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Developing of some inequalities on golden Riemannian manifolds endowed with semi-symmetric connections

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Abstract. The objective of this paper is to obtain generalised Wintgen inequalities for submanifolds that are immersed in golden Riemannian manifolds endowed with semi-symmetric metric and semi-symmetric non-metric connections by employing mathematical operators.

1. Introduction

Let $\mathcal{M}^2$ represents any surface in the Euclidean space E^4, then Wintgen inequality can be asserted as follows [25]

$$\|\mathcal{H}\|^2 \geq \mathcal{K} + |\mathcal{K}^{\perp}|, \tag{1}$$

in above case, $\mathcal{H}$ stands for the squared norm of mean curvature, $\mathcal{K}$, $\mathcal{K}^{\perp}$ indicates Gauss and normal curvature of $\mathcal{M}^2$, respectively. In addition to this, equality sign holds in (1) provided ellipse of the curvature becomes exactly a circle.

Further, the inequality (1) was investigated independently and generalized to the case of surfaces of any co-dimension in real space forms by ([24],[15])

$$\mathcal{K} - c \leq \|\mathcal{H}\|^2 - |\mathcal{K}^{\perp}|.$$

Let ρ represent the normalized scalar curvature. Then, the generalized Wintgen inequality is reproduced in [12] with

$$\|\mathcal{H}\|^2 \geq \rho^{\perp} - c + \rho,$$

here $\rho^{\perp}$ means the normalized normal scalar curvature. This one had been termed as DDVV conjecture. In the recent years, DDVV inequalities appeared for various ambient manifolds and a survey can be found in [6].

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On the other side, the semi-symmetric linear connection has been studied in different ways since its introduction in 1924 [13] and that paved the way of studying differentiable manifolds with new settings. Hayden [16] has the credit of defining semi-symmetric metric connection onto manifold endowed with Riemannian metric. Imai [18], Yano [27] investigated several interesting properties of Riemannian manifold equipped with semi-symmetric metric connection . Nakao [23] generalized the results of Imai and established Gauss like and Codazzi-Mainardi like equations. In 1925, Agashe and Chafle ([2],[1]), investigated Riemannian manifolds endowed with a semi-symmetric non-metric connection. Optimal inequalities have also been derived for various manifolds with semi-symmetric connection (see [5]).

It is to be noted that polynomial structures were investigated on manifolds in the early 1970s due to Goldberg, Yano and Petridis [14] and structure of golden type was discussed in [11] producing several interesting results. Recently, submanifolds of slant type in golden Riemannian manifolds has been taken to study in ([3],[8],[10], [19] etc.).

Here, the generalized Wintgen inequalities are investigated for golden Riemannian manifolds equipped with semi-symmetric connections. We also investigate inequalities for different slant cases as application of main theorems.

Following are proved:

Theorem 1.1 For any θ-slant submanifold $\mathcal{S}^n$ isometrically immersed in locally golden product space form $\overline{\mathcal{S}}^m$ endowed with semi-symmetric metric connection . We have

$$\begin{aligned} \rho_{\mathcal{S}} \quad &\leq \quad \|\mathcal{H}\|^2 - 2\rho + \frac{1}{5}(c_1 + c_2)\Big\{3 - \frac{2}{n} tr\varphi + \frac{2}{n(n-1)}[tr^2\varphi - (trT + n)\cos^2\theta]\Big\} \\ &+ \quad \frac{1}{\sqrt{5}n}(c_p - c_q)(2tr\varphi - n) - \frac{4}{n} tr\beta. \end{aligned} \tag{2}$$

Moreover, (2) satisfies equality case iff in view of some orthonormal frames $\{u_1, \ldots, u_n\}$ and $\{u_{n+1}, \ldots, u_m\}$, $\mathbb{S}$ reduces to

$$\mathbb{S}_{n+1} = \begin{pmatrix} ð_1 & g & 0 & \ldots & 0 & 0 \\ g & ð_1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & ð_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & ð_1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & ð_1 \end{pmatrix}, \tag{3}$$

$$\mathbb{S}_{n+2} = \begin{pmatrix} ð_2 + \text{Ͽ} & 0 & 0 & \ldots & 0 & 0 \\ 0 & ð_2 - \text{Ͽ} & 0 & \ldots & 0 & 0 \\ 0 & 0 & ð_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & ð_2 & 0 \\ 0 & 0 & 0 & \ldots & 0 & ð_2 \end{pmatrix}, \tag{4}$$

$$\mathbb{S}_{n+3} = \begin{pmatrix} ð_3 & 0 & 0 & \ldots & 0 & 0 \\ 0 & ð_3 & 0 & \ldots & 0 & 0 \\ 0 & 0 & ð_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & ð_3 & 0 \\ 0 & 0 & 0 & \ldots & 0 & ð_3 \end{pmatrix}, \qquad \mathbb{S}_{n+4} = \cdots = \mathbb{S}_m = 0, \tag{5}$$

where $ð_1, ð_2, ð_3$ and $\eth$ are real functions on $\mathcal{S}$.

