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Ricci soliton in an (ϵ)-para-Sasakian manifold admitting conharmonic curvature tensor

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Abstract. In this paper our aim is to investigate Ricci soliton in an (ϵ)-para-Sasakian manifold admitting conharmonic curvature tensor satisfying the conditions $\mathfrak{R}(\zeta, \mathcal{G}_1).\mathcal{H} = 0$, $\mathcal{S}(\zeta, \mathcal{G}_1).\mathcal{H} = 0$, $\widetilde{\mathcal{P}}(\zeta, \mathcal{G}_1).\mathcal{H} = 0$ and $\mathcal{H}(\zeta, \mathcal{G}_1).\widetilde{\mathcal{P}} = 0$. Under these conditions taking ζ as space-like or time-like vector field, it is shown that Ricci soliton is expanding, steady and shrinking according as $\Theta > 0, \Theta = 0$ and $\Theta < 0$, respectively. Finally, we give an example of an (ϵ)-para-Sasakian manifold which justify our results.

1. Introduction

The study of Ricci solitons is a very fascinating topic in differential geometry and physics. Ricci soliton is a natural generalization of Einstein metric and is also a self-similar solution to Hamilton's Ricci flow [18, 20]. Hamilton initiated the concept of Ricci flow [19] to find a canonical metric on smooth manifolds. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as under:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2\mathfrak{R}_{ij}.$$

A Ricci soliton arises as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton $(g, \mathcal{V}, \Theta)$ on a Riemannian manifold $(\mathfrak{M}, g)$ is defined as follows:

$$\mathfrak{L}_{\mathcal{V}} g(\mathcal{G}_1, \mathcal{G}_2) + 2\mathcal{S}(\mathcal{G}_1, \mathcal{G}_2) + 2\Theta g(\mathcal{G}_1, \mathcal{G}_2) = 0, \tag{1}$$

$\forall\ \mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{X}(\mathfrak{M})$, where $\mathfrak{X}(\mathfrak{M})$ is a set of all smooth vector fields on $\mathfrak{M}$, $\mathcal{S}$ is a Ricci tensor, $\mathfrak{L}_{\mathcal{V}}$ is the Lie-derivative operator along the vector field $\mathcal{V}$ on $\mathfrak{M}$ and Θ is a real number. If the potential vector field $\mathcal{V}$

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vanishes identically then the Ricci soliton becomes trivial and in this case the manifold is an Einstein one. The Ricci soliton is said to be shrinking, steady and expanding according to $\Theta < 0$, $\Theta = 0$ and $\Theta > 0$, respectively. In 2008, Sharma investigated the Ricci solitons in contact geometry [32]. Later on, Ricci solitons in contact metric manifolds have been studied by several authors like Bagewadi et al. [1, 2], Bejan and Crasmareanu [3], Blaga [5], Chandra et al. [7], Chen and Deshmukh [8], Deshmukh et al. [9], Chenxu and Meng [21], Nagaraja and Premalatha [24]. Recently, Ricci solitons have been studied by several authors like De et al. [11, 13–15, 31].

The conharmonic curvature tensor $\mathcal{H}$ of type $(1,3)$ on a Riemannian manifold $(\mathfrak{M}^n, g)$ is defined by [12]

$$\begin{aligned}\mathcal{H}(\mathcal{G}_1,\mathcal{G}_2)\mathcal{G}_3 &= \mathfrak{R}(\mathcal{G}_1,\mathcal{G}_2)\mathcal{G}_3 - \frac{1}{n-2}[\mathcal{S}(\mathcal{G}_2,\mathcal{G}_3)\mathcal{G}_1 \\ &\quad -\mathcal{S}(\mathcal{G}_1,\mathcal{G}_3)\mathcal{G}_2 + g(\mathcal{G}_2,\mathcal{G}_3)Q\mathcal{G}_1 \\ &\quad -g(\mathcal{G}_1,\mathcal{G}_3)Q\mathcal{G}_2]. \end{aligned} \tag{2}$$

The pseudo projective curvature tensor on a Riemannian manifold $(\mathfrak{M}^n, g)$ $(n > 2)$ of type $(1, 3)$ as follows [25]

$$\begin{aligned}\widetilde{\mathcal{P}}(\mathcal{G}_1,\mathcal{G}_2)\mathcal{G}_3 &= a\mathfrak{R}(\mathcal{G}_1,\mathcal{G}_2)\mathcal{G}_3 + b[\mathcal{S}(\mathcal{G}_2,\mathcal{G}_3)\mathcal{G}_1 - \mathcal{S}(\mathcal{G}_1,\mathcal{G}_3)\mathcal{G}_2] \\ &\quad -\frac{\mathfrak{r}}{n}\left(\frac{a}{n-1}+b\right)[g(\mathcal{G}_2,\mathcal{G}_3)\mathcal{G}_1 - g(\mathcal{G}_1,\mathcal{G}_3)\mathcal{G}_2], \end{aligned} \tag{3}$$

where a and b are constants such that $a, b \neq 0$. If $a = 1$ and $b = -\frac{1}{n-1}$, then (3) takes the form

$$\begin{aligned}\widetilde{\mathcal{P}}(\mathcal{G}_1,\mathcal{G}_2)\mathcal{G}_3 &= \mathfrak{R}(\mathcal{G}_1,\mathcal{G}_2)\mathcal{G}_3 - \frac{1}{n-1}[\mathcal{S}(\mathcal{G}_2,\mathcal{G}_3)\mathcal{G}_1 - \mathcal{S}(\mathcal{G}_1,\mathcal{G}_3)\mathcal{G}_2] \\ &= \mathcal{P}(\mathcal{G}_1,\mathcal{G}_2)\mathcal{G}_3, \end{aligned} \tag{4}$$

where $\mathcal{P}$ is the projective curvature tensor [23].

On the other hand, T. Takahashi [33] originated almost contact manifolds equipped with an associated pseudo-Riemannian metric. In particular, he investigated Sasakian manifolds equipped with an associated pseudo-Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as (ϵ)-almost contact metric manifolds and (ϵ)-Sasakian manifolds, respectively [4, 16, 17]. Further, K. Matsumoto [9] replaced the structure vector field ζ by $-\zeta$ in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure and called it a Lorentzian almost paracontact manifold. In 2010, Tripathi et al. [34] investigated the study of (ϵ)-almost paracontact manifolds, which is not necessarily Lorentzian. In particular, they studied (ϵ)-para-Sasakian manifolds with the structure vector field ζ is spacelike or timelike according as $\epsilon = 1$ or $\epsilon = -1$.

Motivated by all these studies present paper is organized as follows: Section 2 is devoted to the fundamentals of (ϵ)-para-Sasakian manifolds and some curvature tensors. Ricci soliton in an (ϵ)-para-Sasakian manifolds satisfying the conditions $\mathfrak{R}(\zeta,\mathcal{G}_1).\mathcal{H} = 0$, $\mathcal{S}(\zeta,\mathcal{G}_1).\mathcal{H} = 0$, $\widetilde{P}(\zeta,\mathcal{G}_1).\mathcal{H} = 0$ and $\mathcal{H}(\zeta,\mathcal{G}_1).\widetilde{P} = 0$ are studied in the sections 3, 4, 5 and 6, respectively. Finally, in the last section we have given an example of an (ϵ)-para-Sasakian manifold.

2. Preliminaries

An n-dimensional manifold $\mathfrak{M}$ admits an almost paracontact structure (φ, ζ, η), where φ is a $(1,1)$ tensor field, ζ is a structure vector field, η is a 1-form if [28, 29]

$$\varphi^2(\mathcal{G}_1) = \mathcal{G}_1 - \eta(\mathcal{G}_1)\zeta, \quad \eta(\zeta) = 1, \quad \varphi\zeta = 0, \quad \eta(\varphi\mathcal{G}_1) = 0. \tag{5}$$

Let g be a pseudo-Riemannian metric such that

$$g(\varphi\mathcal{G}_1, \varphi\mathcal{G}_2) = g(\mathcal{G}_1,\mathcal{G}_2) - \epsilon\eta(\mathcal{G}_1)\eta(\mathcal{G}_2), \tag{6}$$

where $\epsilon = +1$ or -1. Then $\mathfrak{M}$ is called an (ϵ)-almost paracontact metric manifold equipped with an (ϵ)-almost paracontact metric structure $(\varphi, \zeta, \eta, g, \epsilon)$ [34]. From (6), we have

$$g(\varphi\mathcal{G}_1, \mathcal{G}_2) = g(\mathcal{G}_1, \varphi\mathcal{G}_2), \tag{7}$$

$$\eta(\mathcal{G}_1) = \epsilon g(\mathcal{G}_1, \zeta), \tag{8}$$

$\forall\ \mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{X}(\mathfrak{M})$; $\mathfrak{X}(\mathfrak{M})$ is a set of all smooth vector fields on $\mathfrak{M}$. From (8), we have

$$g(\zeta, \zeta) = \epsilon, \tag{9}$$

i.e. the structure vector field ζ is never lightlike.

Definition 2.1. *An (ϵ)-almost paracontact metric manifold is called (ϵ)-para-Sasakian manifold if [34]*

$$(\nabla_{\mathcal{G}_1}\varphi)\mathcal{G}_2 = -g(\varphi\mathcal{G}_1, \varphi\mathcal{G}_2)\zeta - \epsilon\eta(\mathcal{G}_2)\varphi^2\mathcal{G}_1, \tag{10}$$

where ∇ denotes the Levi-Civita connection with respect to g.

$\mathfrak{M}$ is the usual para-Sasakian manifold if $\epsilon = 1$ and g is a Riemannian metric [27, 28, 30]. $\mathfrak{M}$ becomes a Lorentzian para-Sasakian manifold [22] if $\epsilon = -1$, g Lorentzian metric and ζ is replaced by $-\zeta$.

