ge 0. \end{aligned}$$

Hence (5) is true for k. By induction, (5) holds true for all $k \ge 0$. □

Theorem 3.2. *For the equation (1), if $B-I-C$ is a regular M-matrix and $B-X$ is a nonsingular M-matrix, then the sequence $\{X_k\}$ generated by (4) is well-defined, monotonically increasing and converges to X, where X is the minimal nonnegative solution.*

Proof. We have shown in Lemma 3.1 that the sequence $\{X_k\}$ generated by (4) is well-defined, monotonically increasing and is bounded from above. Thus it has a limit X^*. Taking limit on both side of (4), we know that X^* is a nonnegative solution of (1). By Lemma 3.1, we have $X^* \le X$. On the other hand, since X is the minimal nonnegative solution, we have $X \le X^*$. Thus $X^* = X$. □

Theorem 3.3. *The convergent rate of (4) is given by*

$$\limsup_{k\to\infty} \sqrt[k]{\|X - X_k\|} \le \rho((B-X)^{-1}) \cdot \rho(X).$$

Proof. We have

$$\begin{aligned} X - X_k &= (B-X)^{-1}C - (B-X_{k-1})^{-1}C \\ &= (B-X)^{-1}(X - X_{k-1})(B-X_{k-1})^{-1}C \\ &\le (B-X)^{-1}(X - X_{k-1})(B-X)^{-1}C \\ &= (B-X)^{-1}(X - X_{k-1})X \\ &\le \cdots \\ &= (B-X)^{-k}(X - X_0)X^k \end{aligned}$$

After taking norm and then k-th square root, we can get

$$\sqrt[k]{\|X - X_k\|} \le \sqrt[k]{\|(B-X)^{-k}\|} \cdot \sqrt[k]{\|X - X_0\|} \cdot \sqrt[k]{\|X^k\|}.$$

Taking limit on both side and noting that $\lim_{k\to\infty} \sqrt[k]{\|A^k\|} = \rho(A)$, we have

$$\limsup_{k\to\infty} \sqrt[k]{\|X - X_k\|} \le \rho((B-X)^{-1}) \cdot \rho(X).$$

□

4. Numerical examples

In this section we use some numerical examples to verify the theories and the numerical methods presented in this paper. We will compare the numerical behaviours of Newton method (3), Fixed-point iteration method (2), and Bernoulli method (4), denoted by Newton, FP, and Bernoulli respectively. In addition, we will present numerical results in terms of the numbers of iterations (IT), CPU time (CPU, in seconds) and the residue (RES), where

$$RES := \frac{\|X^2 - BX + C\|_\infty}{\|C\|_\infty}.$$

In our implementations all iterations are performed in Matlab (R2012a) on a personal computer with 2 GHz CPU and 16 GB of memory and are terminated when the current iterate satisfies $RES < 10^{-6}$ or the number of iterations is more than 3000.

Example 4.1. Consider equation (1) with

$$B = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

In this example, $B - I - C$ is a regular M-matrix and the minimal nonnegative solution is

$$X = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$$

The numerical results are summarized in Table 1. From Table 1, we can conclude that all the three methods can compute the solution as required accuracy. In addition, Newton method has the best numerical behaviours.

Table 1: Numerical results of Example 4.1

Method	IT	RES
Newton	5	1.1642e-10
Bernoulli	18	9.5368e-07
FP	30	8.3995e-07

Example 4.2. Consider equation (1) with

$$B = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix}, \quad C = I.$$

This example is from [1], where $B - I - C$ is a nonsingular M-matrix and is near to singular when n is large. For different sizes of n, the numerical results are summarized in Table 2. From Table 2, we can conclude that though the number of iterations and CPU time for all the three methods increase as n increases, all the three methods can compute the solution as required accuracy. So all the three methods are feasible. In addition, Newton method needs the lest CPU time and the lest iteration numbers.

Example 4.3. Consider equation (1) with

$$B = \begin{pmatrix} 5 & -1 & & & \\ -1 & 5 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 5 & -1 \\ & & & -1 & 5 \end{pmatrix}, \quad C = I.$$

Table 2: Numerical results of Example 4.2

n	Method	IT	CPU	RES
100	Newton	8	0.0361	5.9804e-10
	Bernoulli	136	0.0293	9.8108e-07
	FP	264	0.0574	9.9903e-07
200	Newton	8	0.1688	4.1669e-07
	Bernoulli	228	0.1956	9.6992e-07
	FP	447	0.4340	9.9356e-07
300	Newton	9	0.5222	1.2665e-08
	Bernoulli	302	0.7151	9.9731e-07
	FP	597	1.6047	9.9236e-07
400	Newton	9	1.2093	1.0261e-07
	Bernoulli	367	2.0646	9.8517e-07
	FP	725	4.5574	9.9707e-07
500	Newton	9	1.9882	3.2685e-07
	Bernoulli	423	4.0132	9.9192e-07
	FP	838	9.5393	9.9519e-07

This example is a modification of Example 4.2. For different sizes of n, the numerical results are summarized in Table 3. From Table 3, we can conclude that all the three methods can compute the solution as required accuracy. In particular, Newton method needs the most CPU time in this example, while Bernoulli method is a little cheaper.

Table 3: Numerical results of Example 4.3

n	Method	IT	CPU	RES
100	Newton	4	0.0170	2.3446e-13
	Bernoulli	8	0.0025	1.4977e-07
	FP	10	0.0033	4.4914e-07
200	Newton	4	0.0795	2.7085e-13
	Bernoulli	8	0.0088	1.4977e-07
	FP	10	0.0109	4.4914e-07
500	Newton	4	0.8969	3.4633e-13
	Bernoulli	8	0.1136	1.4977e-07
	FP	10	0.1845	4.4914e-07
800	Newton	4	4.1688	4.0366e-13
	Bernoulli	8	0.8154	1.4977e-07
	FP	10	1.1216	4.4914e-07
1000	Newton	4	8.0529	4.4677e-13
	Bernoulli	8	1.5944	1.4977e-07
	FP	10	1.9230	4.4914e-07

From the above three examples we can conclude that in general Newton method can converge quickly, but is a little expensive. Bernoulli method needs more iterations than Newton method usually, but may be cheaper than Newton method in some cases.

5. Conclusions

We studied a class of quadratic matrix equations in this paper. Under a weak condition, we proved the existence of minimal nonnegative solution for this quadratic matrix equation. In addition, some numerical methods are proposed to solve this quadratic matrix equation. Theoretical analysis and numerical examples have shown the validations of the theories and the numerical methods in this paper.

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