Theorem 1.2 For any θ-slant submanifold $\mathcal{S}^n$ isometrically immersed in locally golden product space form $\overline{\mathcal{S}}^m$ endowed with a semi-symmetric non-metric connection. We have

$$\begin{aligned}\rho_{\mathcal{S}} &\leq \|\mathcal{H}\|^2 - 2\rho + \frac{1}{5}(c_p + c_q)\Big\{3 + \frac{2}{n(n-1)}[tr^2\varphi - (trT + n)\cos^2\theta] - \frac{2}{n}tr\varphi\Big\} \\ &- \frac{4}{n}tr\overline{\beta} + \frac{1}{\sqrt{5}n}(c_p - c_q)(4tr\varphi - 2n) - 4\overline{\phi}(\mathcal{H}). \end{aligned} \tag{6}$$

Moreover, (6) satisfies equality iff for some orthonormal frames $\{u_1, \ldots, u_n\}$ and $\{u_{n+1}, \ldots, u_m\}$, $\mathbb{S}$ takes the form of

$$\mathbb{S}_{n+1} = \begin{pmatrix} ð_1 & g & 0 & \ldots & 0 & 0 \\ g & ð_1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & ð_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & ð_1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & ð_1 \end{pmatrix}, \tag{7}$$

$$\mathbb{S}_{n+2} = \begin{pmatrix} ð_2 + \eth & 0 & 0 & \ldots & 0 & 0 \\ 0 & ð_2 - \eth & 0 & \ldots & 0 & 0 \\ 0 & 0 & ð_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & ð_2 & 0 \\ 0 & 0 & 0 & \ldots & 0 & ð_2 \end{pmatrix}, \tag{8}$$

$$\mathbb{S}_{n+3} = \begin{pmatrix} ð_3 & 0 & 0 & \ldots & 0 & 0 \\ 0 & ð_3 & 0 & \ldots & 0 & 0 \\ 0 & 0 & ð_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & ð_3 & 0 \\ 0 & 0 & 0 & \ldots & 0 & ð_3 \end{pmatrix}, \qquad \mathbb{S}_{n+4} = \cdots = \mathbb{S}_m = 0, \tag{9}$$

where $ð_1, ð_2, ð_3$ and $\eth$ are real functions on $\mathcal{S}$.

2. Preliminaries

2.1. Semi-Symmetric Metric Connection

Suppose $(\overline{\mathcal{S}}^m, g)$ represents Riemannian manifold and $\mathcal{T}$ stands for torsion tensor of linear connection ∇^* on $\overline{\mathcal{S}}$ satisfying [27]

$$\mathcal{T}(\ell_2, \ell_3) = \gamma(\ell_3)(\ell_2) - \gamma(\ell_2)(\ell_3), \tag{10}$$

∇^* in above situation is termed as semi-symmetric connection. Further, assume that $\overline{\eta}$ be any vector field and γ be 1-form associated with $\overline{\eta}$ by

$$\gamma(\ell_1) = g(\ell_1, \overline{\eta}).$$

In addition to this, ∇^* becomes semi-symmetric metric connection provided

$$\nabla^* g = 0, \tag{11}$$

and a semi-symmetric non-metric connection when

$$\nabla^* g \neq 0. \tag{12}$$

In [27], ∇^* semi-symmetric metric connection on $\overline{\mathcal{S}}$ was defined with

$$\nabla^*_{\ell_1}\ell_2 = \gamma(\ell_2)\ell_1 - g(\ell_1,\ell_2)\overline{\eta} + \nabla_{\ell_1}\ell_2,$$

in above case, ∇ represents the Levi-Civita connection of $\overline{\mathcal{S}}$.

Fix curvature tensors of mathematical operators ∇ and ∇^* of $\overline{\mathcal{S}}$ with R and R^*. One can write [18]

$$\begin{aligned} R^*(\ell_1,\ell_2)\ell_3 &= R(\ell_1,\ell_2)\ell_3 + g(\ell_1,\ell_3)K\ell_2 - \beta(\ell_2,\ell_3)\ell_1 \\ &\quad - g(\ell_2,\ell_3)K\ell_1 + \beta(\ell_1,\ell_3)\ell_2, \qquad \forall \ell_i \in T\overline{\mathcal{S}}, \end{aligned} \tag{13}$$

in above situation β represents a $(0,2)$-tensor field given as

$$\beta(\ell_1,\ell_2) = \frac{1}{2}\gamma(\overline{\eta})g(\ell_1,\ell_2) + (\nabla_{\ell_1}\gamma)\ell_2 - \gamma(\ell_1)\gamma(\ell_2)$$

and

$$g(K\ell_1,\ell_2) = \beta(\ell_1,\ell_2).$$

Consider that $\overline{\mathcal{S}}$ be m-dimensional Riemannian manifold equipped with semi-symmetric metric connection and $\mathcal{S}^n$ be submanifold of $\overline{\mathcal{S}}$. Let us fix mathematical operators ∇ and $\overline{\nabla}$ for covariant differentiation in connection with Levi-Civita connection in $\mathcal{S}$ and $\overline{\mathcal{S}}$, respectively. Represent with $\mathbb{S}_N$ the shape operator of $\mathcal{S}$ with respect to $N \in \Gamma(T^\perp\mathcal{S})$. One gets