In an (ϵ)-para-Sasakian manifold, we have [6, 34]

$$\nabla_{\mathcal{G}_1}\zeta = \epsilon\varphi\mathcal{G}_1, \tag{11}$$

$$(\nabla_{\mathcal{G}_1}\eta)\mathcal{G}_2 = \epsilon g(\mathcal{G}_2, \varphi\mathcal{G}_1), \tag{12}$$

$$\mathfrak{R}(\mathcal{G}_1, \mathcal{G}_2)\zeta = \eta(\mathcal{G}_1)\mathcal{G}_2 - \eta(\mathcal{G}_2)\mathcal{G}_1, \tag{13}$$

$$\mathfrak{R}(\zeta, \mathcal{G}_1)\mathcal{G}_2 = -\epsilon g(\mathcal{G}_1, \mathcal{G}_2)\zeta + \eta(\mathcal{G}_2)\mathcal{G}_1, \tag{14}$$

$$\eta(\mathfrak{R}(\mathcal{G}_1, \mathcal{G}_2)\mathcal{G}_3) = \epsilon[g(\mathcal{G}_1, \mathcal{G}_3)\eta(\mathcal{G}_2) - g(\mathcal{G}_2, \mathcal{G}_3)\eta(\mathcal{G}_1)], \tag{15}$$

$$\mathcal{S}(\mathcal{G}_1, \zeta) = -(n-1)\eta(\mathcal{G}_1), \tag{16}$$

$$\mathcal{S}(\mathcal{G}_1, \mathcal{G}_2) = g(Q\mathcal{G}_1, \mathcal{G}_2), \tag{17}$$

$$Q\zeta = -\epsilon(n-1)\zeta, \tag{18}$$

$\forall\ \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \in \mathfrak{X}(\mathfrak{M})$; where ∇, $\mathfrak{R}$, $\mathcal{S}$ and Q denotes the Levi-Civita connection, curvature tensor, Ricci tensor and Ricci operator respectively on $\mathfrak{M}$.

Let $(g, \mathcal{V}, \Theta)$ be a Ricci soliton in an (ϵ)-para-Sasakian manifold. From (11), we have

$$\begin{aligned}(\mathfrak{L}_\zeta g)(\mathcal{G}_1, \mathcal{G}_2) &= g(\nabla_{\mathcal{G}_1}\zeta, \mathcal{G}_2) + g(\mathcal{G}_1, \nabla_{\mathcal{G}_2}\zeta) \\ &= 2\epsilon g(\varphi\mathcal{G}_1, \mathcal{G}_2).\end{aligned} \tag{19}$$

By virtue of (1) and (19), we have

$$\mathcal{S}(\mathcal{G}_1, \mathcal{G}_2) = -[\epsilon g(\varphi\mathcal{G}_1, \mathcal{G}_2) + \Theta g(\mathcal{G}_1, \mathcal{G}_2)]. \tag{20}$$

By virtue of (20), we have

$$Q\mathcal{G}_1 = -[\epsilon\varphi\mathcal{G}_1 + \Theta\mathcal{G}_1]. \tag{21}$$

On replacing $\mathcal{G}_2$ by ζ in (20) and using (5), (8), we have

$$\mathcal{S}(\mathcal{G}_1, \zeta) = -\frac{\Theta}{\epsilon}\eta(\mathcal{G}_1). \tag{22}$$

Contracting (20), we have

$$\mathfrak{r} = -[\epsilon\psi + n\Theta], \tag{23}$$

where $\mathfrak{r}$ is the scalar curvature and $\psi = trace(\varphi)$.
The conharmonic curvature tensor $\mathcal{H}$ of type $(1,3)$ on a Riemannian manifold $(\mathfrak{M}^n, g)$ is defined by [12]

$$\begin{aligned}\mathcal{H}(\mathcal{G}_1, \mathcal{G}_2)\mathcal{G}_3 &= \mathfrak{R}(\mathcal{G}_1, \mathcal{G}_2)\mathcal{G}_3 - \frac{1}{n-2}[\mathcal{S}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{G}_1 \\ &\quad -\mathcal{S}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{G}_2 + g(\mathcal{G}_2, \mathcal{G}_3)Q\mathcal{G}_1 \\ &\quad -g(\mathcal{G}_1, \mathcal{G}_3)Q\mathcal{G}_2].\end{aligned} \tag{24}$$

Taking inner product with ζ in (24) and using (8), (15), (17), (20), (22), we have

$$\begin{aligned}\eta(\mathcal{H}(\mathcal{G}_1, \mathcal{G}_2)\mathcal{G}_3) &= \left(\epsilon - \frac{2\Theta}{(n-2)}\right)\Big[g(\mathcal{G}_1, \mathcal{G}_3)\eta(\mathcal{G}_2) - g(\mathcal{G}_2, \mathcal{G}_3)\eta(\mathcal{G}_1)\Big] \\ &\quad -\frac{\epsilon}{(n-2)}\Big[g(\varphi\mathcal{G}_1, \mathcal{G}_3)\eta(\mathcal{G}_2) - g(\varphi\mathcal{G}_2, \mathcal{G}_3)\eta(\mathcal{G}_1)\Big].\end{aligned} \tag{25}$$

Taking $\mathcal{G}_1 = \zeta$ in (24) and using (8), (14), (20), (21), (22), we have

$$\begin{aligned}\mathcal{H}(\zeta, \mathcal{G}_2)\mathcal{G}_3 &= \left(1 - \frac{2\Theta}{\epsilon(n-2)}\right)\Big\{\eta(\mathcal{G}_3)\mathcal{G}_2 - \epsilon g(\mathcal{G}_2, \mathcal{G}_3)\zeta\Big\} \\ &\quad +\frac{\epsilon}{(n-2)}\Big\{g(\varphi\mathcal{G}_2, \mathcal{G}_3)\zeta - \frac{1}{\epsilon}\eta(\mathcal{G}_3)\varphi\mathcal{G}_2\Big\}.\end{aligned} \tag{26}$$

Taking $\mathcal{G}_2 = \zeta$ in (24) and using (8), (14), (20), (21), (22), we have

$$\begin{aligned}\mathcal{H}(\mathcal{G}_1, \zeta)\mathcal{G}_3 &= \left(1 - \frac{2\Theta}{\epsilon(n-2)}\right)\Big\{\epsilon g(\mathcal{G}_1, \mathcal{G}_3)\zeta - \eta(\mathcal{G}_3)\mathcal{G}_1\Big\} \\ &\quad -\frac{\epsilon}{(n-2)}\Big\{g(\varphi\mathcal{G}_1, \mathcal{G}_3)\zeta - \frac{1}{\epsilon}\eta(\mathcal{G}_3)\varphi\mathcal{G}_1\Big\}.\end{aligned} \tag{27}$$

Replacing $\mathcal{G}_3$ by ζ in (24) and using (8), (13), (21), (22), we have

$$\begin{aligned}\mathcal{H}(\mathcal{G}_1, \mathcal{G}_2)\zeta &= \left(1 - \frac{2\Theta}{\epsilon(n-2)}\right)\Big\{\eta(\mathcal{G}_1)\mathcal{G}_2 - \eta(\mathcal{G}_2)\mathcal{G}_1\Big\} \\ &\quad -\frac{1}{(n-2)}\Big\{\eta(\mathcal{G}_1)\varphi\mathcal{G}_2 - \eta(\mathcal{G}_2)\varphi\mathcal{G}_1\Big\}.\end{aligned} \tag{28}$$

Pseudo projective curvature tensor $\widetilde{\mathcal{P}}$ is defined by [25]

$$\begin{aligned}\widetilde{\mathcal{P}}(\mathcal{G}_1, \mathcal{G}_2)\mathcal{G}_3 &= a\mathfrak{R}(\mathcal{G}_1, \mathcal{G}_2)\mathcal{G}_3 + b[\mathcal{S}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{G}_1 - \mathcal{S}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{G}_2] \\ &\quad -\frac{\mathfrak{r}}{n}\left(\frac{a}{n-1} + b\right)[g(\mathcal{G}_2, \mathcal{G}_3)\mathcal{G}_1 - g(\mathcal{G}_1, \mathcal{G}_3)\mathcal{G}_2].\end{aligned} \tag{29}$$

Taking $\mathcal{G}_1 = \zeta$ in (29) and using (8), (14), (20), (22), we have

$$\begin{aligned}\widetilde{\mathcal{P}}(\zeta, \mathcal{G}_2)\mathcal{G}_3 &= \left[a + \frac{\Theta b}{\epsilon} + \frac{\mathfrak{r}}{\epsilon n}\left(\frac{a}{n-1} + b\right)\right]\Big\{\eta(\mathcal{G}_3)\mathcal{G}_2 - \epsilon g(\mathcal{G}_2, \mathcal{G}_3)\zeta\Big\} \\ &\quad -\epsilon b g(\varphi\mathcal{G}_2, \mathcal{G}_3)\zeta.\end{aligned} \tag{30}$$

Taking $\mathcal{G}_3 = \zeta$ in (29) and using (8), (13), (22), we have

$$\widetilde{\mathcal{P}}(\mathcal{G}_1, \mathcal{G}_2)\zeta = \left[a + \frac{\Theta b}{\epsilon} + \frac{\mathrm{r}}{\epsilon n}\left(\frac{a}{n-1} + b\right)\right]\left\{\eta(\mathcal{G}_1)\mathcal{G}_2 - \eta(\mathcal{G}_2)\mathcal{G}_1\right\}. \tag{31}$$