$$\overline{\nabla}_{\ell_1}\ell_2 = \nabla_{\ell_1}\ell_2 + h(\ell_1,\ell_2)$$

and

$$\overline{\nabla}_{\ell_1}N = -\mathbb{S}_N\ell_1 + \nabla^\perp_{\ell_1}N,$$

in this case $\nabla^\perp$ denotes connection in $T^\perp\mathcal{S}$. One also has

$$g(\mathbb{S}_N\ell_1,\ell_2) = g(h(\ell_1,\ell_2),N).$$

Let us suppose that $R^\perp$ stands for the Riemannian curvature tensor on $T^\perp\mathcal{S}$. Hence, equation of Gauss is [4]

$$\begin{aligned} R(\ell_1,\ell_2,\ell_3,\ell_4) &= \overline{R}(\ell_1,\ell_2,\ell_3,\ell_4) - g(h(\ell_1,\ell_4),h(\ell_2,\ell_3)) \\ &\quad + g(h(\ell_1,\ell_3),h(\ell_2,\ell_4)), \end{aligned} \tag{14}$$

in above situation $\ell_1,\ell_2,\ell_3,\ \ell_4 \in \Gamma(T\mathcal{S})$, $\overline{R}$ and R indicate curvature tensors of $\overline{\mathcal{S}}$ and $\mathcal{S}$. For any normal vector fields ξ_1 and ξ_2, we write [26]

$$g(\overline{R}(\ell_1,\ell_2)\xi_1,\xi_2) = g(R^\perp(\ell_1,\ell_2)\xi_1,\xi_2) + g([\mathbb{S}_{\xi_1},\mathbb{S}_{\xi_2}]\ell_1,\ell_2), \tag{15}$$

in this case $[\mathbb{S}_{\xi_1},\mathbb{S}_{\xi_2}] = \mathbb{S}_{\xi_1}\mathbb{S}_{\xi_2} - \mathbb{S}_{\xi_2}\mathbb{S}_{\xi_1}$.

In view of (2.1), R^* of Riemannian manifold $\overline{\mathcal{S}}$ equipped with a semi-symmetric metric connection ∇^* is represented as

$$\begin{aligned} R^*(\ell_1,\ell_2)\ell_3 &= R(\ell_1,\ell_2)\ell_3 - \beta(\ell_2,\ell_3)\ell_1 + \beta(\ell_1,\ell_3)\ell_2 \\ &\quad - g(\ell_2,\ell_3)K\ell_1 + g(\ell_1,\ell_3)K\ell_2. \end{aligned} \tag{16}$$

Consider local orthonormal frame $\{u_1, \ldots, u_n\}$ and $\{u_{n+1}, \ldots, u_m\}$ of $\mathcal{S}$ in $\overline{\mathcal{S}}$. Then one has

$$\mathcal{H} = \sum_{i=1}^{n} \frac{1}{n} h(u_i, u_i), \tag{17}$$

and

$$\|h\|^2 = \sum_{1 \le i,j \le n} g\Big(h(u_i, u_j), h(u_i, u_j)\Big). \tag{18}$$

Let $\pi \subset T_p\mathcal{S}, p \in \mathcal{S}$ be the plane section and $\mathcal{K}(\pi)$ be sectional curvature of $\mathcal{S}$ connected with π. Then one can write

$$\tau(p) = \sum_{1 \le i < j \le n} \mathcal{K}(u_i \wedge u_j) \tag{19}$$

and

$$\rho(p) = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} \mathcal{K}(u_i \wedge u_j). \tag{20}$$

Let $\overline{\mathcal{S}}^m$ be Riemannian manifold endowed with semi-symmetric metric connection and $\mathcal{S}$ represents n-dimensional submanifold in $\overline{\mathcal{S}}$. Also assume some orthonormal frames $\{u_1, \ldots, u_n\}$ and $\{u_{n+1}, \ldots, u_m\}$ of $T_p\mathcal{S}$ and $T_p^{\perp}\mathcal{S}$, respectively. Then one writes [21]

$$\rho^{\perp} = \frac{2\tau^{\perp}}{n(n-1)}. \tag{21}$$

In the similar way [28],

$$\mathcal{K}_{\mathcal{S}} = \frac{1}{4} \sum_{r,s=n+1}^{m} (Trace[\mathbb{S}_r, \mathbb{S}_s])^2, \tag{22}$$

where $\mathbb{S}_t$ stands for shape operator of $\mathcal{S}$ in the direction of ξ_t,
$t = n+1, ..., m$.

Next, we represent [22]

$$\rho_{\mathcal{S}} = \frac{2}{n(n-1)} \sqrt{\mathcal{K}_{\mathcal{S}}}. \tag{23}$$

Hence, we write

$$\begin{aligned} \mathcal{K}_{\mathcal{S}} &= \frac{1}{2} \sum_{n+1 \le r < s \le m} (Trace[\mathbb{S}_r, \mathbb{S}_s])^2 \\ &= \sum_{n+1 \le r < s \le m} \sum_{1 \le i < j \le n} g([\mathbb{S}_r, \mathbb{S}_s] u_i, u_j)^2. \end{aligned}$$

Now, one represents $\mathcal{K}_{\mathcal{S}}$ as [22]

$$\mathcal{K}_{\mathcal{S}} = \sum_{n+1 \le r < s \le m} \sum_{1 \le i < j \le n} \Big[\sum_{k=1}^{n} (h^r_{jk} h^s_{ik} - h^r_{ik} h^s_{jk}) \Big]^2. \tag{24}$$

2.2. Semi-Symmetric Non-Metric Connection

Assume $(\overline{\mathcal{S}}^m, g)$ represents any Riemannian manifold and ∇^* stands for linear connection on $\overline{\mathcal{S}}$ and $\mathcal{T}$ be torsion tensor of ∇^*. We have already seen that ∇^* is semi-symmetric connection provided it satisfies (10) and non-metric connection if

$$\nabla^* g \neq 0.$$

In [1], ∇^* semi-symmetric non-metric connection was described as

$$\nabla^*{}_{\ell_1}\ell_2 = \overline{\phi}(\ell_2)\ell_1 + \overline{\nabla}_{\ell_1}\ell_2, \qquad \forall \ell_1, \ell_2 \in \Gamma(T\overline{\mathcal{S}}),$$

in above equation, $\overline{\phi}$ stands for a 1-form.

Assume any Riemannian manifold $\overline{\mathcal{S}}^m$ with semi-symmetric non-metric connection ∇^*. Also suppose that R^* and $\overline{R}$ be curvature tensors of $\overline{\mathcal{S}}$ with respect to mathematical operators ∇^* and $\overline{\nabla}$, respectively. Thus [1]

$$R^*(\ell_1, \ell_2, \ell_3, \ell_4) = \overline{R}(\ell_1, \ell_2, \ell_3, \ell_4) - \overline{\beta}(\ell_2, \ell_3)g(\ell_1, \ell_4) + \overline{\beta}(\ell_1, \ell_3)g(\ell_2, \ell_4), \tag{25}$$

in this situation, $\overline{\beta}$ is $(0, 2)$-tensor field written as

$$\overline{\beta}(\ell_1, \ell_2) = (\overline{\nabla}_{\ell_1}\overline{\phi})\ell_2 - \overline{\phi}(\ell_1)\overline{\phi}(\ell_2). \tag{26}$$

Let us also denote the trace of $\overline{\beta}$ by $\overline{\lambda}$.

Now, let $\mathcal{S}^n$ be submanifold of $\overline{\mathcal{S}}^m$ and mathematical operators ∇ and ∇' be induced semi-symmetric non-metric connection and Levi-Civita connection, respectively. Fix R and R' for the curvature tensors on $\mathcal{S}$ with respect to ∇ and ∇'. The Gauss formulas are expressed as

$$\nabla^*{}_{\ell_1}\ell_2 = \nabla_{\ell_1}\ell_2 + h(\ell_1, \ell_2),$$
$$\overline{\nabla}_{\ell_1}\ell_2 = \nabla'_{\ell_1}\ell_2 + h'(\ell_1, \ell_2),$$

in this case, h represents $(0, 2)$-tensor on $\mathcal{S}$, h' means the second fundamental form of $\mathcal{S}$ in $\overline{\mathcal{S}}$. One can also note that [2]

$$h = h'. \tag{27}$$

For a semi-symmetric non-metric connection, one has [2]

$$\begin{aligned} R^*(t_1, t_2, t_3, t_4) \;=\; & R(t_1, t_2, t_3, t_4) - g(h(t_1, t_4), h(t_2, t_3)) \\ & + g(h(t_1, t_3), h(t_2, t_4)) + g(E, h(t_2, t_3))g(t_1, t_4) \\ & - g(E, h(t_1, t_3))g(t_2, t_4), \qquad \forall t_1, t_2, t_3, t_4 \in \Gamma(T\mathcal{S}), \end{aligned} \tag{28}$$

$\mathbb{E}$ represents vector field satisfying

$$g(\mathbb{E}, t_1) = \overline{\phi}(t_1).$$

One also writes

$$\mathcal{H} = \sum_{i=1}^{n} \frac{1}{n} h(u_i, u_i) \tag{29}$$

and

$$\tau = \sum_{1 \le i < j \le n} R(u_i, u_j, u_j, u_i). \tag{30}$$

We also define

$$\rho \;=\; \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} \mathcal{K}(u_i \wedge u_j), \tag{31}$$

in above case, $\mathcal{K}$ means the sectional curvature function on $\mathcal{S}$. Similarly, we can write other formulas with respect to semi-symmetric non-metric connection

2.3. Golden Riemannian manifolds

Consider the Riemannian manifold $(\overline{\mathcal{S}}^m, g)$ and assume $(1,1)$-tensor field $\mathcal{L}$ on $\overline{\mathcal{S}}$. When [3, 11, 14]

$$\begin{aligned} O(\ell_1) &= a_1 I + a_2 \ell_1 + ... + a_n \ell_1^{m-1} + \ell_1^n \\ &= 0, \end{aligned}$$

I being identity transformation and (for $\ell_1 = \mathcal{L}$) $I, \mathcal{L}(p), , ..., \mathcal{L}^{n-2}(p), \mathcal{L}^{n-1}(p)$ are linearly independent at $p \in \overline{\mathcal{S}}$. Then $O(\ell_1)$ is said to be structure polynomial. In addition to this, $O(\ell_1) = \ell_1^2 + I$ produces an almost complex structure and $O(\ell_1) = \ell_1^2 - I$ results an almost product structure.