Taking inner product with ζ in (29) and using (8), (15), (20), we have

$$\begin{aligned} \eta(\widetilde{\mathcal{P}}(\mathcal{G}_1, \mathcal{G}_2)\mathcal{G}_3) &= \left[a\epsilon + \Theta b + \frac{\mathrm{r}}{n}\left(\frac{a}{n-1} + b\right)\right]\left\{g(\mathcal{G}_1, \mathcal{G}_3)\eta(\mathcal{G}_2) - g(\mathcal{G}_2, \mathcal{G}_3)\eta(\mathcal{G}_1)\right\} \\ &\quad + \epsilon b\left\{g(\varphi\mathcal{G}_1, \mathcal{G}_3)\eta(\mathcal{G}_2) - g(\varphi\mathcal{G}_2, \mathcal{G}_3)\eta(\mathcal{G}_1)\right\}. \end{aligned} \tag{32}$$

3. Ricci soliton in an (ϵ)-para-Sasakian manifold satisfying $\mathfrak{R}(\zeta, \mathcal{G}_1).\mathcal{H} = 0$

Let

$$(\mathfrak{R}(\zeta, \mathcal{G}_1).\mathcal{H})(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U} = 0,$$

then we have [26]

$$\begin{aligned} &\mathfrak{R}(\zeta, \mathcal{G}_1)\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U} - \mathcal{H}(\mathfrak{R}(\zeta, \mathcal{G}_1)\mathcal{G}_2, \mathcal{G}_3)\mathcal{U} \\ &-\mathcal{H}(\mathcal{G}_2, \mathfrak{R}(\zeta, \mathcal{G}_1)\mathcal{G}_3)\mathcal{U} - \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathfrak{R}(\zeta, \mathcal{G}_1)\mathcal{U} = 0. \end{aligned} \tag{33}$$

By virtue of (14), we have

$$\begin{aligned} &\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U})\mathcal{G}_1 - \epsilon g(\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U})\zeta \\ &-\eta(\mathcal{G}_2)\mathcal{H}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{U} + \epsilon g(\mathcal{G}_1, \mathcal{G}_2)\mathcal{H}(\zeta, \mathcal{G}_3)\mathcal{U} \\ &-\eta(\mathcal{G}_3)\mathcal{H}(\mathcal{G}_2, \mathcal{G}_1)\mathcal{U} + \epsilon g(\mathcal{G}_1, \mathcal{G}_3)\mathcal{H}(\mathcal{G}_2, \zeta)\mathcal{U} \\ &-\eta(\mathcal{U})\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{G}_1 + \epsilon g(\mathcal{G}_1, \mathcal{U})\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\zeta = 0. \end{aligned} \tag{34}$$

Taking inner product with ζ in (34) and using (5), (8), we have

$$\begin{aligned} \epsilon g(\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) &= \epsilon\eta(\mathcal{G}_1)\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) - \epsilon\eta(\mathcal{G}_2)\eta(\mathcal{H}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{U}) \\ &\quad + g(\mathcal{G}_1, \mathcal{G}_2)\eta(\mathcal{H}(\zeta, \mathcal{G}_3)\mathcal{U}) - \epsilon\eta(\mathcal{G}_3)\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_1)\mathcal{U}) \\ &\quad + g(\mathcal{G}_1, \mathcal{G}_3)\eta(\mathcal{H}(\mathcal{G}_2, \zeta)\mathcal{U}) - \epsilon\eta(\mathcal{U})\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{G}_1) \\ &\quad + g(\mathcal{G}_1, \mathcal{U})\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\zeta). \end{aligned} \tag{35}$$

By virtue of (25), we have

$$\begin{aligned} \epsilon g(\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) &= \left(\epsilon - \frac{2\Theta}{(n-2)}\right)\left\{g(\mathcal{G}_1, \mathcal{G}_3)g(\mathcal{G}_2, \mathcal{U}) - g(\mathcal{G}_1, \mathcal{G}_2)g(\mathcal{G}_3, \mathcal{U})\right\} \\ &\quad - \frac{\epsilon}{(n-2)}\left\{g(\mathcal{G}_1, \mathcal{G}_3)g(\varphi\mathcal{G}_2, \mathcal{U}) - g(\mathcal{G}_1, \mathcal{G}_2)g(\varphi\mathcal{G}_3, \mathcal{U})\right. \\ &\quad \left. - \epsilon g(\varphi\mathcal{G}_2, \mathcal{G}_1)\eta(\mathcal{G}_3)\eta(\mathcal{U}) + \epsilon g(\varphi\mathcal{G}_3, \mathcal{G}_1)\eta(\mathcal{G}_2)\eta(\mathcal{U})\right\}. \end{aligned} \tag{36}$$

By virtue of (24), we have

$$\begin{aligned} \epsilon' \mathfrak{R}(\mathcal{G}_2, \mathcal{G}_3, \mathcal{U}, \mathcal{G}_1) &= \left(\epsilon - \frac{2\Theta}{(n-2)}\right)\left\{g(\mathcal{G}_1, \mathcal{G}_3)g(\mathcal{G}_2, \mathcal{U}) - g(\mathcal{G}_1, \mathcal{G}_2)g(\mathcal{G}_3, \mathcal{U})\right\} \\ &\quad + \frac{\epsilon}{(n-2)}\left\{\mathcal{S}(\mathcal{G}_3, \mathcal{U})g(\mathcal{G}_1, \mathcal{G}_2) - \mathcal{S}(\mathcal{G}_2, \mathcal{U})g(\mathcal{G}_1, \mathcal{G}_3)\right. \\ &\quad + \mathcal{S}(\mathcal{G}_1, \mathcal{G}_2)g(\mathcal{G}_3, \mathcal{U}) - \mathcal{S}(\mathcal{G}_1, \mathcal{G}_3)g(\mathcal{G}_2, \mathcal{U}) \\ &\quad - g(\mathcal{G}_1, \mathcal{G}_3)g(\varphi\mathcal{G}_2, \mathcal{U}) + g(\mathcal{G}_1, \mathcal{G}_2)g(\varphi\mathcal{G}_3, \mathcal{U}) \end{aligned}$$

$$+\epsilon g(\varphi \mathcal{G}_2, \mathcal{G}_1)\eta(\mathcal{G}_3)\eta(\mathcal{U}) - \epsilon g(\varphi \mathcal{G}_3, \mathcal{G}_1)\eta(\mathcal{G}_2)\eta(\mathcal{U})\Big\}. \tag{37}$$

Contracting (37) with respect to $\mathcal{G}_1$ and $\mathcal{U}$, we have

$$\mathcal{S}(\mathcal{G}_2, \mathcal{G}_3) = 0. \tag{38}$$

By virtue of (20) and (38), we have

$$\epsilon g(\varphi \mathcal{G}_2, \mathcal{G}_3) + \Theta g(\mathcal{G}_2, \mathcal{G}_3) = 0. \tag{39}$$

Putting $\mathcal{G}_2 = \mathcal{G}_3 = \zeta$ in (39) and using (5), (9), we have

$$\Theta = 0. \tag{40}$$

Hence we have the following theorem:

Theorem 3.1. *A Ricci soliton (g, ζ, Θ) in an (ϵ)-para-Sasakian manifold satisfying $\mathfrak{R}(\zeta, \mathcal{G}_1).\mathcal{H} = 0$ is always steady.*

4. Ricci soliton in an (ϵ)-para-Sasakian manifold satisfying $\mathcal{S}(\zeta, \mathcal{G}_1).\mathcal{H} = 0$

Let

$$\mathcal{S}(\zeta, \mathcal{G}_1).\mathcal{H} = 0, \tag{41}$$

then we have [26]

$$\begin{aligned}(\mathcal{S}(\mathcal{G}_1, \zeta).\mathcal{H})(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U} &= ((\mathcal{G}_1 \wedge_{\mathcal{S}} \zeta)\mathcal{H})(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U} \\ &= (\mathcal{G}_1 \wedge_{\mathcal{S}} \zeta)\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U} + \mathcal{H}((\mathcal{G}_1 \wedge_{\mathcal{S}} \zeta)(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) \\ &\quad + \mathcal{H}(\mathcal{G}_2, (\mathcal{G}_1 \wedge_{\mathcal{S}} \zeta)\mathcal{G}_3)\mathcal{U} + \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)(\mathcal{G}_1 \wedge_{\mathcal{S}} \zeta)\mathcal{U},\end{aligned} \tag{42}$$

where the endomorphism $(\mathcal{G}_1 \wedge_{\mathcal{S}} \mathcal{G}_2)$ is defined as

$$(\mathcal{G}_1 \wedge_{\mathcal{S}} \mathcal{G}_2)\mathcal{G}_3 = \mathcal{S}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{G}_1 - \mathcal{S}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{G}_2. \tag{43}$$

Using (43) in (42), we have

$$\begin{aligned}(\mathcal{S}(\mathcal{G}_1, \zeta).\mathcal{H})(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U} &= \mathcal{S}(\zeta, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U})\mathcal{G}_1 - \mathcal{S}(\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U})\zeta \\ &\quad + \mathcal{S}(\zeta, \mathcal{G}_2)\mathcal{H}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{U} - \mathcal{S}(\mathcal{G}_1, \mathcal{G}_2)\mathcal{H}(\zeta, \mathcal{G}_3)\mathcal{U} \\ &\quad + \mathcal{S}(\zeta, \mathcal{G}_3)\mathcal{H}(\mathcal{G}_2, \mathcal{G}_1)\mathcal{U} - \mathcal{S}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{H}(\mathcal{G}_2, \zeta)\mathcal{U} \\ &\quad + \mathcal{S}(\zeta, \mathcal{U})\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{G}_1 - \mathcal{S}(\mathcal{G}_1, \mathcal{U})\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\zeta.\end{aligned} \tag{44}$$