Additionally, φ ($(1,1)$-tensor field) satisfying the equality [3, 14]

$$\varphi^2 = \varphi + I,$$

is known as golden structure on $\overline{\mathcal{S}}$. Moreover, g becomes φ-compatible if

$$g(\varphi\ell_1, \ell_2) = g(\ell_1, \varphi\ell_2) \qquad \forall \ell_1, \ell_2 \in \Gamma(T\overline{\mathcal{S}}). \tag{32}$$

A golden Riemannian manifold $(\overline{\mathcal{S}}, g, \varphi)$ endows golden structure φ with φ-compatible Riemannian metric g [3, 11]. Setting $\varphi\ell_1$ in place of ℓ_1 in (32), one obtains

$$\begin{aligned} g(\varphi\ell_1, \varphi\ell_2) &= g(\varphi^2\ell_1, \ell_2) \\ &= g(\varphi\ell_1, \ell_2) + g(\ell_1, \ell_2) \qquad \forall \ell_1, \ell_2 \in \Gamma(T\overline{\mathcal{S}}). \end{aligned}$$

Let φ stands for golden structure and $\mathcal{L}$ be almost product structure. Then $\mathcal{L}$ produces

$$\varphi = \frac{1}{2}(\sqrt{5}\mathcal{L} + I)$$

and φ induces $\mathcal{L}$ [3, 11]

$$\mathcal{L} = \frac{1}{\sqrt{5}}(2\varphi - I).$$

Further, $(\overline{\mathcal{S}}, g, \varphi)$ is known as locally golden if with respect to Levi-Civita connection, φ becomes parallel.

Assume that $(\mathcal{S}, g)$ is a submanifold of $(\overline{\mathcal{S}}, g, \varphi)$. Then, we express

$$\varphi\ell_2 = P\ell_2 + Q\ell_2, \forall(\mathcal{Y}) \in \Gamma(TS)$$

in this case $P\ell_2$ stands for tangential component and $Q\ell_2$ represents normal components of $\varphi\ell_2$.

A submanifold $(\mathcal{S}, g)$ immersed in $(\overline{\mathcal{S}}, g, \varphi)$ is known as slant when any nonzero vector $\ell_1 \in T_p\mathcal{S}$, $p \in \mathcal{S}$, the angle $\theta(\ell_1)$ between $T_p\mathcal{S}$ and $\varphi\ell_1$ is independent of $p \in \mathcal{S}$ and $\ell_1 \in T_p\mathcal{S}$. We have these cases for $\mathcal{S}$:

- $\theta = 0$ (φ-invariant)
- $\theta = \frac{\pi}{2}$ (φ-anti-invariant)
- proper slant when it is neither invariant nor anti-invariant.

Lemma 2.1 [3] For any submanifold $(\mathcal{S}^n, g)$ of Riemannian manifold with golden structure $(\overline{\mathcal{S}}^m, g, \varphi)$. We have:

1. $\mathcal{S}$ is slant $\iff \exists\, \mu \in [0,1]$ satisfying $P^2 = \mu(I + \varphi)$. Additionally, $\mu = cos^2\theta$, for slant angle θ.
2. $\mathcal{S}$ is slant $\iff \exists\, \mu \in [0,1]$ satisfying $\varphi^2 = \frac{1}{\mu}P^2$. In this case, $\mu = cos^2\theta$.
3. $g(P\ell_1, P\ell_2) = cos^2\theta(g(\ell_1, P\ell_2) + g(\ell_1, \ell_2))$.
4. $g(Q\ell_1, Q\ell_2) = sin^2\theta(g(P\ell_1, \ell_2) + g(\ell_1, \ell_2))$, $\quad \forall \ell_1, \ell_2 \in \Gamma(T\mathcal{S})$.

Now, consider real-space forms $\mathcal{S}_p$ and $\mathcal{S}_q$. For locally golden product space form ($\overline{\mathcal{S}} = \mathcal{S}_p(c_p) \times \mathcal{S}_q(c_q), g, \varphi$), one has the Riemannian curvature tensor R [9]:

$$\begin{aligned} R(\ell_1,\ell_2)\ell_3 &= \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10}[g(\varphi\ell_2,\ell_3)\ell_1 - g(\varphi\ell_1,\ell_3)\ell_2 \\ &+ g(\ell_2,\ell_3)\varphi\ell_1 - g(\ell_1,\ell_3)\varphi\ell_2] \\ &+ \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q}{10}[g(\ell_2,\ell_3)\ell_1 - g(\ell_1,\ell_3)\ell_2] \\ &+ \frac{c_p + c_q}{5}[g(\varphi\ell_2,\ell_3)\varphi\ell_1 - g(\varphi\ell_1,\ell_3)\varphi\ell_2]. \end{aligned} \tag{33}$$