Assuming $(\mathcal{S}(\mathcal{G}_1, \zeta).\mathcal{H})(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U} = 0$, then above equation takes the form

$$\begin{aligned}&\mathcal{S}(\zeta, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U})\mathcal{G}_1 - \mathcal{S}(\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U})\zeta \\ &+ \mathcal{S}(\zeta, \mathcal{G}_2)\mathcal{H}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{U} - \mathcal{S}(\mathcal{G}_1, \mathcal{G}_2)\mathcal{H}(\zeta, \mathcal{G}_3)\mathcal{U} \\ &+ \mathcal{S}(\zeta, \mathcal{G}_3)\mathcal{H}(\mathcal{G}_2, \mathcal{G}_1)\mathcal{U} - \mathcal{S}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{H}(\mathcal{G}_2, \zeta)\mathcal{U} \\ &+ \mathcal{S}(\zeta, \mathcal{U})\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{G}_1 - \mathcal{S}(\mathcal{G}_1, \mathcal{U})\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\zeta = 0.\end{aligned} \tag{45}$$

Taking inner product with ζ in (45), we have

$$\begin{aligned}&\mathcal{S}(\zeta, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U})\eta(\mathcal{G}_1) - \mathcal{S}(\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) \\ &+ \mathcal{S}(\zeta, \mathcal{G}_2)\eta(\mathcal{H}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{U}) - \mathcal{S}(\mathcal{G}_1, \mathcal{G}_2)\eta(\mathcal{H}(\zeta, \mathcal{G}_3)\mathcal{U}) \\ &+ \mathcal{S}(\zeta, \mathcal{G}_3)\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_1)\mathcal{U}) - \mathcal{S}(\mathcal{G}_1, \mathcal{G}_3)\eta(\mathcal{H}(\mathcal{G}_2, \zeta)\mathcal{U}) \\ &+ \mathcal{S}(\zeta, \mathcal{U})\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{G}_1) - \mathcal{S}(\mathcal{G}_1, \mathcal{U})\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\zeta) = 0.\end{aligned} \tag{46}$$

By virtue of equations (20) and (22), we have

$$\begin{aligned}
&\Theta g(\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) + \epsilon g(\varphi\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) \\
&-\epsilon\Theta\eta(\mathcal{G}_1)\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) - \epsilon\Theta\eta(\mathcal{G}_2)\eta(\mathcal{H}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{U}) \\
&+\epsilon g(\varphi\mathcal{G}_1, \mathcal{G}_2)\eta(\mathcal{H}(\zeta, \mathcal{G}_3)\mathcal{U}) + \Theta g(\mathcal{G}_1, \mathcal{G}_2)\eta(\mathcal{H}(\zeta, \mathcal{G}_3)\mathcal{U}) \\
&-\epsilon\Theta\eta(\mathcal{G}_3)\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_1)\mathcal{U}) + \epsilon g(\varphi\mathcal{G}_1, \mathcal{G}_3)\eta(\mathcal{H}(\mathcal{G}_2, \zeta)\mathcal{U}) \\
&+\Theta g(\mathcal{G}_1, \mathcal{G}_3)\eta(\mathcal{H}(\mathcal{G}_2, \zeta)\mathcal{U}) - \epsilon\Theta\eta(\mathcal{U})\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{G}_1) \\
&+\epsilon g(\varphi\mathcal{G}_1, \mathcal{U})\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\zeta) + \Theta g(\mathcal{G}_1, \mathcal{U})\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\zeta) = 0.
\end{aligned} \tag{47}$$

By virtue of (25), above equation takes the form

$$\begin{aligned}
&\Theta g(\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) + \epsilon g(\varphi\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) \\
&-\left(\epsilon - \frac{2\Theta}{(n-2)}\right)\Big\{2\epsilon\Theta g(\mathcal{G}_2, \mathcal{U})\eta(\mathcal{G}_1)\eta(\mathcal{G}_3) \\
&-2\epsilon\Theta g(\mathcal{G}_3, \mathcal{U})\eta(\mathcal{G}_1)\eta(\mathcal{G}_2) - g(\varphi\mathcal{G}_1, \mathcal{G}_2)\eta(\mathcal{U})\eta(\mathcal{G}_3) \\
&+\epsilon g(\varphi\mathcal{G}_1, \mathcal{G}_2)g(\mathcal{G}_3, \mathcal{U}) + \Theta g(\mathcal{G}_1, \mathcal{G}_2)g(\mathcal{G}_3, \mathcal{U}) \\
&-\epsilon g(\varphi\mathcal{G}_1, \mathcal{G}_3)g(\mathcal{G}_2, \mathcal{U}) + g(\varphi\mathcal{G}_1, \mathcal{G}_3)\eta(\mathcal{G}_2)\eta(\mathcal{U}) \\
&-\Theta g(\mathcal{G}_1, \mathcal{G}_3)g(\mathcal{G}_2, \mathcal{U})\Big\} + \frac{\epsilon}{(n-2)}\Big\{2\epsilon\Theta g(\varphi\mathcal{G}_2, \mathcal{U})\eta(\mathcal{G}_1)\eta(\mathcal{G}_3) \\
&-2\epsilon\Theta g(\varphi\mathcal{G}_3, \mathcal{U})\eta(\mathcal{G}_1)\eta(\mathcal{G}_2) + \epsilon g(\varphi\mathcal{G}_1, \mathcal{G}_2)g(\varphi\mathcal{G}_3, \mathcal{U}) \\
&+\Theta g(\mathcal{G}_1, \mathcal{G}_2)g(\varphi\mathcal{G}_3, \mathcal{U}) - \epsilon g(\varphi\mathcal{G}_1, \mathcal{G}_3)g(\varphi\mathcal{G}_2, \mathcal{U}) \\
&-\Theta g(\mathcal{G}_1, \mathcal{G}_3)g(\varphi\mathcal{G}_2, \mathcal{U}) + \epsilon\Theta g(\varphi\mathcal{G}_2, \mathcal{G}_1)\eta(\mathcal{G}_3)\eta(\mathcal{U}) \\
&-\epsilon\Theta g(\varphi\mathcal{G}_3, \mathcal{G}_1)\eta(\mathcal{G}_2)\eta(\mathcal{U})\Big\} = 0.
\end{aligned} \tag{48}$$

Using (24) in (48), we have

$$\begin{aligned}
&\Theta'\mathfrak{R}(\mathcal{U}, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) + \epsilon'\mathfrak{R}(\mathcal{U}, \varphi\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) \\
&-\left(\epsilon - \frac{2\Theta}{(n-2)}\right)\Big\{2\epsilon\Theta g(\mathcal{G}_2, \mathcal{U})\eta(\mathcal{G}_1)\eta(\mathcal{G}_3) \\
&-2\epsilon\Theta g(\mathcal{G}_3, \mathcal{U})\eta(\mathcal{G}_1)\eta(\mathcal{G}_2) - g(\varphi\mathcal{G}_1, \mathcal{G}_2)\eta(\mathcal{U})\eta(\mathcal{G}_3) \\
&+\epsilon g(\varphi\mathcal{G}_1, \mathcal{G}_2)g(\mathcal{G}_3, \mathcal{U}) + \Theta g(\mathcal{G}_1, \mathcal{G}_2)g(\mathcal{G}_3, \mathcal{U}) \\
&-\epsilon g(\varphi\mathcal{G}_1, \mathcal{G}_3)g(\mathcal{G}_2, \mathcal{U}) + g(\varphi\mathcal{G}_1, \mathcal{G}_3)\eta(\mathcal{G}_2)\eta(\mathcal{U}) \\
&-\Theta g(\mathcal{G}_1, \mathcal{G}_3)g(\mathcal{G}_2, \mathcal{U})\Big\} - \frac{1}{(n-2)}\Big\{\Theta\mathcal{S}(\mathcal{G}_3, \mathcal{U})g(\mathcal{G}_1, \mathcal{G}_2) \\
&-\Theta\mathcal{S}(\mathcal{G}_2, \mathcal{U})g(\mathcal{G}_1, \mathcal{G}_3) + \Theta\mathcal{S}(\mathcal{G}_1, \mathcal{G}_2)g(\mathcal{G}_3, \mathcal{U}) \\
&-\Theta(\mathcal{G}_1, \mathcal{G}_3)g(\mathcal{G}_2, \mathcal{U}) + \epsilon\mathcal{S}(\mathcal{G}_3, \mathcal{U})g(\varphi\mathcal{G}_1, \mathcal{G}_2) \\
&-\epsilon\mathcal{S}(\mathcal{G}_2, \mathcal{U})g(\varphi\mathcal{G}_1, \mathcal{G}_3) + \epsilon\mathcal{S}(\mathcal{G}_2, \varphi\mathcal{G}_1)g(\mathcal{G}_3, \mathcal{U}) \\
&-\epsilon\mathcal{S}(\mathcal{G}_3, \varphi\mathcal{G}_1)g(\mathcal{G}_2, \mathcal{U}) - 2\epsilon^2\Theta g(\mathcal{G}_2, \varphi\mathcal{U})\eta(\mathcal{G}_1)\eta(\mathcal{G}_3) \\
&+2\epsilon^2\Theta g(\mathcal{G}_3, \varphi\mathcal{U})\eta(\mathcal{G}_1)\eta(\mathcal{G}_2) - \epsilon^2 g(\varphi\mathcal{G}_1, \mathcal{G}_2)g(\mathcal{G}_3, \varphi\mathcal{U}) \\
&-\epsilon\Theta g(\mathcal{G}_1, \mathcal{G}_2)g(\mathcal{G}_3, \varphi\mathcal{U}) + \epsilon^2 g(\varphi\mathcal{G}_1, \mathcal{G}_3)g(\mathcal{G}_2, \varphi\mathcal{U}) \\
&+\epsilon\Theta g(\mathcal{G}_1, \mathcal{G}_3)g(\mathcal{G}_2, \varphi\mathcal{U}) - \epsilon^2\Theta g(\mathcal{G}_2, \varphi\mathcal{G}_1)\eta(\mathcal{G}_3)\eta(\mathcal{U}) \\
&+\epsilon^2\Theta g(\mathcal{G}_3, \varphi\mathcal{G}_1)\eta(\mathcal{G}_2)\eta(\mathcal{U})\Big\} = 0.
\end{aligned} \tag{49}$$

Contracting (49) with respect to $\mathcal{G}_2$ and $\mathcal{G}_3$, we have

$$\Theta\mathcal{S}(\mathcal{U}, \mathcal{G}_1) + \epsilon\mathcal{S}(\mathcal{U}, \varphi\mathcal{G}_1) = 0. \tag{50}$$

Using (20) in (50) and putting $\mathcal{U} = \mathcal{G}_1 = \zeta$, we have

$$\Theta = 0. \tag{51}$$

Hence we have the following theorem:

Theorem 4.1. *A Ricci soliton (g, ζ, Θ) in an (ϵ)-para-Sasakian manifold satisfying $\mathcal{S}(\zeta, \mathcal{G}_1).\mathcal{H} = 0$ is always steady.*

5. Ricci soliton in an (ϵ)-para-Sasakian manifold satisfying $\widetilde{\mathcal{P}}(\zeta, \mathcal{G}_1).\mathcal{H} = 0$

Let

$$\widetilde{\mathcal{P}}(\zeta, \mathcal{G}_1).\mathcal{H} = 0, \tag{52}$$

where $\widetilde{\mathcal{P}}$ is pseudo-projective curvature tensor.

$$(\widetilde{\mathcal{P}}(\zeta, \mathcal{G}_1).\mathcal{H})(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U} = 0. \tag{53}$$

It follows that [26]

$$\begin{aligned}&\widetilde{\mathcal{P}}(\zeta, \mathcal{G}_1)\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U} - \mathcal{H}(\widetilde{\mathcal{P}}(\zeta, \mathcal{G}_1)\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}\\ &-\mathcal{H}(\mathcal{G}_2, \widetilde{\mathcal{P}}(\zeta, \mathcal{G}_1)\mathcal{G}_3)\mathcal{U} - \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\widetilde{\mathcal{P}}(\zeta, \mathcal{G}_1)\mathcal{U} = 0.\end{aligned} \tag{54}$$

Using (30) in (54), we have

$$\begin{aligned}&\mathcal{K}_1[\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U})\mathcal{G}_1 - \epsilon g(\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U})\zeta\\ &-\eta(\mathcal{G}_2)\mathcal{H}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{U} + \epsilon g(\mathcal{G}_1, \mathcal{G}_2)\mathcal{H}(\zeta, \mathcal{G}_3)\mathcal{U}\\ &-\eta(\mathcal{G}_3)\mathcal{H}(\mathcal{G}_2, \mathcal{G}_1)\mathcal{U} + \epsilon g(\mathcal{G}_1, \mathcal{G}_3)\mathcal{H}(\mathcal{G}_2, \zeta)\mathcal{U}\\ &-\eta(\mathcal{U})\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{G}_1 + \epsilon g(\mathcal{G}_1, \mathcal{U})\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\zeta]\\ &-b[\epsilon g(\varphi\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U})\zeta - \epsilon g(\varphi\mathcal{G}_1, \mathcal{G}_2)\mathcal{H}(\zeta, \mathcal{G}_3)\mathcal{U}\\ &-\epsilon g(\varphi\mathcal{G}_1, \mathcal{G}_3)\mathcal{H}(\mathcal{G}_2, \zeta)\mathcal{U} - \epsilon g(\varphi\mathcal{G}_1, \mathcal{U})\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\zeta] = 0,\end{aligned} \tag{55}$$

where $\mathcal{K}_1 = \left[a + \frac{\Theta b}{\epsilon} + \frac{r}{\epsilon n}\left(\frac{a}{n-1} + b\right)\right]$.
Taking inner product with ζ in (55) and using (8), (9), we have

$$\begin{aligned}&\mathcal{K}_1\Big[\frac{1}{\epsilon}\eta(\mathcal{G}_1)\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) - \epsilon^2 g(\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U})\\ &-\frac{1}{\epsilon}\eta(\mathcal{G}_2)\eta(\mathcal{H}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{U}) + g(\mathcal{G}_1, \mathcal{G}_2)\eta(\mathcal{H}(\zeta, \mathcal{G}_3)\mathcal{U})\\ &-\frac{1}{\epsilon}\eta(\mathcal{G}_3)\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_1)\mathcal{U}) + g(\mathcal{G}_1, \mathcal{G}_3)\eta(\mathcal{H}(\mathcal{G}_2, \zeta)\mathcal{U})\\ &-\frac{1}{\epsilon}\eta(\mathcal{U})\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{G}_1) + g(\mathcal{G}_1, \mathcal{U})\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\zeta)\Big]\\ &-b[\epsilon^2 g(\varphi\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) - g(\varphi\mathcal{G}_1, \mathcal{G}_2)\eta(\mathcal{H}(\zeta, \mathcal{G}_3)\mathcal{U})\\ &-g(\varphi\mathcal{G}_1, \mathcal{G}_3)\eta(\mathcal{H}(\mathcal{G}_2, \zeta)\mathcal{U}) - g(\varphi\mathcal{G}_1, \mathcal{U})\eta(\mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\zeta)] = 0.\end{aligned} \tag{56}$$

Using (25) in above equation, we have

$$\begin{aligned}\epsilon^2\mathcal{K}_1 g(\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) \quad = \quad &-b\epsilon^2 g(\varphi\mathcal{G}_1, \mathcal{H}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) + \mathcal{K}_1\Big[\Big(\epsilon - \frac{2\Theta}{n-2}\Big)\\ &\Big\{g(\mathcal{G}_1, \mathcal{G}_3)g(\mathcal{G}_2, \mathcal{U}) - g(\mathcal{G}_1, \mathcal{G}_2)g(\mathcal{G}_3, \mathcal{U})\Big\}\end{aligned}$$

$$-\frac{\epsilon}{n-2}\Big\{g(\mathcal{G}_1,\mathcal{G}_3)g(\varphi\mathcal{G}_2,\mathcal{U})-g(\mathcal{G}_1,\mathcal{G}_2)g(\varphi\mathcal{G}_3,\mathcal{U})$$
$$-\frac{1}{\epsilon}g(\varphi\mathcal{G}_2,\mathcal{G}_1)\eta(\mathcal{G}_3)\eta(\mathcal{U})+\frac{1}{\epsilon}g(\varphi\mathcal{G}_3,\mathcal{G}_1)\eta(\mathcal{G}_2)\eta(\mathcal{U})\Big\}\Big]$$
$$-b\Big[\Big(\epsilon-\frac{2\Theta}{n-2}\Big)\Big\{-\frac{1}{\epsilon}g(\varphi\mathcal{G}_1,\mathcal{G}_2)\eta(\mathcal{G}_3)\eta(\mathcal{U})$$
$$+g(\mathcal{G}_3,\mathcal{U})g(\varphi\mathcal{G}_1,\mathcal{G}_2)-g(\mathcal{G}_2,\mathcal{U})g(\varphi\mathcal{G}_1,\mathcal{G}_3)$$
$$+\frac{1}{\epsilon}g(\varphi\mathcal{G}_1,\mathcal{G}_3)\eta(\mathcal{G}_2)\eta(\mathcal{U})\Big\}+\frac{\epsilon}{n-2}\Big\{g(\varphi\mathcal{G}_1,\mathcal{G}_3)g(\varphi\mathcal{G}_2,\mathcal{U})$$
$$-g(\varphi\mathcal{G}_1,\mathcal{G}_2)g(\varphi\mathcal{G}_3,\mathcal{U})\Big\}\Big]. \tag{57}$$