Further, if $\overline{\mathcal{S}}$ is equipped with semi-symmetric metric connection. Then curvature tensor of $\overline{\mathcal{S}}$ is

$$\begin{aligned} R^*(\ell_1,\ell_2)\ell_3 &= \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q}{10}[g(\ell_2,\ell_3)\ell_1 - g(\ell_1,\ell_3)\ell_2] \\ &+ \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10}[g(\varphi\ell_2,\ell_3)\ell_1 - g(\varphi\ell_1,\ell_3)\ell_2 \\ &+ g(\ell_2,\ell_3)\varphi\ell_1 - g(\ell_1,\ell_3)\varphi\ell_2] \\ &- \beta(\ell_2,\ell_3)\ell_1 - g(\ell_2,\ell_3)K\ell_1 \\ &+ \frac{c_p + c_q}{5}[g(\varphi\ell_2,\ell_3)\varphi\ell_1 - g(\varphi\ell_1,\ell_3)\varphi\ell_2] \\ &+ \beta(\ell_1,\ell_3)\ell_2 + g(\ell_1,\ell_3)K\ell_2 \end{aligned} \tag{34}$$

where (16) and (33) have been used.

If $\overline{\mathcal{S}}^m$ is equipped with semi-symmetric non-metric connection. Then taking into use (25) and (33), one expresses

$$\begin{aligned} R^*(\ell_1,\ell_2)\ell_3 &= \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q}{10}[g(\ell_2,\ell_3)\ell_1 - g(\ell_1,\ell_3)\ell_2] \\ &+ \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10}[g(\varphi\ell_2,\ell_3)\ell_1 - g(\varphi\ell_1,\ell_3)\ell_2 \\ &+ g(\ell_2,\ell_3)\varphi\ell_1 - g(\ell_1,\ell_3)\varphi\ell_2] + \overline{\beta}(\ell_1,\ell_3)g(\ell_2,\ell_4) \\ &+ \frac{c_p + c_q}{5}[g(\varphi\ell_2,\ell_3)\varphi\ell_1 - g(\varphi\ell_1,\ell_3)\varphi\ell_2] \\ &- \overline{\beta}(\ell_2,\ell_3)g(\ell_1,\ell_4). \end{aligned} \tag{35}$$

3. Main Proofs

Theorem 1:
Proof: In the light of (34), one obtains

$$\begin{aligned}\sum_{1\leq i<j\leq n} R(u_i,u_j,u_j,u_i) &= \frac{(\mp\sqrt{5}+3)c_p+(\pm\sqrt{5}+3)c_q}{10}[g(u_j,u_j)g(u_i,u_i)\\ &- g(u_i,u_j)g(u_j,u_i)]\\ &+ \frac{(\pm\sqrt{5}-1)c_p+(\mp\sqrt{5}-1)c_q}{10}[g(\varphi u_j,u_j)g(u_i,u_i)\\ &- g(\varphi u_i,u_j)g(u_j,u_i)+g(u_j,u_j)g(\varphi u_i,u_i)\\ &- g(u_i,u_j)g(\varphi u_j,u_i)]\\ &+ \frac{c_p+c_q}{5}[g(\varphi u_j,u_j)g(\varphi u_i,u_i)-g(\varphi u_i,u_j)g(\varphi u_j,u_i)]\\ &- \beta(u_j,u_j)g(u_i,u_i)+\beta(u_i,u_j)g(u_j,u_i)\\ &- g(u_j,u_j)g(Ku_i,u_i)+g(h(u_i,u_j),h(u_j,u_i))\\ &- g(h(u_i,u_i),h(u_j,u_j))+g(u_i,u_j)g(Ku_j,u_i)\end{aligned} \tag{36}$$

where Gauss equation has been used. With the help of Lemma 2.3, one obtains

$$\begin{aligned}\sum_{1\leq i<j\leq n} R(u_i,u_j,u_j,u_i) &= \frac{1}{4}(c_p+c_q)\frac{n(n-1)}{5}\Big\{6-\frac{4}{n}tr\varphi\\ &+ \frac{4}{n(n-1)}[tr^2\varphi-(trT+n)\cos^2\theta]\Big\}+B_1\\ &+ \frac{1}{4}\frac{(n-1)}{\sqrt{5}}(c_p-c_q)(4tr\varphi-2n)-2(n-1)tr\beta,\end{aligned} \tag{37}$$

here $B_1=\sum_{\alpha=n+1}^{m}\sum_{1\leq i<j\leq n}\left[h_{ii}^{\alpha}h_{jj}^{\alpha}-(h_{ij}^{\alpha})^2\right]$.