Using (24) in (57), we have

$$\begin{aligned}\epsilon^2\mathcal{K}_1 g(\mathfrak{R}(\mathcal{U},\mathcal{G}_1)\mathcal{G}_2,\mathcal{G}_3) &= -b\epsilon^2 g(\mathfrak{R}(\mathcal{U},\varphi\mathcal{G}_1)\mathcal{G}_2,\mathcal{G}_3)\\
&+\mathcal{K}_1\Big[\Big(\epsilon-\frac{2\Theta}{n-2}\Big)\Big\{g(\mathcal{G}_1,\mathcal{G}_3)g(\mathcal{G}_2,\mathcal{U})\\
&-g(\mathcal{G}_1,\mathcal{G}_2)g(\mathcal{G}_3,\mathcal{U})\Big\}+\frac{\epsilon}{n-2}\Big\{\epsilon\mathcal{S}(\mathcal{G}_3,\mathcal{U})g(\mathcal{G}_1,\mathcal{G}_2)\\
&-\epsilon\mathcal{S}(\mathcal{G}_2,\mathcal{U})g(\mathcal{G}_1,\mathcal{G}_3)+\epsilon\mathcal{S}(\mathcal{G}_2,\mathcal{G}_1)g(\mathcal{G}_3,\mathcal{U})\\
&-\epsilon\mathcal{S}(\mathcal{G}_3,\mathcal{G}_1)g(\mathcal{G}_2,\mathcal{U})-g(\mathcal{G}_1,\mathcal{G}_3)g(\varphi\mathcal{G}_2,\mathcal{U})\\
&+g(\mathcal{G}_1,\mathcal{G}_2)g(\varphi\mathcal{G}_3,\mathcal{U})+\frac{1}{\epsilon}g(\varphi\mathcal{G}_2,\mathcal{G}_1)\eta(\mathcal{G}_3)\eta(\mathcal{U})\\
&-\frac{1}{\epsilon}g(\varphi\mathcal{G}_3,\mathcal{G}_1)\eta(\mathcal{G}_2)\eta(\mathcal{U})\Big\}\Big]\\
&-b\Big[\Big(\epsilon-\frac{2\Theta}{n-2}\Big)\Big\{-\frac{1}{\epsilon}g(\varphi\mathcal{G}_1,\mathcal{G}_2)\eta(\mathcal{G}_3)\eta(\mathcal{U})\\
&+g(\mathcal{G}_3,\mathcal{U})g(\varphi\mathcal{G}_1,\mathcal{G}_2)-g(\mathcal{G}_2,\mathcal{U})g(\varphi\mathcal{G}_1,\mathcal{G}_3)\\
&+\frac{1}{\epsilon}g(\varphi\mathcal{G}_1,\mathcal{G}_3)\eta(\mathcal{G}_2)\eta(\mathcal{U})\Big\}\\
&+\frac{\epsilon}{n-2}\Big\{g(\varphi\mathcal{G}_1,\mathcal{G}_3)g(\varphi\mathcal{G}_2,\mathcal{U})\\
&-g(\varphi\mathcal{G}_1,\mathcal{G}_2)g(\varphi\mathcal{G}_3,\mathcal{U})-\epsilon\mathcal{S}(\mathcal{G}_3,\mathcal{U})g(\varphi\mathcal{G}_1,\mathcal{G}_2)\\
&+\epsilon\mathcal{S}(\mathcal{G}_2,\mathcal{U})g(\varphi\mathcal{G}_1,\mathcal{G}_3)-\epsilon\mathcal{S}(\mathcal{G}_2,\varphi\mathcal{G}_1)g(\mathcal{G}_3,\mathcal{U})\\
&+\epsilon\mathcal{S}(\mathcal{G}_3,\varphi\mathcal{G}_1)g(\mathcal{G}_2,\mathcal{U})\Big\}\Big].\end{aligned} \tag{58}$$

Contracting (58) with respect to $\mathcal{G}_2$ and $\mathcal{G}_3$, we have

$$\mathcal{K}_1\mathcal{S}(\mathcal{U},\mathcal{G}_1)+b\mathcal{S}(\mathcal{U},\varphi\mathcal{G}_1)=0. \tag{59}$$

Using (20) in (59) and putting $\mathcal{U}=\mathcal{G}_1=\zeta$, we have

$$\Theta\mathcal{K}_1=0. \tag{60}$$

where

$$\mathcal{K}_1=\Big[a+\frac{\Theta b}{\epsilon}+\frac{\mathfrak{r}}{\epsilon n}\Big(\frac{a}{n-1}+b\Big)\Big]$$

Hence we have the following theorems:

Theorem 5.1. *A Ricci soliton (g,ζ,Θ) in an (ϵ)-para-Sasakian manifold satisfying $\widetilde{\mathcal{P}}(\zeta,\mathcal{G}_1).\mathcal{H}=0$ is either steady or expanding.*

Theorem 5.2. *A Ricci soliton* (g, ζ, Θ) *in an* (ϵ)*-para-Sasakian manifold satisfying* $\widetilde{\mathcal{P}}(\zeta, \mathcal{G}_1).\mathcal{H} = 0$ *is either steady or shrinking.*

6. Ricci soliton in an (ϵ)-para-Sasakian manifold satisfying $\mathcal{H}(\zeta, \mathcal{G}_1).\widetilde{\mathcal{P}} = 0$

The condition

$$\mathcal{H}(\zeta, \mathcal{G}_1).\widetilde{\mathcal{P}} = 0, \tag{61}$$

implies that

$$(\mathcal{H}(\zeta, \mathcal{G}_1).\widetilde{\mathcal{P}})(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U} = 0. \tag{62}$$

It follows that [26]

$$\begin{aligned} &\mathcal{H}(\zeta, \mathcal{G}_1)\widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U} - \widetilde{\mathcal{P}}(\mathcal{H}(\zeta, \mathcal{G}_1)\mathcal{G}_2, \mathcal{G}_3)\mathcal{U} \\ &-\widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{H}(\zeta, \mathcal{G}_1)\mathcal{G}_3)\mathcal{U} - \widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{H}(\zeta, \mathcal{G}_1)\mathcal{U} = 0. \end{aligned} \tag{63}$$

Using equation (26) in (63), we have

$$\begin{aligned} &\mathcal{K}_2\Big\{\eta(\widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U})\mathcal{G}_1 - \epsilon g(\mathcal{G}_1, \widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U})\zeta \\ &-\eta(\mathcal{G}_2)\widetilde{\mathcal{P}}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{U} + \epsilon g(\mathcal{G}_1, \mathcal{G}_2)\widetilde{\mathcal{P}}(\zeta, \mathcal{G}_3)\mathcal{U} \\ &-\eta(\mathcal{G}_3)\widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_1)\mathcal{U} + \epsilon g(\mathcal{G}_1, \mathcal{G}_3)\widetilde{\mathcal{P}}(\mathcal{G}_2, \zeta)\mathcal{U} \\ &-\eta(\mathcal{U})\widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{G}_1 + \epsilon g(\mathcal{G}_1, \mathcal{U})\widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\zeta\Big\} \\ &+\frac{\epsilon}{(n-2)}\Big\{g(\varphi\mathcal{G}_1, \widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U})\zeta - g(\varphi\mathcal{G}_1, \mathcal{G}_2)\widetilde{\mathcal{P}}(\zeta, \mathcal{G}_3)\mathcal{U} \\ &-g(\varphi\mathcal{G}_1, \mathcal{G}_3)\widetilde{\mathcal{P}}(\mathcal{G}_2, \zeta)\mathcal{U} - g(\varphi\mathcal{G}_1, \mathcal{U})\widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\zeta \\ &+\frac{1}{\epsilon}\eta(\mathcal{G}_2)\widetilde{\mathcal{P}}(\varphi\mathcal{G}_1, \mathcal{G}_3)\mathcal{U} + \frac{1}{\epsilon}\eta(\mathcal{G}_3)\widetilde{\mathcal{P}}(\mathcal{G}_2, \varphi\mathcal{G}_1)\mathcal{U} \\ &+\frac{1}{\epsilon}\eta(\mathcal{U})\widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\varphi\mathcal{G}_1\Big\} = 0, \end{aligned} \tag{64}$$

where $\mathcal{K}_2 = \left(1 - \frac{2\Theta}{\epsilon(n-2)}\right)$.

Taking inner product with ζ in (64), we have

$$\begin{aligned} &\mathcal{K}_2\Big\{\frac{1}{\epsilon}\eta(\mathcal{G}_1)\eta(\widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) - \epsilon^2 g(\mathcal{G}_1, \widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) \\ &-\frac{1}{\epsilon}\eta(\mathcal{G}_2)\eta(\widetilde{\mathcal{P}}(\mathcal{G}_1, \mathcal{G}_3)\mathcal{U}) + g(\mathcal{G}_1, \mathcal{G}_2)\eta(\widetilde{\mathcal{P}}(\zeta, \mathcal{G}_3)\mathcal{U}) \\ &-\frac{1}{\epsilon}\eta(\mathcal{G}_3)\eta(\widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_1)\mathcal{U}) + g(\mathcal{G}_1, \mathcal{G}_3)\eta(\widetilde{\mathcal{P}}(\mathcal{G}_2, \zeta)\mathcal{U}) \\ &-\frac{1}{\epsilon}\eta(\mathcal{U})\eta(\widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{G}_1) + g(\mathcal{G}_1, \mathcal{U})\eta(\widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\zeta)\Big\} \\ &+\frac{\epsilon}{(n-2)}\Big\{\epsilon g(\varphi\mathcal{G}_1, \widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\mathcal{U}) - \frac{1}{\epsilon}g(\varphi\mathcal{G}_1, \mathcal{G}_2)\eta(\widetilde{\mathcal{P}}(\zeta, \mathcal{G}_3)\mathcal{U}) \\ &-\frac{1}{\epsilon}g(\varphi\mathcal{G}_1, \mathcal{G}_3)\eta(\widetilde{\mathcal{P}}(\mathcal{G}_2, \zeta)\mathcal{U}) - \frac{1}{\epsilon}g(\varphi\mathcal{G}_1, \mathcal{U})\eta(\widetilde{\mathcal{P}}(\mathcal{G}_2, \mathcal{G}_3)\zeta) \\ &+\frac{1}{\epsilon^2}\eta(\mathcal{G}_2)\eta(\widetilde{\mathcal{P}}(\varphi\mathcal{G}_1, \mathcal{G}_3)\mathcal{U}) + \frac{1}{\epsilon^2}\eta(\mathcal{G}_3)\eta(\widetilde{\mathcal{P}}(\mathcal{G}_2, \varphi\mathcal{G}_1)\mathcal{U}) \end{aligned}$$

$$+\frac{1}{\epsilon^2}\eta(\mathcal{U})\eta(\widetilde{\mathcal{P}}(\mathcal{G}_2,\mathcal{G}_3)\varphi\mathcal{G}_1)\Big\} = 0. \tag{65}$$

Using (32) in (65), we have

$$\begin{aligned}\mathcal{K}_2\epsilon^2 g(\mathcal{G}_1,\widetilde{\mathcal{P}}(\mathcal{G}_2,\mathcal{G}_3)\mathcal{U}) &= \frac{\epsilon^2}{(n-2)}g(\varphi\mathcal{G}_1,\widetilde{\mathcal{P}}(\mathcal{G}_2,\mathcal{G}_3)\mathcal{U}) + \mathcal{K}_2\Big[\mathcal{K}_3\Big\{g(\mathcal{G}_1,\mathcal{G}_3)g(\mathcal{G}_2,\mathcal{U})\\ &-g(\mathcal{G}_1,\mathcal{G}_2)g(\mathcal{G}_3,\mathcal{U})\Big\} + b\Big\{\epsilon g(\mathcal{G}_1,\mathcal{G}_3)g(\varphi\mathcal{G}_2,\mathcal{U})\\ &-\epsilon g(\mathcal{G}_1,\mathcal{G}_2)g(\varphi\mathcal{G}_3,\mathcal{U}) - g(\varphi\mathcal{G}_2,\mathcal{G}_1)\eta(\mathcal{G}_3)\eta(\mathcal{U})\\ &+g(\varphi\mathcal{G}_3,\mathcal{G}_1)\eta(\mathcal{G}_2)\eta(\mathcal{U})\Big\}\Big] + \frac{\mathcal{K}_3}{(n-2)}\Big\{g(\mathcal{G}_3,\mathcal{U})g(\varphi\mathcal{G}_1,\mathcal{G}_2)\\ &-g(\mathcal{G}_2,\mathcal{U})g(\varphi\mathcal{G}_1,\mathcal{G}_3)\Big\} + \frac{b}{(n-2)}\Big\{\epsilon g(\varphi\mathcal{G}_1,\mathcal{G}_2)g(\varphi\mathcal{G}_3,\mathcal{U})\\ &-\epsilon g(\varphi\mathcal{G}_1,\mathcal{G}_3)g(\varphi\mathcal{G}_2,\mathcal{U}) + g(\varphi\mathcal{G}_1,\varphi\mathcal{G}_2)\eta(\mathcal{G}_3)\eta(\mathcal{U})\\ &-g(\varphi\mathcal{G}_1,\varphi\mathcal{G}_3)\eta(\mathcal{G}_2)\eta(\mathcal{U})\Big\},\end{aligned} \tag{66}$$

where $\mathcal{K}_3 = \Big[a\epsilon + \Theta b + \frac{\mathrm{r}}{n}\Big(\frac{a}{n-1} + b\Big)\Big]$.
Using (29) in (66), we have

$$\begin{aligned}\mathcal{K}_2 a\epsilon^2 g(\mathfrak{R}(\mathcal{U},\mathcal{G}_1)\mathcal{G}_2,\mathcal{G}_3) &= \frac{a}{(n-2)}\epsilon^2 g(\mathfrak{R}(\mathcal{U},\varphi\mathcal{G}_1)\mathcal{G}_2,\mathcal{G}_3)\\ &+\mathcal{K}_2\Big[\mathcal{K}_3\Big\{g(\mathcal{G}_1,\mathcal{G}_3)g(\mathcal{G}_2,\mathcal{U}) - g(\mathcal{G}_1,\mathcal{G}_2)g(\mathcal{G}_3,\mathcal{U})\Big\}\\ &+b\Big\{\epsilon^2\mathcal{S}(\mathcal{G}_2,\mathcal{U})g(\mathcal{G}_1,\mathcal{G}_3) - \epsilon^2\mathcal{S}(\mathcal{G}_3,\mathcal{U})g(\mathcal{G}_1,\mathcal{G}_2)\\ &+\epsilon g(\mathcal{G}_1,\mathcal{G}_3)g(\varphi\mathcal{G}_2,\mathcal{U}) - \epsilon g(\mathcal{G}_1,\mathcal{G}_2)g(\varphi\mathcal{G}_3,\mathcal{U})\\ &-g(\mathcal{G}_1,\varphi\mathcal{G}_2)\eta(\mathcal{G}_3)\eta(\mathcal{U}) + g(\varphi\mathcal{G}_3,\mathcal{G}_1)\eta(\mathcal{G}_2)\eta(\mathcal{U})\Big\}\\ &+\frac{\mathrm{r}}{n}\Big[\frac{a}{n-1} + b\Big]\Big\{\epsilon^2 g(\mathcal{G}_1,\mathcal{G}_2)g(\mathcal{G}_3,\mathcal{U})\\ &-\epsilon^2 g(\mathcal{G}_1,\mathcal{G}_3)g(\mathcal{G}_2,\mathcal{U})\Big\}\Big] + \frac{1}{(n-2)}\Big[\mathcal{K}_3\Big\{g(\mathcal{G}_3,\mathcal{U}).\\ &g(\varphi\mathcal{G}_1,\mathcal{G}_2) - g(\mathcal{G}_2,\mathcal{U})g(\varphi\mathcal{G}_1,\mathcal{G}_3)\Big\}\\ &+b\Big\{\epsilon^2\mathcal{S}(\mathcal{G}_3,\mathcal{U})g(\varphi\mathcal{G}_1,\mathcal{G}_2) - \epsilon^2\mathcal{S}(\mathcal{G}_2,\mathcal{U})g(\varphi\mathcal{G}_1,\mathcal{G}_3)\\ &+\epsilon g(\varphi\mathcal{G}_1,\mathcal{G}_2)g(\varphi\mathcal{G}_3,\mathcal{U}) - \epsilon g(\varphi\mathcal{G}_1,\mathcal{G}_3)g(\varphi\mathcal{G}_2,\mathcal{U})\\ &+g(\varphi\mathcal{G}_1,\varphi\mathcal{G}_2)\eta(\mathcal{G}_3)\eta(\mathcal{U}) - g(\varphi\mathcal{G}_1,\varphi\mathcal{G}_3)\eta(\mathcal{G}_2)\eta(\mathcal{U})\Big\}\\ &-\frac{\mathrm{r}}{n}\Big[\frac{a}{n-1} + b\Big]\Big\{\epsilon^2 g(\mathcal{G}_3,\mathcal{U})g(\varphi\mathcal{G}_1,\mathcal{G}_2)\\ &-\epsilon^2 g(\mathcal{G}_2,\mathcal{U})g(\varphi\mathcal{G}_1,\mathcal{G}_3)\Big\}\Big].\end{aligned} \tag{67}$$

Contracting (67) with respect to $\mathcal{G}_2$ and $\mathcal{G}_3$, we have

$$\mathcal{K}_2\mathcal{S}(\mathcal{U},\mathcal{G}_1) - \frac{1}{(n-2)}\mathcal{S}(\mathcal{U},\varphi\mathcal{G}_1) = 0. \tag{68}$$

Using (20) in (68) and putting $\mathcal{U} = \mathcal{G}_1 = \zeta$, we have

$$\Theta\mathcal{K}_2 = 0, \tag{69}$$

where

$$\mathcal{K}_2 = \left[1 - \frac{2\Theta}{\epsilon(n-2)}\right].$$

Hence we have the following theorems:

Theorem 6.1. *A Ricci soliton (g, ζ, Θ) in an (ϵ)-para-Sasakian manifold satisfying $\mathcal{H}(\zeta, \mathcal{G}_1).\widetilde{\mathcal{P}} = 0$ is either steady or expanding.*

Theorem 6.2. *A Ricci soliton (g, ζ, Θ) in an (ϵ)-para-Sasakian manifold satisfying $\mathcal{H}(\zeta, \mathcal{G}_1).\widetilde{\mathcal{P}} = 0$ is either steady or shrinking.*

7. Example of an (ϵ)-para-Sasakian manifold

Let us consider the 5-dimensional manifold $\mathfrak{M}=\left\{(t_1, t_2, t_3, t_4, t_5) \in \mathbb{R}^5 : t_5 > 0\right\}$, where $(t_1, t_2, t_3, t_4, t_5)$ are the standard coordinates in $\mathbb{R}^5$. Let $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4$ and ς_5 be the vector fields on $\mathfrak{M}$ [10] given by

$$\varsigma_1 = \epsilon e^{-t_5}\frac{\partial}{\partial t_1}, \quad \varsigma_2 = \epsilon e^{-t_5}\frac{\partial}{\partial t_2}, \quad \varsigma_3 = \epsilon e^{-t_5}\frac{\partial}{\partial t_3}, \quad \varsigma_4 = \epsilon e^{-t_5}\frac{\partial}{\partial t_4}, \quad \varsigma_5 = \epsilon\frac{\partial}{\partial t_5} = \zeta.$$

These vectors are linearly independent at each point of $\mathfrak{M}$. Let g be the indefinite Riemannian metric defined by

$$g(\varsigma_i, \varsigma_j) = 0, \forall \quad i \neq j, \quad i, j = 1, 2, 3, 4, 5$$

and

$$g(\varsigma_1, \varsigma_1) = g(\varsigma_2, \varsigma_2) = g(\varsigma_3, \varsigma_3) = g(\varsigma_4, \varsigma_4) = 1, \quad g(\varsigma_5, \varsigma_5) = \epsilon. \tag{70}$$

Let η be the 1-form defined by $\eta(\mathcal{G}_1) = \epsilon g(\mathcal{G}_1, \varsigma_5)$ for any $\mathcal{G}_1 \in \mathfrak{X}(\mathfrak{M})$. Let φ be the $(1, 1)$ tensor field on $\mathfrak{M}$ defined by

$$\varphi\varsigma_1 = \varsigma_1, \quad \varphi\varsigma_2 = \varsigma_2, \quad \varphi\varsigma_3 = \varsigma_3, \quad \varphi\varsigma_4 = \varsigma_4, \quad \varphi\varsigma_5 = 0. \tag{71}$$