We also know that

$$2\tau=\sum_{1\leq i<j\leq n} R(u_i,u_j,u_j,u_i), \tag{38}$$

that produces

$$\begin{aligned}2\tau &= \frac{1}{4}(c_p+c_q)\frac{n(n-1)}{5}\Big\{6+\frac{4}{n(n-1)}[tr^2\varphi-(trT+n)\cos^2\theta]-\frac{4}{n}tr\varphi\Big\}\\ &+ \frac{1}{4}\frac{(n-1)}{\sqrt{5}}(c_p-c_q)(4tr\varphi-2n)-2(n-1)tr\beta+B_1.\end{aligned} \tag{39}$$

Let $A_1=\sum_{\alpha=n+1}^{m}\sum_{1\leq i<j\leq n}(h_{ii}^{\alpha}-h_{jj}^{\alpha})^2$ and $A_2=\sum_{\alpha=n+1}^{m}\sum_{1\leq i<j\leq n}h_{ii}^{\alpha}h_{jj}^{\alpha}$, then

$$\begin{aligned}n^2\|\mathcal{H}\|^2 &= \sum_{\alpha=n+1}^{m}\Big(\sum_{i=1}^{n}h_{ii}^{\alpha}\Big)^2\\ &= \frac{1}{n-1}A_1+\frac{2n}{n-1}A_2.\end{aligned} \tag{40}$$

One can also note [20]

$$B_2\leq A_3+\frac{1}{2n}A_1, \tag{41}$$

inwhere $B_2 = \left\{ \sum_{n+1\leq\alpha<\beta\leq m-n} \sum_{1\leq i<j\leq n} [\sum_{k=1}^{n} (h^{\alpha}_{jk} h^{\beta}_{ik} - h^{\alpha}_{ik} h^{\beta}_{jk})]^2 \right\}^{\frac{1}{2}}$ and $A_3 = \sum_{\alpha=n+1}^{m} \sum_{1\leq i<j\leq n} (h^{\alpha}_{ij})^2$.

Taking into consideration (40), (41) and (24), it results

$$B_1 \leq \frac{n-1}{2n} \left\{ n^2 \|\mathcal{H}\|^2 - n^2 \rho_N \right\}. \tag{42}$$

Finally, taking help of (23) and (39), we reach to

$$\begin{aligned} \rho_N - \|\mathcal{H}\|^2 \quad &\leq \quad \frac{1}{10}(c_p + c_q)\Big\{6 + \frac{4}{n(n-1)}[tr^2\varphi - (trT + n)\cos^2\theta] - \frac{4}{n} tr\varphi\Big\} \\ &- \quad \frac{4}{n} tr\beta + \frac{1}{2\sqrt{5}n}(c_p - c_q)(4tr\varphi - 2n) - 2\rho, \end{aligned}$$

in this equation (42) has been used and thereby establishing the required result.

Proof of Theorem 1:

Proof: Using (25),(28) and (35), one writes

$$\begin{aligned} \sum_{1\leq i<j\leq n} R(u_i, u_j, u_j, u_i) \quad &= \quad n(1-n)\overline{\phi}(\mathcal{H}) + \frac{1}{4}(c_p + c_q)\frac{n(n-1)}{5}\Big\{6 - \frac{4}{n} tr\varphi \\ &+ \quad \frac{4}{n(n-1)}[tr^2\varphi - (trT + n)\cos^2\theta]\Big\} + B_1 \\ &+ \quad \frac{1}{4}\frac{(n-1)}{\sqrt{5}}(c_p - c_q)(4tr\varphi - 2n) - (n-1)tr\overline{\beta}, \end{aligned} \tag{43}$$

wherein Lemma 2.3 has also been considered.

It is also known that

$$2\tau = \sum_{1\leq i<j\leq n} R(u_i, u_j, u_j, u_i), \tag{44}$$

that produces

$$\begin{aligned} 2\tau \quad &= \quad n(1-n)\overline{\phi}(\mathcal{H}) + \frac{1}{4}\frac{(n-1)}{\sqrt{5}}(c_p - c_q)(4tr\varphi - 2n) \\ &+ \quad \frac{1}{4}(c_p + c_q)\frac{n(n-1)}{5}\Big\{6 + \frac{4}{n(n-1)}[tr^2\varphi - (trT + n)\cos^2\theta] - \frac{4}{n} tr\varphi\Big\} \\ &+ \quad (1-n)tr\overline{\beta} + B_1. \end{aligned} \tag{45}$$

One can write

$$n^2\|\mathcal{H}\|^2 \quad = \quad \frac{1}{n-1}A_1 + \frac{2n}{n-1}A_2.$$

One can also note [20]

$$B_2 \leq \frac{1}{2n}A_1 + A_3. \tag{46}$$

Taking into consideration (46), (46) and (22), it results

$$B_1 \leq \frac{n-1}{2n}\Big[n^2\|\mathcal{H}\|^2 - n^2\rho_{\mathcal{S}}\Big]. \tag{47}$$

Let $W_1 = \frac{4}{n} tr\overline{\beta}$ and $W_2 = 4\overline{\phi}(\mathcal{H})$. Then, taking help of (20) and (45), we reach to

$$\begin{aligned} \rho_{\mathcal{S}} - \|\mathcal{H}\|^2 \quad &\leq \quad \frac{1}{10}(c_p + c_q)\Big\{6 + \frac{4}{n(n-1)}[tr^2\varphi - (trT + n)\cos^2\theta] - \frac{4}{n} tr\varphi\Big\} \\ &+ \quad \frac{1}{2\sqrt{5}n}(c_p - c_q)(4tr\varphi - 2n) - W_1 - W_2 - 2\rho, \end{aligned}$$

where (47) has been used and thereby establishing the required result.