Now for $\mathcal{G}_1 = \mathcal{G}_1^1\varsigma_1 + \mathcal{G}_1^2\varsigma_2 + \mathcal{G}_1^3\varsigma_3 + \mathcal{G}_1^4\varsigma_4 + \mathcal{G}_1^5\varsigma_5$, using linearity of φ and g, we have

$$\begin{aligned} &\eta(\varsigma_5) = \eta(\zeta) = 1, \quad \varphi^2(\mathcal{G}_1) = \mathcal{G}_1 - \eta(\mathcal{G}_1)\varsigma_5 = (\mathcal{G}_1^1\varsigma_1 + \mathcal{G}_1^2\varsigma_2 + \mathcal{G}_1^3\varsigma_3 + \mathcal{G}_1^4\varsigma_4), \\ &g(\varphi\mathcal{G}_1, \varphi\mathcal{G}_2) = g(\mathcal{G}_1, \mathcal{G}_2) - \epsilon\eta(\mathcal{G}_1)\eta(\mathcal{G}_2), \end{aligned} \tag{72}$$

where $\mathcal{G}_1^1, \mathcal{G}_1^2, \mathcal{G}_1^3, \mathcal{G}_1^4$ and $\mathcal{G}_1^5$ are scalars, $\forall\ \mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{X}(\mathfrak{M})$. Hence for $\varsigma_5 = \zeta$, the quartet $(\varphi, \zeta, \eta, g, \epsilon)$ defines an indefinite almost paracontact metric quartet on $\mathfrak{M}$. Then we have

$$\begin{aligned} &[\varsigma_1, \varsigma_2] = [\varsigma_1, \varsigma_3] = [\varsigma_1, \varsigma_4] = [\varsigma_2, \varsigma_3] = [\varsigma_2, \varsigma_4] = [\varsigma_3, \varsigma_4] = 0, \\ &[\varsigma_1, \varsigma_5] = \epsilon\varsigma_1, [\varsigma_2, \varsigma_5] = \epsilon\varsigma_2, [\varsigma_3, \varsigma_5] = \epsilon\varsigma_3, [\varsigma_4, \varsigma_5] = \epsilon\varsigma_4. \end{aligned} \tag{73}$$

Now using the Koszul's formula for Levi-Civita connection ∇ with respect to g, i.e.,

$$\begin{aligned} 2g(\nabla_{\mathcal{G}_1}\mathcal{G}_2, \mathcal{G}_3) = \ &\mathcal{G}_1 g(\mathcal{G}_2, \mathcal{G}_3) + \mathcal{G}_2 g(\mathcal{G}_1, \mathcal{G}_3) - \mathcal{G}_3 g(\mathcal{G}_1, \mathcal{G}_2) \\ &- g(\mathcal{G}_1, [\mathcal{G}_2, \mathcal{G}_3]) - g(\mathcal{G}_2, [\mathcal{G}_1, \mathcal{G}_3]) + g(\mathcal{G}_3, [\mathcal{G}_1, \mathcal{G}_2]). \end{aligned} \tag{74}$$

For arbitrary vector fields $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \in \mathfrak{X}(\mathfrak{M})$.
By virtue of (74), we have

$$\begin{array}{lllll} \nabla_{\varsigma_1}\varsigma_1 = -\varsigma_5, & \nabla_{\varsigma_1}\varsigma_2 = 0, & \nabla_{\varsigma_1}\varsigma_3 = 0, & \nabla_{\varsigma_1}\varsigma_4 = 0, & \nabla_{\varsigma_1}\varsigma_5 = \epsilon\varsigma_1, \\ \nabla_{\varsigma_2}\varsigma_1 = 0, & \nabla_{\varsigma_2}\varsigma_2 = -\varsigma_5, & \nabla_{\varsigma_2}\varsigma_3 = 0, & \nabla_{\varsigma_2}\varsigma_4 = 0, & \nabla_{\varsigma_2}\varsigma_5 = \epsilon\varsigma_2, \\ \nabla_{\varsigma_3}\varsigma_1 = 0, & \nabla_{\varsigma_3}\varsigma_2 = 0, & \nabla_{\varsigma_3}\varsigma_3 = -\varsigma_5, & \nabla_{\varsigma_3}\varsigma_4 = 0, & \nabla_{\varsigma_3}\varsigma_5 = \epsilon\varsigma_3, \\ \nabla_{\varsigma_4}\varsigma_1 = 0, & \nabla_{\varsigma_4}\varsigma_2 = 0, & \nabla_{\varsigma_4}\varsigma_3 = 0, & \nabla_{\varsigma_4}\varsigma_4 = -\varsigma_5, & \nabla_{\varsigma_4}\varsigma_5 = \epsilon\varsigma_4, \\ \nabla_{\varsigma_5}\varsigma_1 = 0, & \nabla_{\varsigma_5}\varsigma_2 = 0, & \nabla_{\varsigma_5}\varsigma_3 = 0, & \nabla_{\varsigma_5}\varsigma_4 = 0, & \nabla_{\varsigma_5}\varsigma_5 = 0. \end{array} \tag{75}$$

It follows that $\nabla_{\mathcal{G}_1}\xi = \epsilon\varphi\mathcal{G}_1$. Hence the manifold is an (ϵ)-para-Sasakian manifold. The Riemannian curvature tensor $\mathfrak{R}$ is given as

$$\mathfrak{R}(\mathcal{G}_1, \mathcal{G}_2)\mathcal{G}_3 = \nabla_{\mathcal{G}_1}\nabla_{\mathcal{G}_2}\mathcal{G}_3 - \nabla_{\mathcal{G}_2}\nabla_{\mathcal{G}_1}\mathcal{G}_3 - \nabla_{[\mathcal{G}_1,\mathcal{G}_2]}\mathcal{G}_3. \tag{76}$$

By virtue of (76) we obtain the non-vanishing components of the curvature tensor as follows:

$$\begin{array}{c} \mathfrak{R}(\varsigma_1,\varsigma_2)\varsigma_2 = \mathfrak{R}(\varsigma_1,\varsigma_3)\varsigma_3 = \mathfrak{R}(\varsigma_1,\varsigma_4)\varsigma_4 = -\epsilon\varsigma_1, \quad \mathfrak{R}(\varsigma_1,\varsigma_5)\varsigma_5 = -\varsigma_1, \\ \mathfrak{R}(\varsigma_1,\varsigma_2)\varsigma_1 = \mathfrak{R}(\varsigma_3,\varsigma_2)\varsigma_3 = \mathfrak{R}(\varsigma_4,\varsigma_2)\varsigma_4 = \epsilon\varsigma_2, \quad \mathfrak{R}(\varsigma_2,\varsigma_5)\varsigma_5 = -\varsigma_2, \\ \mathfrak{R}(\varsigma_1,\varsigma_3)\varsigma_1 = \mathfrak{R}(\varsigma_4,\varsigma_3)\varsigma_4 = \mathfrak{R}(\varsigma_2,\varsigma_3)\varsigma_2 = \epsilon\varsigma_3, \quad \mathfrak{R}(\varsigma_3,\varsigma_5)\varsigma_5 = -\varsigma_3, \\ \mathfrak{R}(\varsigma_1,\varsigma_5)\varsigma_1 = \mathfrak{R}(\varsigma_2,\varsigma_5)\varsigma_2 = \mathfrak{R}(\varsigma_3,\varsigma_5)\varsigma_3 = \mathfrak{R}(\varsigma_4,\varsigma_5)\varsigma_4 = \epsilon\varsigma_5, \\ \mathfrak{R}(\varsigma_1,\varsigma_4)\varsigma_1 = \mathfrak{R}(\varsigma_2,\varsigma_4)\varsigma_2 = \mathfrak{R}(\varsigma_3,\varsigma_4)\varsigma_3 = \epsilon\varsigma_4, \quad \mathfrak{R}(\varsigma_5,\varsigma_4)\varsigma_5 = \varsigma_4. \end{array} \tag{77}$$

Along with $\mathfrak{R}(\varsigma_i,\varsigma_i)\varsigma_i = 0; \forall\, i = 1,2,3,4,5$.
In view of the above results we obtain the non-vanishing components of the Ricci tensor as follows:

$$\mathcal{S}(\varsigma_j,\varsigma_k) = \sum_{i=1}^{5} g(\mathfrak{R}(\varsigma_i,\varsigma_j)\varsigma_k,\varsigma_i).$$

It follows that

$$\mathcal{S}(\varsigma_1,\varsigma_1) = \mathcal{S}(\varsigma_2,\varsigma_2) = \mathcal{S}(\varsigma_3,\varsigma_3) = \mathcal{S}(\varsigma_4,\varsigma_4) = -4\epsilon, \quad \mathcal{S}(\varsigma_5,\varsigma_5) = -4. \tag{78}$$

Along with $\mathcal{S}(\varsigma_j,\varsigma_k) = 0, \forall\, j,k = 1,2,3,4,5\ (j \neq k)$. From the above equation (78), we can verify the equation (16).
By virtue of (20), (71) and (78), we have

$$\begin{aligned} \mathcal{S}(\varsigma_5,\varsigma_5) &= -\epsilon g(\varphi\varsigma_5,\varsigma_5) - \Theta g(\varsigma_5,\varsigma_5) \\ &= -\epsilon g(0,\varsigma_5) - \Theta\epsilon \\ &= -\Theta\epsilon. \end{aligned} \tag{79}$$

On equating both the values of $\mathcal{S}(\varsigma_5,\varsigma_5)$, we have

$$\begin{aligned} -\Theta\epsilon &= -4 \\ \Theta &= \frac{4}{\epsilon}. \end{aligned} \tag{80}$$

It follows that the Ricci soliton (g, ζ, Θ) is either expanding or shrinking according as $\epsilon = 1$ or -1, respectively. Hence from given example, it is clear that theorems 5.1, 5.2, 6.1 and 6.2 are satisfied.

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