4. Some Applications of main theorems

As an application of Theorem 1, one obtains these generalized Wintgen inequalities.

Corollary 4.1 For any invariant submanifold $\mathcal{S}^n$ isometrically immersed in $\overline{\mathcal{S}}^m$. We have

$$\begin{aligned}\rho_{\mathcal{S}} &\leq \|\mathcal{H}\|^2 - 2\rho + \frac{1}{5}(c_1 + c_2)\Big\{3 - \frac{2}{n}tr\varphi + \frac{2}{n(n-1)}[tr^2\varphi - (trT + n)]\Big\} \\ &+ \frac{1}{\sqrt{5}n}(c_p - c_q)\,(2tr\varphi - n) - \frac{4}{n}tr\beta. \end{aligned} \tag{48}$$

Moreover, for some orthonormal frames $\{u_1, \ldots, u_n\}$ and $\{u_{n+1}, \ldots, u_m\}$ and some real functions $\eth_1, \eth_2, \eth_3$ and $\backepsilon$ on $\mathcal{S}$, the equality in (48) holds iff $\mathbb{S}$ looks like (3), (4) and (5).

Corollary 4.2 For any anti-invariant submanifold $\mathcal{S}^n$ isometrically immersed in $\overline{\mathcal{S}}^m$. We have

$$\begin{aligned}\rho_{\mathcal{S}} &\leq \|\mathcal{H}\|^2 - 2\rho + \frac{1}{5}(c_1 + c_2)\Big\{3 - \frac{2}{n}tr\varphi + \frac{2}{n(n-1)}tr^2\varphi\Big\} \\ &+ \frac{1}{\sqrt{5}n}(c_p - c_q)\,(2tr\varphi - n) - \frac{4}{n}tr\beta. \end{aligned} \tag{49}$$

In addition to this, for some orthonormal frames $\{u_1, \ldots, u_n\}$ and $\{u_{n+1}, \ldots, u_m\}$ and some real functions $\eth_1, \eth_2, \eth_3$ and $\backepsilon$ on $\mathcal{S}$, the equality in (49) holds iff $\mathbb{S}$ appears to be like (3), (4) and (5).

As an application of Theorem 1, one obtains these generalized Wintgen inequalities.

Corollary 4.3 For any invariant submanifold $\mathcal{S}^n$ immersed in $\overline{\mathcal{S}}^m$. We have

$$\begin{aligned}\rho_{\mathcal{S}} &\leq \|\mathcal{H}\|^2 - 2\rho + \frac{1}{5}(c_p + c_q)\Big\{3 + \frac{2}{n(n-1)}[tr^2\varphi - (trT + n)] - \frac{2}{n}tr\varphi\Big\} \\ &+ \frac{1}{\sqrt{5}n}(c_p - c_q)\,(4tr\varphi - 2n) - W_1 - W_2. \end{aligned} \tag{50}$$

Moreover, (50) satisfies equality iff for some orthonormal frames $\{u_1, \ldots, u_n\}$ and $\{u_{n+1}, \ldots, u_m\}$ and some real functions $\eth_1, \eth_2, \eth_3$ and $\backepsilon$ on $\mathcal{S}$, $\mathbb{S}$ takes the form of (7), (8) and (9).

Corollary 4.4 For any anti-invariant submanifold $\mathcal{S}^n$ isometrically immersed in $\overline{\mathcal{S}}^m$. We have

$$\begin{aligned}\rho_{\mathcal{S}} &\leq \|\mathcal{H}\|^2 - 2\rho + \frac{1}{5}(c_p + c_q)\Big\{3 + \frac{2}{n(n-1)}tr^2\varphi - \frac{2}{n}tr\varphi\Big\} \\ &+ \frac{1}{\sqrt{5}n}(c_p - c_q)\,(4tr\varphi - 2n) - W_1 - W_2. \end{aligned} \tag{51}$$

Moreover, (51) satisfies equality iff for some orthonormal frames $\{u_1, \ldots, u_n\}$ and $\{u_{n+1}, \ldots, u_m\}$ and some real functions $\eth_1, \eth_2, \eth_3$ and $\backepsilon$ on $\mathcal{S}$, $\mathbb{S}$ takes the form of (7), (8) and (9).

Some More Applications:

- Theorems 1 and 1 generalize main result of [7].
- Putting $\theta = 0$ and $\theta = \frac{\pi}{2}$ in Theorems 1 and 1, we can write other results of this article.
- We can also discuss these results for other structures defined on Riemannian manifold $\overline{\mathcal{S}}$ [17].
  1. for $p = 2, q = 1$, the silver ratio $\sigma_{2,1} = 1 + \sqrt{2}$,
  2. the bronze ratio $\sigma_{3,1} = \frac{3+\sqrt{13}}{2}$ $(p = 3, q = 1)$,
  3. for $p = 4, q = 1$, the subtle mean $\sigma_{4,1} = 2 + \sqrt{5}$,
  4. the copper ratio $\sigma_{1,2} = 2$ $(p = 1, q = 2)$ etc.

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